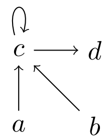
Logical reasoning and programming, lab session 10

(November 30, 2020)

10.1 We have a language that contains only one binary predicate symbol $\in$ and we have an interpretation $\mathcal{M} = (D, i)$ such that $D = \{a, b, c, d\}$ and $i(\in)$ is given by the following diagram:



Meaning that $x \in y$ iff there is an arrow from x to y. Decide whether the following formulae are valid in $\mathcal{M}$:

(a) $\exists X \forall Y (\neg(Y \in X))$,

(b) $\exists X \forall Y (Y \in X)$,

(c) $\exists X \forall Y (Y \in X \leftrightarrow Y \in Y)$,

(d) $\exists X \forall Y (Y \in X \leftrightarrow \neg(Y \in Y))$.

10.2 Show that the following formulae are valid and provide counter-examples for the opposite implications:

(a) $\forall X p(X) \vee \forall X q(X) \to \forall X (p(X) \vee q(X))$,

(b) $\exists X (p(X) \wedge q(X)) \to \exists X p(X) \wedge \exists X q(X)$,

(c) $\exists X \forall Y p(X, Y) \to \forall Y \exists X p(X, Y)$,

(d) $\forall X p(X) \to \exists X p(X)$.

10.3 Show that the "exists unique" quantifier $\exists!$ does not commute with $\exists$, $\forall$, nor $\exists!$.

10.4 Decide whether for any formula φ holds:

(a) $\varphi \equiv \forall \varphi$,

(b) $\varphi \equiv \exists \varphi$,

(c) $\models \varphi$ iff $\models \forall \varphi$,

(d) $\models \varphi$ iff $\models \exists \varphi$,

where $\forall \varphi$ ($\exists \varphi$) is the universal (existential) closure of φ. If not, does at least one implication hold?

10.5 Find a set of formulae Γ and a formula φ such that $\Gamma \models \varphi$ and $\Gamma \models \neg \varphi$.

10.6 Show that for any set of formulae Γ and a formula φ holds if $\Gamma \models \varphi$, then $\forall \Gamma \models \varphi$, where $\forall \Gamma = \{\, \forall \psi \colon \psi \in \Gamma \,\}$. Does the opposite direction hold?

10.7 Produce equivalent formulae in prenex form:

(a) $\forall X (p(X) \to \forall Y (q(X, Y) \to \neg \forall Z r(Y, Z)))$,

(b) $\exists X p(X,Y) \rightarrow (q(X) \rightarrow \neg\forall Z p(X,Z))$,

(c) $\exists X p(X,Y) \rightarrow (q(X) \rightarrow \neg\exists Z p(X,Z))$,

(d) $p(X,Y) \rightarrow \exists Y(q(Y) \rightarrow (\exists X q(X) \rightarrow r(Y)))$,

(e) $\forall Y p(Y) \rightarrow (\forall X q(X) \rightarrow r(Z))$.

10.8 In **10.7** you could obtain in some cases various prefixes; the order of quantifiers can be different. Are all these variants correct?

10.9 Can we produce a formula equivalent to **10.7**e with just one quantifier?

10.10 Produce Skolemized formulae equisatisfiable with those in **10.7**. Try to produce as simple as possible Skolem functions.

10.11 Skolemize the following formula

$$\forall X(p(a) \vee \exists Y(q(Y) \wedge \forall Z(p(Y,Z) \vee \exists U q(X,Y)))) \vee \exists W q(a,W).$$

Why is it possible in this particular case to do that without producing an equivalent formula in prenex form?

10.12 Unify the following pairs of formulae:

(a) $\{p(X,Y) \doteq p(Y,f(Z))\}$,

(b) $\{p(a,Y,f(Y)) \doteq p(Z,Z,U)\}$,

(c) $\{p(X,g(X)) \doteq p(Y,Y)\}$,

(d) $\{p(X,g(X),Y) \doteq p(Z,U,g(U))\}$,

(e) $\{p(g(X),Y) \doteq p(Y,Y), p(Y,Y) \doteq p(U,f(W))\}$.

10.13 What is the size of the maximal term that is produced when you try to unify

$$\{f(g(X_1,X_1),g(X_2,X_2),\ldots,g(X_{n-1},X_{n-1})) \doteq f(X_2,X_3,\ldots,X_n\}.$$

10.14 The compactness theorem in First-Order Logic says that a set of sentences has a model iff every finite subset of it has a model. Use this theorem to show that the transitive closure is not definable in FOL.