

Adversarial Search

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Games, man vs. algorithm

- ▶ Deep Blue
- ▶ Alpha Go
- ▶ Deep Stack
- ▶ Why Games, actually?

Games are interesting for AI *because* they are hard (to solve).

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More: Adversarial Learning

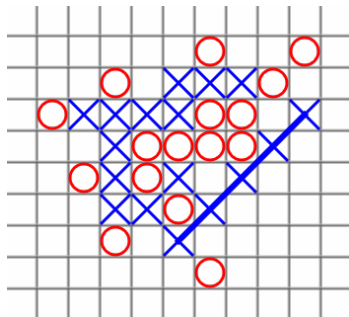


Video: Adversing visual segmentation

Vision for Robotics and Autonomous Systems, <http://cyber.felk.cvut.cz/vras>, video at YT: <https://youtu.be/KvdZmtVguOo>

Elements of the game

- ▶ s_0 : The initial state
- ▶ $\text{PLAYER}(s)$. Which player has to move in s .
- ▶ $\text{ACTIONS}(s)$. What are the legal moves?
- ▶ $\text{RESULT}(s, a)$. Transition, result of a move.
- ▶ $\text{TERMINAL-TEST}(s)$. Game over?
- ▶ $\text{TERMINAL-UTILITY}(s, p)$. What is the prize? Examples for some games ...

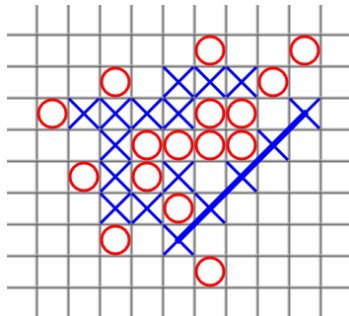


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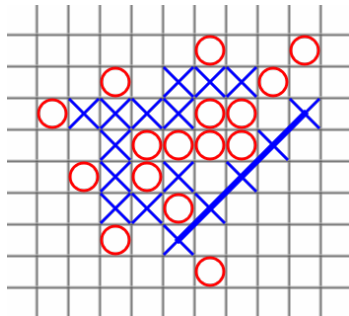


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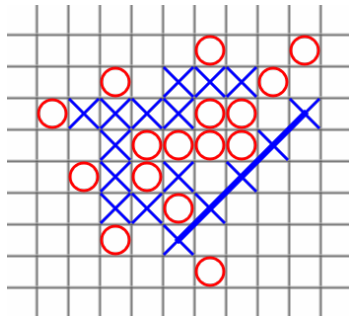


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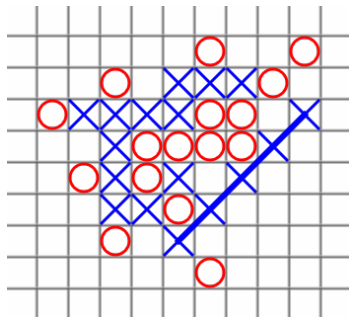


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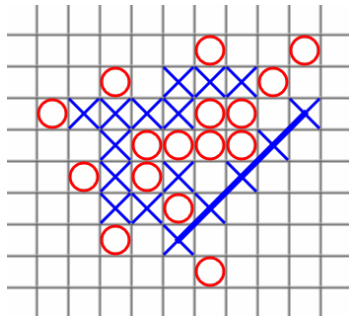


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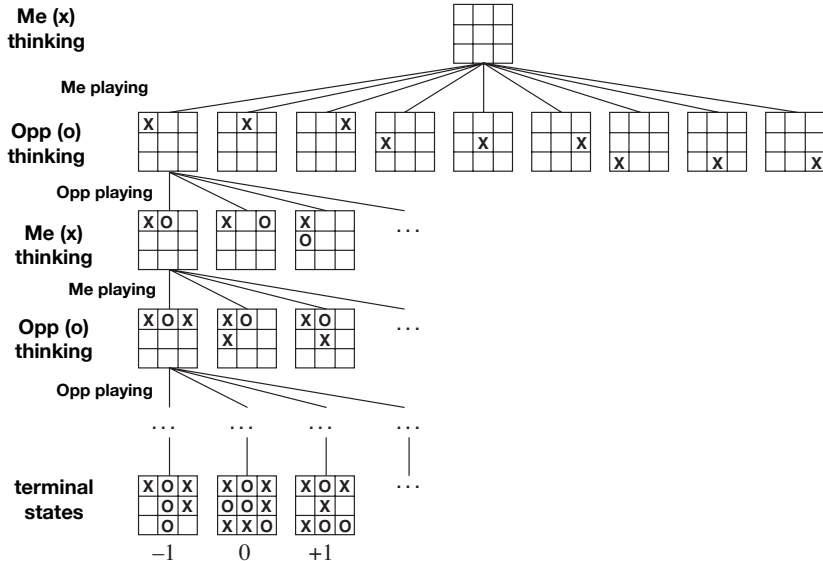
Terminal utility: Zero-Sum and General games

- ▶ Zero-sum: players have opposite utilities (values)
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- ▶ General game: cooperations, competition, ...

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Game Tree(s)



State Value $V(s)$

$V(s)$ – value V of a state s : The best utility achievable from this state.

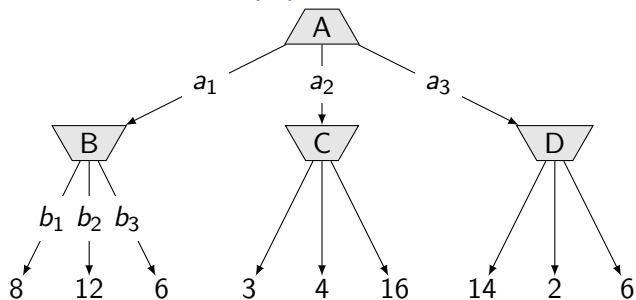
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What is the Value of the root $V(A)$?



$V(s)$ – value V of a state s : The best utility achievable from this state.

A, B, C, D - states of the game. I begin, values represent values of terminal states, more is better for me - think about the (my) money prize. Assume (strictly) rational players.

A: $V(A) = 6$

B: $V(A) = 3$

C: $V(A) = 2$

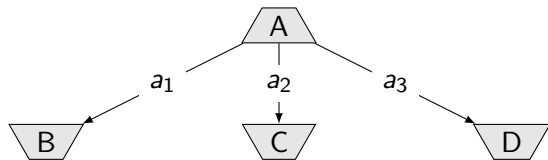
D: $V(A) = 16$

Two-ply game: **max** for me, **min** for the opponent.



