Classifiers: Naïve Bayes, evaluation

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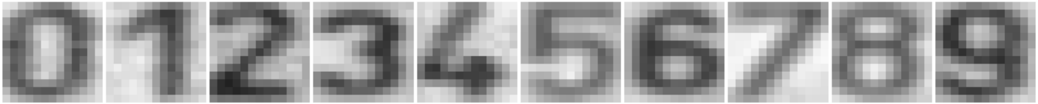
Example: Digit recognition/classification



- **Input:** 8-bit image 13 × 13, pixel intensities $0 - 255$. (0 means black, 255 means white)
- **Output:** Digit $0 - 9$. Decision about the class, classification.
- **Features:** Pixel intensities ...

Classification as a special case of statistical decision theory

- Attribute vector $\vec{x} = [x_1, x_2, \ldots]^\top$: pixels 1, 2, ...
- **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \ldots 9\}$.**
- State = actual class, Decision = recognized class
- Loss function:

$$l(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{l(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

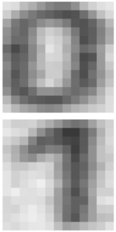
$$\delta^*(\vec{x}) = \arg\min_d \left(1 - P(d|\vec{x})\right) = \arg\max_d P(d|\vec{x})$$

Bayes classification in practice; $P(s|\vec{x}) = ?$

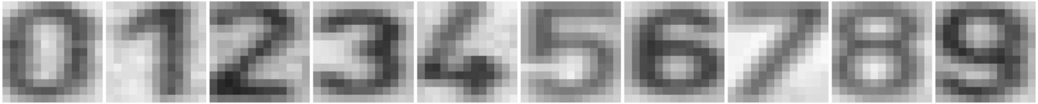
- Usually, we are not given $P(s|\vec{x})$
- It has to be estimated from already classified examples – training data
- For discrete $\vec{x}$, training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \ldots (\vec{x}_l, s_l)$
  - every $(\vec{x}_i, s)$ is drawn independently from $P(\vec{x}, s)$, i.e. sample i does not depend on $1, \cdots, i-1$
  - so-called i.i.d (independent, identically distributed) multiset
- Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) = \frac{P(\vec{x}, s)}{P(\vec{x})} \approx \frac{\#\text{ examples where } \vec{x}_i = \vec{x} \textbf{ and } s_i = s}{\#\text{ examples where } \vec{x}_i = \vec{x}}$$

- Hard in practice:
  - To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of $\vec{x}$.
    - e.g. with the number of pixels in images
    - curse of dimensionality
    - denominator often 0



How many images?



8-bit image 13 × 13, pixel intensities $0 - 255$. (0 means black, 255 means white)

A: 169^{256}

B: 256^{169}

C: 13^{13}

D: 169×256

E: different quantity

Naïve Bayes classification

- For efficient classification we must thus rely on additional assumptions.
- In the exceptional case of statistical independence between components of $\vec{x}$ for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

- Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots =$$

- No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s.
- No need to estimate $P(\vec{x})$. (Why?)
- $P(s)$ may be provided apriori.
- naïve = when used despite statistical dependence

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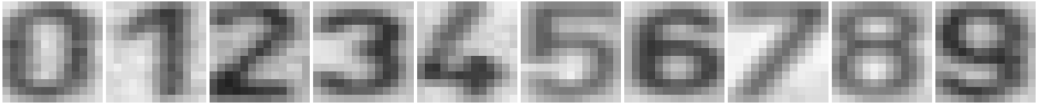

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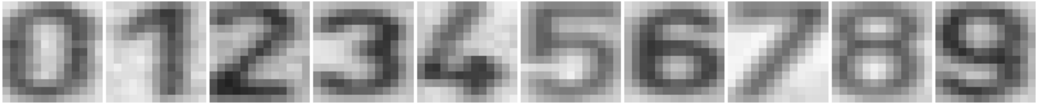
Collect data , . . .

- $P(\vec{x})$. What is the dimension of $\vec{x}$? How many possible images?
- Learn $P(\vec{x}|s)$ per each class (digit).
- Classify $s^* = \operatorname{argmax}_s P(s|\vec{x})$.

