Probabilistic classification

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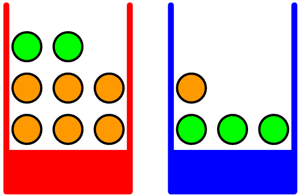
April 21, 2021

(Re-)introduction uncertainty/probability

- Markov Decision Processes (MDP) – uncertainty about outcome of actions
- Now: uncertainty may be also associated with states
  - Different states may have different prior probabilities.
  - The states $s \in \mathcal{S}$ may not be directly observable .
  - They need to be inferred from features $x \in \mathcal{X}$.
- This is addressed by the rules of probability *(such as Bayes theorem)* and leads on to
  - Bayesian classification
  - Bayesian decision making

Probability example: Picking fruits

- red box: 2 apples, 6 oranges
- blue box: 3 apples, 1 orange

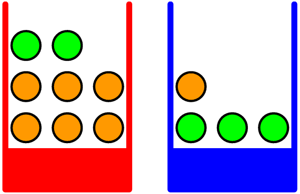


- Scenario: Pick a box—say red box in 40% cases—, *then* pick a fruit at random.
- (Frequent) questions:
  - What is the overall probability that the selection procedure will pick an apple?
  - Given that we have chosen an orange, what is the probability that the box we chose was the blue one?

Example from Chapter 1.2 [1]

Picking fruits. What is the probability that . . . ?

- red box: 2 apples, 6 oranges
- blue box: 3 apples, 1 orange



Procedure: Pick a box (say red box in 40% cases), then pick a fruit at random.

What is the probability that the selection procedure will pick an apple?

A: $11/20$

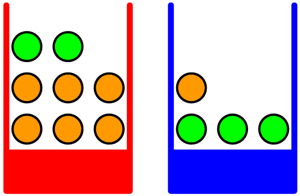
B: $6/8$

C: $1/2$

D: Different value.

Picking fruits. What is the probability that . . . ?

- red box: 2 apples, 6 oranges
- blue box: 3 apples, 1 orange



Procedure: Pick a box (say red box in 40% cases), then pick a fruit at random.

Given that we have chosen an orange, what is the probability that the box we chose was the blue one?

A: $1/4$

B: $3/5$

C: $1/3$

D: Different value.

Rules of probability and notation I

- random variables X, Y
- x_i where $i = 1, ..., M$ – values taken by variable X
- y_j where $j = 1, ..., L$ – values taken by variable Y
- $P(X = x_i, Y = y_i)$ – probability that X takes the value x_i *and* Y takes y_i – joint probability
- $P(X = x_i)$ – probability that X takes the value x_i
- Sum rule of probability :
  - $P(X = x_i) = \sum_{j=1}^{L} P(X = x_i, Y = y_j)$
  - $P(X = x_i)$ is sometimes called marginal probability – obtained by marginalizing / summing out the other variables
  - general rule, compact notation: $P(X) = \sum_Y P(X, Y)$

Rules of probability and notation II

- Conditional probability : $P(Y = y_j | X = x_i)$
- Product rule of probability :
  - $P(X = x_i, Y = y_i) = P(Y = y_j | X = x_i) P(X = x_i)$
  - general rule, compact notation: $P(X, Y) = P(Y|X)P(X)$
- Bayes theorem :
  - from $P(X, Y) = P(Y, X)$ and product rule

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

$$posterior = \frac{likelihood \times prior}{evidence}$$

- Independence : $P(X, Y) = P(X)P(Y)$

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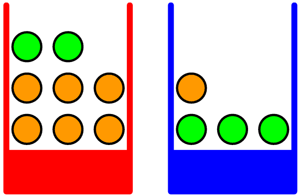
Boxes and Fruits: posterior? likelihood? prior? evidence?

$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}}$$



Connect with lines:

- posterior
  after observation
- likelihood
  of an observation
- prior
  before observation
- evidence
  total observations

- $P(B)$
- $P(F)$
- $P(F \mid B)$
- $P(B \mid F)$

Decision example: Insure or not? (from late 1980s) [4]

A doctor calls: "Your HIV test is positive, 999/1000 you will die in 10 years. I'm sorry . . . ". Insurance company does not want to insure a married couple.

