DCGI

DEPARTMENT OF COMPUTER GRAPHICS AND INTERACTION

TRIANGULATIONS

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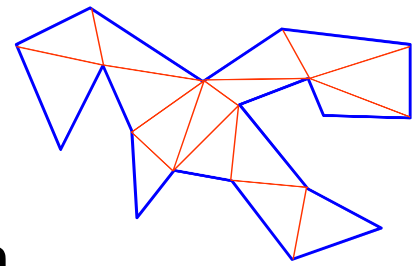
https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg] and [Mount]

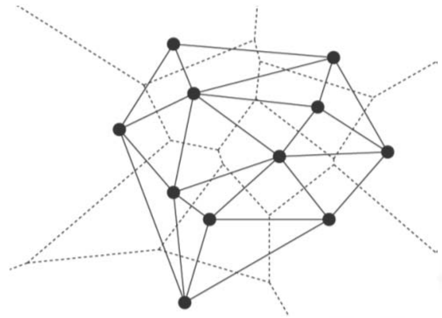
Version from 16.11.2020

Talk overview

- Polygon triangulation
  - Monotone polygon triangulation
  - Monotonization of non-monotone polygon

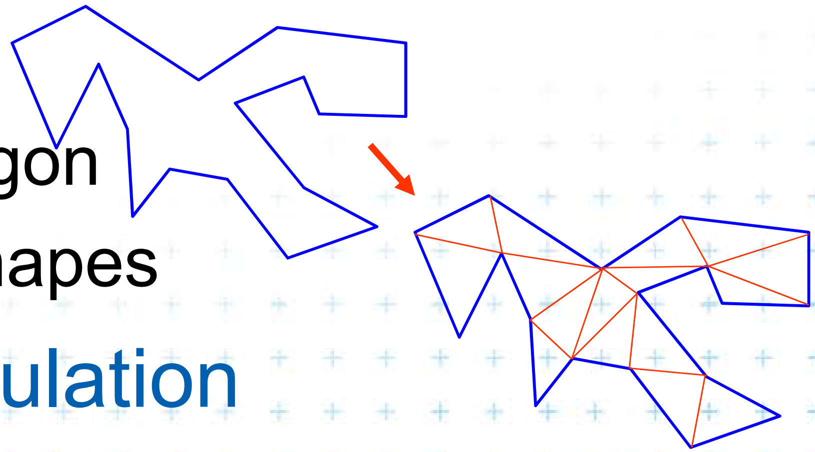


- Delaunay triangulation (DT) of points
  - Input: set of 2D points
  - Properties
  - Incremental Algorithm
  - Relation of DT in 2D and lower envelope (CH) in 3D and
    relation of VD in 2D to upper envelope in 3D



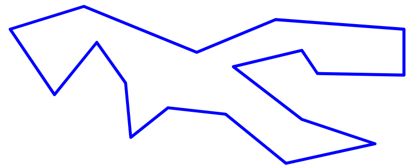
Polygon triangulation problem

- Triangulation (in general)
  = subdividing a spatial domain into simplices
- Application
  – decomposition of complex shapes into simpler shapes
  – art gallery problem (how many cameras and where)
- We will discuss
  – Triangulation of a simple polygon
  – without demand on triangle shapes
- Complexity of polygon triangulation
  – O(*n*) alg. exists [Chazelle91], but it is too complicated
  – practical algorithms run in O(*n* log *n*)



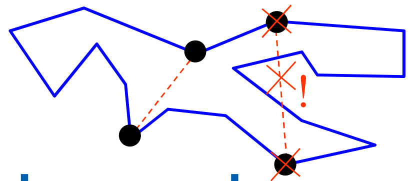
Terminology

Simple polygon



= region enclosed by a closed polygonal chain that does not intersect itself

Visible points



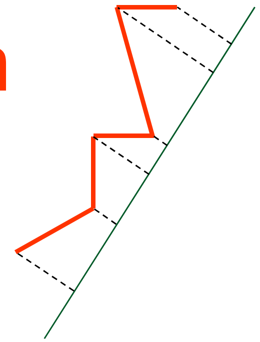
= two points on the boundary are visible if the interior of the line segment joining them lies entirely in the interior of the polygon

Diagonal

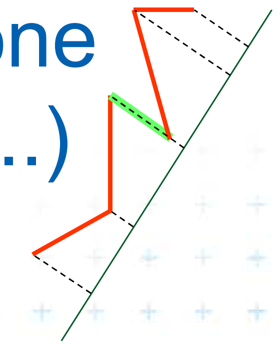
= line segment joining any pair of visible vertices

Terminology

- A polygonal chain C is strictly monotone with respect to line L, if any line orthogonal to L intersects C in at most one *point*



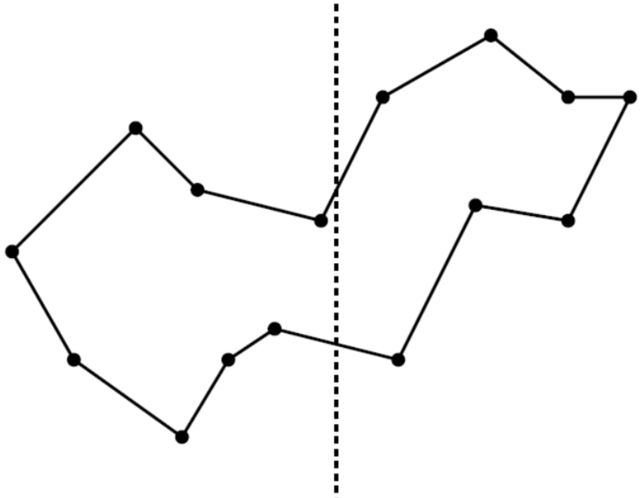
- A chain C is monotone with respect to line L, if any line orthogonal to L intersects C in at most one *connected component* (point, line segment,...)



- Polygon P is monotone with respect to line L, if its boundary (bnd(P), ∂P) can be split into two chains, each of which is monotone with respect to L

Terminology

- Horizontally monotone polygon
  = monotone with respect to *x*-axis
  - Can be tested in $O(n)$
  - Find leftmost and rightmost point in $O(n)$
  - Split boundary to upper and lower chain
  - Walk left to right, verifying that x-coord are non-decreasing



x–monotone polygon

[Mount]

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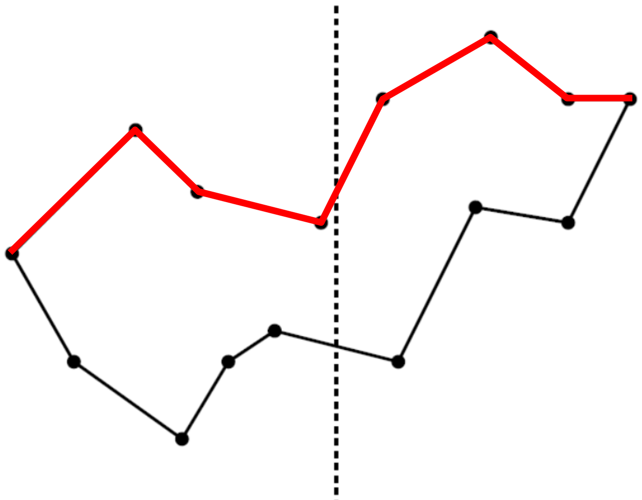
x–monotone polygon

[Mount]

