12. (Min,+)-problems for graphical models

MAP inference for GRFs

- $X = \{X_j \in F \mid j \in V\}$ a random field of features (observable)
- $S = \{S_i \in K \mid i \in V\}$ — " — of hidden states

Assume, their joint p.d. is a GRF w.r.t. the system $\mathcal{C}$ of subsets of V

$$p(x,s) = \frac{1}{Z} \exp\Big[\sum_{c \in \mathcal{C}} u_c(x_c, s_c)\Big]$$

Inference: Given $x \in F^V$ infer $s \in K^V$ w.r.t. 0/1 loss $\Rightarrow$ MAP

$$s^* \in \operatorname*{argmax}_{s \in K^V} p(x,s) = \operatorname*{argmax}_{s \in K^V} \sum_{c \in \mathcal{C}} u_c(x_c, s_c)$$

- discrete optimisation problem for $|V|$ variables
- objective function $\triangleq$ sum of functions, each depending on a subset of variables

Particular case: $\mathcal{C}$ is the structure of a graph (V,E)

Solve the task

$$s^* \in \operatorname*{argmin}_{s \in K^V} U(s) = \operatorname*{argmin}_{s \in K^V} \Big[\sum_{i \in V} u_i(s_i) + \sum_{ij \in E} u_{ij}(s_i, s_j)\Big]$$

- Easy to solve if (V,E) is acyclic
- NP complete in general (MaxClique)

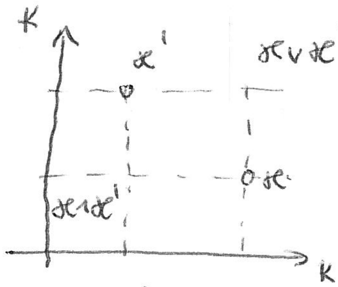
Options:

- search for tractable subclasses
- search for approximation algorithms

Submodular (Min,+)-problems

- Let K be completely ordered and denote min, max w.r.t. this order by $\wedge, \vee$
- K^n is a distributive lattice $\hat{=}$ poset with operations „infimum" and „supremum"

$$x, x' \in K^n$$
$$x \wedge x' = (x_1 \wedge x_1', \dots, x_n \wedge x_n')$$
$$x \vee x' = (x_1 \vee x_1', \dots, x_n \vee x_n')$$



Definition 1 Let K be completely ordered. A real valued function $u: K^n \to \mathbb{R}$ is submodular if

$$u(x \wedge x') + u(x \vee x') \leq u(x) + u(x')$$

holds $\forall\, x, x' \in K^n$.

Remarks

(1) if „$\leq$" replaced by „$\geq$" $\Rightarrow$ supermodular function

(2) any function $u: K \to \mathbb{R}$ is submodular and supermodular

(3) any function $u: K^2 \to \mathbb{R}$ can be decomposed into a sum of a super- and submodular part

(4) if $|K| = 2$, then K^V is a Boolean lattice and any $x \in K^V$ can be identified with a subset of V: $\{i \in V \mid x_i = 1\}$ (we assume $K = \{0, 1\}$)

Examples

(1) Let $K = \{0, 1\}$ be ordered. The function $u: K^2 \to \mathbb{R}$ defined by $u(k, k') = |k - k'|$ is submodular

(2) Let $K = \{0, 1, 2, \dots, m\}$ be ordered. Consider functions $u: K^2 \to \mathbb{R}$

$$u(k, k') = |k - k'| \quad \text{is submodular}$$
$$u(k, k') = \mathbb{1}\{k \neq k'\} \quad \text{is not submodular}$$

(3) Let $K = \{0, 1, 2, .., m\}$ be ordered. Consider the functions $u: K^n \to \mathbb{R}$ defined by $u(k_1, .., k_n) = \max_i k_i - \min_i k_i$. It is submodular.

Theorem 1 (Iwata, Fleisher, Fujishige)

Any submodular function on $\{0,1\}^n$ can be minimised with complexity $\mathcal{O}((n^6 \mu + n^7) \log n)$, where μ denotes the time required for computing the function value. ▫

Theorem 2 (Schlesinger, Flach, 2006)

If all arity 2 functions $u_{ij} : K^2 \to \mathbb{R}$ of a (Min,+)-problem on a graph are submodular w.r.t. some ordering of K, then the (Min,+)-problem is equivalent to a MinCut problem and solvable with complexity $\mathcal{O}(V^2 E K^4)$. ▫

Transforming a submodular (Min,+)-problem on a graph (V,E) into a MinCut problem: Here, we assume $|K| = 2$ for simplicity

(1) Express the (Min,+)-problem in canonical form, using equivalent transformations

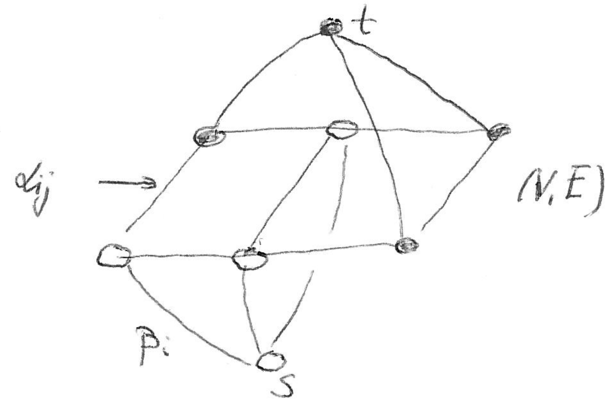
$$\sum_{ij \in E} \alpha_{ij} |s_i - s_j| + \sum_{i \in V} \beta_i s_i \to \min_{s \in K^V},$$

where $s_i = 0, 1$. Submodularity ensures $\alpha_{ij} \geq 0 \;\; \forall \{i,j\} \in E$

(2) Rewrite the linear terms: Let $V_+ = \{i \in V \mid \beta_i \geq 0\}$, $V_- = V \setminus V_+$

$$\sum_{i \in V} \beta_i s_i = \sum_{i \in V_+} \beta_i s_i + \sum_{i \in V_-} |\beta_i| (1 - s_i) + \text{const}$$

(3) The task is now equivalent to an st-MinCut problem with positive edge weights



$\tilde{V} = V \cup \{s,t\}$

$\tilde{E} = E \cup E_+ \cup E_-$

$E_+ = \{\{s,i\} \mid i \in V_+\}$

$E_- = \{\{t,i\} \mid i \in V_-\}$

(14) Solve it by MinCut $\Leftrightarrow$ MaxFlow, e.g.

- augmenting path alg.
- pre-flow push alg.
- V. Kolmogorov's alg.