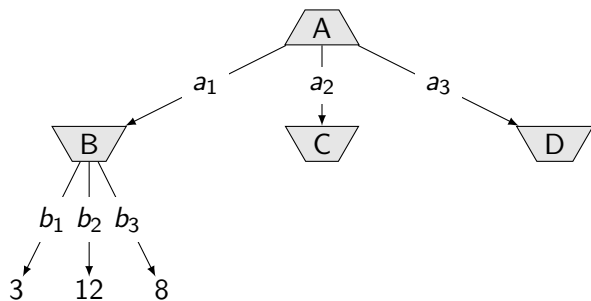
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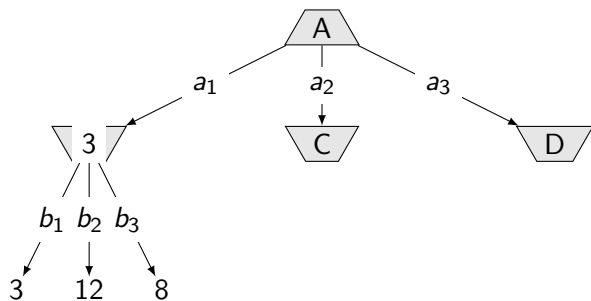
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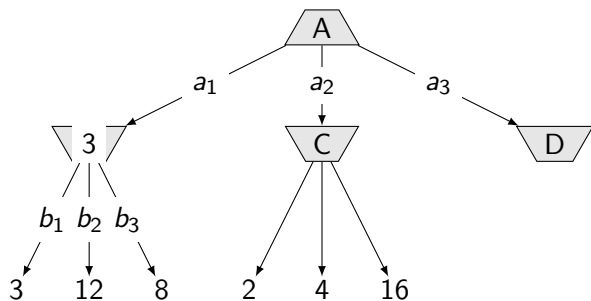
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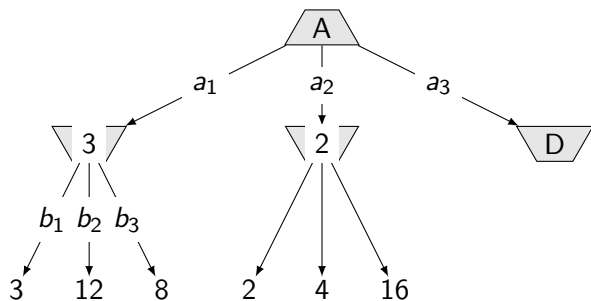
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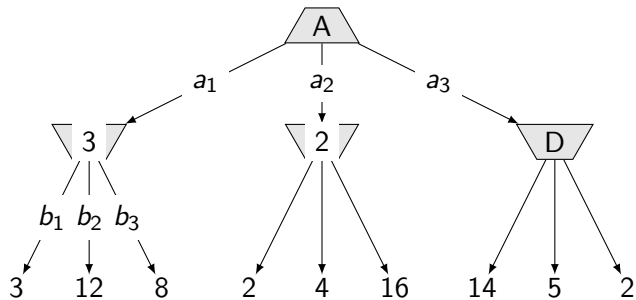
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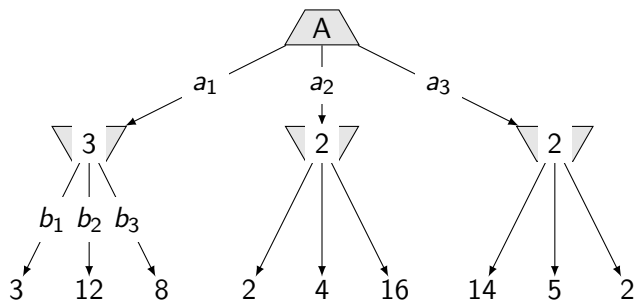
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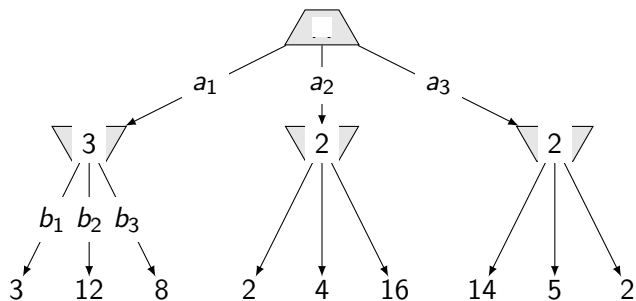
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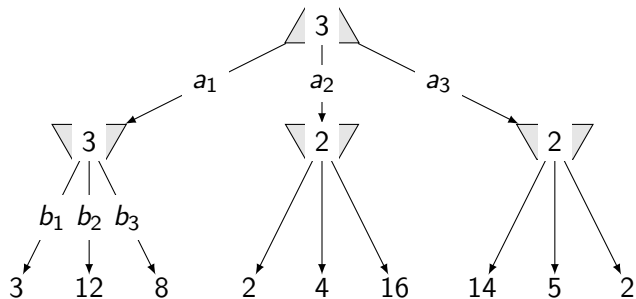
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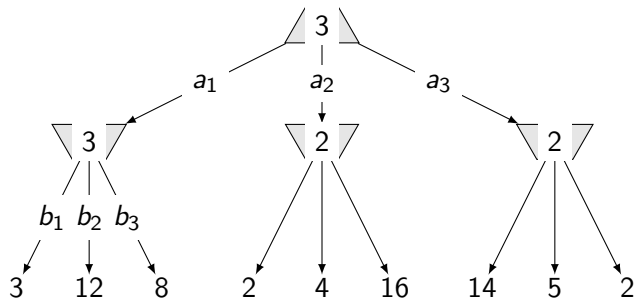
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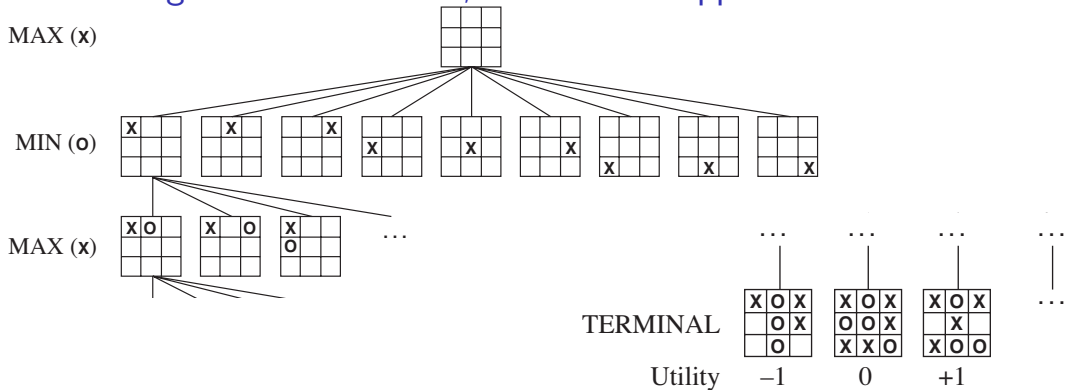
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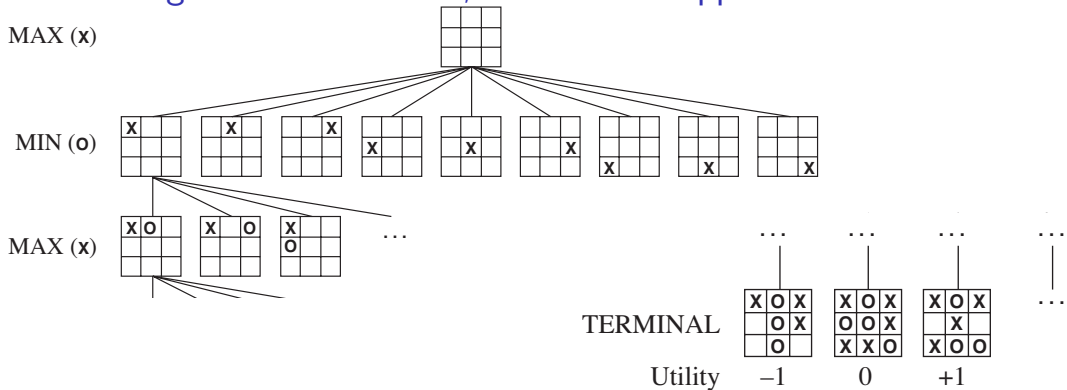
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Zero-Sum game: **max** for me, **min** for the opponent.



$$\begin{aligned}
 \text{MINIMAX}(s) = & \\
 & \text{UTILITY}(s) \quad \text{if } \text{TERMINAL-TEST}(s) \\
 & \max_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s, a)) \quad \text{if } \text{PLAYER}(s) = \text{MAX} \\
 & \min_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s, a)) \quad \text{if } \text{PLAYER}(s) = \text{MIN}
 \end{aligned}$$