Terminology

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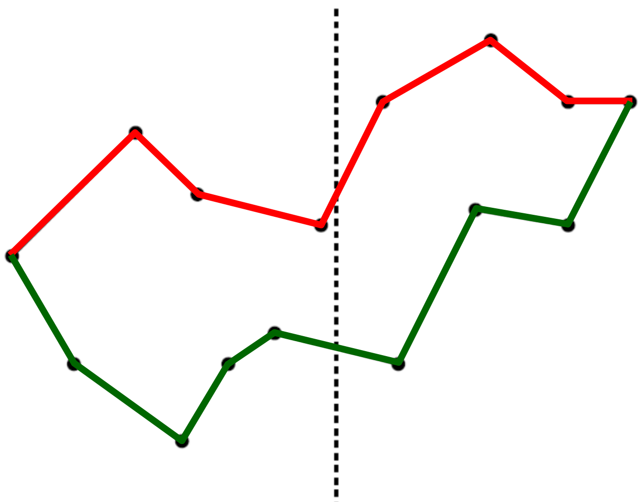


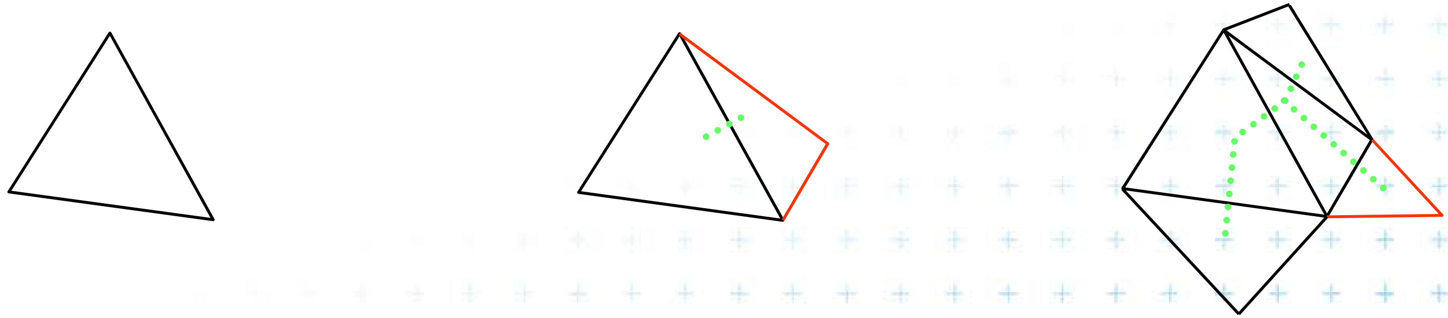
x–monotone polygon

[Mount]

Terminology

- Every simple polygon can be triangulated
- Simple polygon with n vertices consists of
  - exactly $n - 2$ triangles
  - exactly $n - 3$ diagonals
  - Each diagonal is added once
    $\Rightarrow O(n)$ sweep line algorithm exist

Proof by induction



$n = 3 \Rightarrow 0$ diagonal $n = 4 \Rightarrow 1$ diagonal $n := n + 1 \Rightarrow n + 1 - 3$ diagonals

$n - 3$ $n + 1 = 7 \Rightarrow 4$ diagonals)

Simple polygon triangulation

- Simple polygon can be triangulated in 2 steps:
  1. Partition the polygon into x-monotone pieces
  2. Triangulate all monotone pieces

  (we will discuss the steps in the reversed order)

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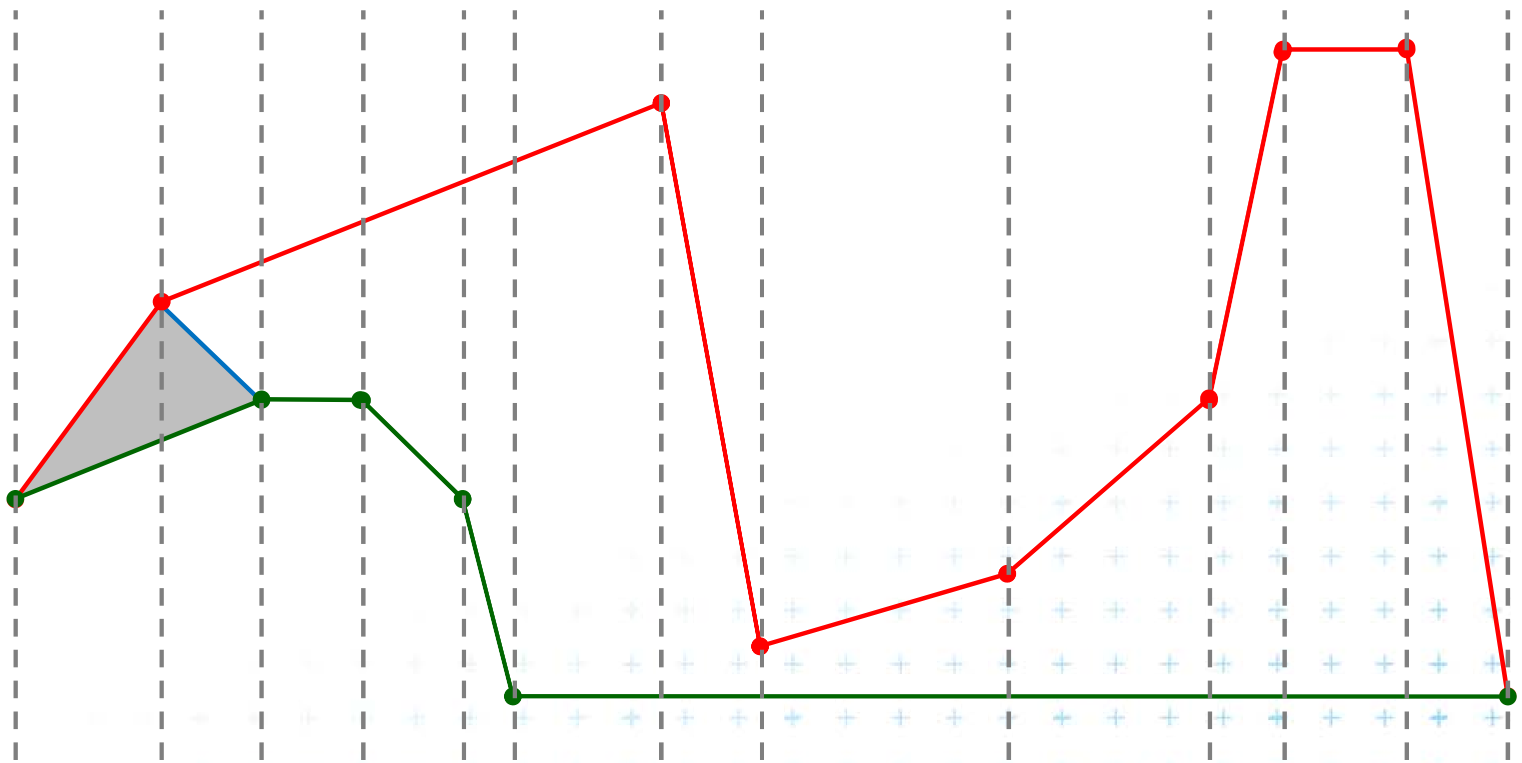
x- monotone polygon triangulation principle

- Sweep left to right - in $O(n)$ steps
- Triangulate everything you can by adding diagonals between visible points (left from the sweep line)



