Sequential decisions under uncertainty
Policy iteration

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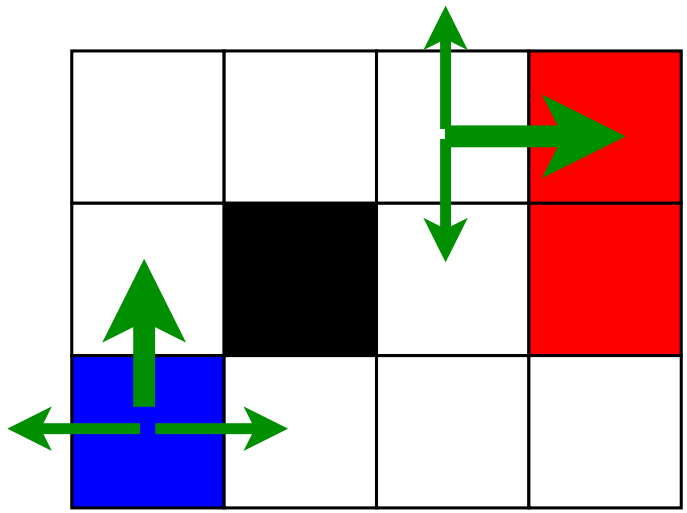
Unreliable actions in observable grid world

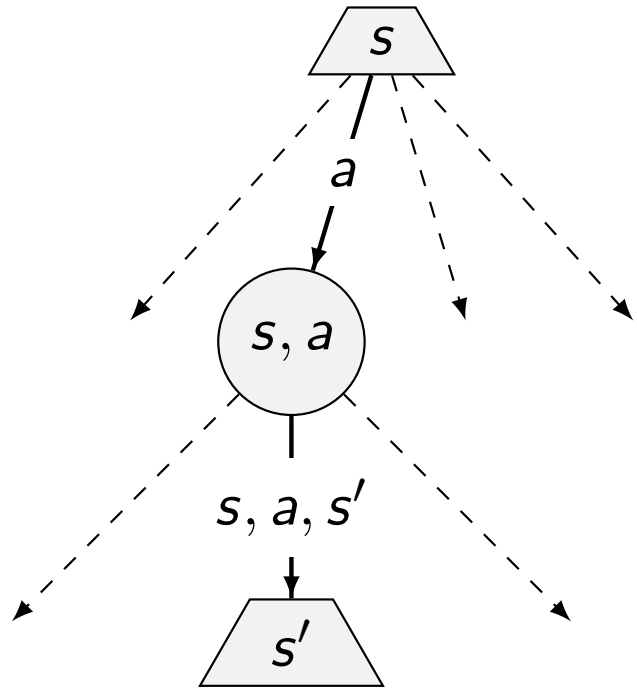
- Walls block movement – agent/robot stays in place.
- Actions do not always go as planned.
- Agent receives rewards each time step:
  - Small "living" reward/penalty.
  - Big rewards/penalties at the end.
- Goal: maximize sum of (discounted rewards)



MDPs recap

Markov decision processes (MDPs):

- Set of states S
- Set of actions A
- Transitions $P(s'|s,a)$ or $T(s,a,s')$
- Rewards $R(s)$; and discount γ



MDP quantities:

- Policy $\pi(s) : S \rightarrow A$
- Utility – sum of (discounted) rewards.
- Values – expected future utility from a state (max-node)
- Q-Values – expected future utility from a q-state (chance-node)

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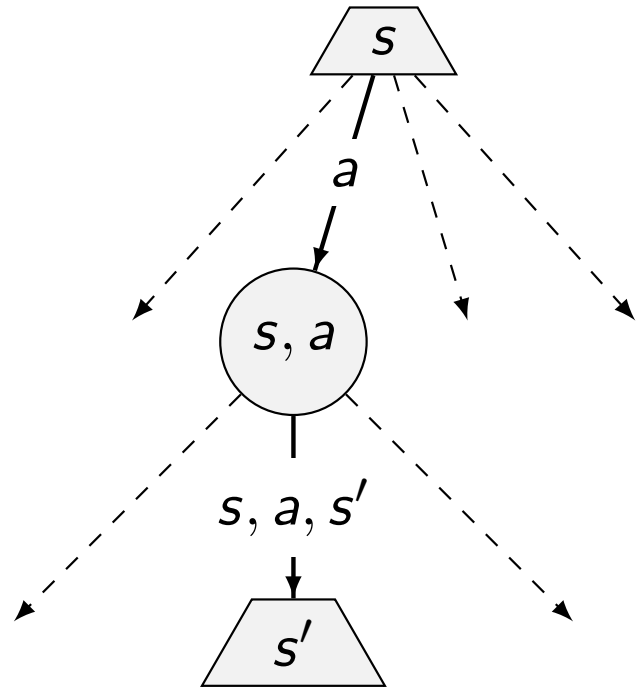
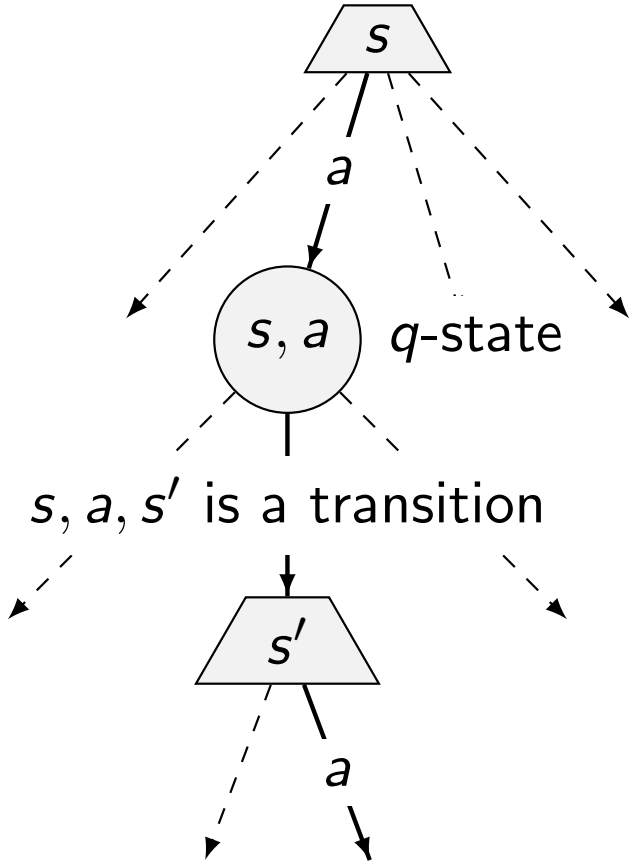