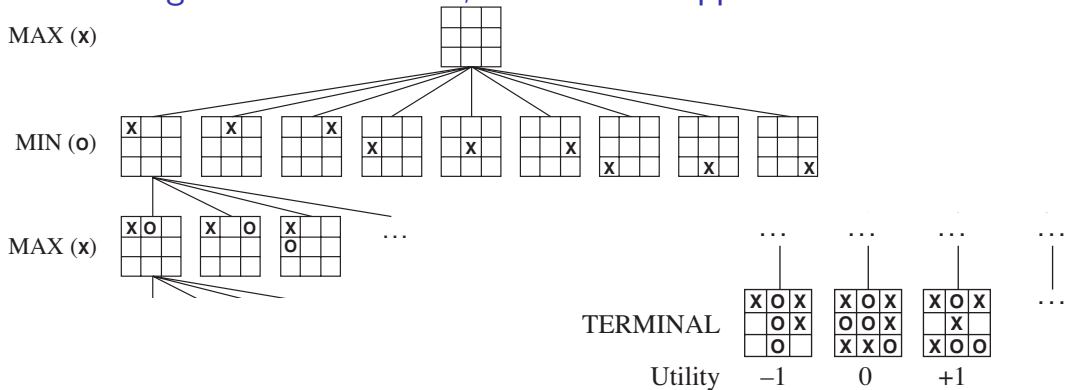
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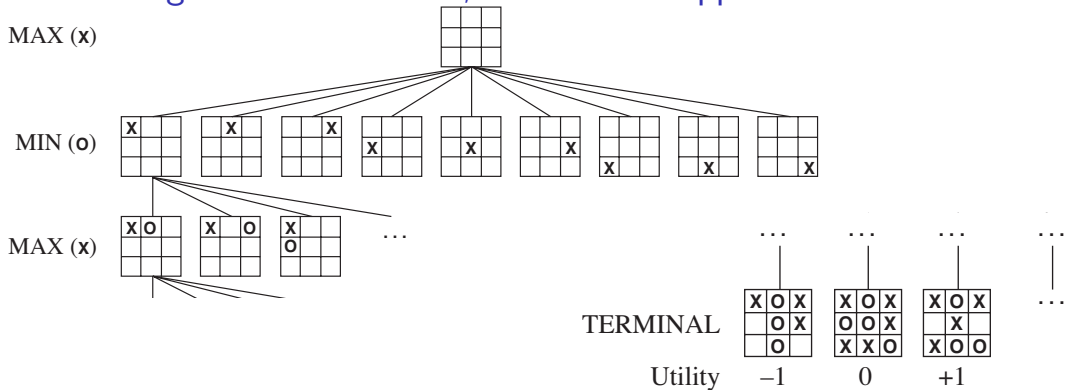
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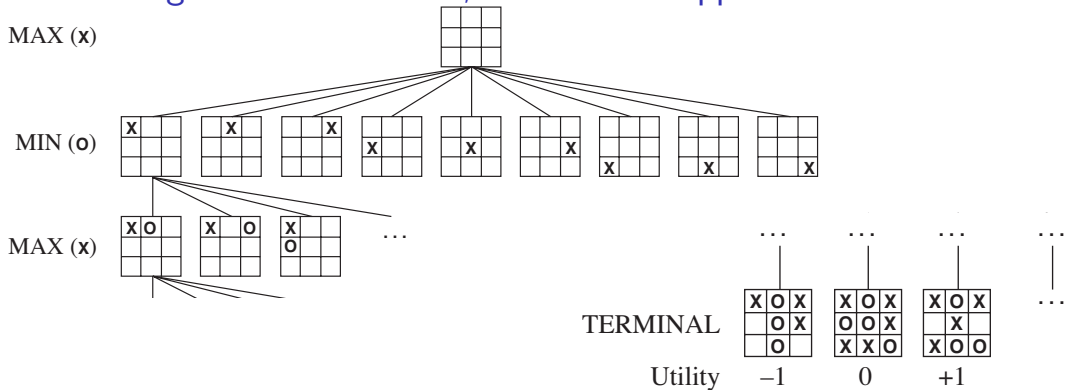
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$$\text{MINIMAX}(s) = \begin{cases} \text{UTILITY}(s) & \text{if } \text{TERMINAL-TEST}(s) \\ \max_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s, a)) & \text{if } \text{PLAYER}(s) = \text{MAX} \\ \min_{a \in \text{ACTIONS}(s)} \text{MINIMAX}(\text{RESULT}(s, a)) & \text{if } \text{PLAYER}(s) = \text{MIN} \end{cases}$$

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Minimax algorithm

function MINIMAX(state) **returns** an action

```
    return argmaxa ∈ Actions(s) MIN-VALUE(RESULT(state, a))
```

