Sequential decisions under uncertainty
Policy iteration

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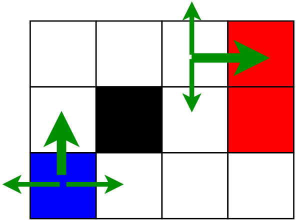
March 29, 2021

Unreliable actions in observable grid world

- Walls block movement – agent/robot stays in place.
- Actions do not always go as planned.
- Agent receives rewards each time step:
  - Small "living" reward/penalty.
  - Big rewards/penalties at the end.
- Goal: maximize sum of (discounted) rewards



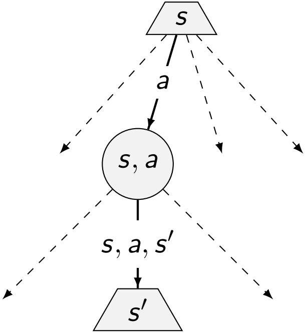
MDPs recap

Markov decision processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s,a)$ or $T(s,a,s')$
- Rewards $r(s,a,s')$; and discount γ

MDP quantities:

- Policy $\pi(s) : \mathcal{S} \rightarrow \mathcal{A}$
- Utility – sum of (discounted) rewards.
- Values – expected future utility from a state (max-node), $v(s)$
- *Q*-Values – expected future utility from a *q*-state (chance-node), $q(s,a)$



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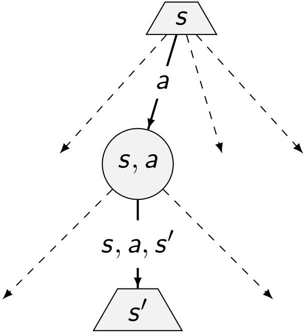


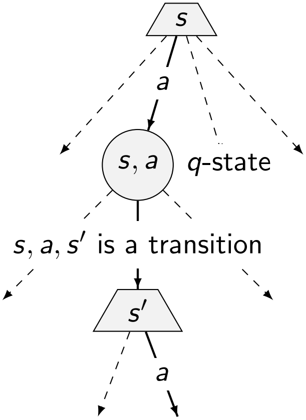
Optimal quantities

- The optimal policy: $\pi^*(s)$ – optimal action from state s
- Expected utility/return of a policy.

$$U^\pi(S_t) = \mathrm{E}^\pi\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

Best policy π^* maximizes above.

- The value of a state s: $v^*(s)$ – expected utility starting in s and acting optimally.
- The value of a q-state (s, a): $q^*(s, a)$ - expected utility having taken a from state s and acting optimally thereafter.



s, a, s' is a transition

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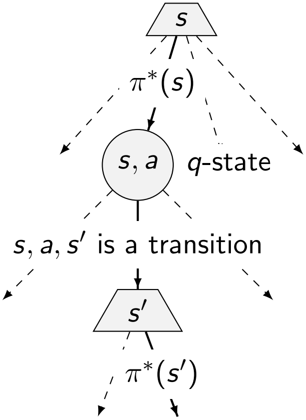

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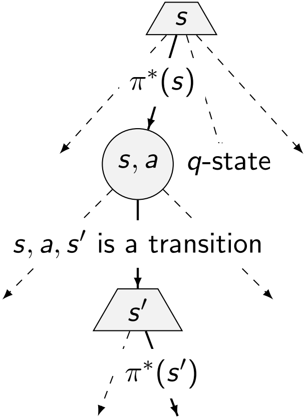
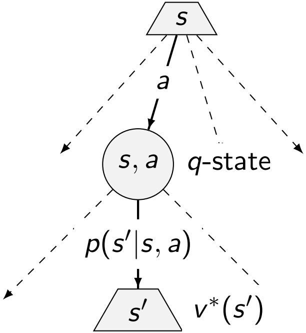


v^* and q^*

The value of a q-state (s, a):

$$q^*(s, a) = \sum_{s'} p(s'|a, s)\left[r(s, a, s') + \gamma\, v^*(s')\right]$$



The value of a state s:

$$v^*(s) = \max_a q^*(s, a)$$

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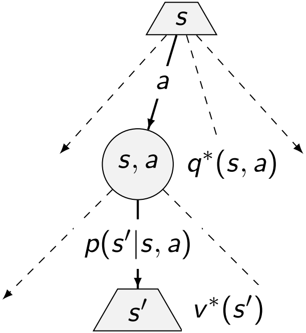

