Value Iteration

and extensions

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PUI Tutorial
Week 9

Outline

- Review of MDP concepts
- Value Iteration algorithm
- VI extensions

Value function of a policy

Look at the following definition of a value function of a policy for inifnite-horizon MDP. It contains multiple mistakes, correct them on a piece of paper:

Def: Value function of a policy for infinite-horizon MDP

Assume infinite horizon MDP with $\gamma \in [0, 100]$. Then let Value function of a policy π for every state $s \in S$ be defined as

$$V^\pi(s) = \sum_{s' \in S} R(s, \pi(s), s')R(s, \pi(s), s') + \gamma\pi(s')$$

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$$V^{\pi}(s) = \sum_{s' \in S} T(s, \pi(s), s')[R(s, \pi(s), s') + \gamma V^{\pi}(s')]$$

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Value Iteration algorithm

Basic algorithm for finding solution of Bellman Equations iteratively:

1. initialize V_0 arbitrarily for each state, e.g to 0, set $n = 0$
2. Set $n = n + 1$.
3. Compute Bellman Backup, i.e. for each $s \in S$:
    1. $V_n(s) = \max_{a \in A} \sum_{s' \in S} T(s, a, s')[R(s, a, s') + \gamma V_{n-1}(s')]$
4. GOTO 2.

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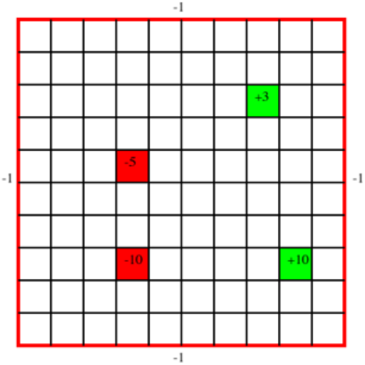
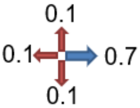
Basic version uses 2 arrays to store state values.

VI example - gridworld

Gridworld domain. Assuming $\gamma = 1$.
Using VI and initializing $\forall s \; V_0(s) = 0$, calculate V_1, V_2, V_3 for the nine states around the +10 tile.
Domain rules:

- Moving into edges gives -1 reward
- Moving onto marked tiles gives corresponding reward





Example from https://artint.info

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4. GOTO 2.

Question: Does it converge? How fast? When do we stop?

Residual - definition

Def: Residual

Residual of value function V_n *from* V_{n+1} *at state* $s \in S$ is defined by:

$$Res^{V_n}(s) = |V_n(s) - V_{n+1}(s)|$$

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Residual of value function V *from* V' is given by:

$$Res^{V_n} = ||V_n - V_{n+1}||_\infty = \max_s |V_n(s) - V_{n+1}(s)|$$

VI stopping criterion

Stopping criterion: When residual of consecutive value functions is below low value of ϵ:

$$||V_n - V_{n+1}|| < \epsilon$$

However, this does not imply ϵ distance of value of greedy policy from optimal value function.

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In case of discounted $(\gamma < 1)$ infinite-horizon MDPs:

$$V_n, V^* \text{as above} \implies \forall s \; |V_n(s) - V^*(s)| < 2\frac{\epsilon\gamma}{1-\gamma}$$

VI with stopping criterion

Algorithm 1: Value Iteration

```
initialize V_0 arbitrarily for each state, e.g to 0 while Res^V > ϵ do
    pick some state s
    Bellman backup V(s) ← max_{a∈A} ∑_{s'∈S} T(s, a, s')[R(s, a, s') + γV(s')]
    Update residual at s Res^V(s) = |V_old(s) − V_new(s)|
return greedy policy π^V;
```

VI with stopping criterion

Algorithm 2: Value Iteration

initialize V_0 arbitrarily for each state, e.g to 0 **while** $Res^V > \epsilon$ **do**
    pick some state s
    Bellman backup $V(s) \leftarrow \max_{a \in A} \sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V(s')]$
    Update residual at s $Res^V(s) = |V_{\text{old}}(s) - V_{\text{new}}(s)|$
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Question: What is the greedy policy?

