

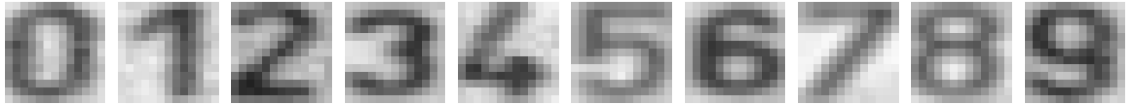
# Classifiers: Naïve Bayes, evaluation

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thanks to Daniel Novák and Filip Železný, Ondřej Drbohlav

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## Example: Digit recognition/classification



- ▶ **Input:** 8-bit image  $13 \times 13$ , pixel intensities 0 – 255. (0 means black, 255 means white)
- ▶ **Output:** Digit 0 – 9. Decision about the class, classification.
- ▶ **Features:** Pixel intensities ...

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### Notes

Digit recognition is a very classical example of classification problem. It has been used for decades, and it is used till today, see e.g. [MNIST demo at PyTorch](#)

# Classification as a special case of statistical decision theory

- ▶ Attribute vector  $\vec{x} = [x_1, x_2, \dots]^T$ : pixels 1, 2, ...
- ▶ **State set  $\mathcal{S}$  = decision set  $\mathcal{D} = \{0, 1, \dots, 9\}$ .**
- ▶ **State = actual class, Decision = recognized class**
- ▶ Loss function:

$$l(s, d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg \min_d \sum_s \underbrace{l(s, d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg \min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously  $\sum_s P(s|\vec{x}) = 1$ , then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg \min_d \left( 1 - P(d|\vec{x}) \right) = \arg \max_d P(d|\vec{x})$$

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## Notes

We are using different word – *classification* instead of *decision* but the reasoning and methods can be well applied in both. In classification problem we usually treat all mistakes – wrong classifications – equally painful, contrary to decision problem – remember “What to cook for dinner” problem?

# Bayes classification in practice; $P(s|\vec{x}) = ?$

- ▶ Usually, we are not given  $P(s|\vec{x})$
- ▶ It has to be estimated from already classified examples – **training data**
- ▶ For discrete  $\vec{x}$ , training examples  $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \dots, (\vec{x}_I, s_I)$ 
  - ▶ every  $(\vec{x}_i, s_i)$  is drawn independently from  $P(\vec{x}, s)$ , i.e. sample  $i$  does not depend on  $1, \dots, i-1$
  - ▶ so-called i.i.d (independent, identically distributed) multiset
- ▶ Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) = \frac{P(\vec{x}, s)}{P(\vec{x})} \approx \frac{\# \text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\# \text{ examples where } \vec{x}_i = \vec{x}}$$

- ▶ Hard in practice:

- ▶ To reliably estimate  $P(s|\vec{x})$ , the number of examples grows exponentially with the number of elements of  $\vec{x}$ .
  - ▶ e.g. with the number of pixels in images
  - ▶ curse of dimensionality
  - ▶ denominator often 0



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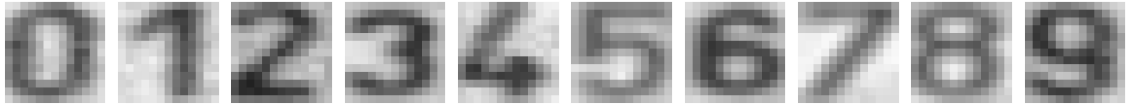
## Notes

Why hard? Way too many various  $\vec{x}$ .

What is the difference between set and multiset?

Reminder about math notation. In literature, vectors are mostly denoted by bold lower case  $\mathbf{x}$ . In lectures, we use  $\vec{x}$  to match notation used on blackboard. It is difficult to write bold with a chalk.