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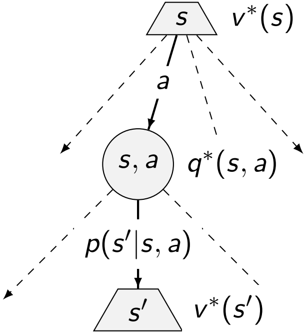
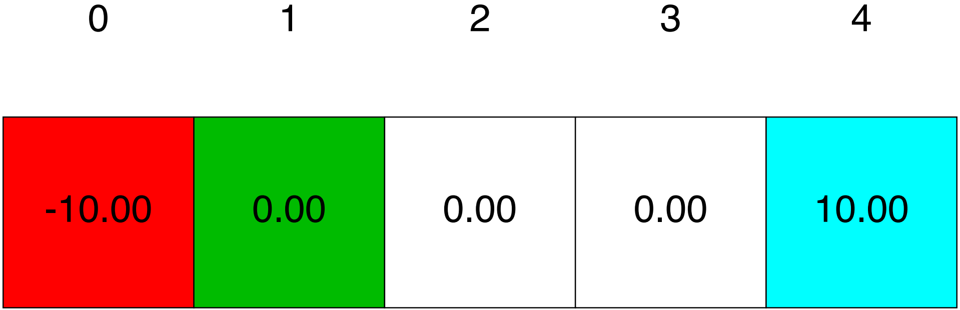


Maze: $V_0 = [0, 0, 0]^\top$, $r(s) = -1$, deterministic robot, $\mathcal{A} = \{\leftarrow, \uparrow, \downarrow, \rightarrow\}$, $\gamma = 1$

0	1	2	3	4
-10.00	0.00	0.00	0.00	10.00

$$\begin{aligned} q^*(s, a) &= \sum_{s'} p(s'|a, s)\left[r(s, a, s') + \gamma\, v^*(s')\right] \\ v^*(s) &= \max_a\, q^*(s, a) \end{aligned}$$

What will be V^* after first sweep? $V_1^* = [v_1^*(1), v_1^*(2), v_1^*(3)]^\top$?



0	1	2	3	4
-10.00	0.00	0.00	0.00	10.00

Sweep is meant as the Bellmann update for all states: $V_1^* = BV_0^*$. $r(s) = -1$. Assume sync version of the algorithm.

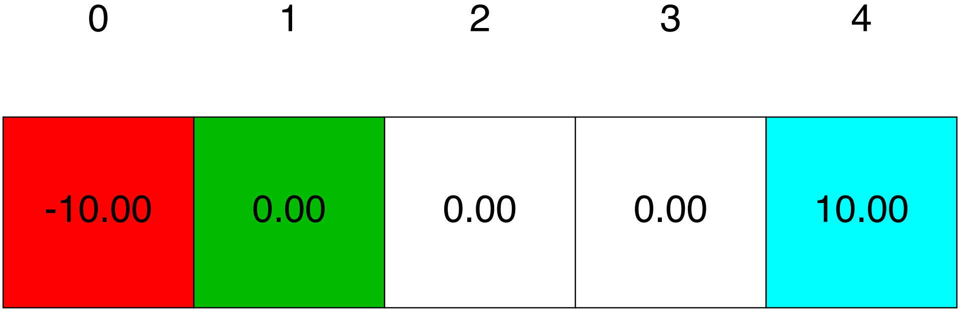
A: $V_1^* = [-1, -1, 9]^\top$

B: $V_1^* = [0, 8, 9]^\top$

C: $V_1^* = [-1, 0, 0]^\top$

D: $V_1^* = [-11, 8, 9]^\top$

What will be V^* after second sweep? $V_2^* = [v_2^*(1), v_2^*(2), v_2^*(3)]^\top$?



0	1	2	3	4
🟥 -10.00	🟩 0.00	0.00	0.00	10.00

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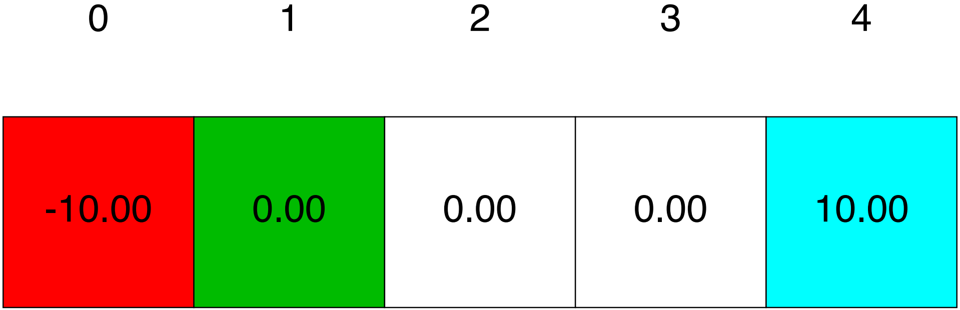
A: $V_2^* = [-1, -1, 9]^\top$