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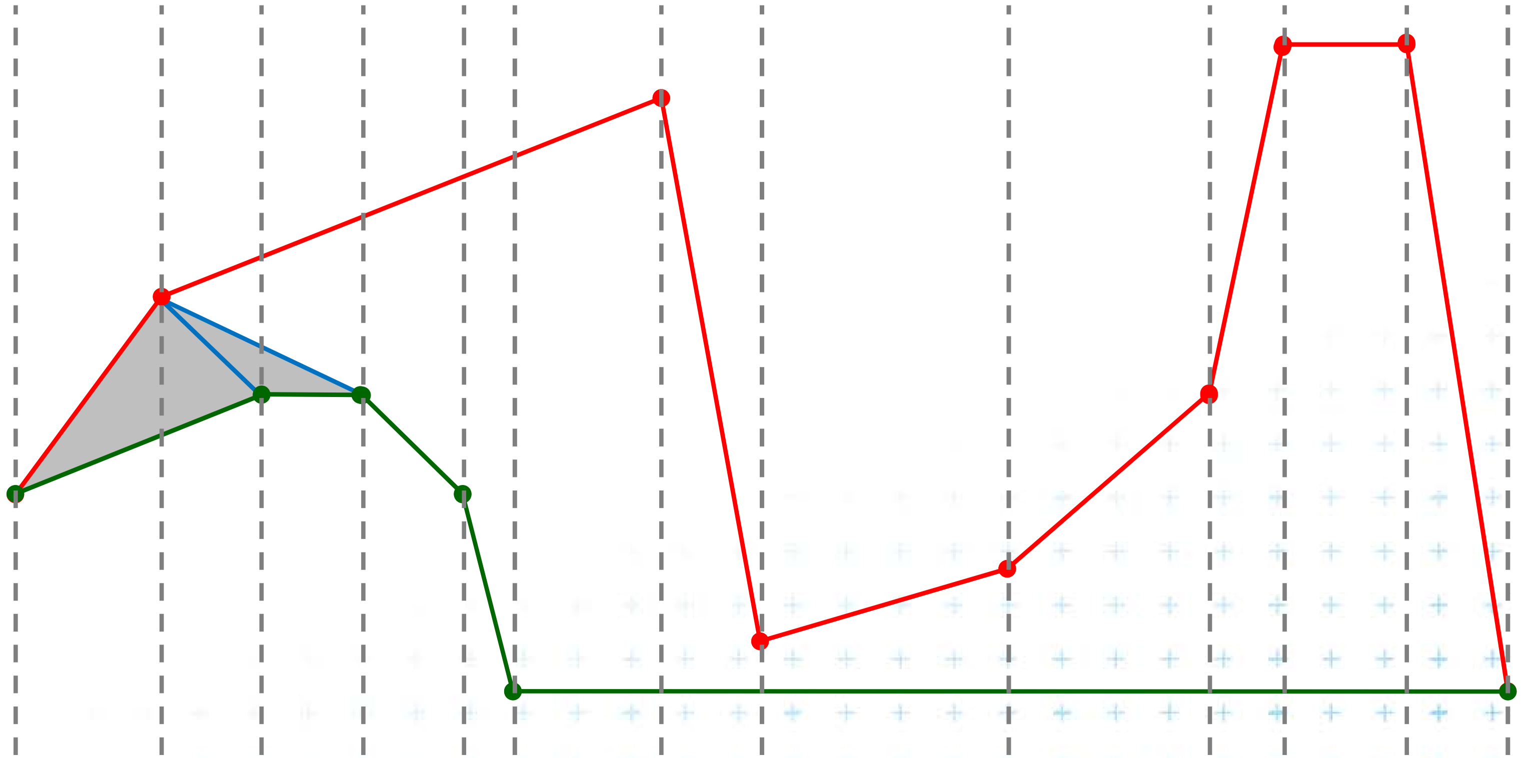

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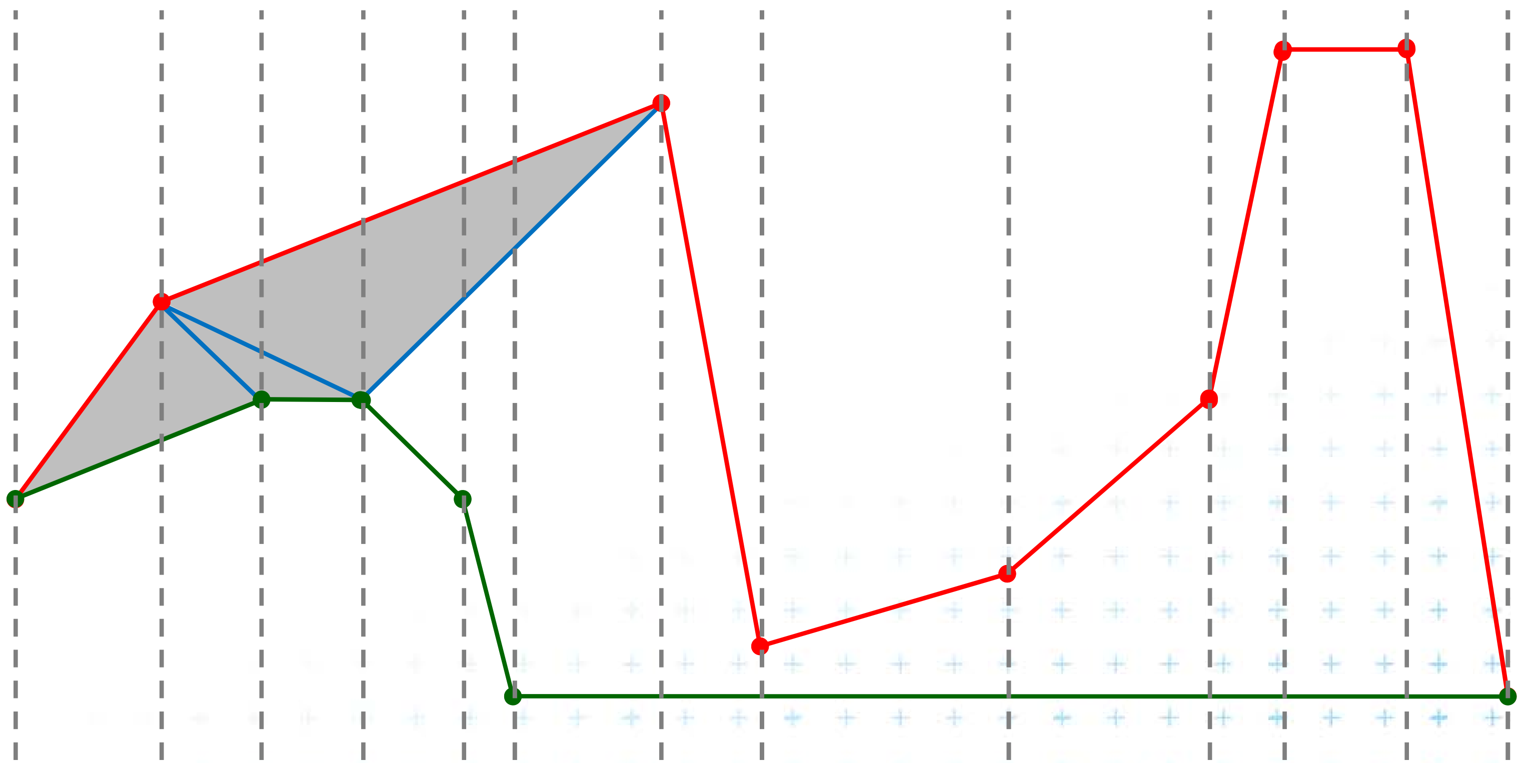