- Was the doctor right?
- Was the insurance company rational?

What the doctor (and the company) knew:

- HIV test falsely positive only in 1 case out of 1000.

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What is the probability the man is infected?

A: $\frac{1}{1000}$

B: $\frac{999}{1000}$

C: Don't know yet, more info needed, but less than $\frac{1}{2}$

D: Don't know yet, more info needed, but more than $\frac{1}{2}$

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A doctor calls: "Your HIV test is positive, 999/1000 you will die in 10 years. I'm sorry . . . ".
Insurance company does not want to insure a married couple.

- Was the doctor right?
- Was the insurance company rational?

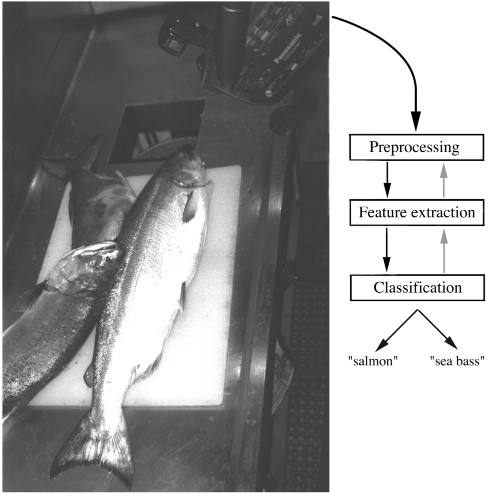
What the doctor (and the company) knew:

- HIV test falsely positive only in 1 case out of 1000.
- Heterosexual male, has family, no drugs, no risk behavior.

Decision: guilty or not? (people of CA vs Collins, 1968) [4]

- Robbery, LA 1964, fuzzy evidence of the offenders:
  - female, around 65 kg
  - wearing something dark
  - hair of light color, between light and dark blond, in a ponytail
- At the same time, additional evidence close to the crime scene:
  - loud scream, yelling, looking at the this direction ...
  - a woman sitting into a yellow car
  - car starts immediately and passes close to the additional witness
  - a black man with beard and moustache was driving
- No more evidence
- Testimony of both the victim and the witness not unambiguous (didn't recognize suspects)
- Still, the suspects were sentenced to jail.

Classification example: What's the fish?



- Factory for fish processing
- 2 classes $s_{1,2}$:
  - salmon
  - sea bass
- Features $\vec{x}$: length, width, lightness etc. from a camera

Fish – classification using probability

$$posterior = \frac{likelihood \times prior}{evidence}$$

- Notation for classification problem
  - Classes $s_j \in \mathcal{S}$ (e.g., salmon, sea bass)
  - Features $x_i \in \mathcal{X}$ or feature vectors ($\vec{x}_i$) (also called attributes)
- Optimal classification of $\vec{x}$:

$$\delta^*(\vec{x}) = \arg\max_j P(s_j|\vec{x})$$

- We thus choose the most probable class for a given feature vector .
- Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j|\vec{x}) = \frac{P(\vec{x}|s_j)P(s_j)}{P(\vec{x})}$$

- Can we do (classify) better?

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Bayes classification in practice

- Usually, we are not given $P(s|\vec{x})$
- It has to be estimated from already classified examples – training data
- For discrete $\vec{x}$, training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \dots (\vec{x}_l, s_l)$
  - so-called i.i.d (independent, identically distributed) multiset
  - every $(\vec{x}_i, s)$ is drawn independently from $P(\vec{x}, s)$
- Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) \approx \frac{\#\text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\#\text{ examples where } \vec{x}_i = \vec{x}}$$

- Hard in practice:
  - To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of $\vec{x}$.
    - e.g. with the number of pixels in images
    - curse of dimensionality
    - denominator often 0