Optimal quantities

- The optimal policy: $\pi^*(s)$ – optimal action from state s
- Expected utility of a policy.

$$U^{\pi} = \mathrm{E}\left[\sum_{t=0}^{\infty} \gamma^t R(s_t)\right]$$

Best policy π^* maximizes above.



- The value of a state s: $V^*(s)$ – expected utility starting in s and acting optimally.
- The value of a q-state (s, a): $Q^*(s, a)$ - expected utility having taken a from state s and acting optimally thereafter.

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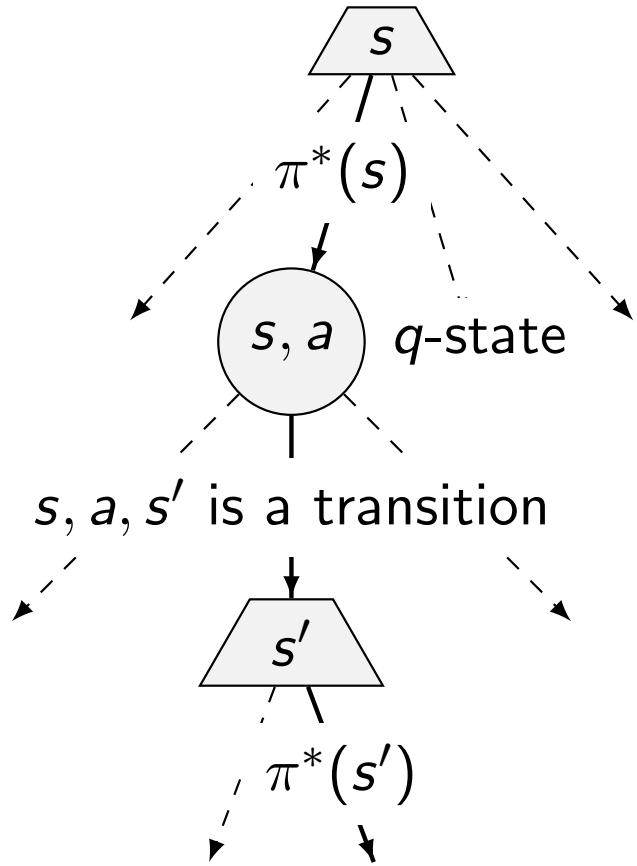

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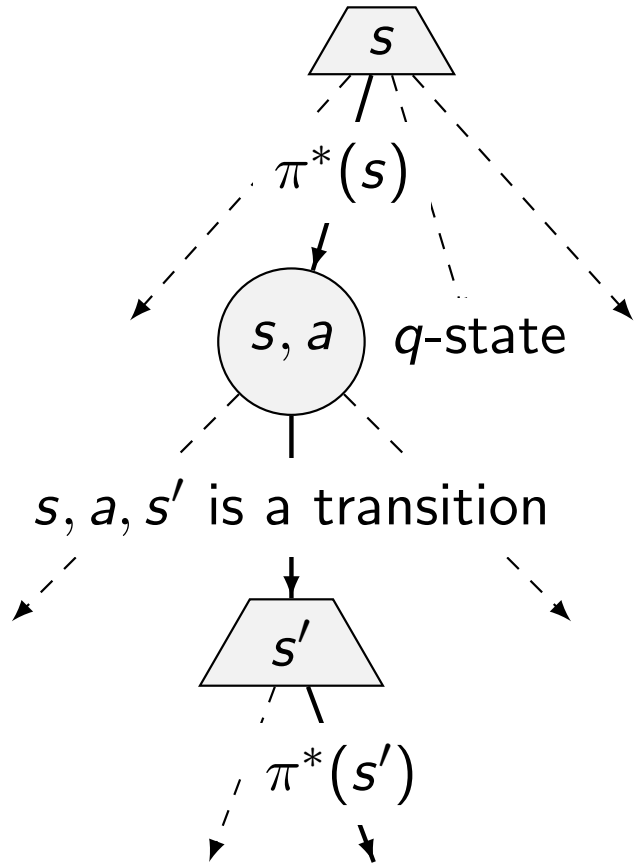

V^* and Q^*

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  $Q^*(s, a) = \sum_{s'} T(s, a, s') V^*(s')$
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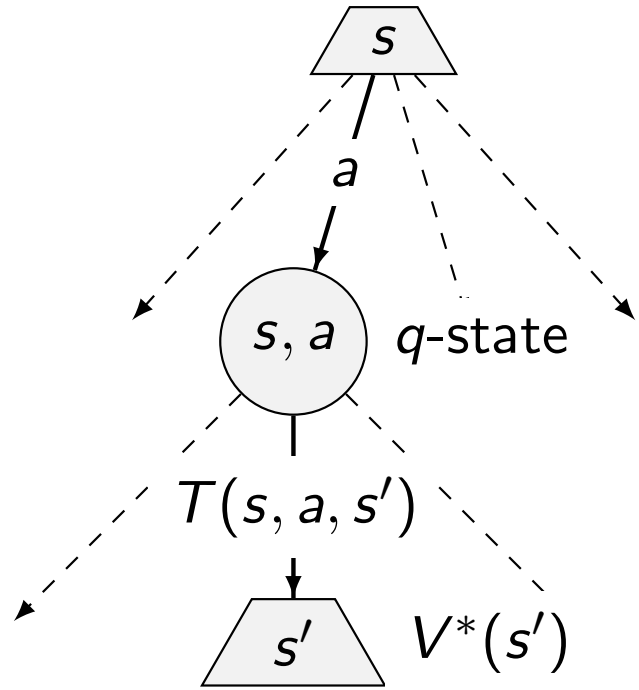