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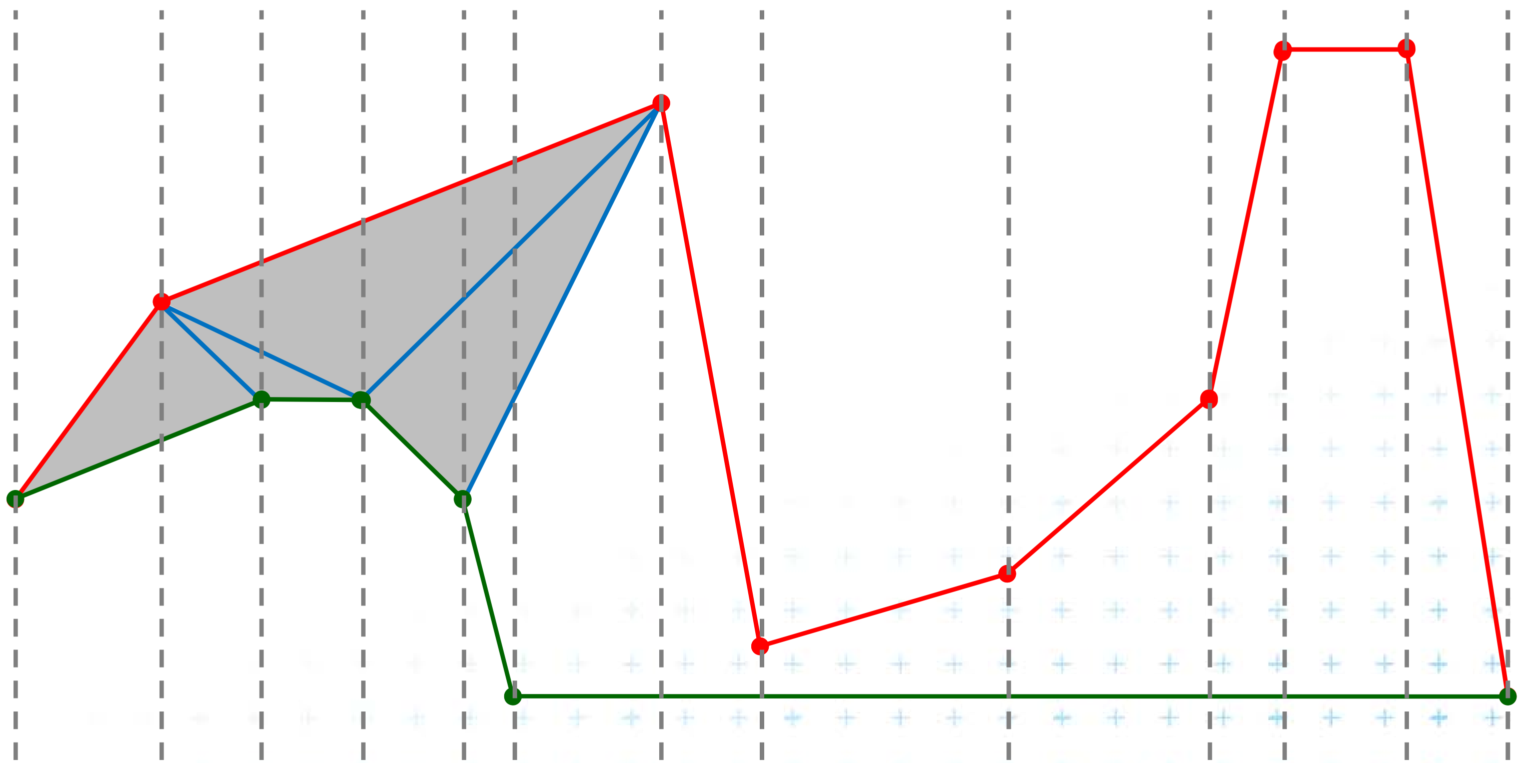

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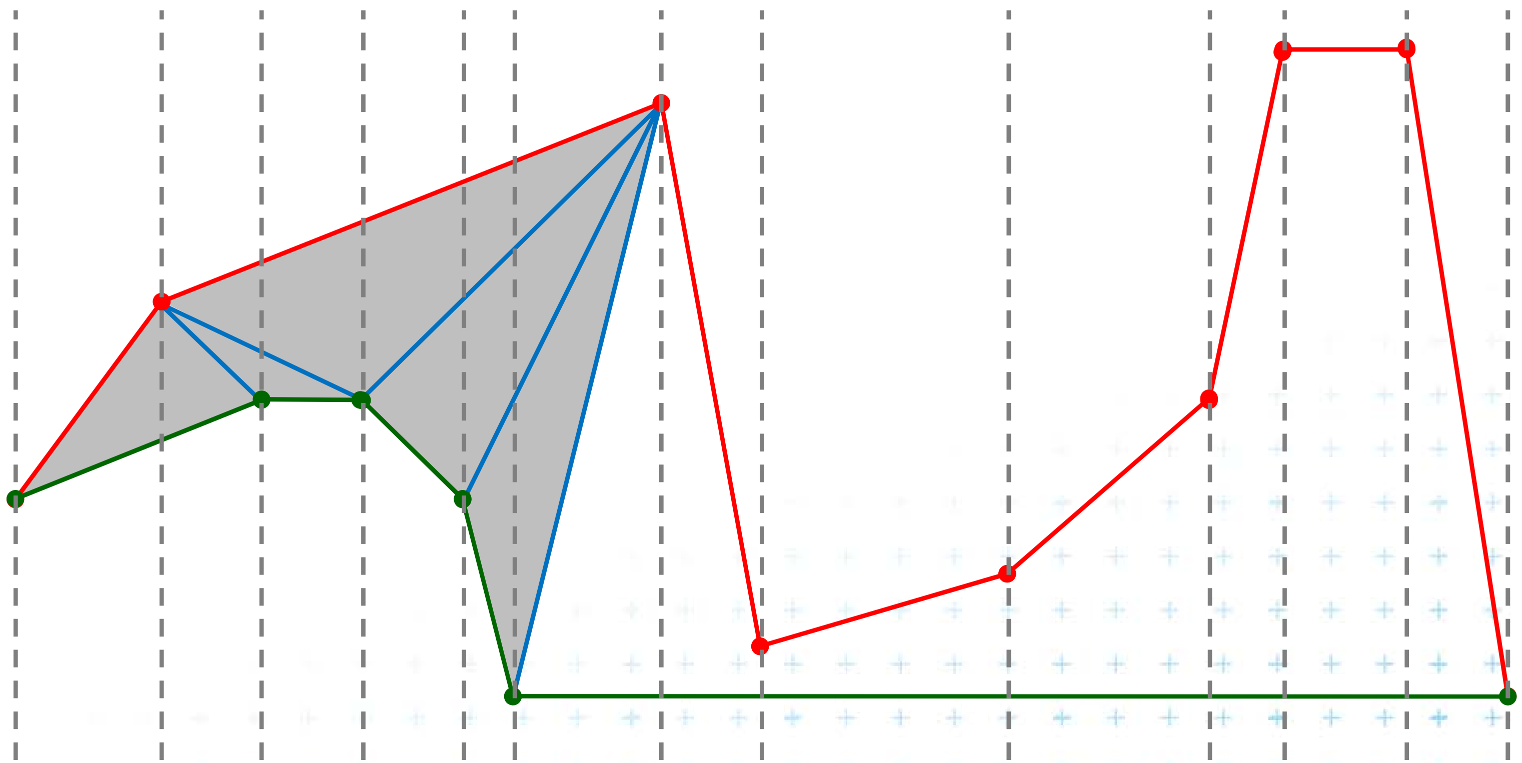