end function

function MIN-VALUE(state) **returns** a utility value v

```
    if TERMINAL-TEST(state) then return UTILITY(state)
```

```
    end if
```

```
     $v \leftarrow \infty$ 
```

```
    for all ACTIONS(state) do
```

```
         $v \leftarrow \min(v, \text{MAX-VALUE}(\text{RESULT}(\text{state}, a)))$ 
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```
    end for
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function MAX-VALUE(state) **returns** a utility value v

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    if TERMINAL-TEST(state) then return UTILITY(state)
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A two ply game, down to terminal and back again ...

function MINIMAX(s) **returns** a

$\operatorname{argmax}_{a \in \text{Actions}(s)} \text{MINVAL}(\text{RES}(s, a))$

end function

function MINVAL(s) **returns** v

if TERMINAL(s) **then** UTIL(s)

end if

$v \leftarrow \infty$

for all ACTIONS(s) **do**

$v \leftarrow \min(v, \text{MAXVAL}(\text{RES}(s, a)))$

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end if

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for all ACTIONS(s) **do**

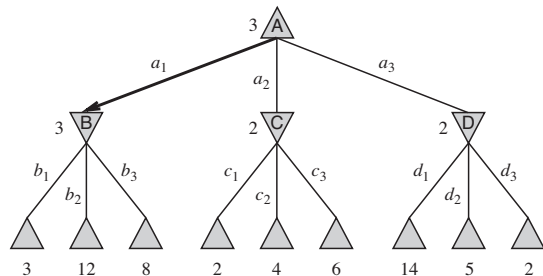
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end function

MAX

MIN



A two ply game, recursive run

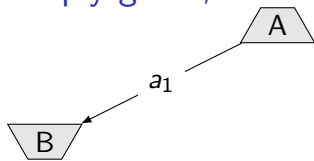


Is it like DFS or BFS?

What is the complexity? How many nodes to visit?

Can we do better? How?

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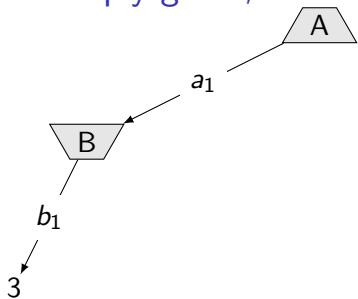


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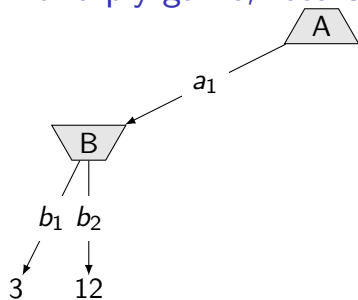


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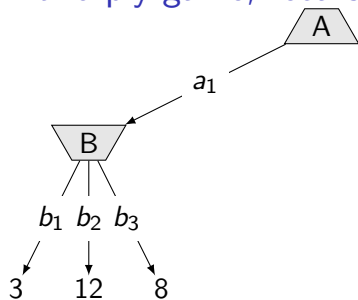


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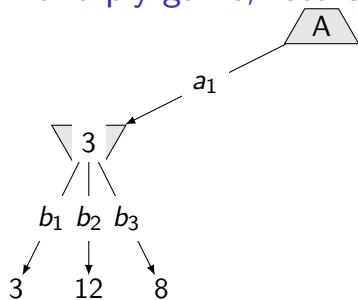


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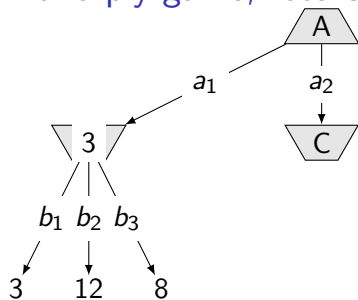


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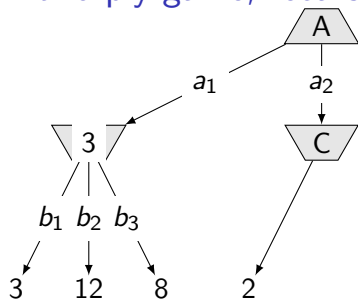


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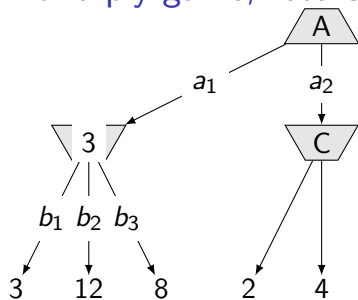


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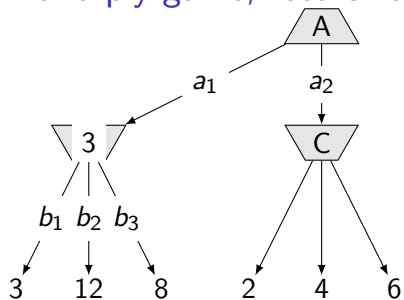