B: $V_2^* = [-1, 8, 9]^\top$

C: $V_2^* = [-2, 8, 9]^\top$

D: $V_2^* = [7, 8, 9]^\top$

What will be the $q^*(s, a)$ values after second value-iter sweep?



0	1	2	3	4
-10.00	0.00	0.00	0.00	10.00

Pick the option which is *wrong*:

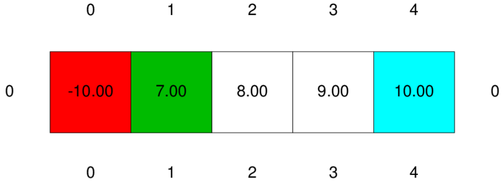
A: $q^*(1, \uparrow) = -2$

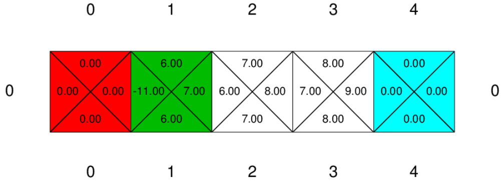
B: $q^*(1, \rightarrow) = -2$

C: $q^*(2, \rightarrow) = -2$

D: $q^*(3, \leftarrow) = -2$

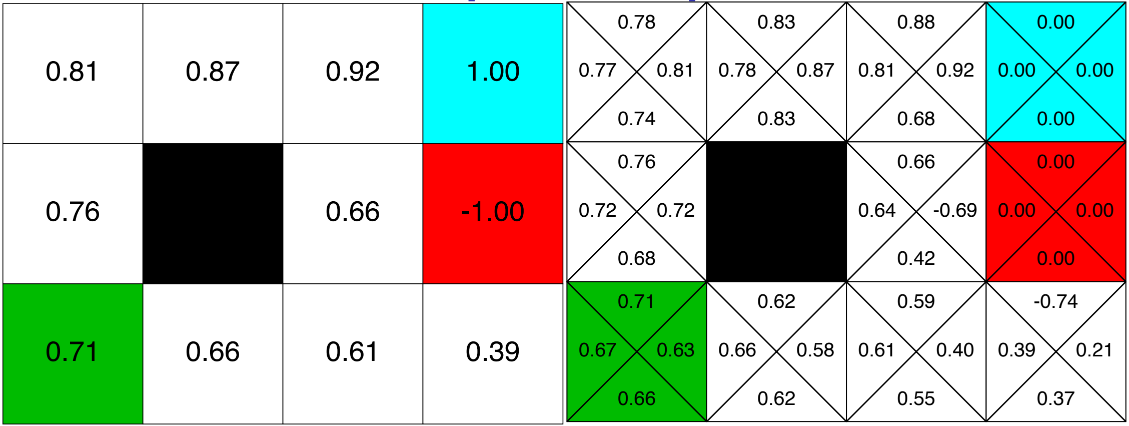
Maze: v^* vs. q^*, deterministic robot, $\mathcal{A} = \{\leftarrow, \uparrow, \downarrow, \rightarrow\}$





$$
\begin{aligned}
q^*(s,a) &= \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s')\right] \\
v^*(s) &= \max_a q^*(s,a)
\end{aligned}
$$

Maze: v^* vs. q^*, $\gamma = 1$, $T = [0.8, 0.1, 0.1, 0]$



0.81	0.87	0.92	1.00
0.76	■	0.66	-1.00
0.71	0.66	0.61	0.39

0.78 0.77 0.81 0.74	0.83 0.78 0.87 0.83	0.88 0.81 0.92 0.68	0.00 0.00 0.00 0.00
0.76 0.72 0.72 0.68	■	0.66 0.64 -0.69 0.42	0.00 0.00 0.00 0.00
0.71 0.67 0.63 0.66	0.62 0.66 0.58 0.62	0.59 0.61 0.40 0.55	-0.74 0.39 0.21 0.37