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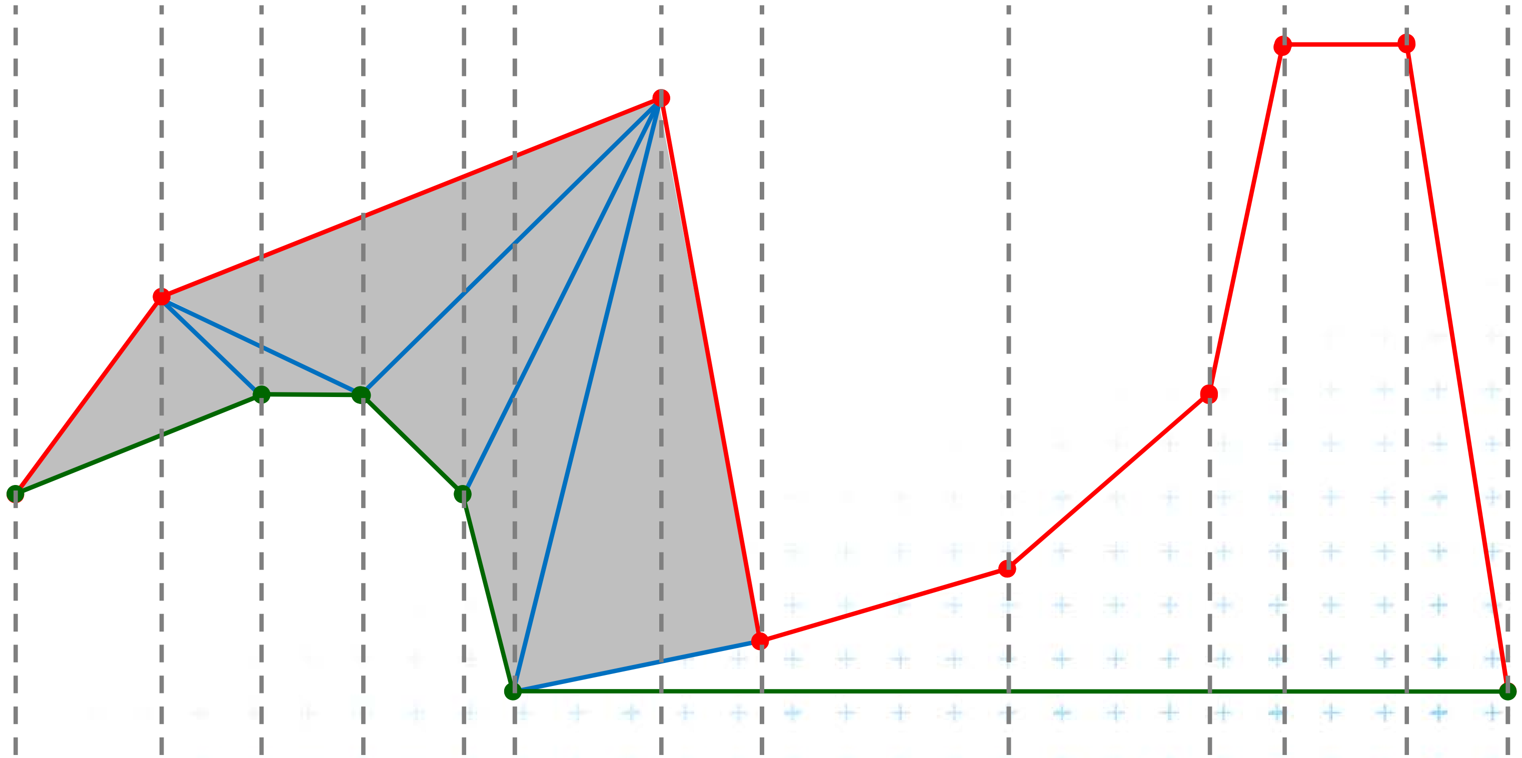

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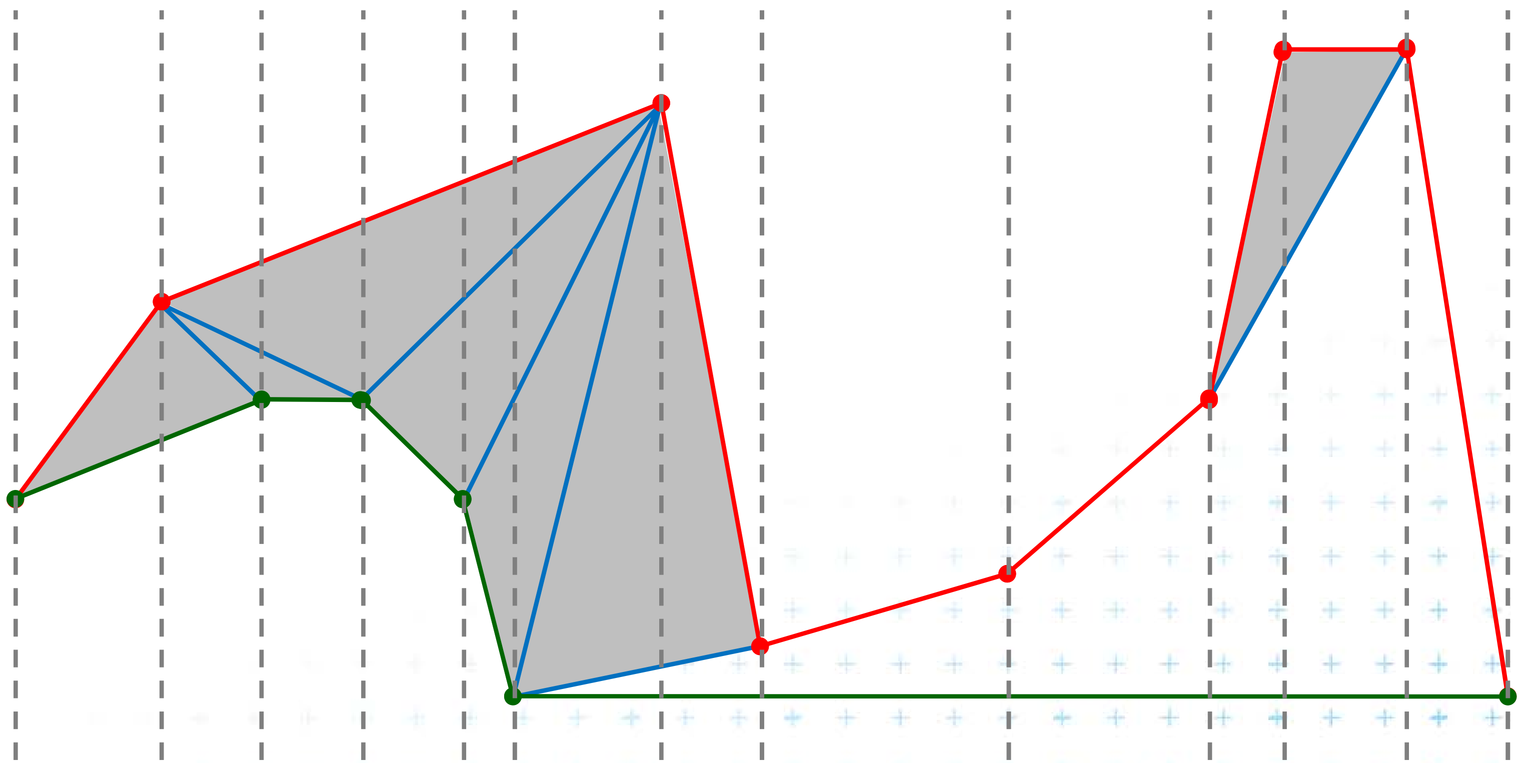