## How many images?



8-bit image  $13 \times 13$ , pixel intensities 0 – 255. (0 means black, 255 means white)

- A:  $169^{256}$
- B:  $256^{169}$
- C:  $13^{13}$
- D:  $169 \times 256$
- E: different quantity

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### Notes

Think about simple binary  $10 \times 10$  image -  $\vec{x}$  contains 0, 1, position matters. What is the total number of unique images? Think binary,  $1 \times 8$  binary image?

And a minimal dataset would contain each possible image at least once! We must be ready for any image (like for any state).

# Naïve Bayes classification

- ▶ For efficient classification we must thus rely on additional assumptions.
- ▶ In the exceptional case of **statistical independence** between components of  $\vec{x}$  for each class  $s$  it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

- ▶ Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots =$$

- ▶ No combinatorial curse in estimating  $P(s)$  and  $P(x[i]|s)$  separately for each  $i$  and  $s$ .
- ▶ No need to estimate  $P(\vec{x})$ . (Why?)
- ▶  $P(s)$  may be provided apriori.
- ▶ **naïve** = when used despite statistical dependence

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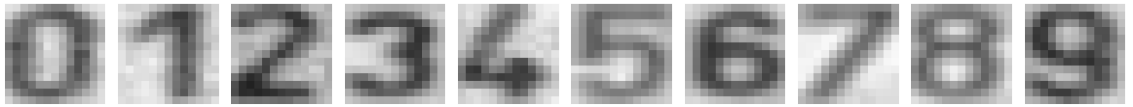
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## Notes

Why naïve at all? Consider  $N$ - dimensional space,  $8 - bit$  values. Instead of problem  $2^{8N}$  we have  $2^8 \times N$  problem.

Think about statistical independence. Example1: person's weight and height. Are they independent? Example2: pixel values in images.

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- ▶ **Features:** Pixel intensities ...

Collect data ...

- ▶  $P(\vec{x})$ . What is the dimension of  $\vec{x}$ ? How many possible images?
- ▶ Learn  $P(\vec{x}|s)$  per each class (digit).
- ▶ Classify  $s^* = \operatorname{argmax}_s P(s|\vec{x})$ .

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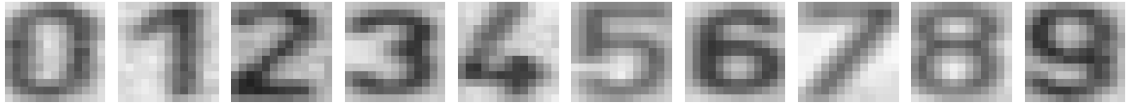
### Notes

We can create many more features than just pixel intensities. But first things first.

We are assuming all errors are equally important - minimizing the number of wrong decisions.

Dimension of  $\vec{x}$  is  $13 \times 13 = 169$ . There are  $256^{169}$  possible images. (we already know)

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Collect **data** , ...

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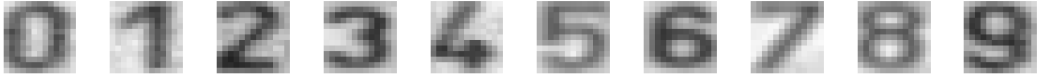
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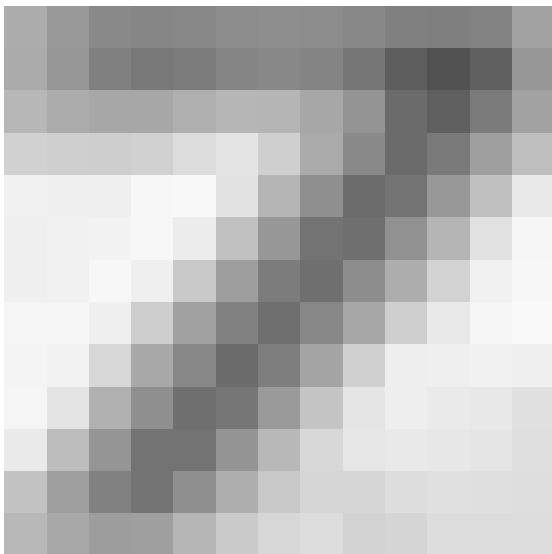
From images to  $\vec{x}$