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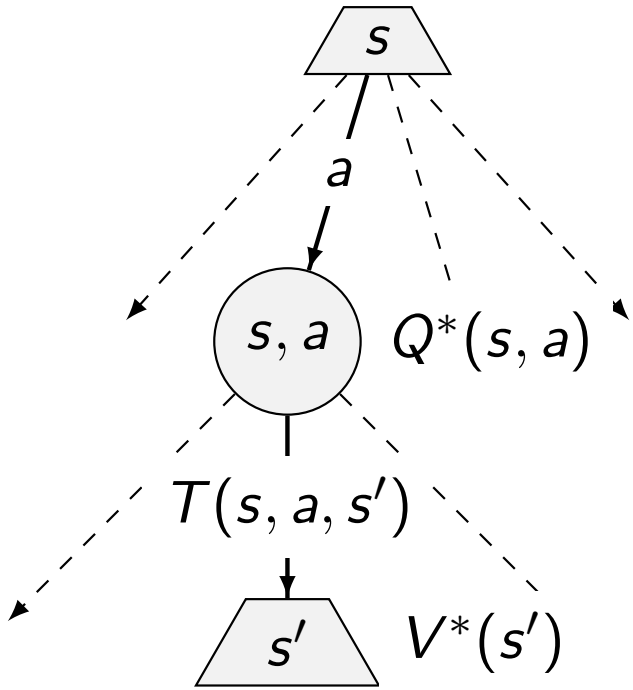

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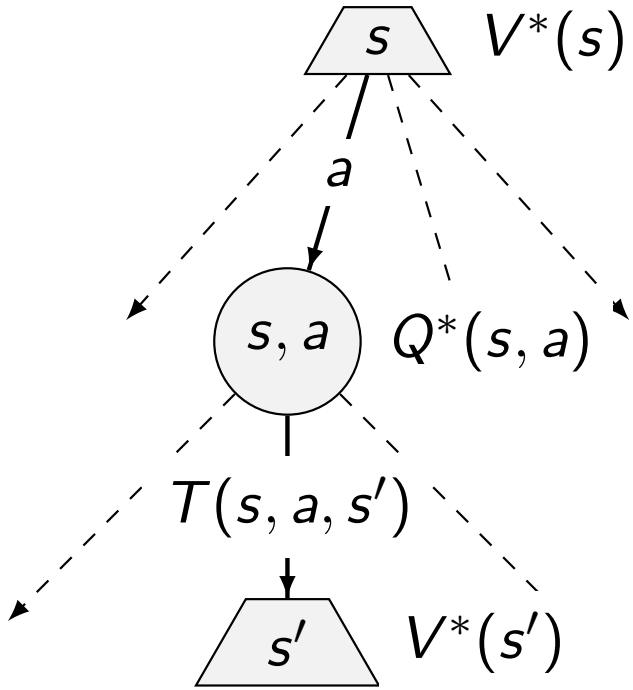
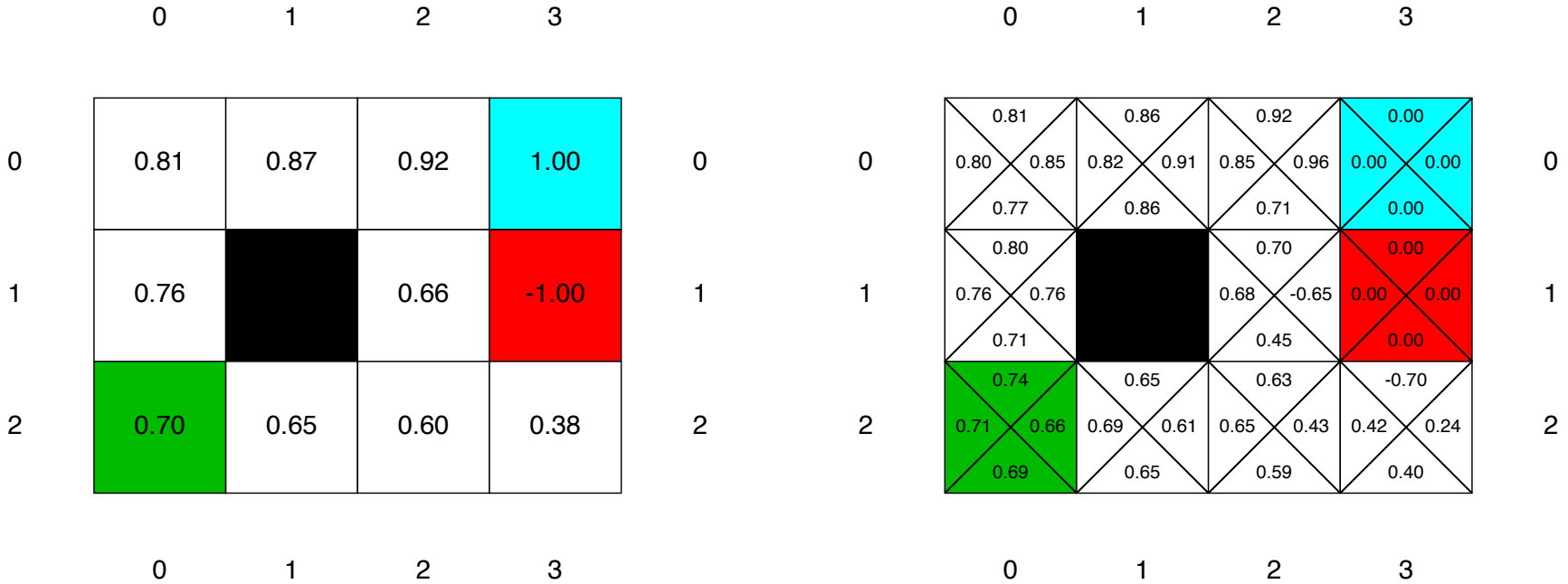