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Naïve Bayes classification

- For efficient classification we must thus rely on additional assumptions.
- In the exceptional case of statistical independence between $\vec{x}$ components for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

- Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots =$$

- No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s.
- No need to estimate $P(\vec{x})$. (Why?)
- $P(s)$ may be provided apriori.
- naïve = when used despite statistical dependence

Decision making under uncertainty

- An important feature of intelligent systems
  - make the best possible decision
  - in uncertain conditions
- **Example**: Take a tram OR subway from A to B?
  - Tram: timetables imply a quicker route, but adherence uncertain.
  - Subway: longer route, but adherence almost certain.
- **Example**: where to route a letter with this ZIP?
  - 15700? 15706? 15200? 15206?
- What is the optimal decision ?
- Both examples fall into the same framework.

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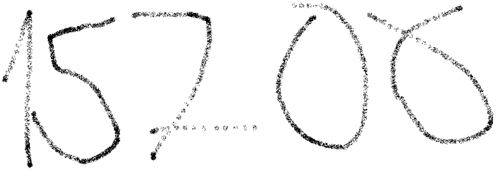

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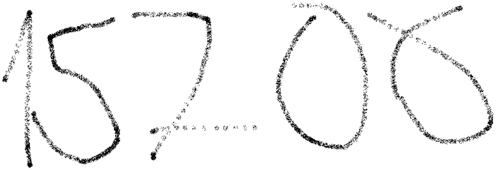

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Example: What to cook for a dinner [3]

- *Wife coming back from work. Husband: what to cook for dinner?*
- 3 dishes (decisions) in his repertoire:
  - *nothing* … **don't bother cooking** ⇒ no work but makes wife upset
  - *pizza* … **microwave a frozen pizza** ⇒ not much work but won't impress
  - *g.T.c.* … **general Tso's chicken** ⇒ will make her day, but very laborious.
- Hassle incurred by the individual options depends wife's feeling
- For each of the 9 possible situation (3 possible decisions × 3 possible states) the hasse is quantified by a loss function $l(d, s)$:

$l(s,d)$	$d = nothing$	$d = pizza$	$d = g.T.c.$
$s = good$	0	2	4
$s = average$	5	3	5
$s = bad$	10	9	6

Wife's state of mind is an uncertain state.

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Example (cont'd), State uncertain, ...

- Husband's experiment. He tells her he accidentally overtaped their wedding video and observes her reaction.
- Anticipates 4 possible reactions:
  - *mild* ... all right, we keep our memories.
  - *irritated* ... how many times do I have to tell you....
  - *upset* ... Why did I marry this guy?
  - *alarming* ... silence
- The reaction is a measurable attribute ("feature") of the mind state.
- From experience, the husband knows how individual reactions are probable in each state of mind; this is captured by the joint distribution $P(x, s)$.

$P(x,s)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$s = good$	0.35	0.28	0.07	0.00
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Decision strategy

- **Decision strategy** : a rule selecting a decision for *any given value* of the measured attribute(s).
- i.e. function $d = \delta(x)$.
- Example of husband's possible strategies:

$\delta(x)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$\delta_1(x) =$	*nothing*	*nothing*	*pizza*	*g.T.c.*
$\delta_2(x) =$	*nothing*	*pizza*	*g.T.c.*	*g.T.c.*
$\delta_3(x) =$	*g.T.c.*	*g.T.c.*	*g.T.c.*	*g.T.c.*
$\delta_4(x) =$	*nothing*	*nothing*	*nothing*	*nothing*

- How many strategies?
- How to define which strategy is best? How to sort them by quality?
- Define the risk of a strategy as a mean (expected) loss value .

$$r(\delta) = \sum_x \sum_s l(s, \delta(x)) P(x, s)$$

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Calculating $r(\delta) = \sum_x \sum_s l(s, \delta(x)) P(x, s)$

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$s = bad$	0.00	0.02	0.05	0.03