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Notes

# Conditional probabilities, likelihoods



- ▶ Apriori digit probabilities  $P(s_k)$
- ▶ Likelihoods for pixels.  $P(x_{r,c} = I_i | s_k)$

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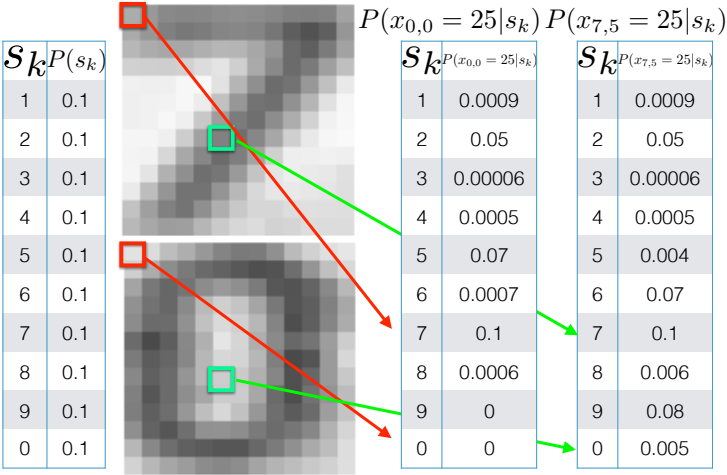
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## Notes

A lexical note, especially for Czech speakers. *probability* as well as *likelihood* can be translated as *pravděpodobnost*. I suggest the following mental model than can work for our purposes.

- **Probability** is related to the future events (unknown outcome). E.g. what is the probability of selecting blue box? What is the probability that a random ZIP Code number begins with 7?
- **Likelihood** refers to past events (known outcome). In my data, how many images of 7 have dark pixel in top right corner? We can think about relative frequency (relativní četnost).

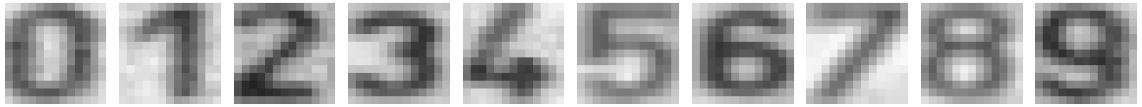
# Conditional likelihoods



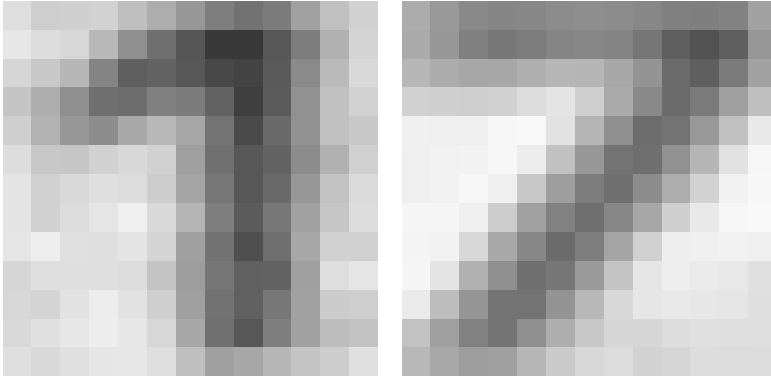
## Notes

For each pixel (position) and possible intensity (image/pixel value) we create such table

## Unseen events



Images  $13 \times 13$ , intensities 0 – 255, 100 exemplars per each class.



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### Notes

Think about the problem of classifying numerals. Some  $P(x_{r,c} = l \mid s) = 0$ . What about an example:

$$P(x_{0,0} = 100 \mid s = 7) = 0.05$$

$$P(x_{0,0} = 101 \mid s = 7) = 0$$

$$P(x_{0,0} = 102 \mid s = 7) = 0.06$$

A new (not in training) query image with  $x_{0,0} = 101$ . How would you classify?