Maze: V^* vs. Q^*



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Value iteration

- Bellman equations characterize the optimal values

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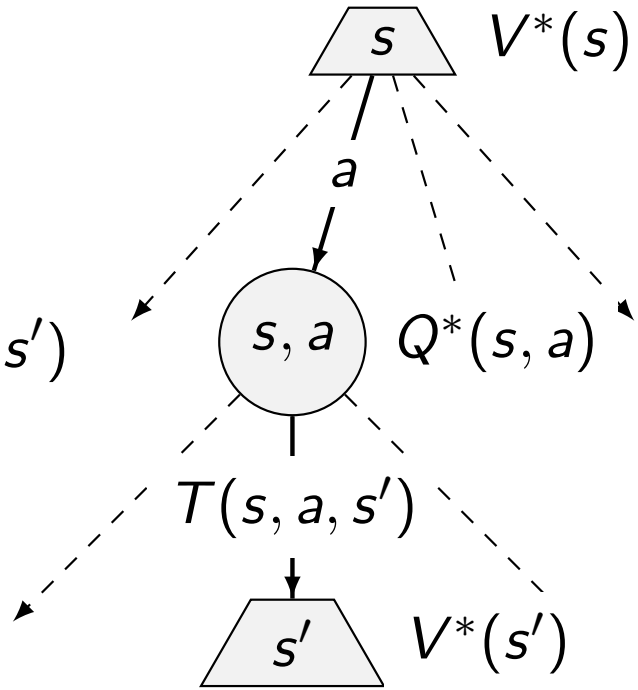

- Value iteration computes them:

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} T(s, a, s') V_k(s')$$

Value iteration is a fixed point solution method.

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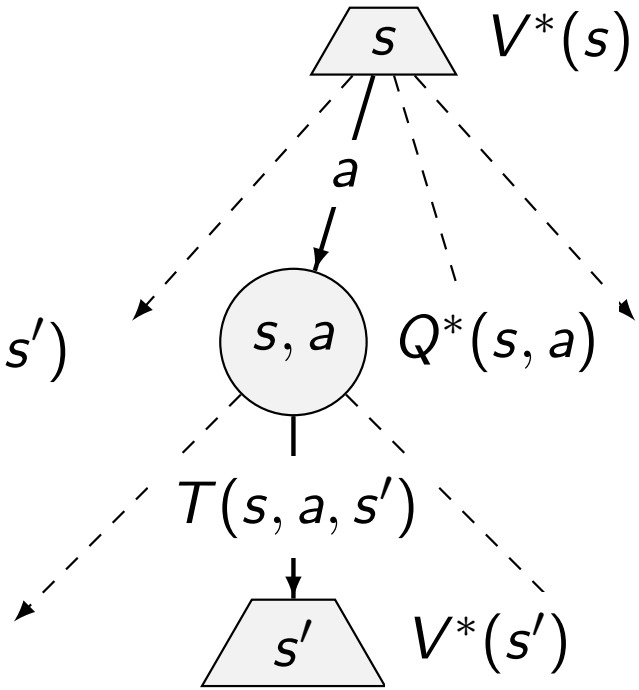

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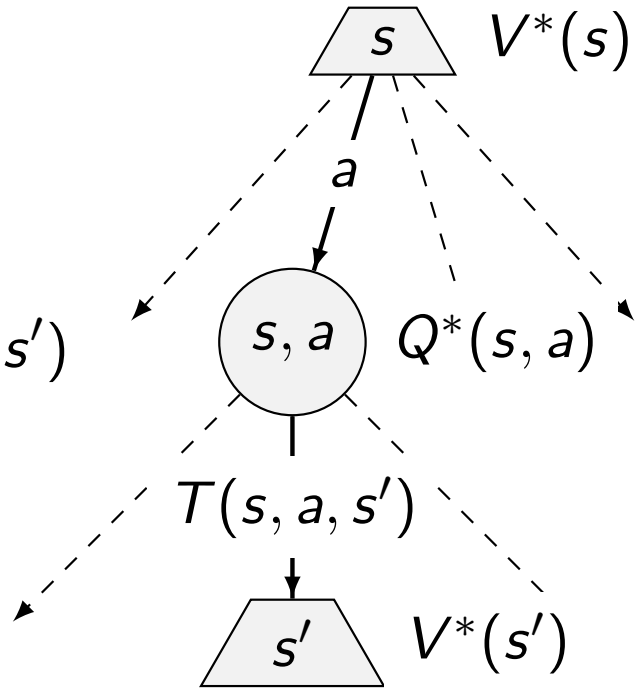

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Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} T(s, a, s') V_k(s')$$

- Think about special cases: deterministic world, $\gamma = 0$, $\gamma = 1$.
- For all s, $V_k(s)$ and $V_{k+1}(s)$ can be seen as expectimax search trees of depth k and $k + 1$

From Values to Policy

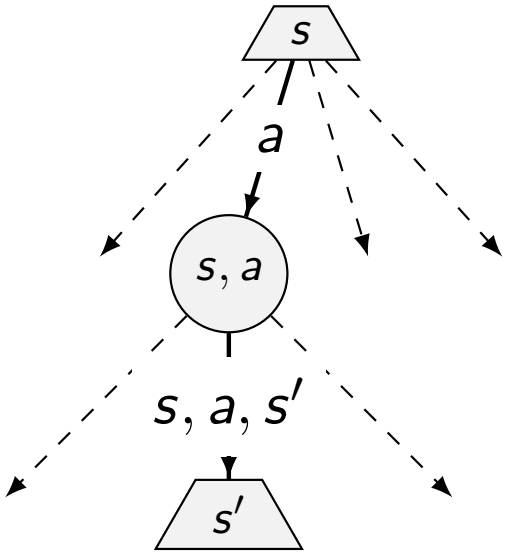
Value iteration

- Start with arbitrary $V_0(s)$
- Compute Bellman update (one ply of expectimax from each state)

