Sequential decisions under uncertainty
Markov Decision Processes (MDP)

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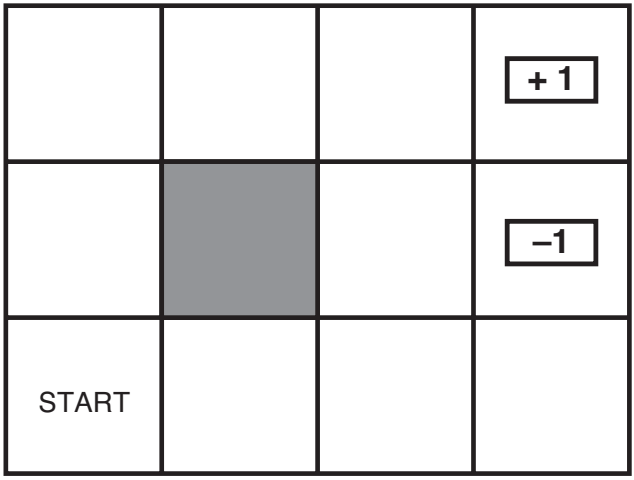
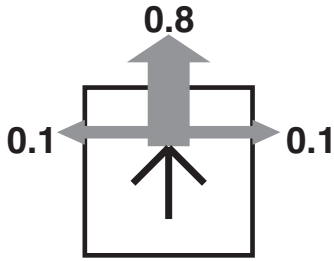
Unreliable actions in observable grid world





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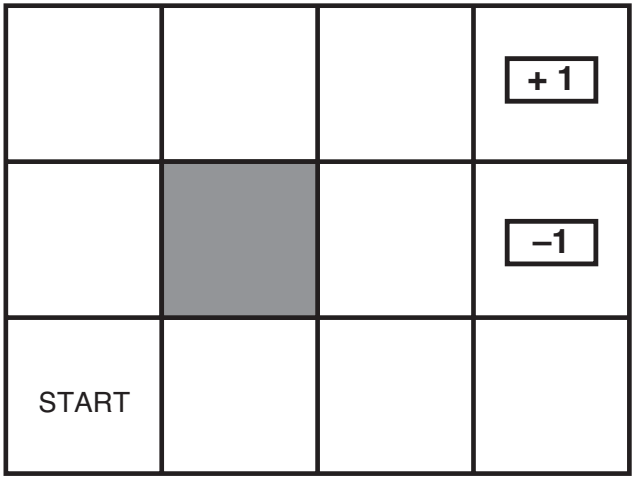

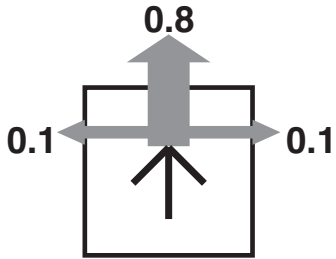

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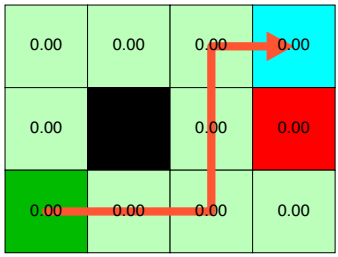
Unreliable (results of) actions

Plan? Policy

- In deterministic world: Plan – sequence of actions from Start to Goal.
- MDPs, we need a *policy* $\pi : \mathcal{S} \to \mathcal{A}$.
- An action for each possible state. Why *each*?
- What is the *best* policy?



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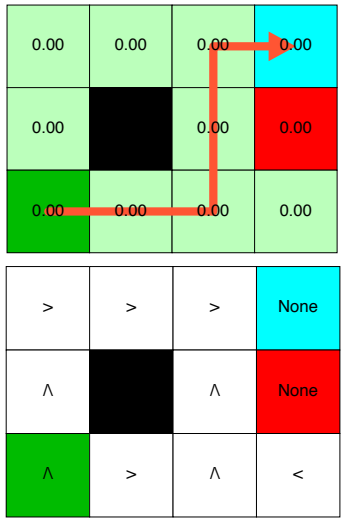