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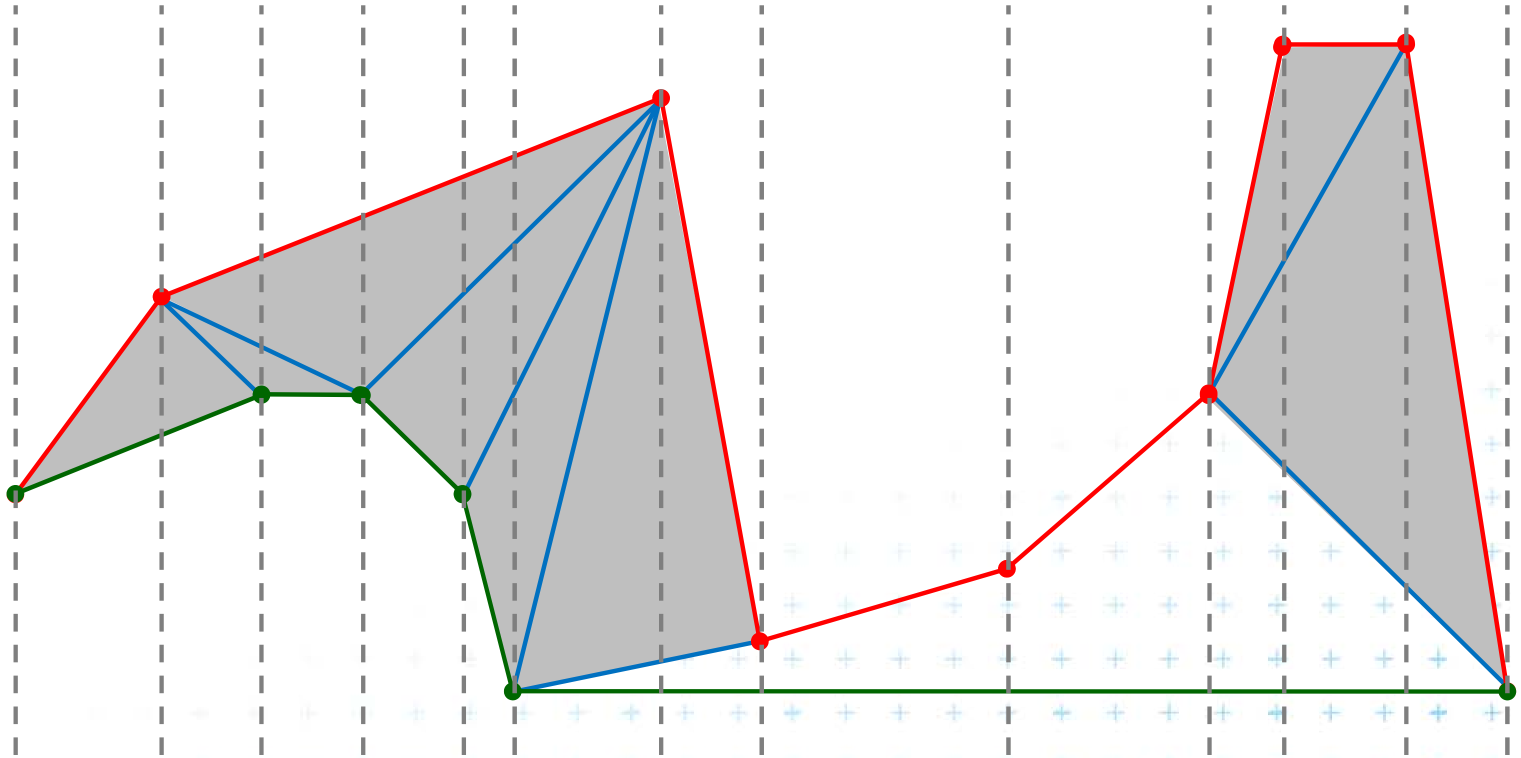

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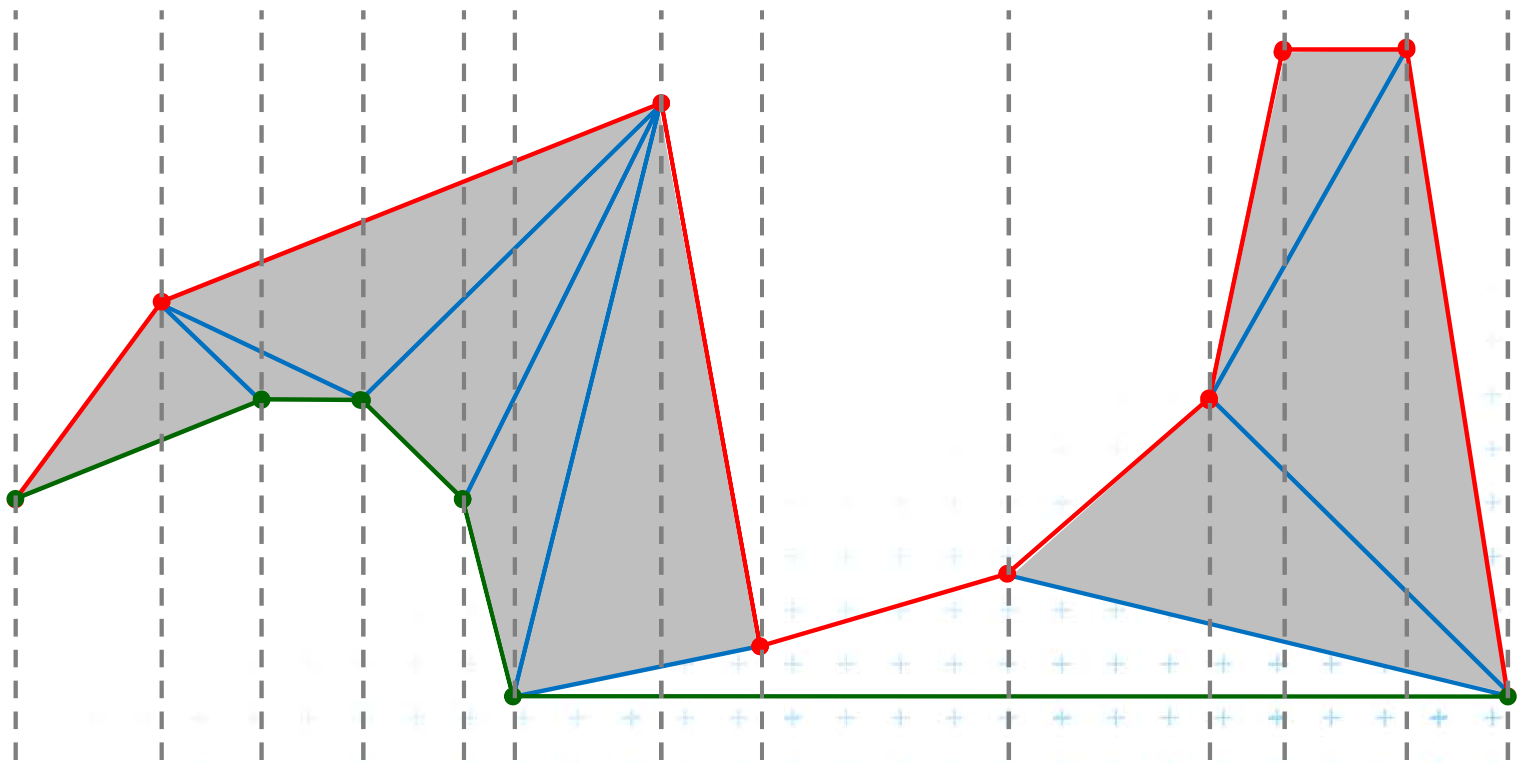

