GRAPHICAL MARKOV MODELS (WS2020)
5. SEMINAR

Assignment 1. Consider the task of finding the most probable sequence of (hidden) states for a (Hidden) Markov model on a chain.
a) Show that the Dynamic Programming approach applied for this task can be interpreted as an equivalent transformation (re-parametrisation) of the model.
b) Show that the transformed functions (potentials) encode an explicit description of *all* optimisers of the problem

Assignment 2. Consider a GRF for binary valued labellings $x\colon V \to \{0,1\}$ of a graph (V,E) given by

$$p(x) = \frac{1}{Z}\exp\Big[\sum_{ij\in E} u_{ij}(x_i,x_j) + \sum_{i\in V} u_i(x_i)\Big].$$

Show that is is always possible to find an equivalent transformation (re-parametrisation)

$$u_{ij} \to \tilde{u}_{ij}, \quad u_i \to \tilde{u}_i$$

such that the new pairwise functions $\tilde{u}_{ij}$ have the form

$$\tilde{u}_{ij}(x_i,x_j) = \alpha_{ij}|x_i - x_j|$$

with some real numbers $\alpha_{ij} \in \mathbb{R}$.

Assignment 3. Transform the *Travelling Salesman Problem* into a $(\min,+)$-problem.

Assignment 4. Prove that a sum of submodular functions is submodular.

Assignment 5. Let K be a completely ordered finite set. We assume w.l.o.g. that $K = \{1,2,\ldots,m\}$. For a function $u\colon K \to \mathbb{R}$ define its discrete "derivative" by $Du(k) = u(k+1) - u(k)$.
a) Let u be a function $u\colon K^2 \to \mathbb{R}$ and denote by D_1 and D_2 the discrete derivatives w.r.t. its first and second argument. Prove the following equality

$$D_1D_2u(k_1,k_2) = u(k_1+1,k_2+1) + u(k_1,k_2) - u(k_1+1,k_2) - u(k_1,k_2+1).$$

Conclude that all mixed derivatives $D_1D_2u(k_1,k_2)$ of a submodular functions are negative.
b) Prove that the condition established in a) is not only necessary but also sufficient for a function to be submodular.

Hint: Start from the observation that the following equality holds for a function of one variable

$$u(k+l) - u(k) = \sum_{i=k}^{k+l-1} Du(i)$$

and generalise it for functions of two variables.
c) Prove that any function $u\colon K^2 \to \mathbb{R}$ can be represented as a sum of a submodular and a supermodular function.

Hint: Consider the mixed derivative $D_1 D_2 u(k_1, k_2)$, decompose it into its negative and positive part and "integrate" them back separately.

Assignment 6. Examine the following functions w.r.t. submodularity
a) $f(k, k') = |k - k'|$, where $k, k' \in \mathbb{Z}$.
b) $f(k, k') = (k - k')^2$, where $k, k' \in \mathbb{Z}$.
c) $f(k_1, \ldots, k_n) = \max_i k_i - \min_i k_i$, where $k_i \in \mathbb{Z}$.