## Unseen event, how to decide?

A new (not in training) query image with  $x_{0,0} = 101$ . How would you classify?

$$P(x_{0,0} = 101 | s_j) = 0, \text{ for all classes}$$

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Notes

# Laplace smoothing (“additive smoothing”)

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem:  $\text{count}(x) = 0$

Pretend you see the (any) sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{N + |X|}$$

where  $N$  is the number of (total) observations;  $|X|$  is the number of possible values  $X$  can take (cardinality).

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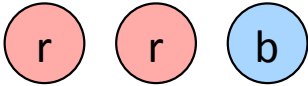
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$$P_{LAP}(X) = ?$$

Observation:



What is  $P_{LAP}(X = \text{red})$  and  $P_{LAP}(X = \text{blue})$ ?

A:  $P_{LAP}(X = \text{red}) = 7/10$ ,  $P_{LAP}(X = \text{blue}) = 3/10$

B:  $P_{LAP}(X = \text{red}) = 2/3$ ,  $P_{LAP}(X = \text{blue}) = 1/3$

C:  $P_{LAP}(X = \text{red}) = 3/5$ ,  $P_{LAP}(X = \text{blue}) = 2/5$

D: None of the above.

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Notes



$$P_{ML}(X) =$$

$$P_{LAP}(X) =$$

originally:

- $P(\text{red}) = 2/3$
- $P(\text{blue}) = 1/3$

after Laplace smoothing – adding one red ball and blue ball to the actual observations:

- $P_{LAP}(\text{red}) = (2 + 1)/(2 + 1 + 1 + 1) = 3/5$
- $P_{LAP}(\text{blue}) = (1 + 1)/(2 + 1 + 1 + 1) = 2/5$

this slide: courtesy of P. Abeel, <http://ai.berkeley.edu>. 21st lecture of CS 188.

# Laplace smoothing - as a hyperparameter $k$

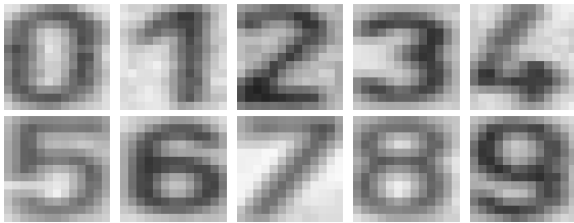
Pretend you see every sample  $k$  extra times:

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{\sum_x [c(x) + k]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{N + k|X|}$$

For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x, s) + k}{c(s) + k|X|}$$



What is  $|X|$  equal to?

A: 10

B: 2

C: 256

D: None of the above

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## Notes

Hyperparameter would be tuned along with your classifier

For  $k = 100$  and blue and red, you would get:

- $P_{\text{LAP}}(\text{red}) = (2 + 100)/(3 + 100 * 2) = 102/203$
- $P_{\text{LAP}}(\text{blue}) = (1 + 100)/(3 + 100 * 2) = 101/203$

In this case, smoothing ("prior") would dominate over the observations - shifting estimate from empirical to uniform.

In the digit recognition from pixels example: 256 intensity values;  $13 \times 13 = 169$  pixels: Applying Laplace smoothing with  $k = 1$  to  $P(x)$  (prior probability of a particular pixel will take an intensity value  $i$ ):  $P(x_{r,c} = i) = (c(x) + 1)/(N + 256)$

Conditional: relevant for the Naïve Bayes case.

# Laplace smoothing - as a hyperparameter $k$

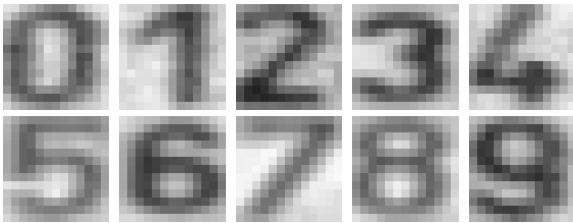
Pretend you see every sample  $k$  extra times:

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What is  $|X|$  equal to?