VI with stopping criterion

Algorithm 3: Value Iteration

initialize V_0 arbitrarily for each state, e.g to 0 **while** $Res^V > \epsilon$ **do**
  pick some state s
  Bellman backup $V(s) \leftarrow \max_{a \in A} \sum_{s' \in S} T(s,a,s')[R(s,a,s') + \gamma V(s')]$
  Update residual at s $Res^V(s) = |V_{\text{old}}(s) - V_{\text{new}}(s)|$

return greedy policy π^V;

Question: What is the greedy policy?

- *Greedy policy* π_n^V is the policy given as argmax of V_n.

Value iteration properties

- Convergence: VI converges from any initialization (unlike PI)
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Value iteration properties

- Convergence: VI converges from any initialization (unlike PI)
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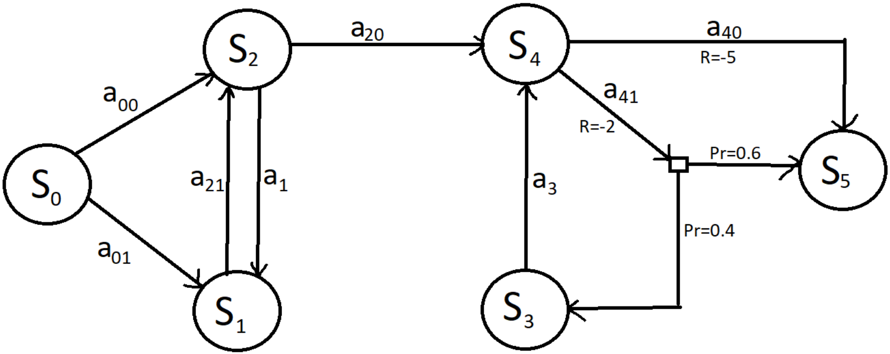
Question: What are the memory requirements of VI?

- Value of each state needs to be stored twice

Another beautiful MDP example



- All undeclared rewards are -1

Task: Initialize VI with negative distance to S_5 and calculate first 3 iterations of VI, with state ordering S_0 to S_5

Asynchronous VI

Algorithm 4: Asynchronous VI

initialize V_0 arbitrarily for each state, e.g to 0 **while** $Res^V > \epsilon$ **do**
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Asynchronous VI

Algorithm 5: Asynchronous VI

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Question: Memory requirements compared to VI?

Asynchronous VI

Algorithm 6: Asynchronous VI

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Question: Memory requirements compared to VI?
Question: Convergence condition?

- Asymptotic as VI under condition that every state visited ∞ often.

Asynchronous VI

Algorithm 7: Asynchronous VI

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5 **return** greedy policy π^V;

Question: Memory requirements compared to VI?

Question: Convergence condition?

- Asymptotic as VI under condition that every state visited ∞ often.

Question: How to pick s in 2.1?

- Simplest is *Gauss-Seidel VI*, that is run AVI over all states iteratively

Prioritized VI

How else can we pick states to update? (Ans: by ordering them in clever ways)

- Build priority queue of states to update $\rightarrow$ *Prioritized Sweeping VI*. Update states in the order of the queue. Priority function:

$$\text{priority}_{PS}(s) \leftarrow \max\{\text{priority}_{PS}(s), \max_{a \in A}\{T(s,a,s')Res^V(s')\}\}$$

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EXAMPLE ON BOARD

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EXAMPLE ON BOARD

Convergence?

- If all states start with non-zero priority
- OR If you interleave regular VI sweeps with Prioritized VI

Partitioned VI

Subdivide state space into partitions. Pre-solve parts independently.

- Example is *Topological VI*

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Topological VI:

1. Find acyclic paritioning (by finding strongly connected components in the graph)
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Question: Why acyclic partitioning?

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EXAMPLE ON BOARD

Thank you for participating in the tutorials :-)

Please fill out the feedback form →

https://forms.gle/wyVaeHbXJYEKdUNX7