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

- Repeat until convergence

Policy extraction - computing actions from Values



- Assume we have $V^*(s)$
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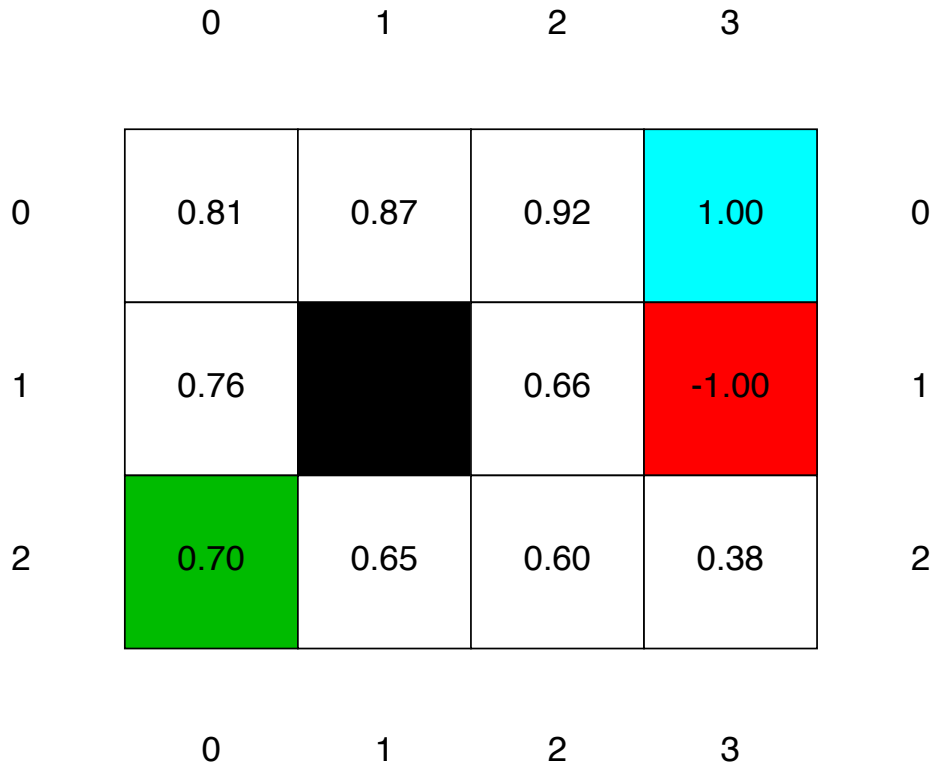

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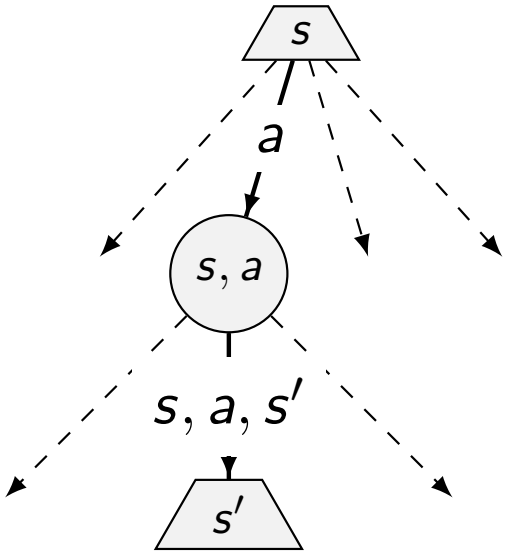