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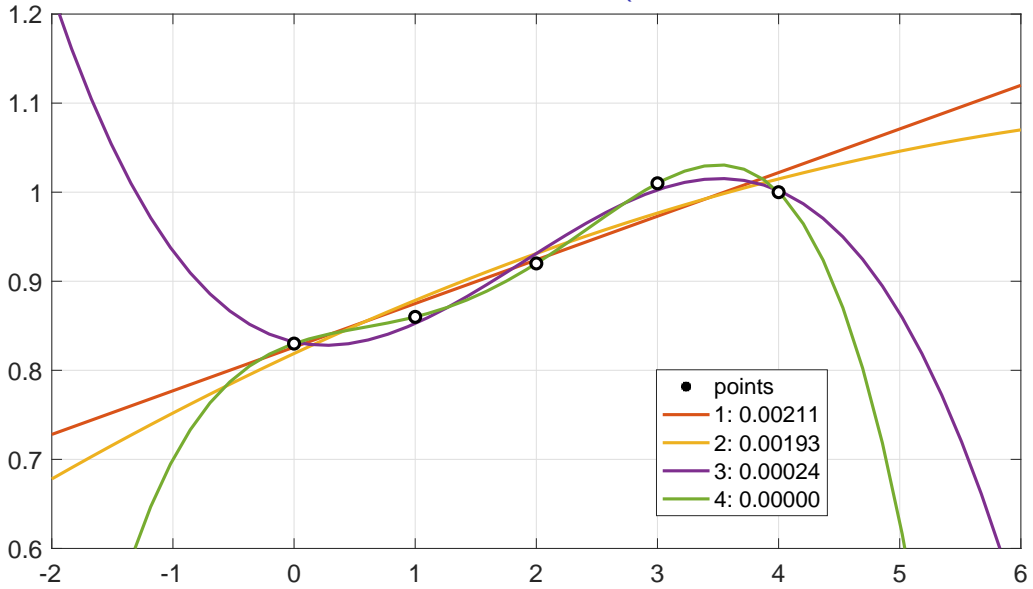
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Conditional: relevant for the Naïve Bayes case.

# What is the right degree of polynomial (hyperparameter of a regressor)



## Notes

See the `tuning_hyper_parameter.m` demo. The small values depict sum of square errors on *training data*.

# Generalization and overfitting

- ▶ **Data: training, validating, testing** . Wanted classifier performs well on what data?
- ▶ Overfitting: too close to training, poor on testing.

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# Training and testing

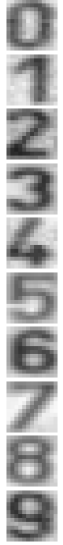
**Data** labeled instances.

- ▶ Training set
- ▶ Held-out (validation) set
- ▶ Testing set.

**Features** : Attribute-value pairs.

**Learning cycle:**

- ▶ **Learn** parameters (e.g. probabilities) on training set.
- ▶ **Tune** hyperparameters on held-out (validation) set.
- ▶ **Evaluate** performance on testing set.



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## Notes

Training set - biggest part.