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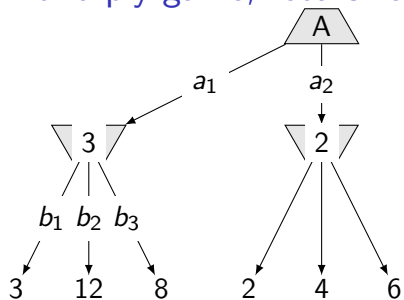


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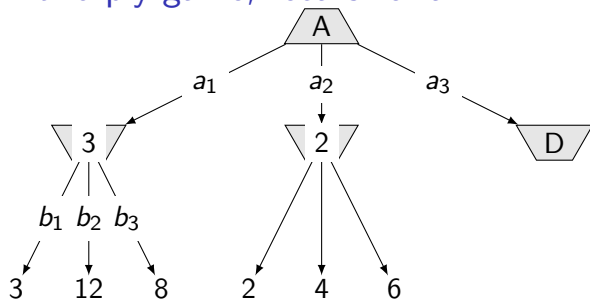


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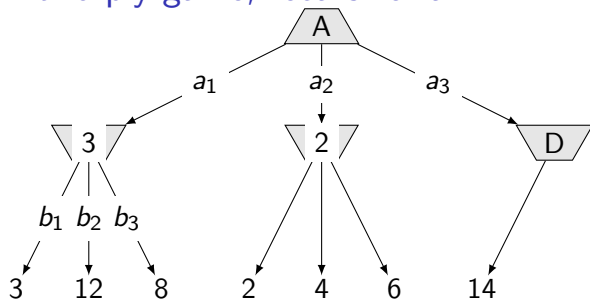


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A two ply game, recursive run

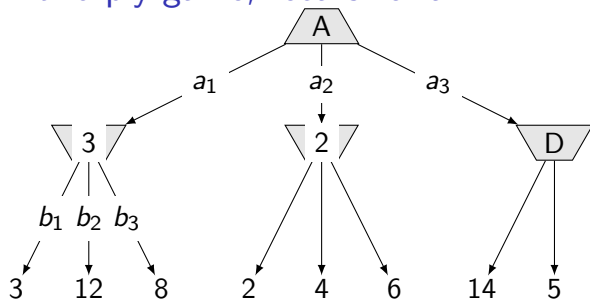


Is it like DFS or BFS?

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A two ply game, recursive run

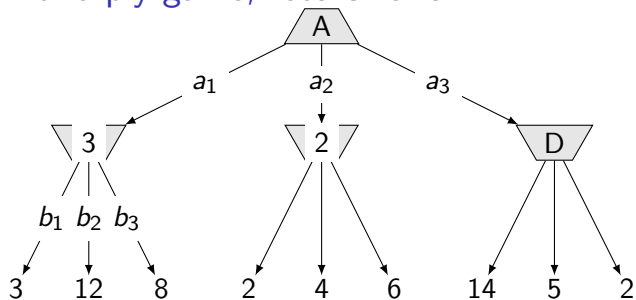


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A two ply game, recursive run

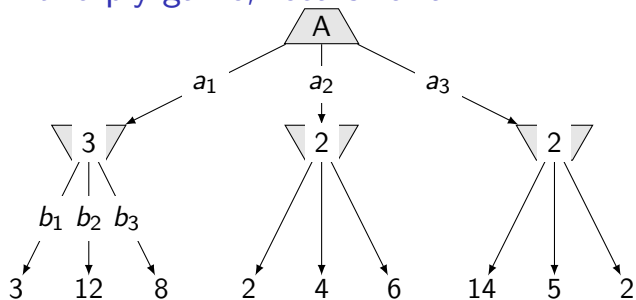


Is it like DFS or BFS?

What is the complexity? How many nodes to visit?

Can we do better? How?

A two ply game, recursive run

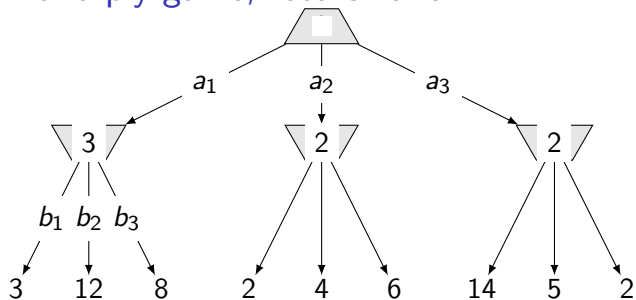


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A two ply game, recursive run

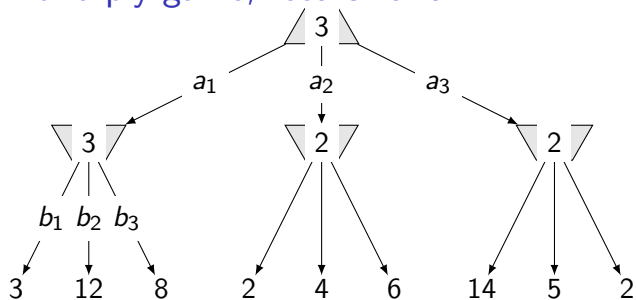


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A two ply game, recursive run

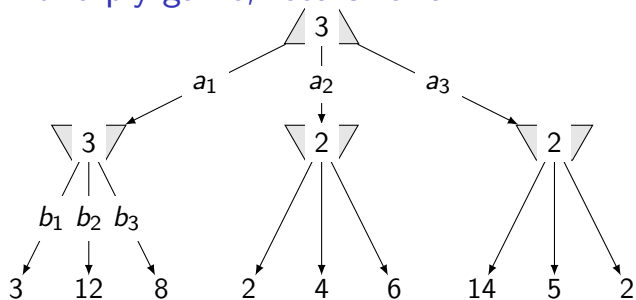


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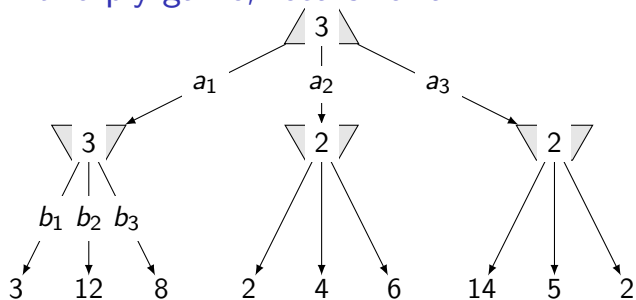


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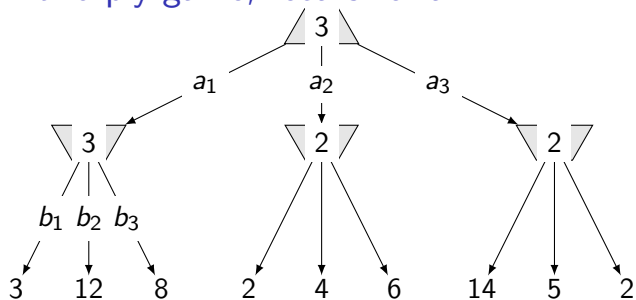


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What is the complexity? How many nodes to visit?

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A two ply game, recursive run



Is it like DFS or BFS?

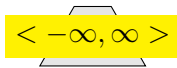
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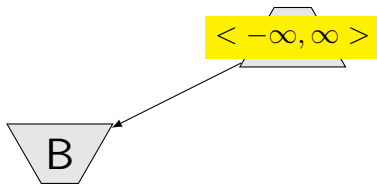
Nodes (sub-trees) worth visiting



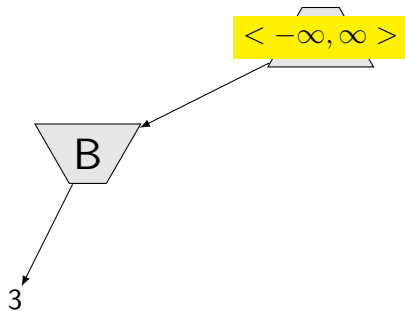
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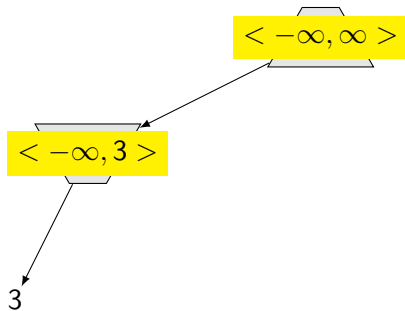
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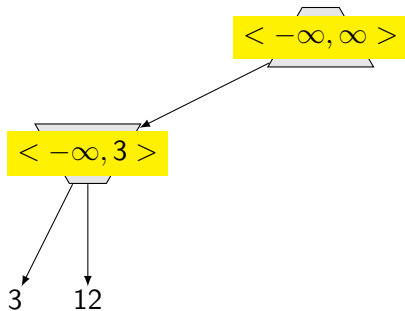
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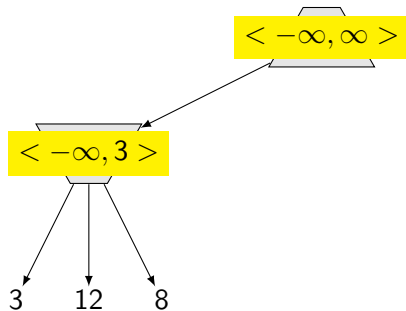
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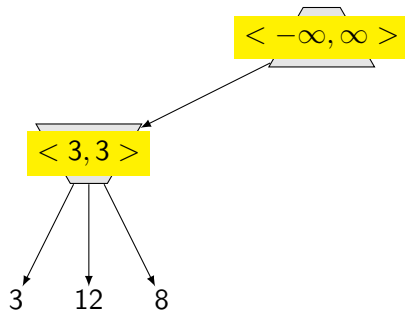
Nodes (sub-trees) worth visiting