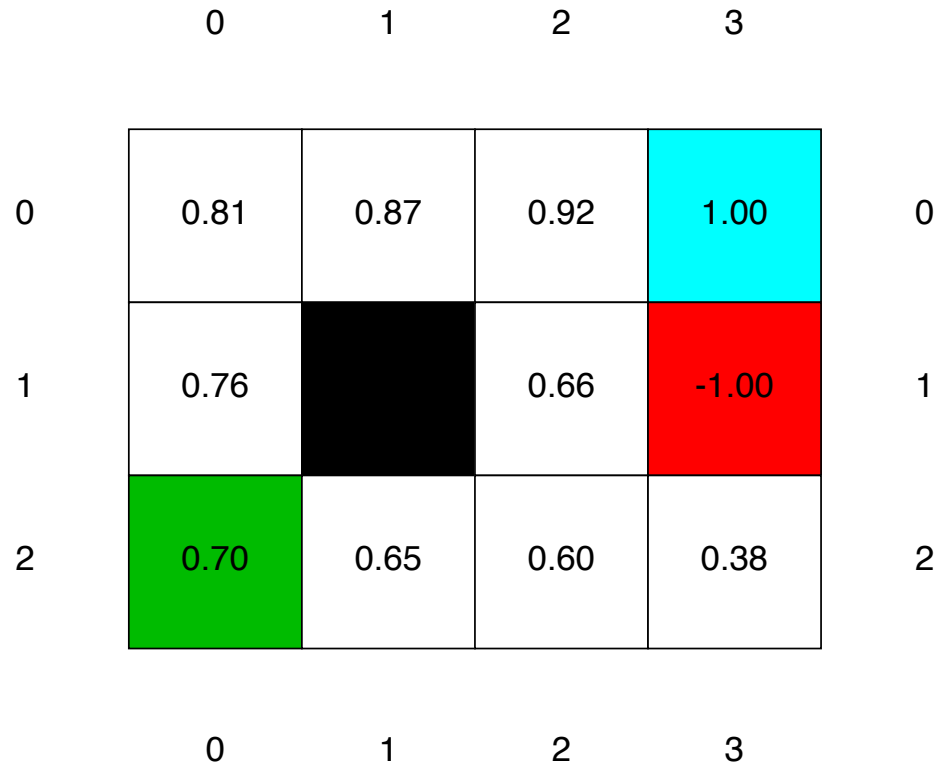

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- Assume we have $Q^*(s, a)$
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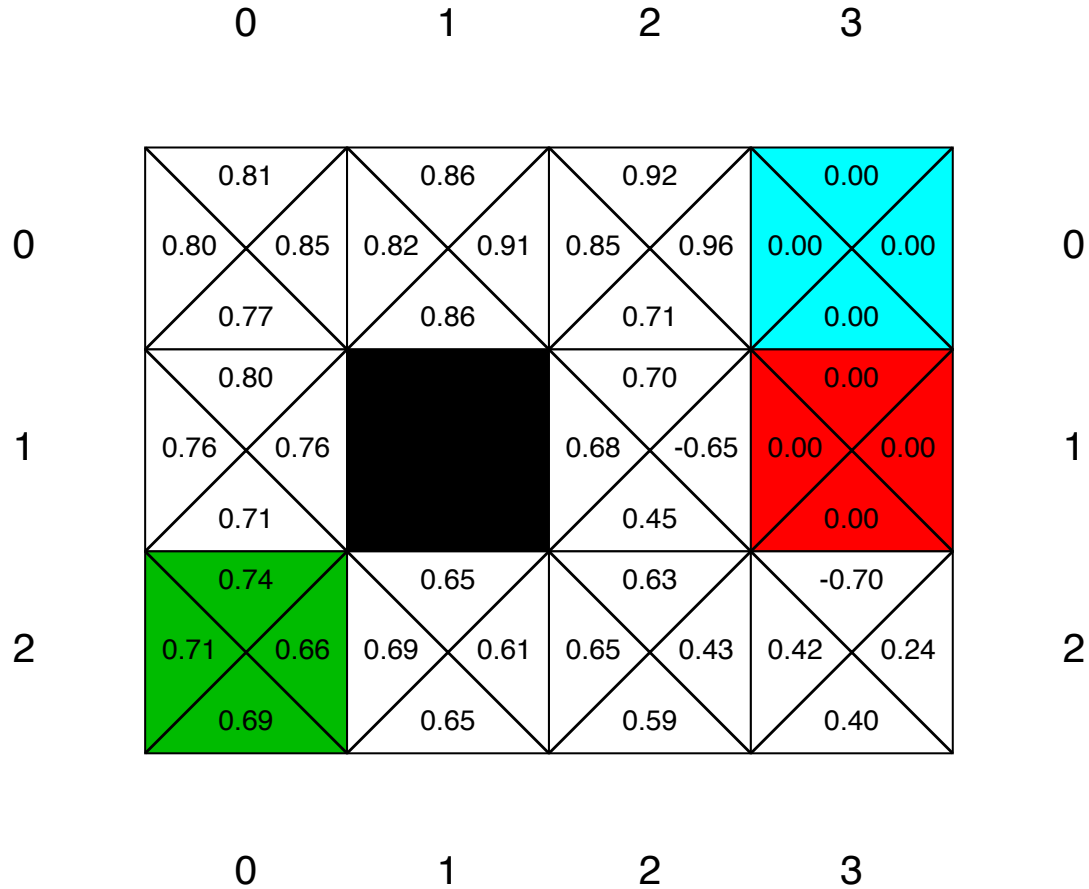
- Just take the max:

$$\pi^*(s) = \max_{a \in A(s)} Q^*(s, a)$$

Actions are easier to extract from q-values.



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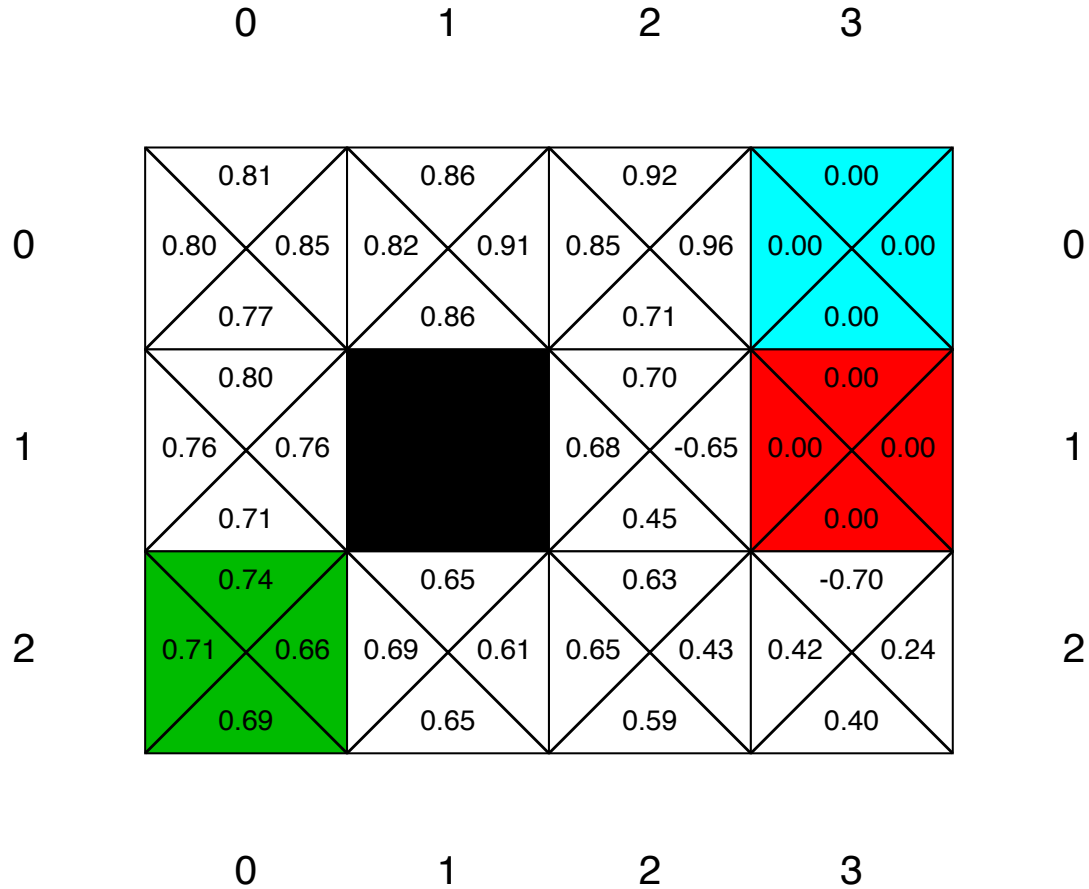