Example: Digit recognition/classification



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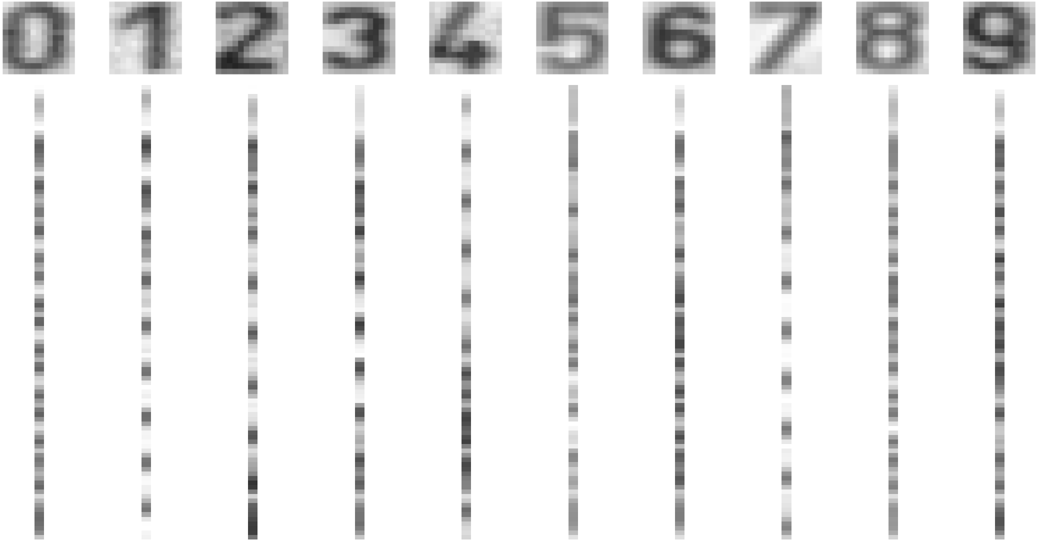
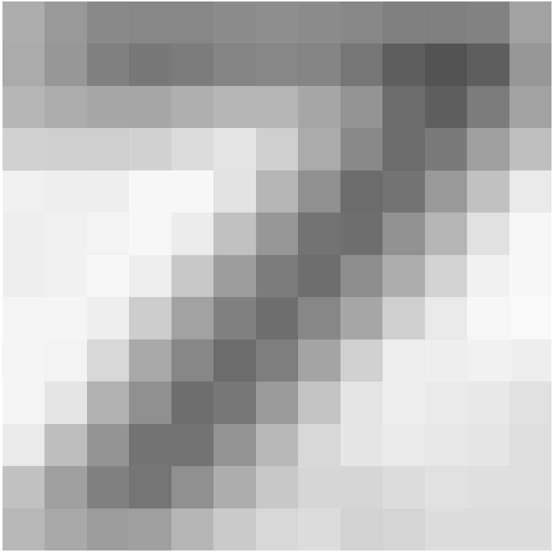
From images to $\vec{x}$



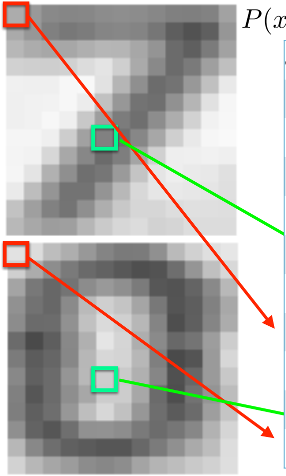
Conditional probabilities, likelihoods



- Apriori digit probabilities $P(s_k)$
- Likelihoods for pixels. $P(x_{r,c} = l_i | s_k)$