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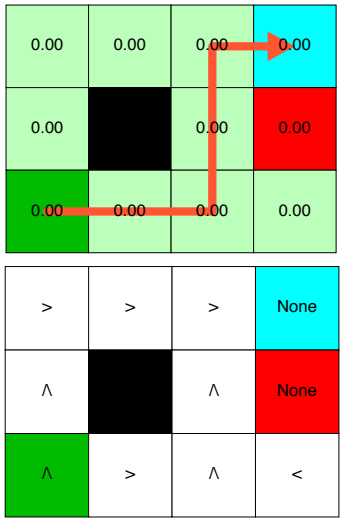
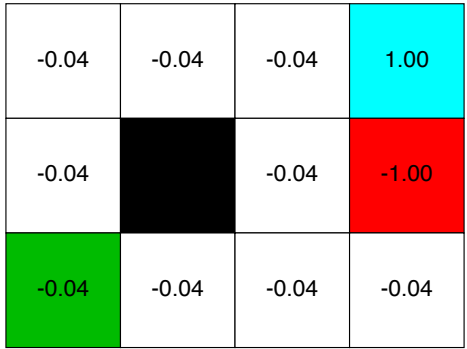


Rewards

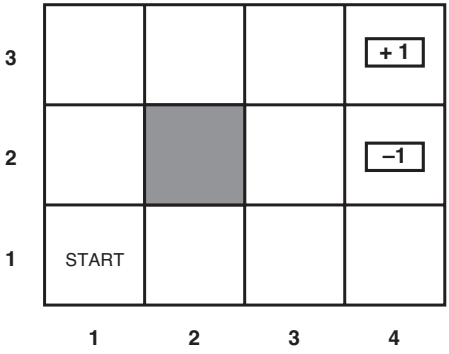


Reward : Robot/Agent takes an action a and it is **immediately** rewarded.

Reward function $r(s)$ (or $r(s,a)$, $r(s,a,s')$)

$$= \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

Markov Decision Processes (MDPs)



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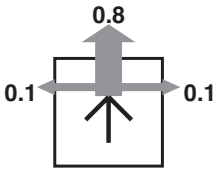

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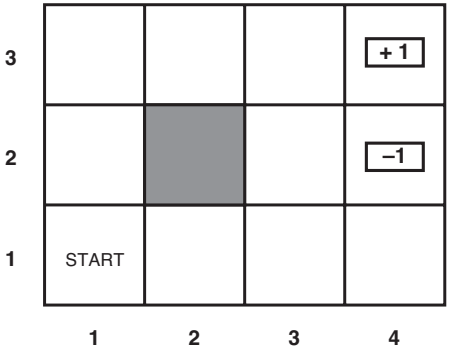

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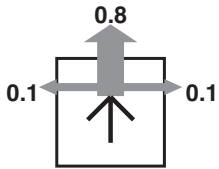

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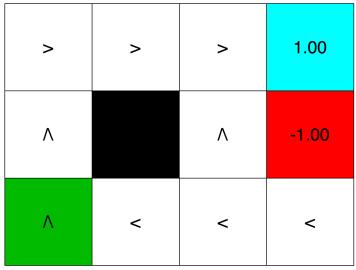
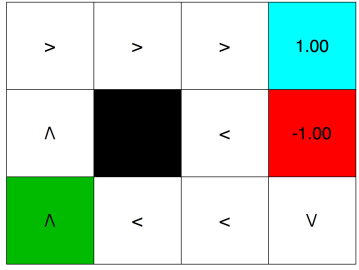
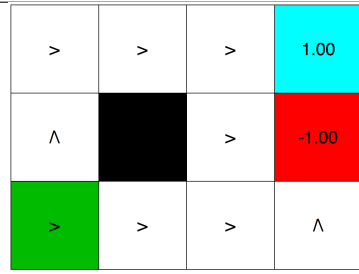
Markovian property

- Given the present state, the future and the past are independent.
- MDP: Markov means action depends only on the current state.
- In search: successor function (transition model) depends on the current state only.

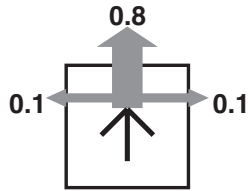
		
A	B	C
$r(s) = \{-2, 1, -1\}$	$r(s) = \{-0.04, 1, -1\}$	$r(s) = \{-0.01, 1, -1\}$
a	b	c

A: A-a, B-b, C-c

B: A-b, B-a, C-c

C: A-b, B-c, C-a

D: A-c, B-a, C-b



Utilities of sequences

- State reward at time/step t, R_t.
- State at time t, S_t. State sequence $[S_0, S_1, S_2, \ldots,]$

Typically, consider stationary preferences on reward sequences:

$$[R, R_1, R_2, R_3, \ldots] \succ [R, R'_1, R'_2, R'_3, \ldots] \Leftrightarrow [R_1, R_2, R_3, \ldots] \succ [R'_1, R'_2, R'_3, \ldots]$$

If stationary preferences :
Utility (h-history)
$U_h([S_0, S_1, S_2, \ldots,]) = R_1 + R_2 + R_3 + \cdots$

If the horizon is finite - limited number of steps - preferences are nonstationary (depends on how many steps left).

