Linear Classifiers II

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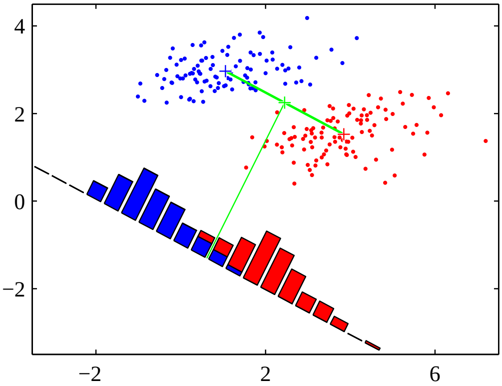
Notes

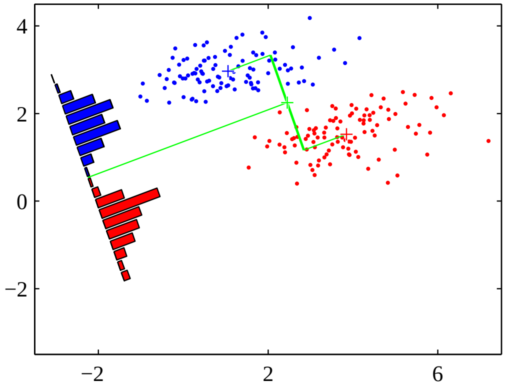
Linear Classifiers - supplement lecture

- Supplement to the lecture about learning Linear Classifiers (perceptron, . . .)
- Better etalons by applying Fischer linear discriminator analysis.
- LSQ formulation of the learning task.

Notes

Fischer linear discriminant





- Dimensionality reduction
- Maximize distance between means, . . .
- . . . and minimize within class variance. (minimize overlap)

Figures from [1]

Notes

Projections to lower dimensions $y = \mathbf{w}^\top \mathbf{x}$

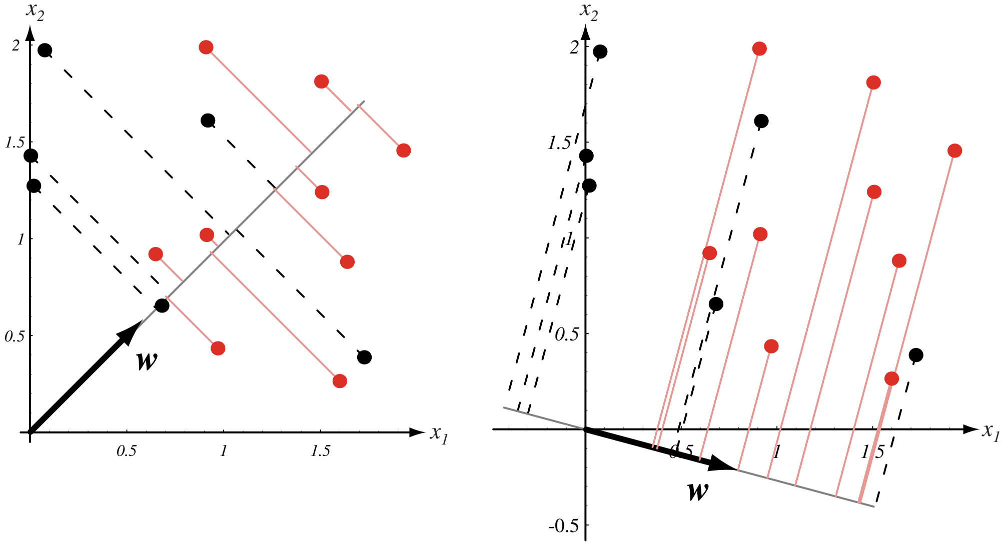


Figure from [2]