Conditional likelihoods

s_k	$P(s_k)$
1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
0	0.1

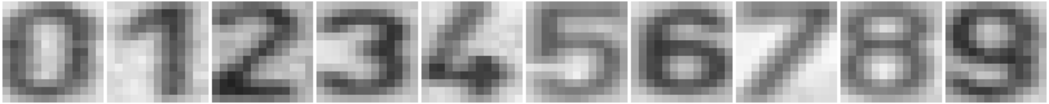


$P(x_{0,0} = 25|s_k)$ $P(x_{7,5} = 25|s_k)$

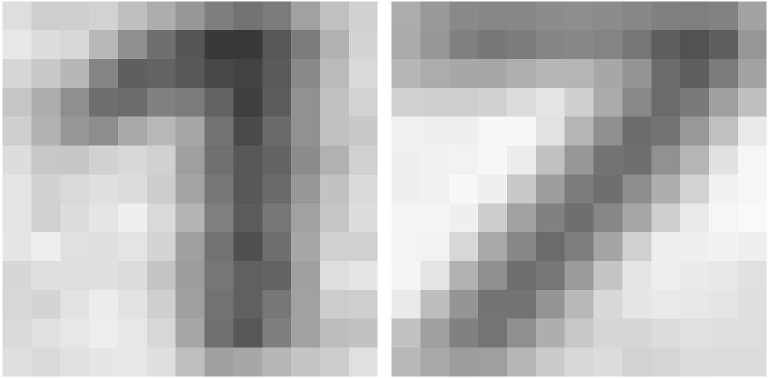
s_k	$P(x_{0,0} = 25 \mid s_k)$
1	0.0009
2	0.05
3	0.00006
4	0.0005
5	0.07
6	0.0007
7	0.1
8	0.0006
9	0
0	0

s_k	$P(x_{7,5} = 25 \mid s_k)$
1	0.0009
2	0.05
3	0.00006
4	0.0005
5	0.004
6	0.07
7	0.1
8	0.006
9	0.08
0	0.005

Unseen events



Images 13 × 13, intensities $0 - 255$, 100 exemplars per each class.



Unseen event, how to decide?

A new (not in training) query image with $x_{0,0} = 101$. How would you classify?

$$P(x_{0,0} = 101 \mid s_j) = 0, \text{ for all classes}$$

Laplace smoothing ("additive smoothing")

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem: $\text{count}(x) = 0$

Pretend you see the (any) sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{N + |X|}$$

where N is the number of (total) observations; $|X|$ is the number of possible values X can take (cardinality).

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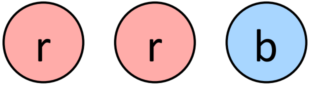
$P_{\text{LAP}}(x) = ?$

Observation:



What is $P_{\text{LAP}}(X = \text{red})$ and $P_{\text{LAP}}(X = \text{blue})$?

A: $P_{\text{LAP}}(X = \text{red}) = 7/10$, $P_{\text{LAP}}(X = \text{blue}) = 3/10$

B: $P_{\text{LAP}}(X = \text{red}) = 2/3$, $P_{\text{LAP}}(X = \text{blue}) = 1/3$

C: $P_{\text{LAP}}(X = \text{red}) = 3/5$, $P_{\text{LAP}}(X = \text{blue}) = 2/5$

D: None of the above.

Laplace smoothing - as a hyperparameter k

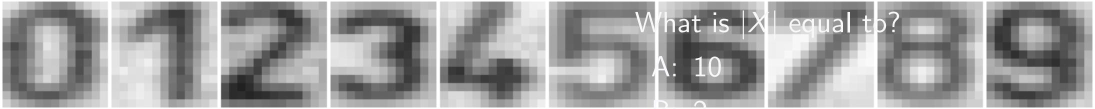
Pretend you see every sample k extra times:

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{\sum_x [c(x) + k]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{N + k|X|}$$

For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x, s) + k}{c(s) + k|X|}$$



What is $|X|$ equal to?

A: 10

B: 2

C: 256

D: None of the above

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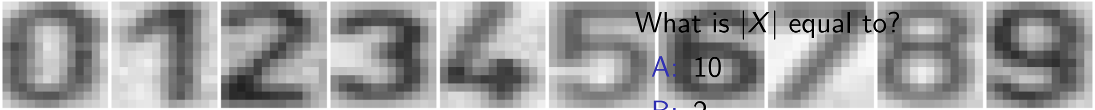