Returns and Episodes

- Executing policy - sequence of states and **rewards**.
- Episode starts at t, ends at T (ending in a terminal state).
- Return (Utility) of the episode (policy execution)

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \cdots + R_T$$

Comparing policies: Finite vs. infinite horizon

Problem: Infinite lifetime $\Rightarrow$ additive utilities are infinite.

- Finite horizon: termination at a fixed time $\Rightarrow$ nonstationary policy, $\pi(s)$ depends on the time left.
- Absorbing (terminal) state.
- Discounted return , $\gamma < 1, R_t \leq R_{\max}$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \cdots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{\max}}{1-\gamma}$$

Returns are successive steps related to each other

$$\begin{aligned} G_t &= R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \gamma^3 R_{t+4} + \cdots \\ &= R_{t+1} + \gamma(R_{t+2} + \gamma^1 R_{t+3} + \gamma^2 R_{t+4} + \cdots) \\ &= R_{t+1} + \gamma G_{t+1} \end{aligned}$$

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MDPs recap

Markov decision processes (MDPs):

- Set of states $\mathcal{S}$
- Set of actions $\mathcal{A}$
- Transitions $p(s'|s,a)$ or $T(s,a,s')$
- Reward function $r(s,a,s')$; and discount γ

MDP quantities:

- (deterministic) Policy $\pi(s)$ – choice of action for each state
- Return (Utility) of an episode (sequence) – sum of (discounted) rewards.

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Value functions

- Executing policy $\pi \rightarrow$ sequence of states (and rewards).
- Utility of a state sequence.
- But actions are unreliable - environment is stochastic.
- Expected return of a policy π.

Starting at time t, i.e. S_t,

$$U^{\pi}(S_t) = \mathrm{E}^{\pi}\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k+1}\right]$$

Value function

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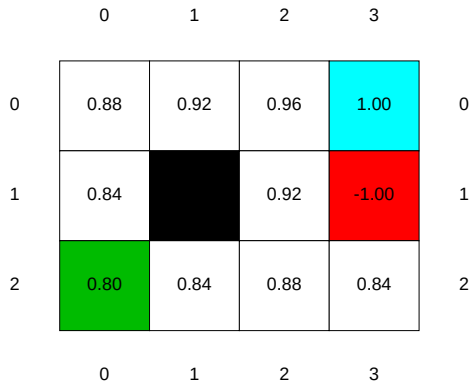

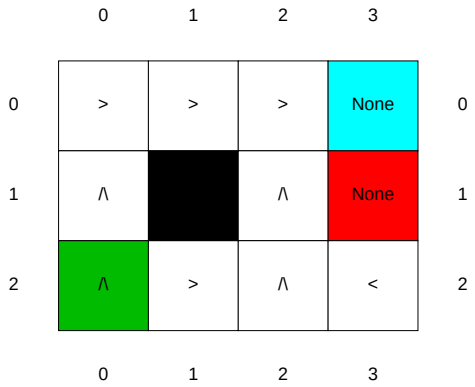

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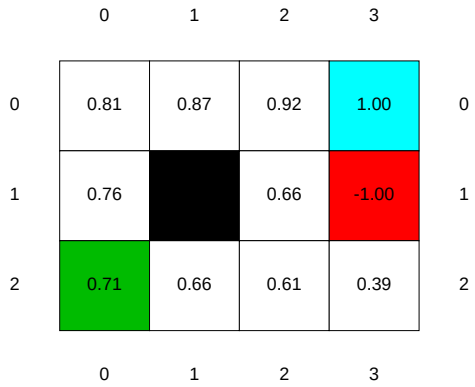

