GVG Exam 01
Solution

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Task 1. *Consider the cameras with the camera projection matrices*

$$\mathtt{P} = \begin{bmatrix} a & 0 & 0 & a \\ 0 & 2 & 0 & 1 \\ a & 0 & a & 1 \end{bmatrix}$$

Find all values of the parameter $a \in \mathbb{R}$ such that P *projects point*

$$\vec{X} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

from the space onto the line $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ in the image.

Solution: The equation of projection is

$$\zeta \vec{x}_\beta = \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}, \quad \zeta \neq 0.$$

The point $\vec{x}_\beta$ belongs to the line $\vec{l}_{\overline{\beta}} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$ if and only if $\vec{l}_{\overline{\beta}}^\top \vec{x}_\beta = 0$, i.e.

$$0 = \vec{l}_{\overline{\beta}}^\top \vec{x}_\beta = \vec{l}_{\overline{\beta}}^\top \zeta \vec{x}_\beta = \vec{l}_{\overline{\beta}}^\top \mathtt{P} \begin{bmatrix} \vec{X}_\delta \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 & a \\ 0 & 2 & 0 & 1 \\ a & 0 & a & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2a \\ 1 \\ 2a+1 \end{bmatrix} = 0$$

$$4a + 2 = 0 \Rightarrow a = -\frac{1}{2}.$$

If we substitute $a = -\frac{1}{2}$ to P, then we see that the point X is projected to the ideal point $\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^\top$ of the line $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}^\top$. □

Task 2. *Consider a camera with $\omega = \mathtt{K}^{-\top}\mathtt{K}^{-1}$ matrix as follows*

$$\omega = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & o_1 \\ 1 & o_1 & 3 \end{bmatrix}$$

with parameter $o_1 \in \mathbb{R}$. Find the values of the parameter such that rays through image points

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

would contain the right angle.

Solution: The constraints on o_1 are

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 1\\ 0 & 1 & o_1\\ 1 & o_1 & 3\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = 0$$

$$\begin{bmatrix}1 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ o_1+1\\ o_1+3\end{bmatrix} = 0$$

$$o_1 + 4 = 0 \Rightarrow o_1 = -4.$$

□

Task 3. *Consider lines $\vec{p} = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^\top$, $\vec{q} = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ and points $\vec{X} = \begin{bmatrix}1 & 0 & 1\end{bmatrix}^\top$, $\vec{Y} = \begin{bmatrix}0 & 1 & 1\end{bmatrix}^\top$ in the real projective plane and their images $\vec{p}' = \begin{bmatrix}1 & 1 & 0\end{bmatrix}^\top$, $\vec{q}' = \begin{bmatrix}0 & 0 & 1\end{bmatrix}^\top$ and $\vec{X}' = \begin{bmatrix}0 & 1 & 1\end{bmatrix}^\top$, $\vec{Y}' = \begin{bmatrix}1 & 0 & 1\end{bmatrix}^\top$ by an unknown homography. Find all matrices $\mathtt{H}$ of all homographies that can map the points and the lines as above.*

Solution: We have the following equations

$$\mathtt{H}^{-\top}\vec{p} = \lambda_1\vec{p}', \quad \mathtt{H}^{-\top}\vec{q} = \lambda_2\vec{q}' \quad \mathtt{H}\vec{X} = \lambda_3\vec{X}' \quad \mathtt{H}\vec{Y} = \lambda_4\vec{Y}', \qquad \lambda_i \neq 0$$

$$\mathtt{H}^\top\vec{p}' = \frac{1}{\lambda_1}\vec{p}, \quad \mathtt{H}^\top\vec{q}' = \frac{1}{\lambda_2}\vec{q}, \quad \mathtt{H}\vec{X} = \lambda_3\vec{X}' \quad \mathtt{H}\vec{Y} = \lambda_4\vec{Y}'$$

If we denote

$$\mathtt{H} = \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix},$$

then we obtain

$$\begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{\lambda_1}\begin{bmatrix}1\\1\\0\end{bmatrix}, \quad \begin{bmatrix}a & d & g\\ b & e & h\\ c & f & i\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix} = \frac{1}{\lambda_2}\begin{bmatrix}0\\0\\1\end{bmatrix}, \quad \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}1\\0\\1\end{bmatrix} = \lambda_3\begin{bmatrix}0\\1\\1\end{bmatrix}, \quad \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix} = \lambda_4\begin{bmatrix}1\\0\\1\end{bmatrix}$$

or,

$$a + d = b + e \neq 0, \;\; c + f = 0,$$
$$g = h = 0, \;\; i \neq 0,$$
$$a + c = 0, \;\; d + f = g + i \neq 0,$$
$$b + c = h + i \neq 0, \;\; e + f = 0.$$

Simplifying it (by replacing g and h with 0) we get

$$a + d = b + e \neq 0, \;\; c + f = 0,$$
$$i \neq 0,$$
$$a + c = 0, \;\; d + f = i,$$
$$b + c = i, \;\; e + f = 0.$$

Expressing a, b, c, d, e via f and i we get

$$a = f, \quad b = f + i, \quad c = -f, \quad d = -f + i, \quad e = -f, \quad i \neq 0.$$