# How to evaluate a classifier? Confusion table

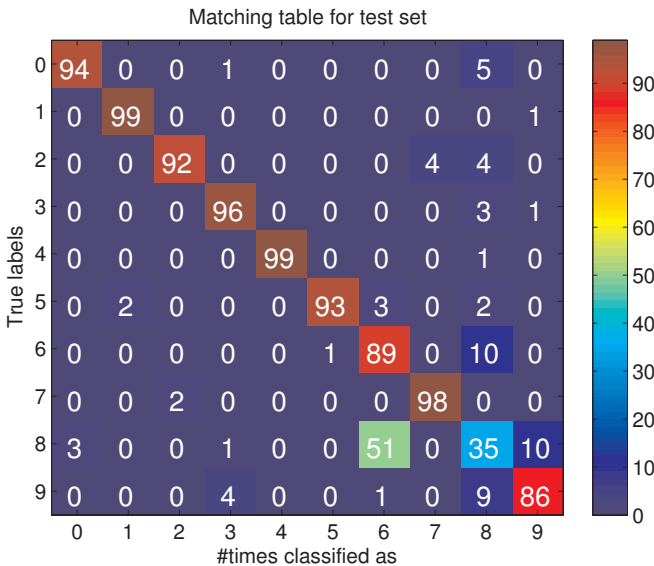


Figure from [5]

## Notes

A result for a one particular classifier and its setting (parameters), one particular testing set



# Precision and Recall, and ...

Consider digit **detection** (is there a digit?) or SPAM/HAM classification.

**Recall** :

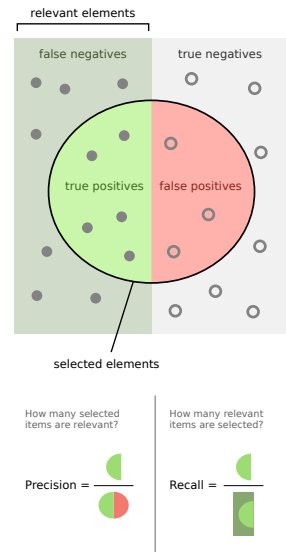
- ▶ How many relevant items are selected?
- ▶ Are we missing some items?
- ▶ Also called: **True positive rate** (TPR), sensitivity, hit rate ...

**Precision**

- ▶ How many selected items are relevant?
- ▶ Also called: Positive predictive value

**False positive rate** (FPR)

- ▶ Probability of false alarm



By Walber - Own work, CC BY-SA 4.0, <https://commons.wikimedia.org/w/index.php?curid=36926283>

## Notes

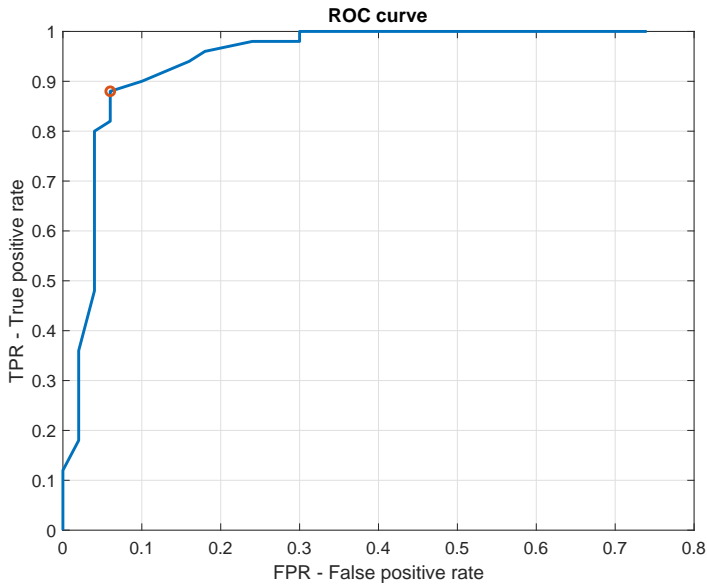
$$\text{TPR} = \frac{\text{TP}}{P} = \frac{\text{TP}}{\text{TP} + \text{FN}}$$

$$\text{Precision} = \frac{\text{TP}}{\text{TP} + \text{FP}}$$

$$\text{FPR} = \frac{\text{FP}}{N} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

Think about TPR vs FPR graph, what is the best classifier?