$\delta(x)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$\delta_1(x) =$	$nothing$	$nothing$	$pizza$	$g.T.c.$
$\delta_2(x) =$	$nothing$	$pizza$	$g.T.c.$	$g.T.c.$
$\delta_3(x) =$	$g.T.c.$	$g.T.c.$	$g.T.c.$	$g.T.c.$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Do we need to evaluate all possible strategies? $P(x, s) = P(s|x)P(x)$

Calculating $r(\delta) = \sum_x \sum_s l(s, \delta(x))P(x, s)$

$l(s, d)$	$d =$ *nothing*	$d =$ *pizza*	$d =$ *g.T.c.*
$s =$ *good*	0	2	4
$s =$ *average*	5	3	5
$s =$ *bad*	10	9	6

$P(x, s)$	$x =$ *mild*	$x =$*irritated*	$x =$ *upset*	$x =$ *alarming*
$s =$ *good*	0.35	0.28	0.07	0.00
$s =$ *average*	0.04	0.10	0.04	0.02
$s =$ *bad*	0.00	0.02	0.05	0.03

$\delta(x)$	$x =$ *mild*	$x =$ *irritated*	$x =$ *upset*	$x =$ *alarming*
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$\delta_3(x) =$	*g.T.c.*	*g.T.c.*	*g.T.c.*	*g.T.c.*
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$\delta_3(x) =$	*g.T.c.*	*g.T.c.*	*g.T.c.*	*g.T.c.*
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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Do we need to evaluate all possible strategies?

$P(x, s) = P(s|x)P(x)$

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$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Do we need to evaluate all possible strategies? $P(x,s) = P(s|x)P(x)$

Bayes optimal strategy

- The Bayes optimal strategy : one minimizing mean risk.

$$\delta^* = \arg\min_{\delta} r(\delta)$$

- From $P(x, s) = P(s|x)P(x)$ (Bayes rule), we have

$$r(\delta) = \sum_x \sum_s l(s, \delta(x))P(x, s) = \sum_s \sum_x l(s, \delta(x))P(s|x)P(x)$$
$$= \sum_x P(x) \underbrace{\sum_s l(s, \delta(x))P(s|x)}_{\text{Conditional risk}}$$

- The optimal strategy is obtained by minimizing the conditional risk *separately* for each x:

$$\delta^*(x) = \arg\min_{d} \sum_s l(s, d)P(s|x)$$

Optimal strategy: $\delta^*(x) = \arg\min_d \sum_s l(s,d)P(s|x)$

$l(s,d)$	$d =$ *nothing*	$d =$ *pizza*	$d =$ *g.T.c.*
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$\delta^*(x) =$	??	??	??	??

Statistical decision making: wrapping up

- **Given**:
  - A set of possible states : $\mathcal{S}$
  - A set of possible decisions : $\mathcal{D}$
  - A loss function $l : \mathcal{D} \times \mathcal{S} \to \Re$
  - The range $\mathcal{X}$ of the attribute
  - Distribution $P(x, s)$, $x \in \mathcal{X}, s \in \mathcal{S}$.
- **Define**:
  - Strategy : function $\delta : \mathcal{X} \to \mathcal{D}$
  - Risk of strategy δ : $r(\delta) = \sum_x \sum_s l(s, \delta(x))P(x, s)$
- **Bayes problem**:
  - Goal: find the optimal strategy $\delta^* = \arg\min_{\delta \in \Delta} r(\delta)$
  - Solution: $\delta^*(x) = \arg\min_d \sum_s l(s, d)P(s|x)$

A special case - Bayesian *classification*

- Bayesian classification is a special case of statistical decision theory:
  - Attribute vector $\vec{x} = (x_1, x_2, \dots)$: pixels 1, 2, ...
  - **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \dots 9\}$.**
  - State = actual class, Decision = recognized class
  - Loss function:

$$l(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{l(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg\min_d [1 - P(d|\vec{x})] = \arg\max_d P(d|\vec{x})$$

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References I

Further reading: Chapter 13 and 14 of [6]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. An interesting insights into how people think and interact with probabilities are presented in [4] (in Czech as [5])

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