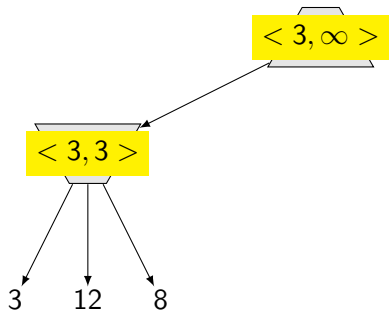
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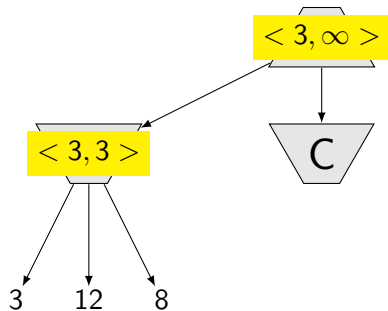
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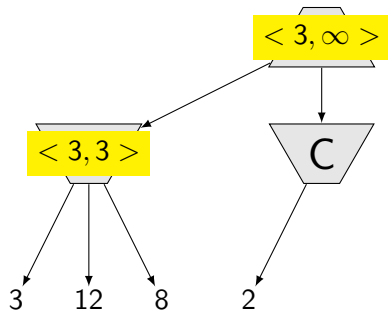
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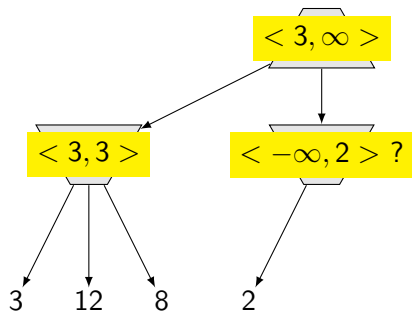
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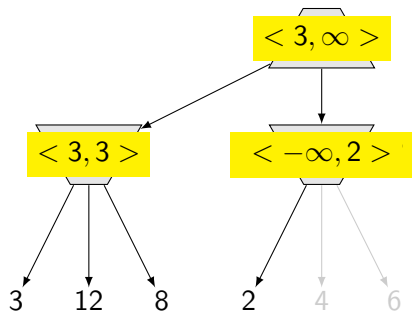
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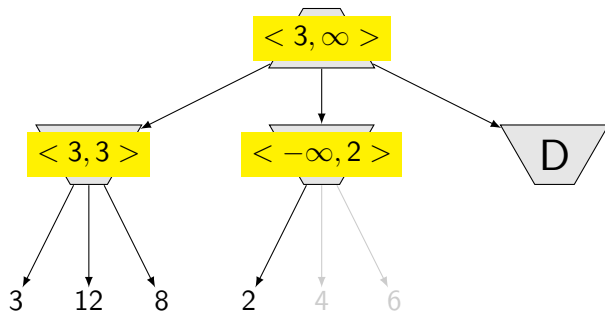
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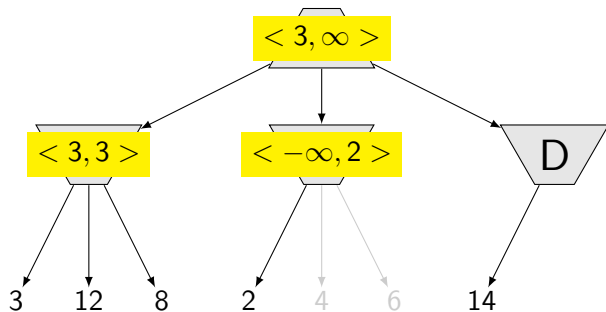
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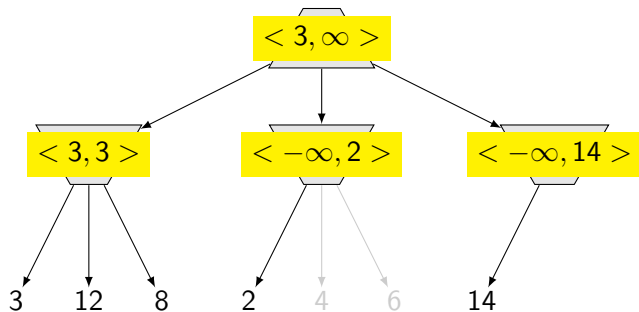
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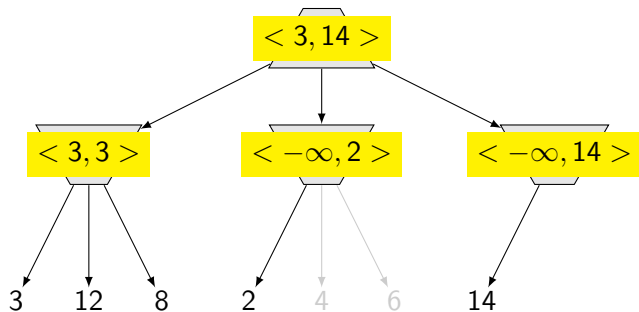
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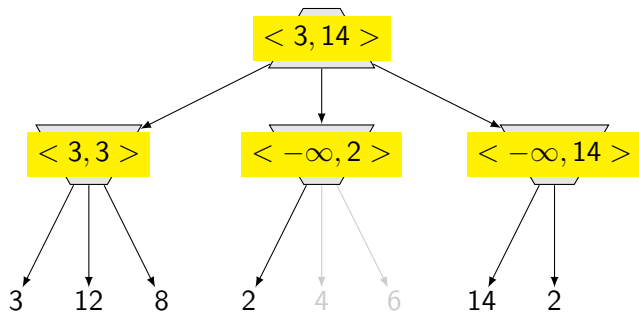
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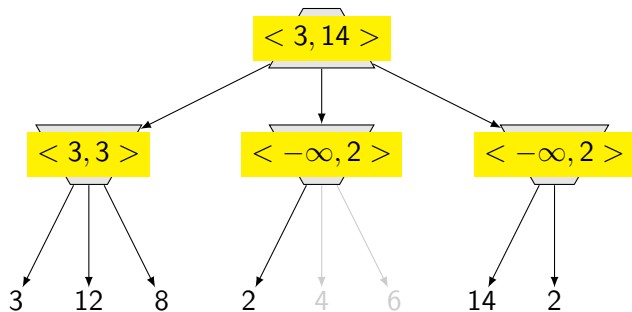
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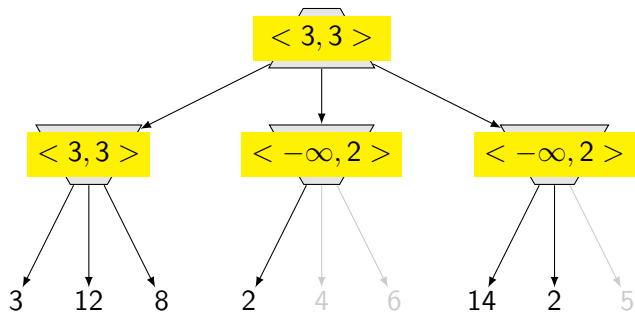
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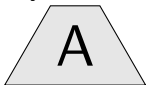
Nodes (sub-trees) worth visiting



α - β pruning

α highest (best) value choice found so far for any choice along MAX

β lowest (best) value choice found so far for any choice along MIN



v value of the state

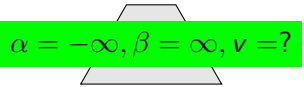
In MIN-VAL: $v \leftarrow 2$

$v \leq \alpha$ then: return v !

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$\alpha = -\infty, \beta = \infty, v = ?$

v value of the state

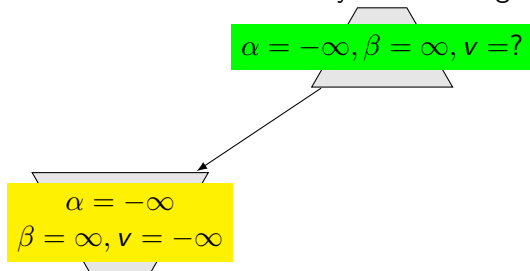
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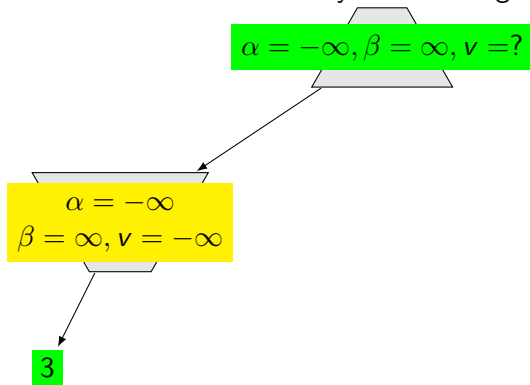
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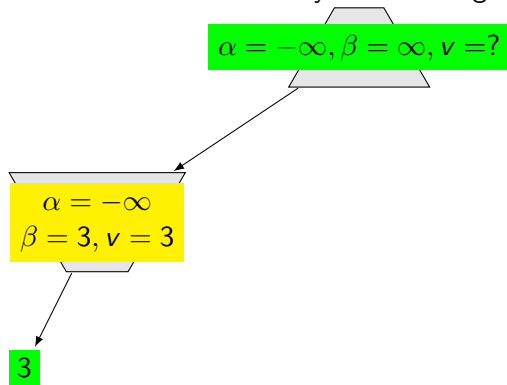
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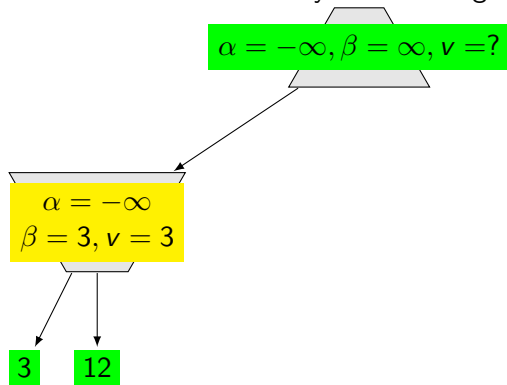


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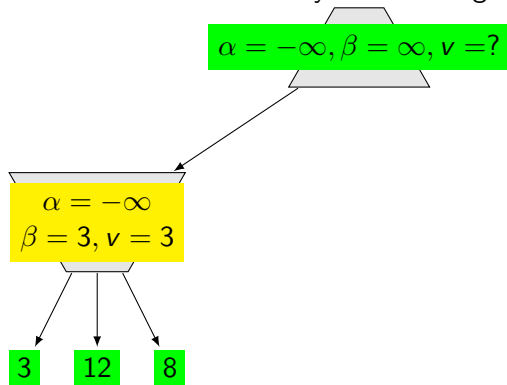
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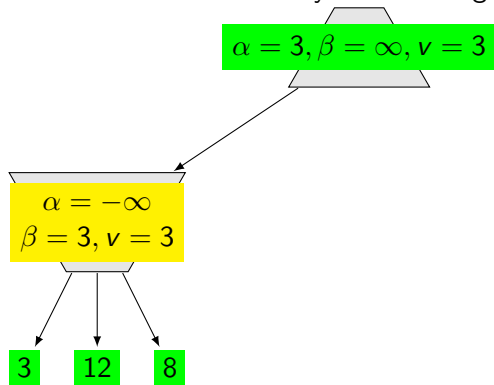
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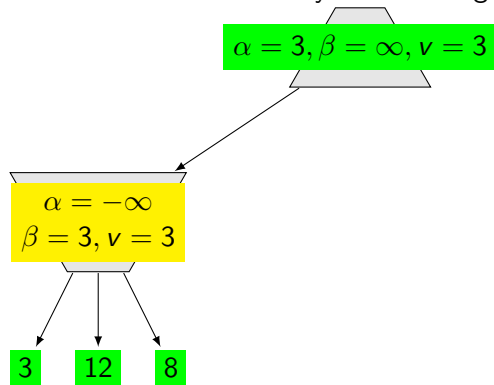