Thus, all possible homographies take the form

$$\left\{ \begin{bmatrix}f & f+i & -f\\ -f+i & -f & f\\ 0 & 0 & i\end{bmatrix} \;\middle|\; f, i \in \mathbb{R}, i \neq 0 \right\}.$$

□

Task 4. *Assume line $\vec{p} = [1, 1, -1]^\top$ and point $\vec{X} = [1, 1, 1]^\top$ in a real projective plane. Find a basis vector $\vec{q}$ of a one-dimensional subspace of $\mathbb{R}^3$ which represents the line q that is parallel to line $\vec{p}$ and passes through point $\vec{X}$.*

Solution: Let $\vec{q} = \begin{bmatrix} a & b & c \end{bmatrix}^\top$. The constraint that p is parallel to q can be expressed as that their intersection is a point at infinity, i.e.

$$\vec{p} \times \vec{q} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} b+c \\ -a-c \\ b-a \end{bmatrix} = \begin{bmatrix} . \\ . \\ 0 \end{bmatrix} \Rightarrow \vec{q} = \begin{bmatrix} a \\ a \\ c \end{bmatrix}.$$

The constraint that q passes through X can be expressed as follows:

$$\vec{p}^\top \vec{X} = \begin{bmatrix} a & a & c \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 2a + c = 0 \Rightarrow \vec{q} = \begin{bmatrix} a \\ a \\ -2a \end{bmatrix}.$$

Thus, one of the possible homogeneous representatives of q is, for example,

$$\vec{q} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

□

Task 5. *Find all fundamental matrices which are compatible with images in two calibrated cameras ($\mathtt{K} = \mathtt{I}$) that translate in the space such that point $\begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ in the first image is in the correspondence with point $\begin{bmatrix} 0 & 0 \end{bmatrix}^\top$ in the second image.*

Solution: By the task, we know that the cameras are calibrated ($\mathtt{K}_1 = \mathtt{K}_2 = \mathtt{I}$) and that they make only translational movement, i.e. $\mathtt{R}_1 = \mathtt{R}_2 = \mathtt{R}$. The fundamental matrix then takes the form

$$\mathtt{F} = \mathtt{R} \left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times \mathtt{R}^\top.$$

We may rewrite the above formula as follows:

$$\mathtt{F} = \mathtt{R} \left[\vec{C}_{2\delta} - \vec{C}_{1\delta} \right]_\times \mathtt{R}^\top = \underbrace{\mathtt{R}\mathtt{R}^\top}_{\mathtt{I}} \left[\mathtt{R}(\vec{C}_{2\delta} - \vec{C}_{1\delta}) \right]_\times = \left[\mathtt{R}(\vec{C}_{2\delta} - \vec{C}_{1\delta}) \right]_\times \tag{1}$$

We claim that all fundamental matrices of the form (1) can be parametrized as follows:

$$\left\{ \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix} \;\middle|\; \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \setminus \{\vec{0}\} \right\}.$$

Obviously, every matrix of the form (1) belongs to the above set, since $\mathtt{F}$ in (1) is described by the cross product matrix. Conversely, any matrix from the above set is realizable as the fundamental matrix from (1): we just take $\mathtt{R} = \mathtt{I}$, $\vec{C}_{1\delta} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$ and $\vec{C}_{2\delta} = \begin{bmatrix} x & y & z \end{bmatrix}^\top$. The condition $\begin{bmatrix} x & y & z \end{bmatrix}^\top \neq \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$ is necessary to have rank $\mathtt{F} = 2$. □

Task 6. *Assume two cameras. The first camera has the camera projection matrix*

$$\mathtt{P}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

Epipole in the second camera equals

$$\vec{e}_{2\beta_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Find all projection matrices $\mathtt{P}_2$.

Solution: We first find the center of the first camera:

$$\vec{C}_{1\delta} = -\mathtt{P}_{:,1:3}^{-1}\,\mathtt{P}_{:,4} = -\begin{bmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}0\\ 0\\ -1\end{bmatrix} = -\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0\\ 0\\ -1\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}.$$

By definition, the epipole in the second camera is the image of the center of the first camera in the second camera:

$$\zeta\vec{e}_{2\beta_2} = \mathtt{P}_2\begin{bmatrix}\vec{C}_{1\delta}\\ 1\end{bmatrix}, \quad \zeta \neq 0$$

$$\zeta\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = \mathtt{P}_2\begin{bmatrix}0\\ 0\\ 1\\ 1\end{bmatrix}, \quad \zeta \neq 0.$$

If we denote

$$\mathtt{P}_2 = \begin{bmatrix}a & b & c & d\\ e & f & g & h\\ i & j & k & l\end{bmatrix}$$

then the constraints on elements of $\mathtt{P}_2$ are the following:

$$\zeta = c + d, \quad 0 = g + h, \quad \zeta = k + l, \quad \zeta \neq 0,$$

or

$$d = \zeta - c, \quad h = -g, \quad l = \zeta - k, \quad \zeta \neq 0.$$

Thus, all the matrices $\mathtt{P}_2$ have the form

$$\mathtt{P}_2 = \begin{bmatrix}a & b & c & \zeta - c\\ e & f & g & -g\\ i & j & k & \zeta - k\end{bmatrix}, \quad \zeta \neq 0,$$

where the left 3×3 block must be an invertible matrix. □