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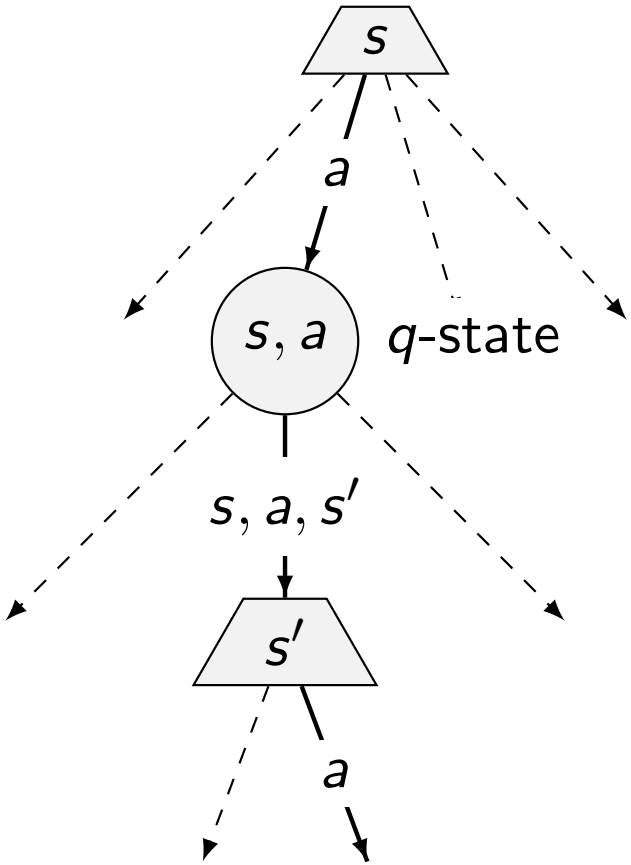
Policy evaluation

- Assume $\pi(s)$ given.
- How to evaluate (compare)?

Fixed policy, do what π says



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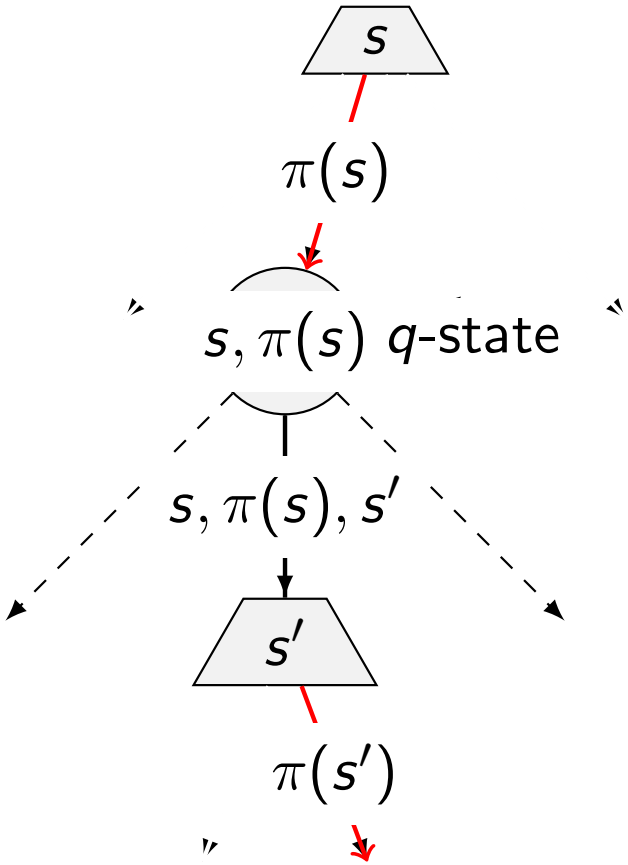

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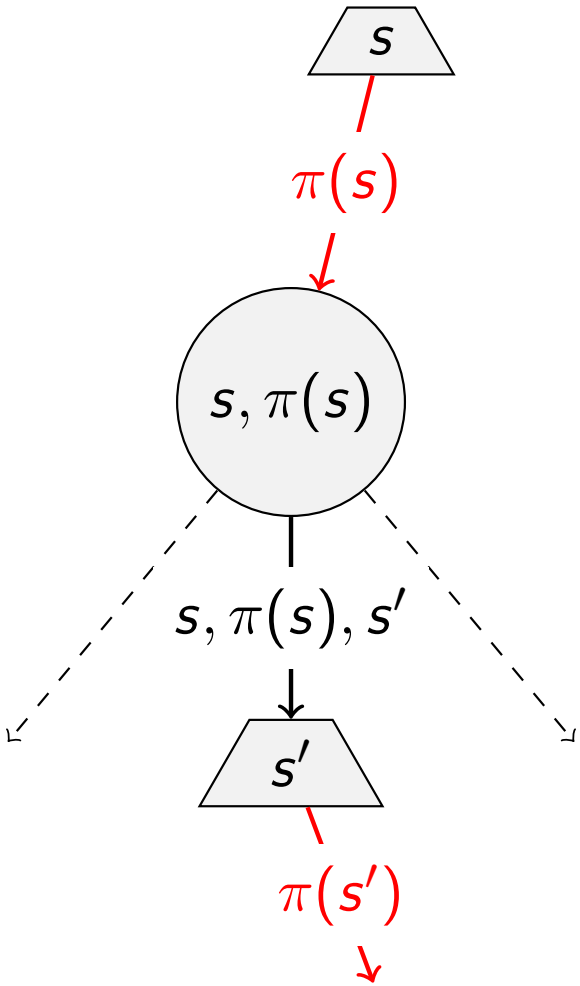