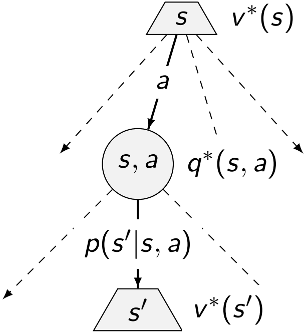
$$q^*(s,a) = \sum_{s'} p(s'|a,s)\left[r(s,a,s') + \gamma\, v^*(s')\right]$$

$$v^*(s) = \max_a q^*(s,a)$$

Value iteration

- Bellman equations **characterize** the optimal values

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma v^*(s')\right]$$



- Value iteration **computes** them:

$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma V_k(s')\right]$$

Value iteration is a fixed point solution method.

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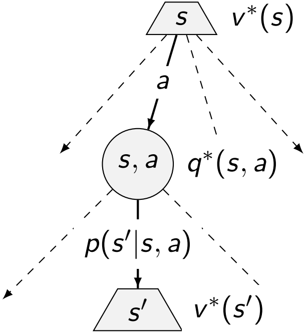

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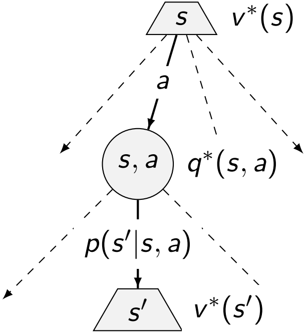

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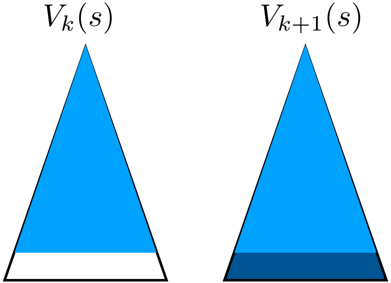
Convergence

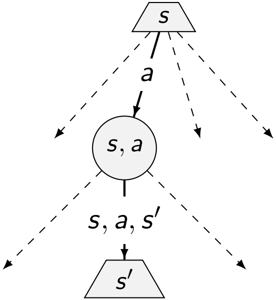
$$V_{k+1}(s) \leftarrow \max_{a \in A(s)} \sum_{s'} p(s'|s,a)\left[r(s,a,s') + \gamma V_k(s')\right]$$

- Thinking about special cases: deterministic world, $\gamma = 0$, $\gamma = 1$.
- For all s, $V_k(s)$ and $V_{k+1}(s)$ can be seen as expectimax search trees of depth k and $k+1$



From Values to Policy

Policy extraction - computing actions from Values



- Assume we have $v^*(s)$
- What is the optimal action?
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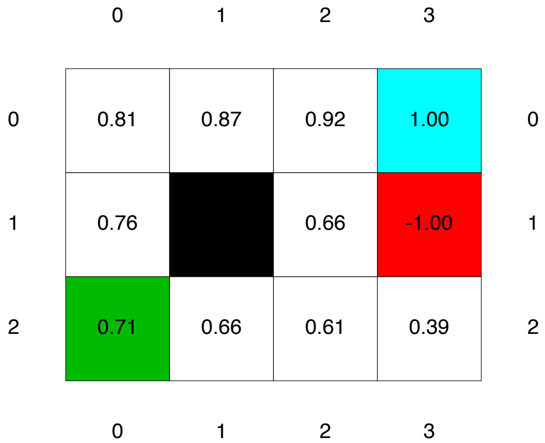

	0	1	2	3	
0	0.81	0.87	0.92	1.00	0
1	0.76		0.66	-1.00	1
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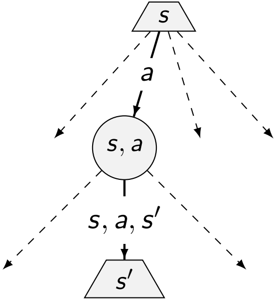

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Policy extraction - computing actions from q-Values

- Assume we have $q^*(s, a)$
- What is the optimal action?
- Just take the (arg) max:

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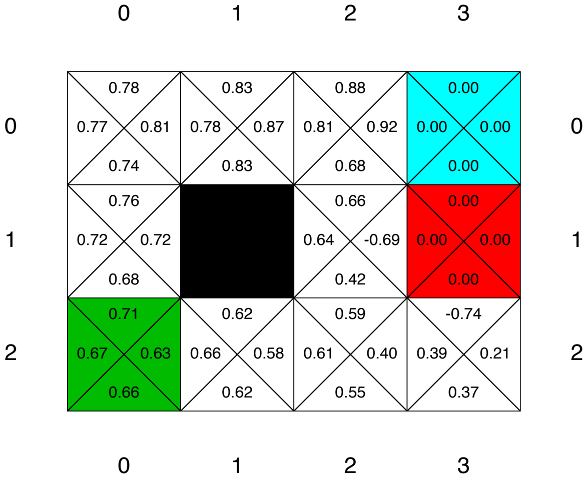
Actions are easier to extract from q-values.