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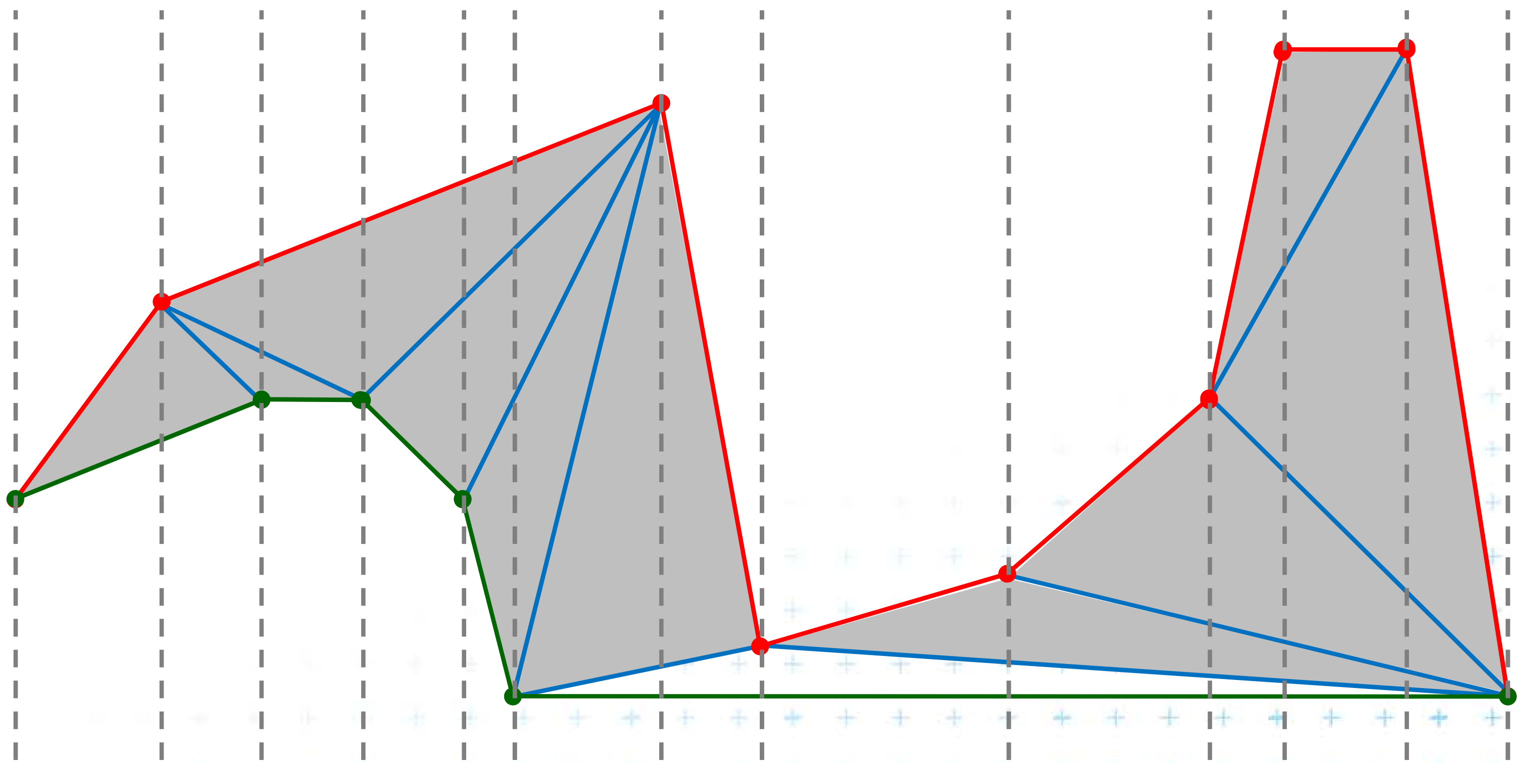

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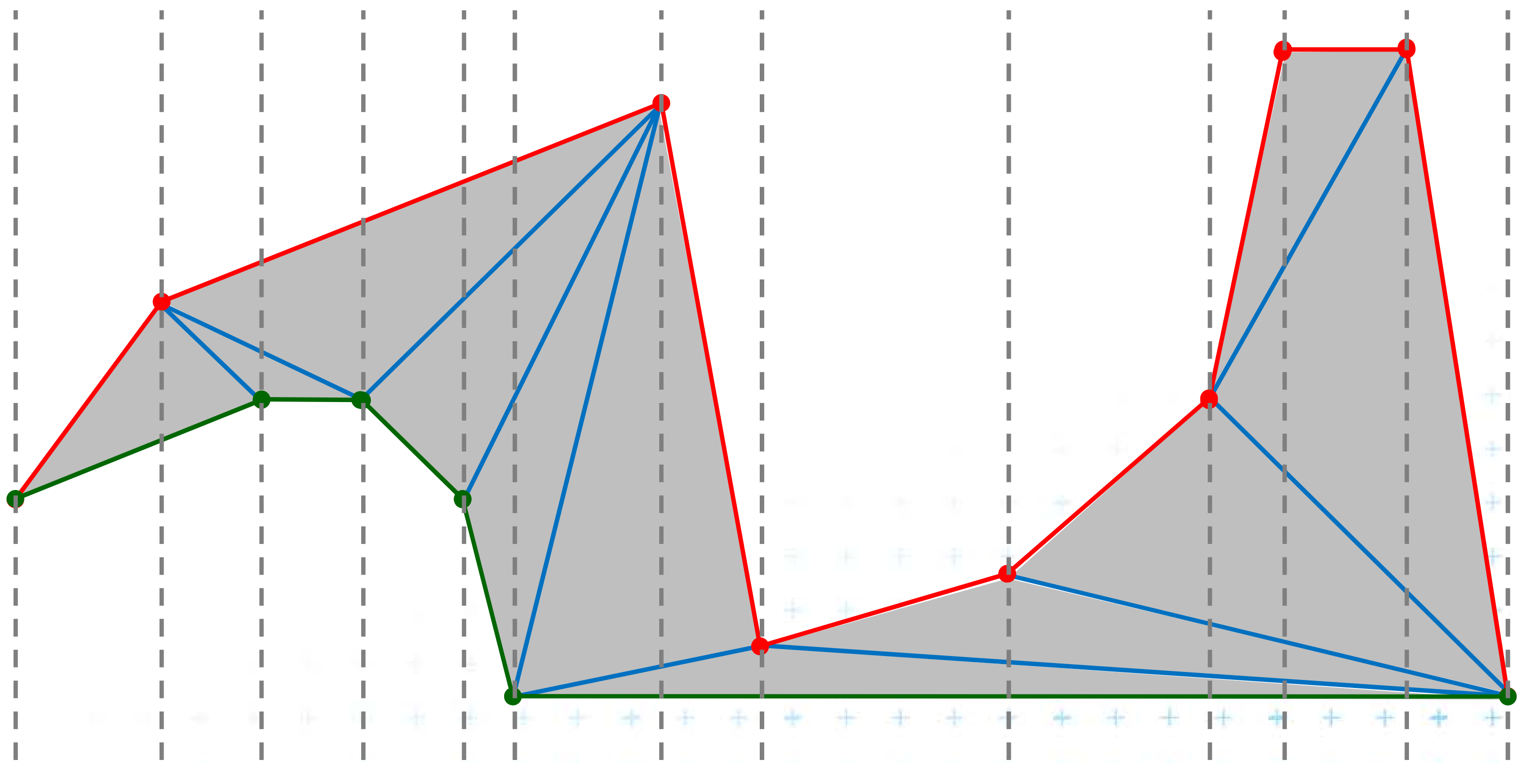
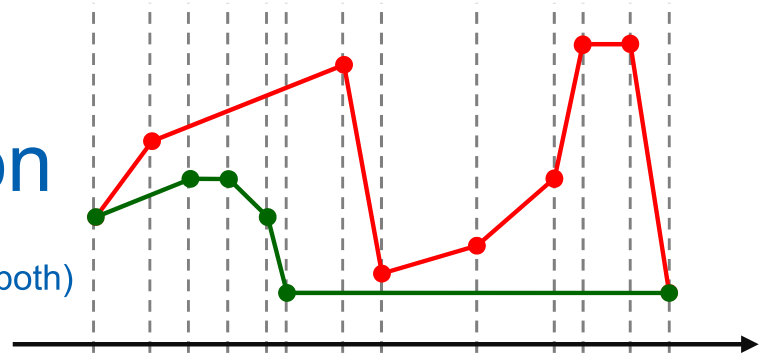