# ROC – Receiver operating characteristics curve



$$\text{TPR} = \frac{\text{TP}}{P} = \frac{\text{TP}}{\text{TP} + \text{FN}}$$
$$\text{FPR} = \frac{\text{FP}}{N} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

## Notes

- How do you slide along the curve?
- What is the meaning of the diagonal?
- What would be the shape of the curve for the ideal/worst classifier?
- How would you compare various curve and select the best classifier?
- Think/read about other ways to evaluate/visualise classification results.

# Discriminant functions $f(\vec{x}, s)$

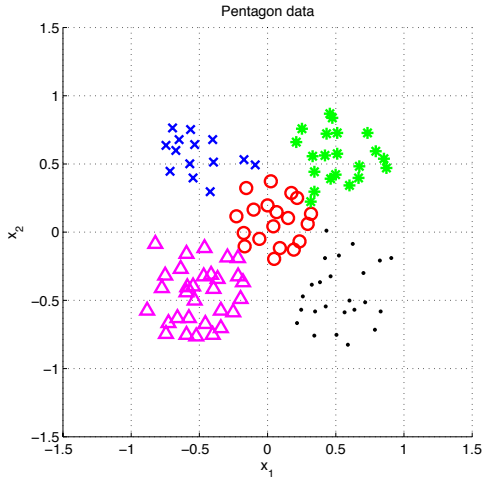
$$s^* = \operatorname{argmax}_{s \in \mathcal{S}} f(\vec{x}, s)$$

Conditional likelihoods:  $\mathcal{N}(\vec{x} | \vec{\mu}_s, \Sigma_s)$

$$\frac{1}{2\pi |\Sigma_s|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\right\}$$

Bayes:

$$s^* = \operatorname{argmax}_{s \in \mathcal{S}} P(s | \vec{x}) = \frac{P(\vec{x} | s)P(s)}{P(\vec{x})}$$



Discriminant function:

$$s^* = \operatorname{argmax}_{s \in \mathcal{S}} f(\vec{x}, s) = P(s) \frac{1}{2\pi |\Sigma_s|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\right\}$$

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## Notes

Normal distribution for general dimensionality D:

$$\mathcal{N}(\vec{x} | \vec{\mu}, \Sigma) = \frac{1}{(2\pi)^{D/2}} \frac{1}{|\Sigma|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu})^\top \Sigma^{-1}(\vec{x} - \vec{\mu})\right\}$$

How about learning  $f(\vec{x}, s)$  directly without explicit modeling of underlying probabilities?

What about  $f(\vec{x}, s) = \vec{w}_s^\top \vec{x} + w_{s0}$

## Towards linear classifier, geometrical thoughts ...

$$f(\vec{x}, s) = P(s) \frac{1}{2\pi |\Sigma_s|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\right\}$$

# Product of many small numbers ...

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

$P(\vec{x})$  not needed, .....

$$\log(P(x[1]|s)P(x[2]|s) \dots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \dots$$

---

## Notes

just try

- `prod(rand(1,100))` and `prod(rand(1,10000))` in Matlab.
- `prod(rand(1,100)) == 0` and `prod(rand(1,10000)) == 0` in Matlab.

or in python console:

- ```
>>> import numpy as np
```
- ```
>>> np.prod(np.random.rand(100))==0
```
- ```
>>> np.prod(np.random.rand(1000))==0
```
- ```
>>> a = np.random.rand(1000)
>>> b = np.random.rand(1000)
>>> np.prod(a)>np.prod(b)
False
>>> np.prod(a)<np.prod(b)
False
>>> np.sum(np.log(a))>np.sum(np.log(b))
True
```

Hitting the limit of number representation.

What is the way out?

$P(\vec{x})$  not needed – does not depend on the class.

Laws of logarithms...

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Laws of logarithms...

# References I

Further reading: Chapter 13 and 14 of [4]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. This lecture has been also inspired by the 21st lecture of CS 188 at <http://ai.berkeley.edu> (e.g., Laplace smoothing). Many Matlab figures created with the help of [3].

[1] Christopher M. Bishop.

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