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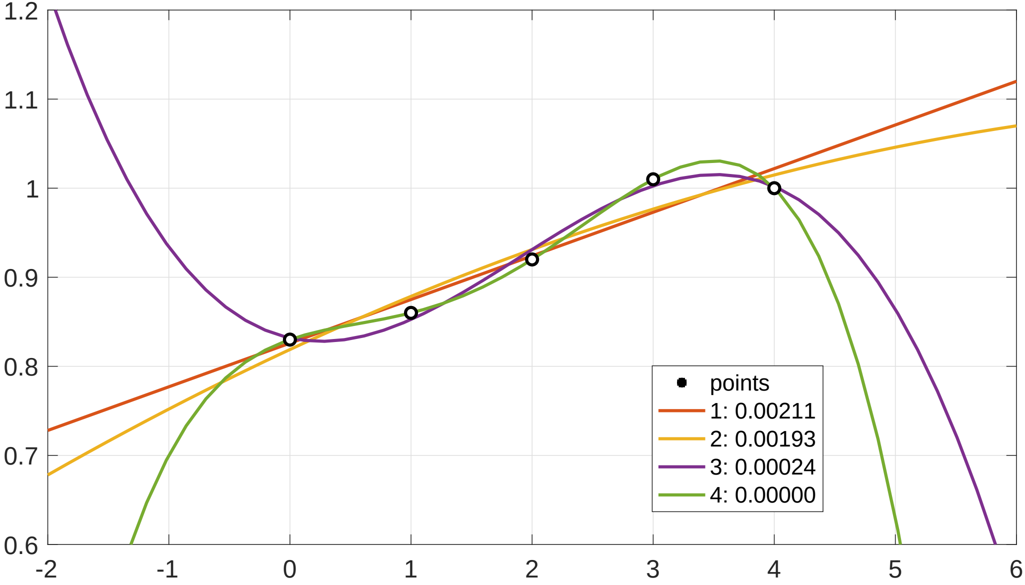
What is the right degree of polynomial (hyperparameter of a regressor)



Generalization and overfiting

- Data: training, validating, testing . Wanted classifier performs well on what data?
- Overfitting: too close to training, poor on testing.

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Training and testing

Data labeled instances.

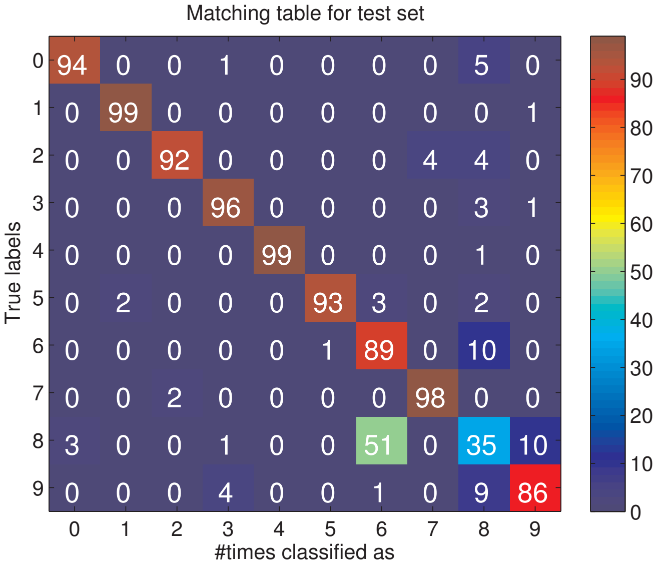
- Training set
- Held-out (validation) set
- Testing set.

Features : Attribute-value pairs.

Learning cycle:

- **Learn** parameters (e.g. probabilities) on training set.
- **Tune** hyperparameters on held-out (validation) set.
- **Evaluate** performance on testing set.

How to evaluate a classifier? Confusion table



	0	1	2	3	4	5	6	7	8	9
0	94	0	0	1	0	0	0	0	5	0
1	0	99	0	0	0	0	0	0	0	1
2	0	0	92	0	0	0	0	4	4	0
3	0	0	0	96	0	0	0	0	3	1
4	0	0	0	0	99	0	0	0	1	0
5	0	2	0	0	0	93	3	0	2	0
6	0	0	0	0	0	1	89	0	10	0
7	0	0	2	0	0	0	0	98	0	0
8	3	0	0	1	0	0	51	0	35	10
9	0	0	0	4	0	0	1	0	9	86

Figure from [5]

Precision and Recall, and ...

Consider digit **detection** (is there a digit?) or SPAM/HAM classification.

Recall :

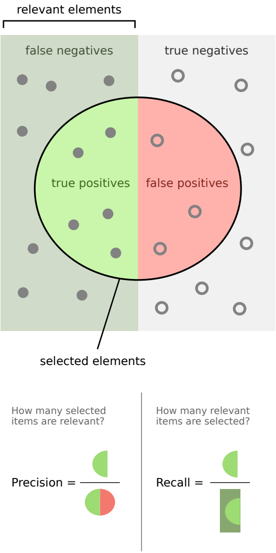
- How many relevant items are selected?
- Are we missing some items?
- Also called: **True positive rate** (TPR), sensitivity, hit rate ...