Notes

Projection to lower dimension $\mathbf{y} = \mathtt{W}^\top \mathbf{x}$

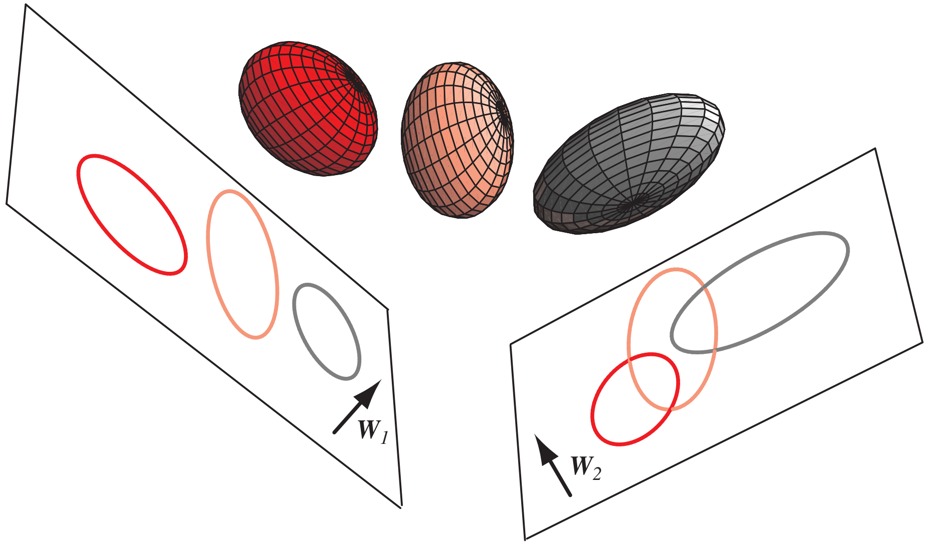
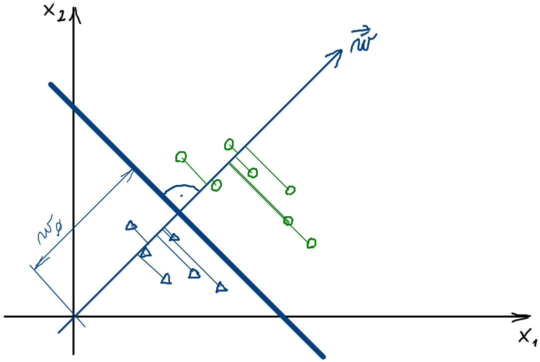
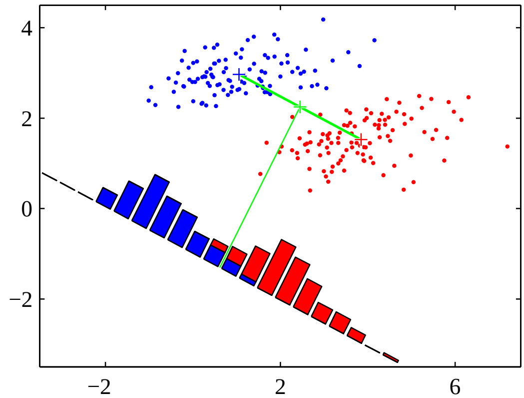


Figure from [2]

Notes

Finding the best projection $y = \mathbf{w}^\top \mathbf{x}$, $y \geq -w_0 \Rightarrow C_1$, otherwise C_2





Notes

This is just to make sure we understand geometric meaning of $\mathbf{w}$, w_0 and the separating hyperplane. Remind the vector notation $\mathbf{w}$ means the same as $\vec{w}$.

Finding the best projection $y = \mathbf{w}^\top \mathbf{x}$, $y \geq -w_0 \Rightarrow C_1$, otherwise C_2

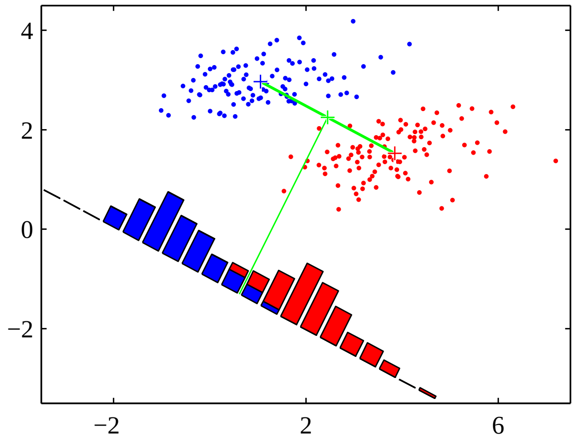
$$m_2 - m_1 = \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1)$$

Within class scatter of projected samples

$$s_i^2 = \sum_{y \in C_i} (y - m_i)^2$$

Fischer criterion:

$$J(\mathbf{w}) = \frac{(m_2 - m_1)^2}{s_1^2 + s_2^2}$$



Notes

Fischer criterion, max or min?

Finding the best projection $y = \mathbf{w}^\top \mathbf{x}$, $y \geq -w_0 \Rightarrow C_1$, otherwise C_2

$$m_2 - m_1 = \mathbf{w}^\top(\mathbf{m}_2 - \mathbf{m}_1)$$

$$s_i^2 = \sum_{y \in C_i} (y - m_i)^2$$

$$J(\mathbf{w}) = \frac{(m_2 - m_1)^2}{s_1^2 + s_2^2}$$

$$\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} = 0$$

$$\mathtt{S}_i = \sum_{x \in C_i} (\mathbf{x} - \mathbf{m}_i)(\mathbf{x} - \mathbf{m}_i)^\top$$

$$\mathtt{S}_W = \mathtt{S}_1 + \mathtt{S}_2$$

$$\mathtt{S}_B = (\mathbf{m}_2 - \mathbf{m}_1)(\mathbf{m}_2 - \mathbf{m}_1)^\top$$

$$J(\mathbf{w}) = \frac{\mathbf{w}^\top \mathtt{S}_B \mathbf{w}}{\mathbf{w}^\top \mathtt{S}_W \mathbf{w}}$$