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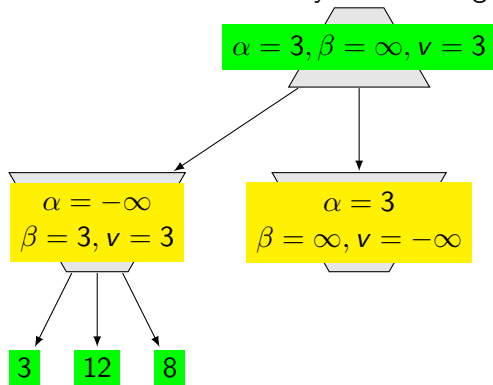
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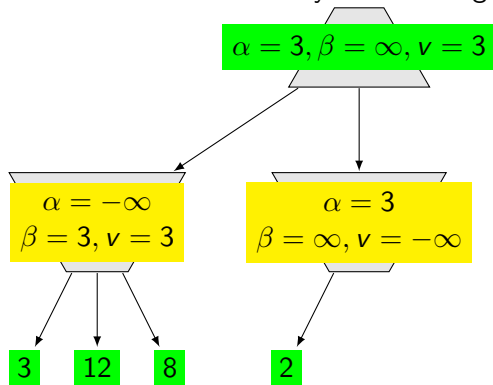
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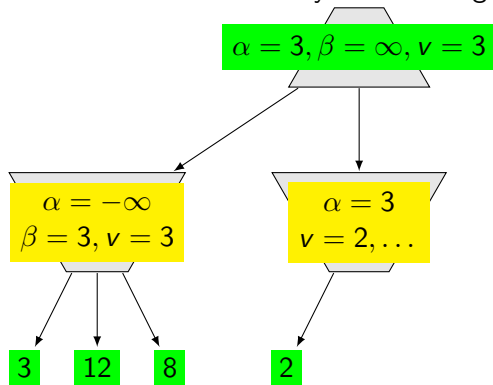
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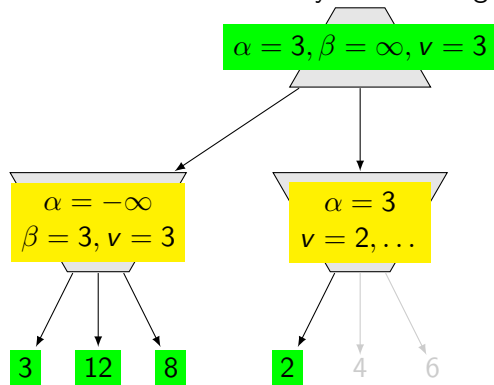
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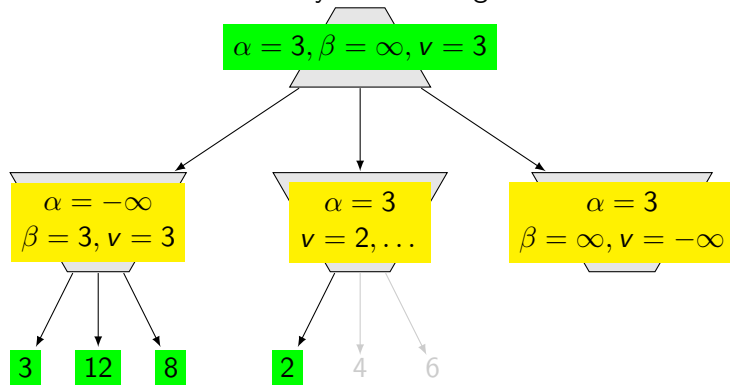
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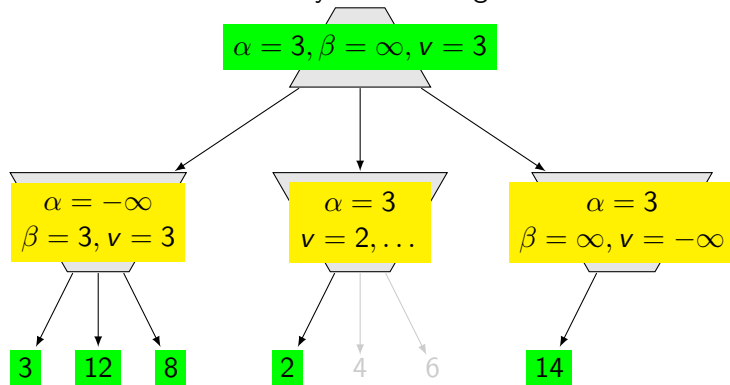
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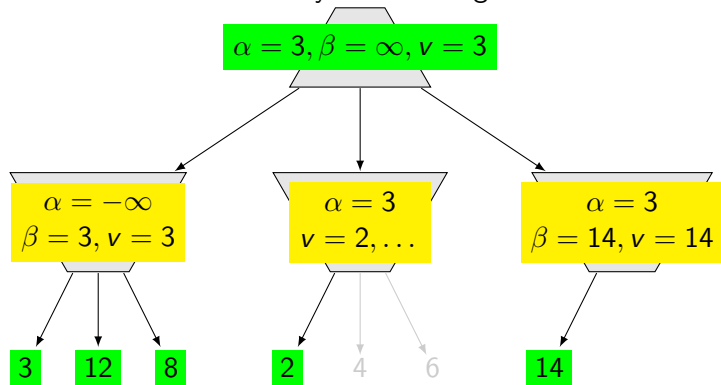


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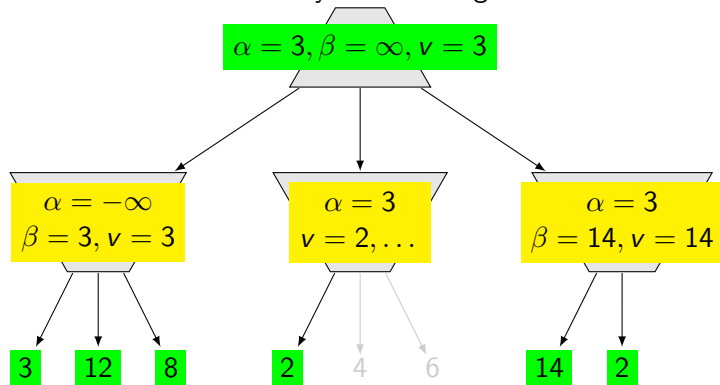


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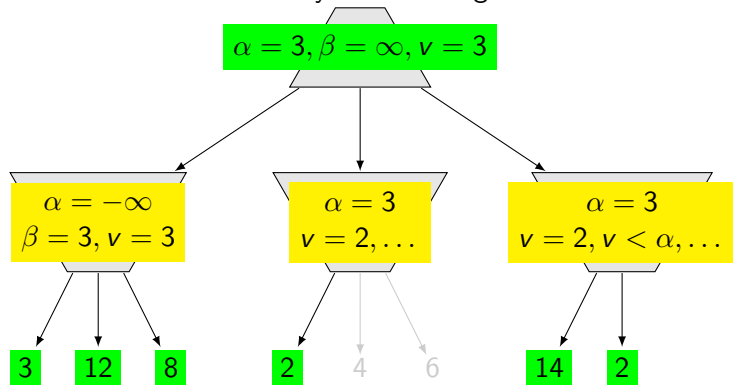
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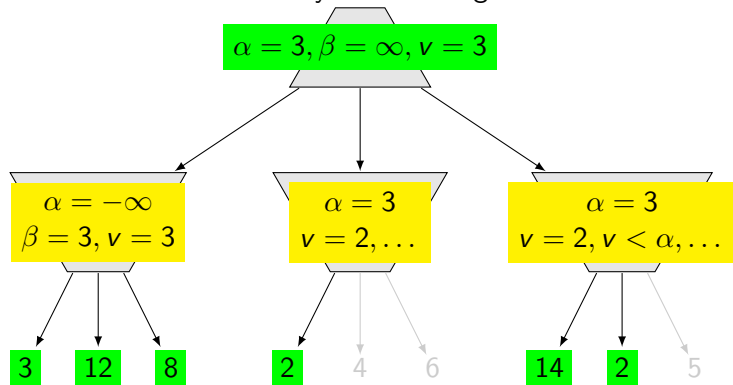
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In MIN-VAL: $v \leftarrow 2$

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α - β prunning – How much can we save?

original: Time: $O(b^m)$

- ▶ how to consider next actions/moves (in what order)?
- ▶ perfect ordering?

α - β saving, sketch ...

function ALPHA-BETA-SEARCH(*state*) **returns** an action

$v \leftarrow \text{MAX-VALUE}(\text{state}, \alpha = -\infty, \beta = \infty)$

return action corresponding to v

end function

function MAX-VALUE(*state*, α , β) **returns** a utility value v

if TERMINAL-TEST(*state*) **return** UTILITY(*state*)

$v \leftarrow -\infty$

for all ACTIONS(*state*) **do**

$v \leftarrow \max(v, \text{MIN-VALUE}(\text{RESULT}(\text{state}, a), \alpha, \beta))$

if $v \geq \beta$ **return** v

$\alpha \leftarrow \max(\alpha, v)$

end for

end function

function MIN-VALUE(*state*, α , β) **returns** a utility value v

if TERMINAL-TEST(*state*) **return** UTILITY(*state*)

$v \leftarrow \infty$

for all ACTIONS(*state*) **do**

$v \leftarrow \min(v, \text{MAX-VALUE}(\text{RESULT}(\text{state}, a), \alpha, \beta))$

if $v \leq \alpha$ **return** v

$\beta \leftarrow \min(\beta, v)$

end for

end function