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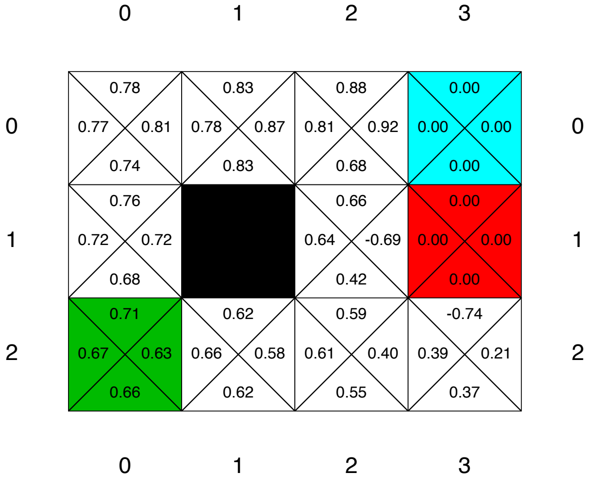

What is wrong with the Value iteration?

$$V_{k+1}(s) \leftarrow \max_{a\in\mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V_k(s') \right]$$

- What is complexity of one iteration - over all S states?
- When does the iteration stop?
- When the does the policy converge?
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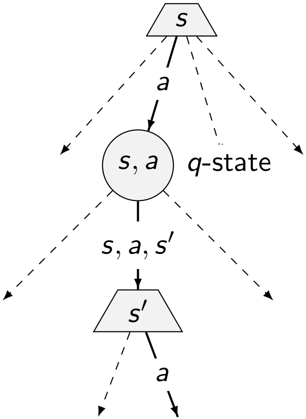
Policy evaluation

- Assume $\pi(s)$ given.
- How to evaluate (compare)?

Fixed policy, do what π says



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- Fixed π for each state → no “max” operator!

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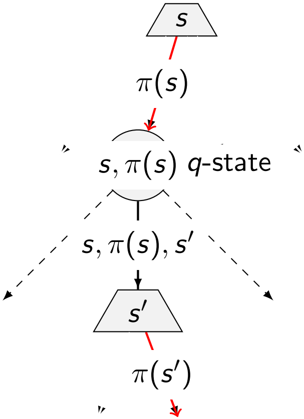

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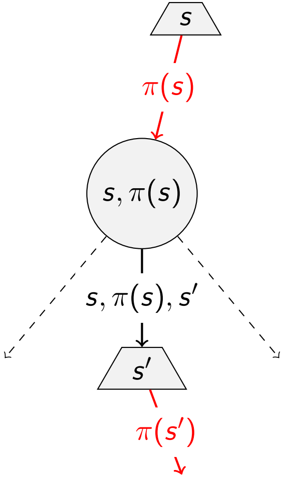
State values under a fixed policy



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$$v^{\pi}(s) = \sum_{s'} p(s' \mid s, \pi(s))\left[r(s, \pi(s), s') + \gamma v^{\pi}(s')\right]$$

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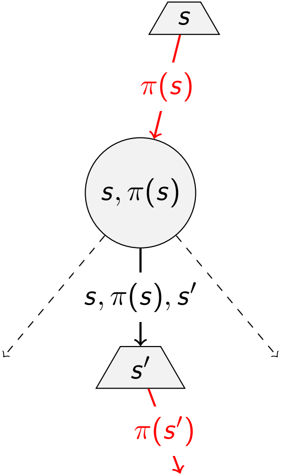

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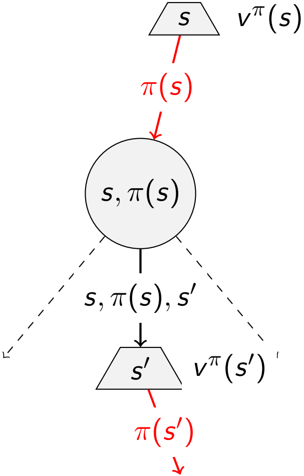