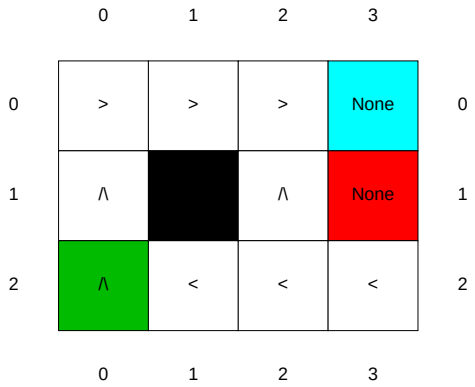

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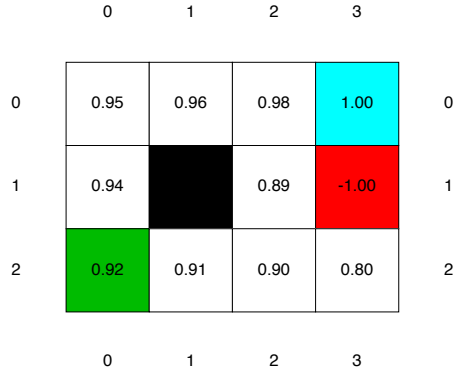

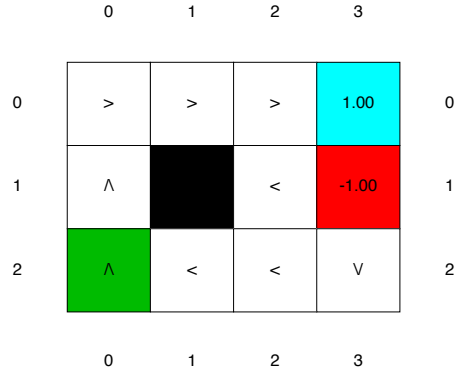

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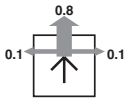
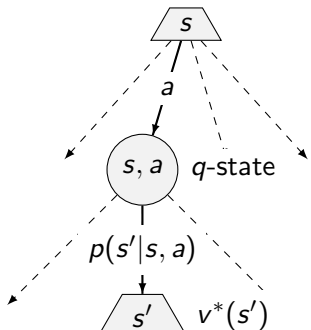


MDP search tree

The value of a q-state (s, a):

$$q^*(s,a) = \sum_{s'} p(s'|a,s) \left[r(s,a,s') + \gamma \, v^*(s') \right]$$



MDP search tree

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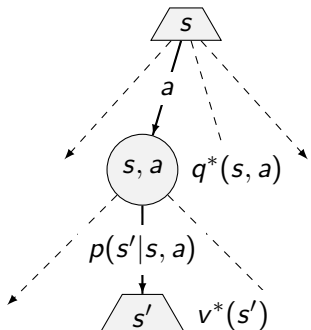

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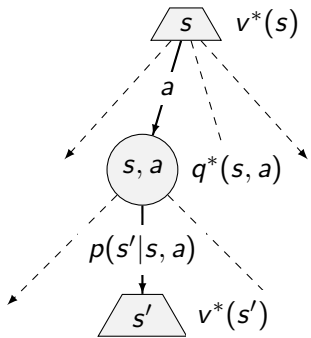
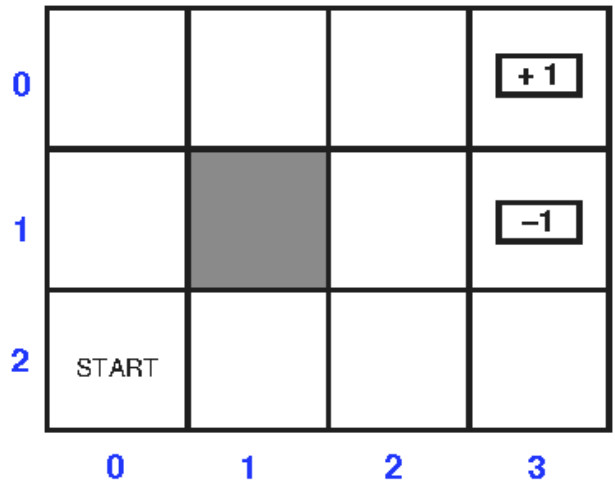
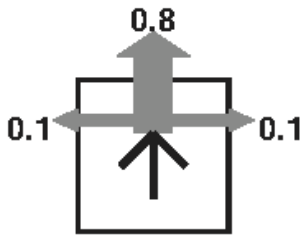


Bellman (optimality) equation

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a,s) \left[r(s,a,s') + \gamma v^*(s') \right]$$





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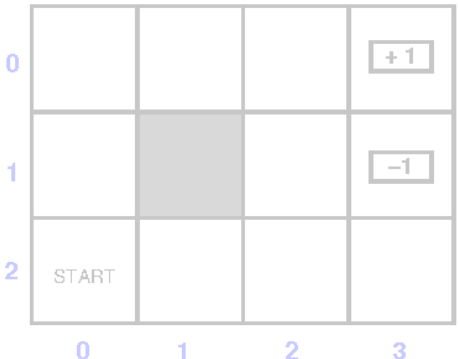

Value iteration

- Start with arbitrary $V_0(s)$ (except for terminals)
- Compute Bellman update (one ply of expectimax from each state)

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$$

- Repeat until convergence

The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent $\Rightarrow$ globally optimal.