```
function ALPHA-BETA-SEARCH(state) returns an action
   $v \leftarrow \text{MAX-VALUE}(\text{state}, \alpha = -\infty, \beta = \infty)$ 
  return action corresponding to  $v$ 
end function
```

```
function MAX-VALUE(state,  $\alpha$ ,  $\beta$ ) returns a utility value  $v$ 
  if TERMINAL-TEST(state) return UTILITY(state)
   $v \leftarrow -\infty$ 
  for all ACTIONS(state) do
     $v \leftarrow \max(v, \text{MIN-VALUE}(\text{RESULT}(\text{state}, a), \alpha, \beta))$ 
    if  $v \geq \beta$  return  $v$ 
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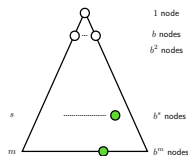
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  end for
end function
```

Recall: Iterative deepening DFS (ID-DFS)

- ▶ Start with `maxdepth = 1`
- ▶ Perform DFS with limited depth. Report success or failure.
- ▶ If failure, forget everything, increase `maxdepth` and repeat DFS.

The “wasting” of resources is not too bad. Recall:

- ▶ Most nodes are at the deepest levels.
- ▶ Asymptotic complexity unchanged.



Bonus for α - β pruning: previous “shallower” iterations can be reused for node ordering.

Imperfect but real-time decisions: iterative deepening

$$\begin{aligned} \text{H-MINIMAX}(s, d) = & \text{EVAL}(s) \quad \text{if } \text{CUTOFF-TEST}(s, d) \\ & \max_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a), d + 1) \quad \text{if } \text{PLAYER}(s) = \text{MAX} \\ & \min_{a \in \text{ACTIONS}(s)} \text{H-MINIMAX}(\text{RESULT}(s, a), d + 1) \quad \text{if } \text{PLAYER}(s) = \text{MIN} \end{aligned}$$

Imperfect but real-time decisions: iterative deepening

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Imperfect but real-time decisions: iterative deepening

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Cutting off search and evaluation functions

Replace

if TERMINAL-TEST(s) **then return** TERMINAL-UTILITY(s)

with:

if CUTOFF-TEST(s,d) **then return** EVAL(s)

Historical note: cutting search off earlier and use of heuristic evaluation functions proposed by Claude Shannon in *Programming a Computer for Playing Chess* (1950).

EVAL(s) – Evaluation functions

(Estimate of) State value for non-terminal states.

We need an easy-to-compute function correlated with “chance of winning”. For chess:

- ▶ $f_1(s)$ Material value for pieces—1 for pawn, 3 for knight/bishop, 5 for rook, 10 for queen. (minus opponent’s pieces)
- ▶ $f_2(s)$ Finetuning: 2 bishops are worth 6.5; knights are worth more in closed positions...
- ▶ Other features worth evaluating: controlling the center of the board, good pawn structure (no double pawns), king safety...
- ▶ $f_i(s) = \dots$ We can create many. How to combine them?

$$\text{EVAL}(s) = w_1 f_1(s) + w_2 f_2(s) + \dots + w_n f_n(s)$$

How to find/compute proper weights?

How to find/create $f_i(s)$?

EVAL(s) – Evaluation functions

(Estimate of) State value for non-terminal states.

We need an easy-to-compute function correlated with “chance of winning”. For chess:

- ▶ $f_1(s)$ Material value for pieces—1 for pawn, 3 for knight/bishop, 5 for rook, 10 for queen. (minus opponent's pieces)
- ▶ $f_2(s)$ Finetuning: 2 bishops are worth 6.5; knights are worth more in closed positions...
- ▶ Other features worth evaluating: controlling the center of the board, good pawn structure (no double pawns), king safety...
- ▶ $f_i(s) = \dots$ We can create many. How to combine them?

$$\text{EVAL}(s) = w_1 f_1(s) + w_2 f_2(s) + \dots + w_n f_n(s)$$

How to find/compute proper weights?

How to find/create $f_i(s)$?

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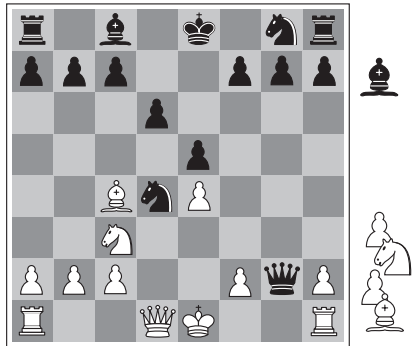
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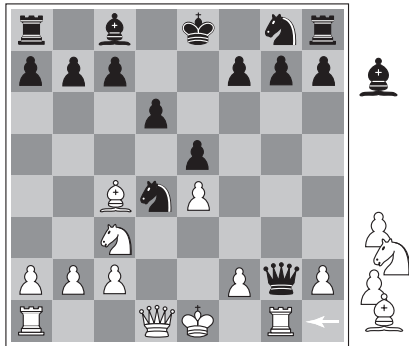
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EVAL(s) – Problems

What if something important happens just after the cut – in the next ply?



(a) White to move



(b) White to move

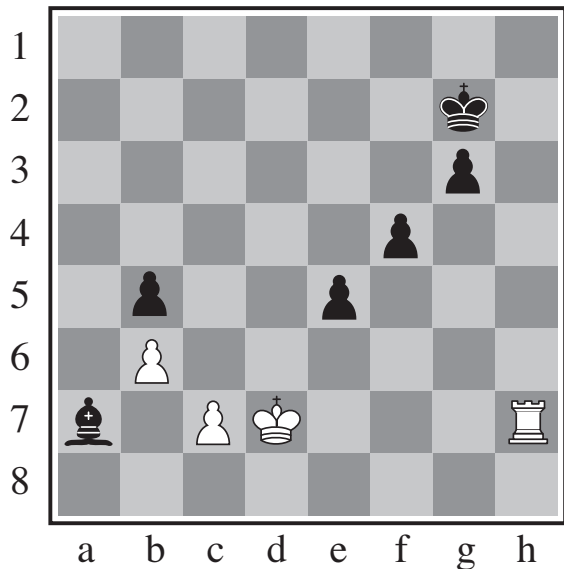
Additional improvements:

- ▶ “Killer moves” —capturing opponent’s pieces, check etc.—should be considered first.
- ▶ *Quiescence search* – EVAL function should be applied only once things calm down. During capturing of pieces, depth should be locally increased.

Horizon effect

Pushing unavoidable loss deeper in tree by a delaying tactics. We know it is useless but does the machine?

See the situation on right. Black is on move, her bishop is surely doomed. However, the inevitable loss can be postponed by moving her pawns and checking the white king. Depending on the searchable depth this may put the loss over the horizon and moving pawns may look promising.



Computer play vs. grandmaster play

- ▶ Computers are better since 1997 (Deep Blue defeating Garry Kasparov).
- ▶ The way they play is still very different: “dumb”, relying on “brute force”.
 - ▶ Deep Blue examined 200M positions per second.
 - ▶ In some cases, depth of search was 40 ply.
- ▶ Grandmasters do not excel in being able to compute very deep—many moves ahead.
 - ▶ They play based on experience: super-effective pruning and evaluation functions.
 - ▶ They consider only 2 to 3 moves in most positions (branching factor).

References

Many images, including the chess plates are from Chapter 5, “Adversarial search” in [1].

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