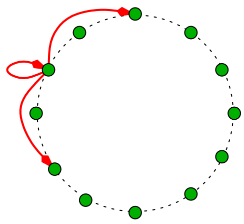
GRAPHICAL MARKOV MODELS
EXAM WS2020 (23P)

Assignment 1 (5p). Consider a random walk on a discretised circle with positions denoted by $k \in \{0, 1, \dots, K-1\}$ (see figure). The transition probabilities are defined as follows. The walker keeps staying at the current position k with probability α or it jumps to one of the two positions $(k \pm 2) \mod K$ with probabilities $(1-\alpha)/2$. Find the conditions under which the corresponding Markov chain model is irreducible and a-periodic. Deduce its stationary distribution.



Assignment 2 (6p). Consider a Hidden Markov Model for pairs of sequences $x \in F^n$, $s \in K^n$. The hidden states are integer numbers from the set $K = \{1, 2, \dots, |K|\}$. We observe a sequence of features x and want to predict the sequence of hidden states s. Assume that the loss function for inference is

$$\ell(s, s') = \sum_{i=1}^{n} (s_i - s'_i)^2.$$

a) Deduce the optimal inference strategy, i.e. the strategy that minimses the expected loss.

b) Find an algorithm for the proposed inference strategy. Give its complexity.

Assignment 3 (6p). Consider the following mixture model for sequences $s = (s_1, \dots, s_n)$ of discrete states $s_i \in K$

$$p(s) = \sum_{m=1}^{M} \beta_m p_m(s),$$

where each $p_m(s)$ is a homogeneous Markov model

$$p_m(s) = p_m(s_1) \prod_{i=2}^{n} p_m(s_i \,|\, s_{i-1}).$$

Neither the mixture weights β_m nor the parameters of the Markov models are known. You are given an i.i.d. training set $\mathcal{T}^\ell = \{s^j \in K^n \mid j = 1, \dots, \ell\}$ of sequences. Explain how to learn all parameters of the mixture by an EM algorithm. Give the complexities of the E-step and the M-step.

Assignment 4 (6p). We want to label the nodes $i \in V$ of an undirected graph (V, E) by integers from the set $K = \{1, 2, \dots, |K|\}$. The cost of a labelling s is defined by

$$\sum_{i \in V} u_i(s_i) + \sum_{ij \in E} (s_i - s_j)^2 + \lambda K(s),$$

with some given unary functions u_i. The last term $K(s)$ denotes the total number of labels occuring in a labelling s, i.e.

$$K(s) = \{k \in K \mid \exists i \in V \text{ s.t. } s_i = k\}.$$

We want to find the labelling with minimal cost.

a) Show that the pairwise functions $u(s_i, s_j) = (s_i - s_j)^2$ are submodular.

b) Explain how to extend the graph (V, E) by auxiallary nodes and edges such that the term $K(s)$ can be expressed as a sum of unary and pairwise functions.

c) Analyse the resulting $(\min, +)$-problem. Can it be solved approximately by using α-expansions?