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The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent $\Rightarrow$ globally optimal.

Value iteration - Complexity of one estimation sweep

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$$

A: $O(AS)$
B: $O(S^2)$
C: $O(AS^2)$
D: $O(A^2S^2)$

Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \leq R(s) \leq R_{\max}$$

Max norm:

$$\|V\| = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \ldots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma}$$

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Convergence cont'd

$V_{k+1} \leftarrow BV_k \ldots B$ as the Bellman update $V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s,a) V_k(s')$

$\|BV_k - BV'_k\| \leq \gamma \|V_k - V'_k\|$

$\|BV_k - V_{\text{true}}\| \leq \gamma \|V_k - V_{\text{true}}\|$

Rewards are bounded, at the beginning then Value error is

$\|V_0 - V_{true}\| \leq \frac{2R_{\max}}{1-\gamma}$

We run N iterations and reduce the error by factor γ in each and want to stop the error is below ϵ:

$\gamma^N 2R_{\max}/(1-\gamma) \leq \epsilon$ Taking logs, we find: $N \geq \frac{\log(2R_{\max}/\epsilon(1-\gamma))}{\log(1/\gamma)}$

To stop the iteration we want to find a bound relating the error to the size of *one* Bellman update for any given iteration.

We stop if

$$\|V_{k+1} - V_k\| \leq \frac{\epsilon(1-\gamma)}{\gamma}$$

then also: $\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$ Proof on the next slide

Convergence cont'd

$\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$ is the same as $\|V_{k+1} - V_\infty\| \leq \epsilon$
Assume $\|V_{k+1} - V_k\| = \text{err}$
In each of the following iteration steps we reduce the error by the factor γ (because $\|BV_k - V_{\text{true}}\| \leq \gamma\|V_k - V_{\text{true}}\|$). Till ∞, the total sum of reduced errors is:

$$\text{total} = \gamma\text{err} + \gamma^2\text{err} + \gamma^3\text{err} + \gamma^4\text{err} + \cdots = \frac{\gamma\text{err}}{(1-\gamma)}$$

We want to have total $< \epsilon$.

$$\frac{\gamma\text{err}}{(1-\gamma)} < \epsilon$$

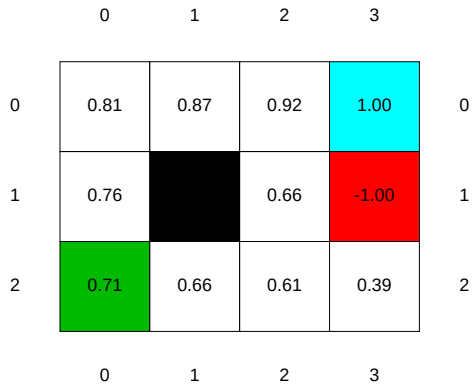
From it follows that

$$\text{err} < \frac{\epsilon(1-\gamma)}{\gamma}$$

Hence we can stop if $\|V_{k+1} - V_k\| < \epsilon(1-\gamma)/\gamma$

Value iteration demo

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V_k(s')$$



Value iteration algorithm

function VALUE-ITERATION(env,ϵ) **returns:** state values V
  input: env - MDP problem, ϵ
  $V' \leftarrow 0$ in all states
  repeat ▷ iterate values until convergence
    $V \leftarrow V'$ ▷ keep the last known values
    $\delta \leftarrow 0$ ▷ reset the max difference
    for each state s **in** S **do**
      $V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s,a) V(s')$
      if $|V'[s] - V[s]| > \delta$ **then** $\delta \leftarrow |V'[s] - V[s]|$
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Sync vs. async Value iteration

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References

Some figures from [1] (chapter 17) but notation slightly changed in order to adapt notation from [2] (chapters 3, 4) which will help us in the Reinforcement Learning part of the course. Note that the book [2] is available on-line.

[1] Stuart Russell and Peter Norvig.
Artificial Intelligence: A Modern Approach.
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[2] Richard S. Sutton and Andrew G. Barto.
Reinforcement Learning; an Introduction.
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