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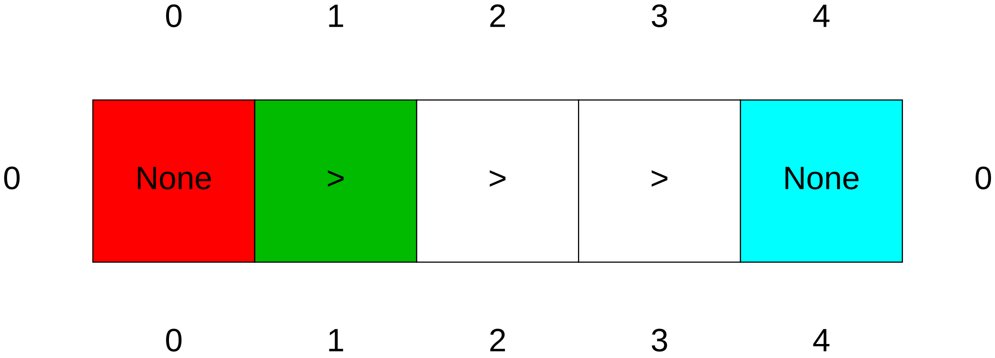
$$v^\pi(s) = \sum_{s'} p(s' \mid s, \pi(s)) \big[r(s, \pi(s), s') + \gamma v^\pi(s') \big]$$

How to compute $v^\pi(s)$?

$$v^\pi(s) = \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, \pi(s), s') + \gamma v^\pi(s') \right]$$



	0	1	2	3	4	
0	None	>	>	>	None	0
	0	1	2	3	4	

Policy iteration

- Start with a random policy.
- Step 1: Evaluate it.
- Step 2: Improve it.
- Repeat steps until policy converges.

Policy iteration

- Policy π evaluation. Solve equations or iterate until convergence.

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} p(s' \mid s, \pi(s)) \left[r(s, \pi(s), s') + \gamma V_k^{\pi_i}(s') \right]$$

- Policy improvement. Look-ahead and keep optimality. Policy extraction from fixed values.

$$\pi_{i+1}(s) = \arg\max_{a \in \mathcal{A}(s)} \sum_{s'} p(s' \mid s, a) \left[r(s, a, s') + \gamma V_k^{\pi_i}(s') \right]$$

Policy iteration algorithm

```
function POLICY-ITERATION(env) returns: policy $\pi$
    input: env - MDP problem
    $\pi(s) \leftarrow$ random $a \in A(s)$ in all states
    $V(s) \leftarrow 0$ in all states
    repeat                                ▷ iterate values until no change in policy
        $V \leftarrow$ POLICY-EVALUATION($\pi$, $V$, env)
        unchanged $\leftarrow$ True
        for each state $s$ in $S$ do
            if $\max_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s') > \sum_{s'} P(s'|s,\pi(s))V(s')$ then
                $\pi(s) \leftarrow \arg\max_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s')$
                unchanged $\leftarrow$ False
            end if
        end for
    until unchanged
end function
```

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    $V(s) \leftarrow 0$ in all states
    repeat                                                  ▷ iterate values until no change in policy
        $V \leftarrow$ POLICY-EVALUATION($\pi$, $V$, env)
        unchanged $\leftarrow$ True
        for each state $s$ in $S$ do
            if $\max_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s') > \sum_{s'} P(s'|s,\pi(s))V(s')$ then
                $\pi(s) \leftarrow \arg\max_{a \in A(s)} \sum_{s'} P(s'|a,s)V(s')$
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Policy iteration algorithm

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Policy vs. Value iteration

- Value iteration.
  - Iteration updates values and policy. (policy only implicitly – can be extracted from values)
  - No track of policy.
- Policy iteration.
  - Update of values is faster – only one action per state.
  - New policy from values (slower).
  - New policy is better or done.
- Both methods belong to Dynamic programming realm.

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References

Further reading: Chapter 17 of [1] however, policy iteration is quite compact there. More detailed discussion can be found in chapter Dynamic programming in [2] with slightly different notation, though. This lecture has been also greatly inspired by the 9th lecture of CS 188 at http://ai.berkeley.edu as it convincingly motivates policy search and offers an alternative convergence proof of the value iteration method.

[1] Stuart Russell and Peter Norvig.
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[2] Richard S. Sutton and Andrew G. Barto.
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Bandits