Precision

- How many selected items are relevant?
- Also called: Positive predictive value

False positive rate (FPR)

- Probability of false alarm



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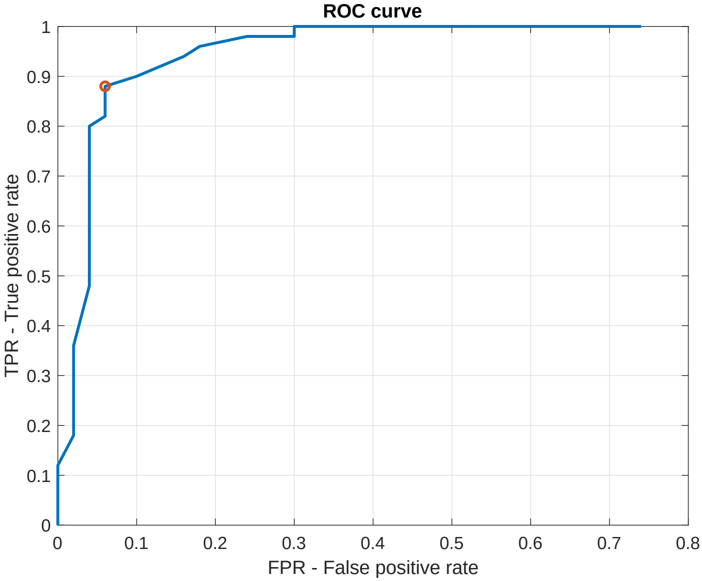
ROC – Receiver operating characteristics curve



$$\mathrm{TPR} = \frac{\mathrm{TP}}{P} = \frac{\mathrm{TP}}{\mathrm{TP} + \mathrm{FN}}$$

$$\mathrm{FPR} = \frac{\mathrm{FP}}{N} = \frac{\mathrm{FP}}{\mathrm{FP} + \mathrm{TN}}$$

Discriminant functions $f(\vec{x}, s)$

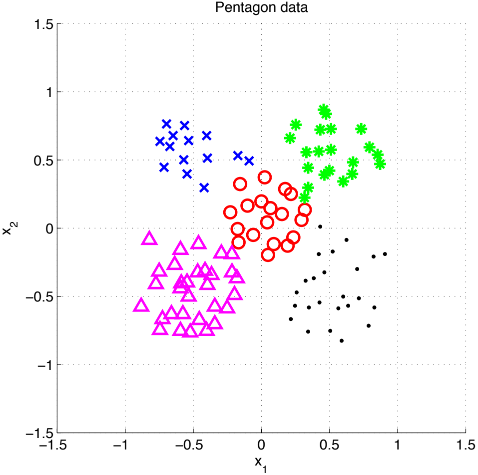
$$s^* = \operatorname*{argmax}_{s \in \mathcal{S}} f(\vec{x}, s)$$

Conditional likelihoods: $\mathcal{N}(\vec{x}|\vec{\mu}_s, \Sigma_s)$

$$\frac{1}{2\pi|\Sigma_s|^{1/2}} \exp\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\}$$

Bayes:

$$s^* = \operatorname*{argmax}_{s \in \mathcal{S}} P(s|\vec{x}) = \frac{P(\vec{x} \mid s)P(s)}{P(\vec{x})}$$



Discriminant function:

$$s^* = \operatorname*{argmax}_{s \in \mathcal{S}} f(\vec{x}, s) = P(s)\frac{1}{2\pi|\Sigma_s|^{1/2}} \exp\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\}$$

Towards linear classifier, geometrical thoughts ...

$$f(\vec{x}, s) = P(s)\frac{1}{2\pi|\Sigma_s|^{1/2}}\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu}_s)^\top\Sigma_s^{-1}(\vec{x}-\vec{\mu}_s)\}$$

Product of many small numbers . . .

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})}P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

$P(\vec{x})$ not needed, ...

$$\log(P(x[1]|s)P(x[2]|s)\cdots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \cdots$$

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References I

Further reading: Chapter 13 and 14 of [4]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. This lecture has been also inspired by the 21st lecture of CS 188 at http://ai.berkeley.edu (e.g., Laplace smoothing). Many Matlab figures created with the help of [3].

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