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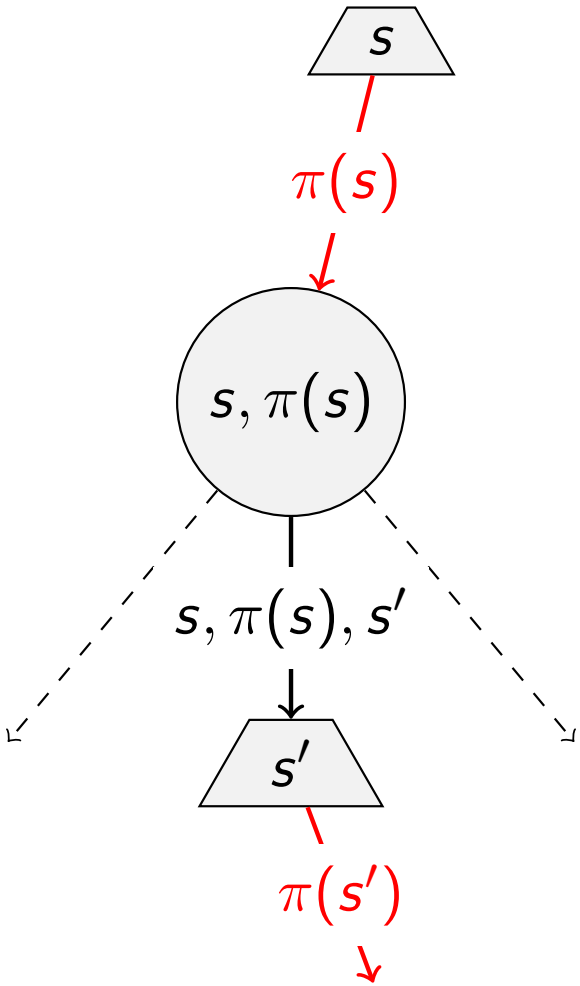

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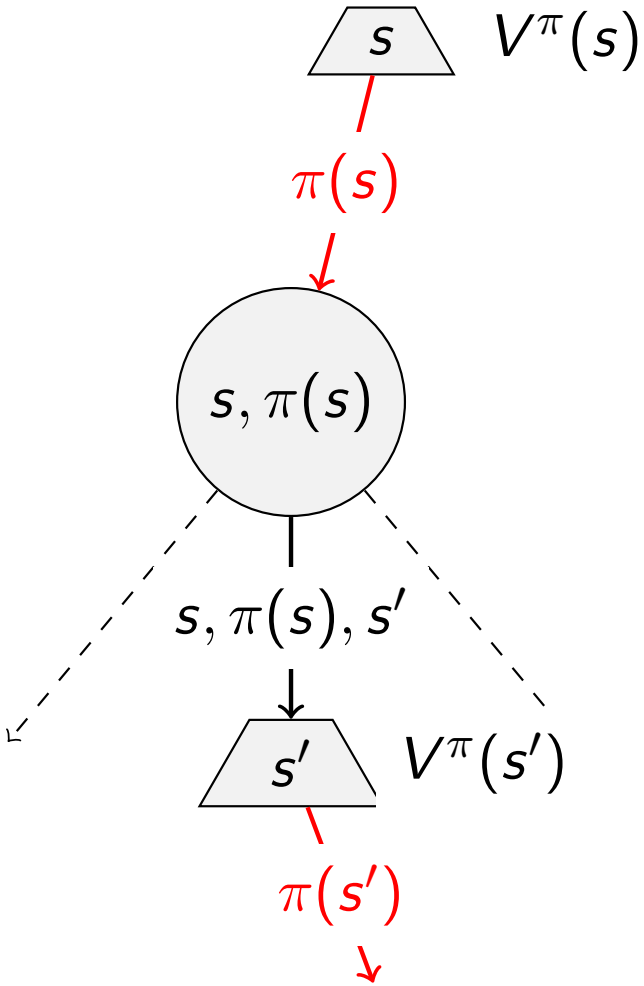

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How to compute $V^{\pi}(s)$?

$$V^{\pi}(s) = R(s) + \gamma \sum_{s'} T(s, \pi(s), s') V^{\pi}(s')$$

Policy iteration

- Start with a random policy.
- Step 1: Evaluate it.
- Step 2: Improve it.
- Repeat steps until policy converges.

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Policy iteration

- Policy evaluation. Solve equations or iterate until convergence.

$$V_{k+1}^{\pi_i}(s) \leftarrow R(s) + \gamma \sum_{s'} T(s, \pi(s), s') V_k^{\pi_i}(s')$$

- Policy improvement. Look-ahead. Policy extraction from fixed values.

$$\pi_{i+1}(s) = \arg\max_{a \in A(s)} \sum_{s'} T(s, a, s') V^{\pi_i}(s')$$

Policy iteration algorithm

function POLICY-ITERATION(env) **returns:** policy π

input: env - MDP problem

$\pi(s) \leftarrow$ random $a \in A(s)$ in all states

$V(s) \leftarrow 0$ in all states

repeat ▷ iterate values until no change

$V \leftarrow$ POLICY-EVALUATION$(\pi, V,$ env$)$

unchanged $\leftarrow$ True

for each state s **in** S **do**

if $\max_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s') > \sum_{s'} P(s'|s,\pi(s))V(s')$ **then**

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  - Iteration updates values and policy. Although policy implicitly – extracted from values
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- Policy iteration.
  - Update utilities is fast – only one action per state.
  - New policy from values (slower)
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- Both methods belong to Dynamic programming realm.

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References

Further reading: Chapter 17 of [1] however, policy iteration is quite compact there. More detailed discussion can be found in chapter Dynamic programming in [2] with slightly different notation, though. This lecture has been also greatly inspired by the 9th lecture of CS 188 at `http://ai.berkeley.edu` as it convincingly motivates policy search and offers an alternative convergence proof of the value iteration method.

[1] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
Prentice Hall, 3rd edition, 2010.
`http://aima.cs.berkeley.edu/`.

[2] Richard S. Sutton and Andrew G. Barto.
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`http://www.incompleteideas.net/book/bookdraft2018jan1.pdf`.

Bandits