Notes

$\mathtt{S}_B$ stands for the *between* class scatter matrix. Remind

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

hence we seek:

$$2\mathtt{S}_B\mathbf{w}(\mathbf{w}^\top \mathtt{S}_W \mathbf{w}) = (\mathbf{w}^\top \mathtt{S}_B \mathbf{w})2\mathtt{S}_W\mathbf{w}$$

the expressions within bracket are (unknown) scalars

$$\mathtt{S}_B\mathbf{w} = \lambda \mathtt{S}_W \mathbf{w}$$

leading to eigenvalue problem

$$\mathtt{S}_W^{-1}\mathtt{S}_B\mathbf{w} = \lambda \mathbf{w}$$

However, $\mathtt{S}_B\mathbf{w}$ is always in direction $(\mathbf{m}_2 - \mathbf{m}_1)$, and scale is not important

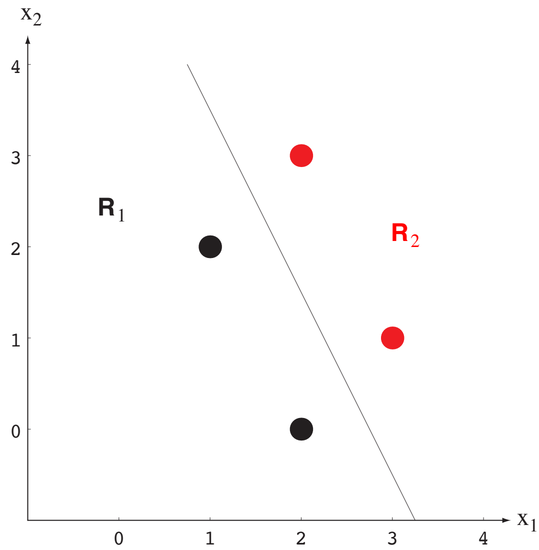
$$\mathbf{w} = \mathtt{S}_W^{-1}(\mathbf{m}_2 - \mathbf{m}_1)$$

LSQ approach to linear classification

$$\mathbf{w} = \begin{bmatrix} w_0 \\ \mathbf{w} \end{bmatrix}$$

$$\mathtt{X}\mathbf{w} = \mathbf{b}$$

$$J(\mathbf{w}) = \|\mathtt{X}\mathbf{w} - \mathbf{b}\|^2$$



Notes

Write dimensions to each symbol, n may stand for the number of points, d for dimensionality of the feature space.
Solving

$$\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} = 0$$

yields $\mathbf{w} = (\mathtt{X}^\top \mathtt{X})^{-1}\mathtt{X}^\top \mathbf{b}$ Try to solve the above figure. We are looking for a separating hyperplane

$$\mathbf{w}^\top \begin{bmatrix} 1 \\ x_1 \\ x_2 \end{bmatrix} = 0$$

and we want points in training set distant from the hyperplane

$$\mathtt{X} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 0 \\ -1 & -3 & -1 \\ -1 & -2 & -3 \end{bmatrix}$$

$$\mathbf{b} = [1\ 1\ 1\ 1]^\top$$

Linear least squares not guaranteed to correctly classify everything on the training set. It's objective function not perfect for classification. Margins $\mathbf{b}$ were set quite arbitrarily.

Outliers can shift the decision boundary.

LSQ approach, better margins $\mathbf{b}$?

$$\mathtt{X} = \begin{bmatrix} \mathtt{1}_1 & \mathtt{X}_1 \\ -\mathtt{1}_2 & -\mathtt{X}_2 \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} \frac{n}{n_1}\mathtt{1}_1 \\ \frac{n}{n_2}\mathtt{1}_2 \end{bmatrix}$$

Notes

After some derivation it can be shown the LSQ solution is equivalent to Fisher linear discriminant
instert into intermediate result when solving $\frac{\partial J(\mathbf{w})}{\partial \mathbf{w}} = 0$

$$\mathtt{X}^\top \mathtt{X}\mathbf{w} = \mathtt{X}^\top \mathbf{b}$$

References I

Further reading: Chapter 4 of [1], or chapter 3 and 5 of [2].

[1] Christopher M. Bishop.
Pattern Recognition and Machine Learning.
Springer Science+Bussiness Media, New York, NY, 2006.
PDF freely downloadable.

[2] Richard O. Duda, Peter E. Hart, and David G. Stork.
Pattern Classification.
John Wiley & Sons, 2nd edition, 2001.

Notes