[Mount]

Event queue

Sweep line event queue

- x-sorted vertices of the polygon with lower/upper flag (2-bits, extremes to both)



Construction – $O(n)$

- Find min x and max x
- Extract lower and upper chain (between min and max x)
  Both are sorted in increasing order of their x-coords
- Merge chains in $O(n)$ keeping lower/upper flag

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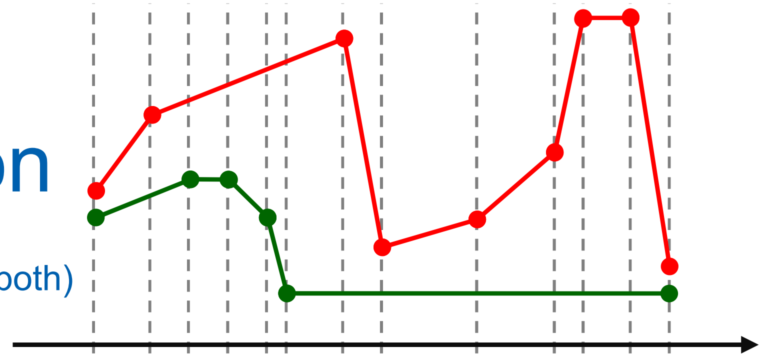

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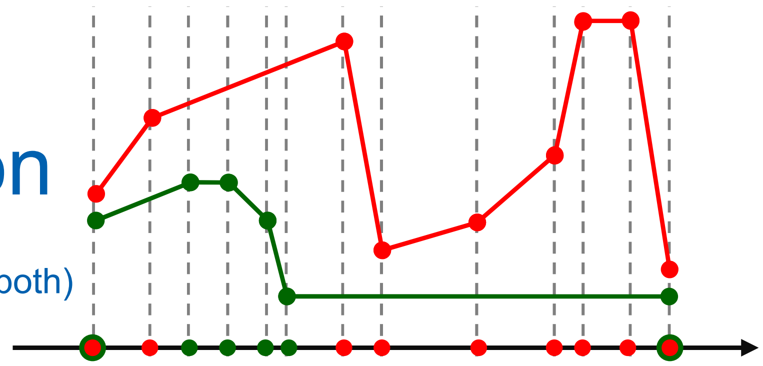

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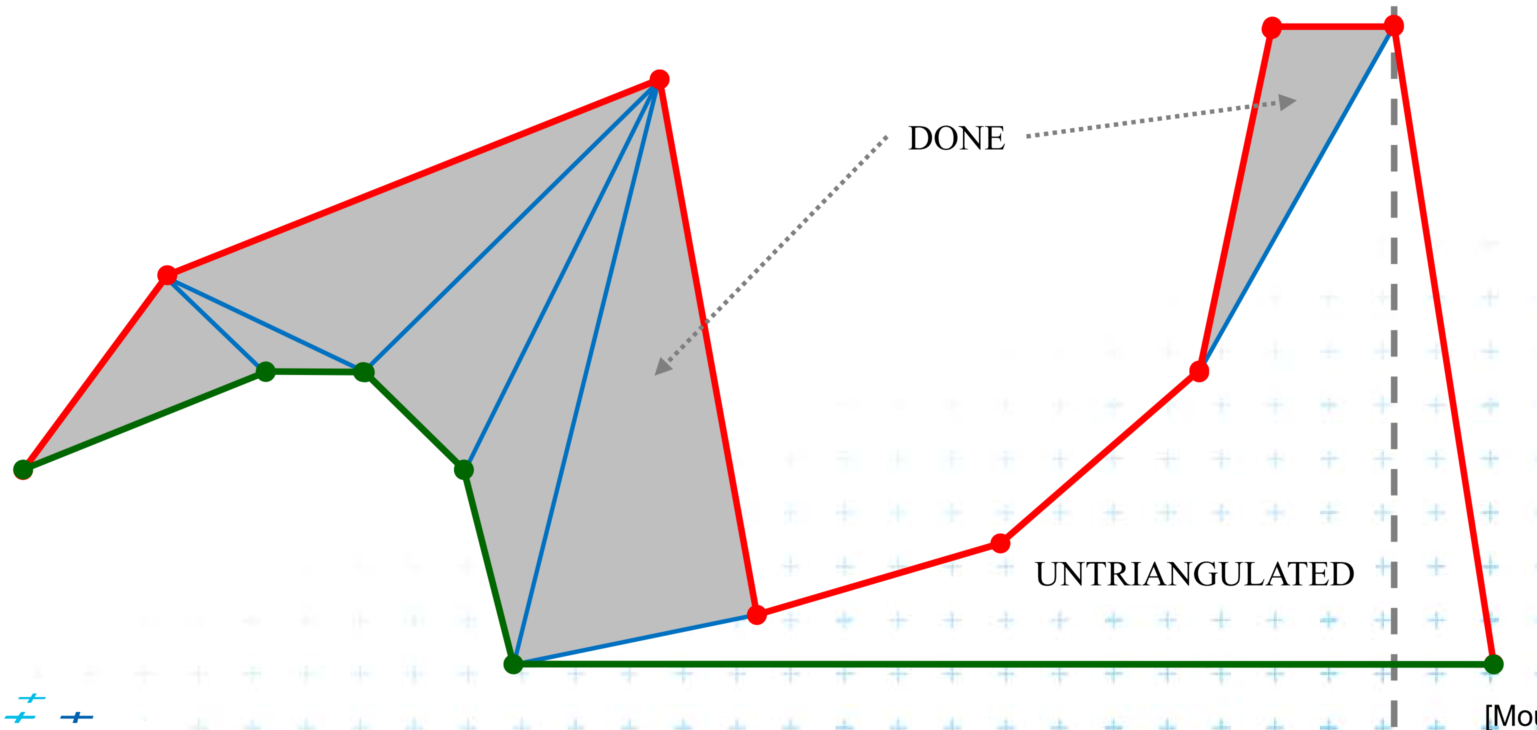
Regions on the left from the sweep line

a) triangulated – points were visible – DONE

b) untriangulated – points were not visible
– characterized by an invariant
(= a condition that is true after each step)



[Mount]

Reflex vertex and reflex chain

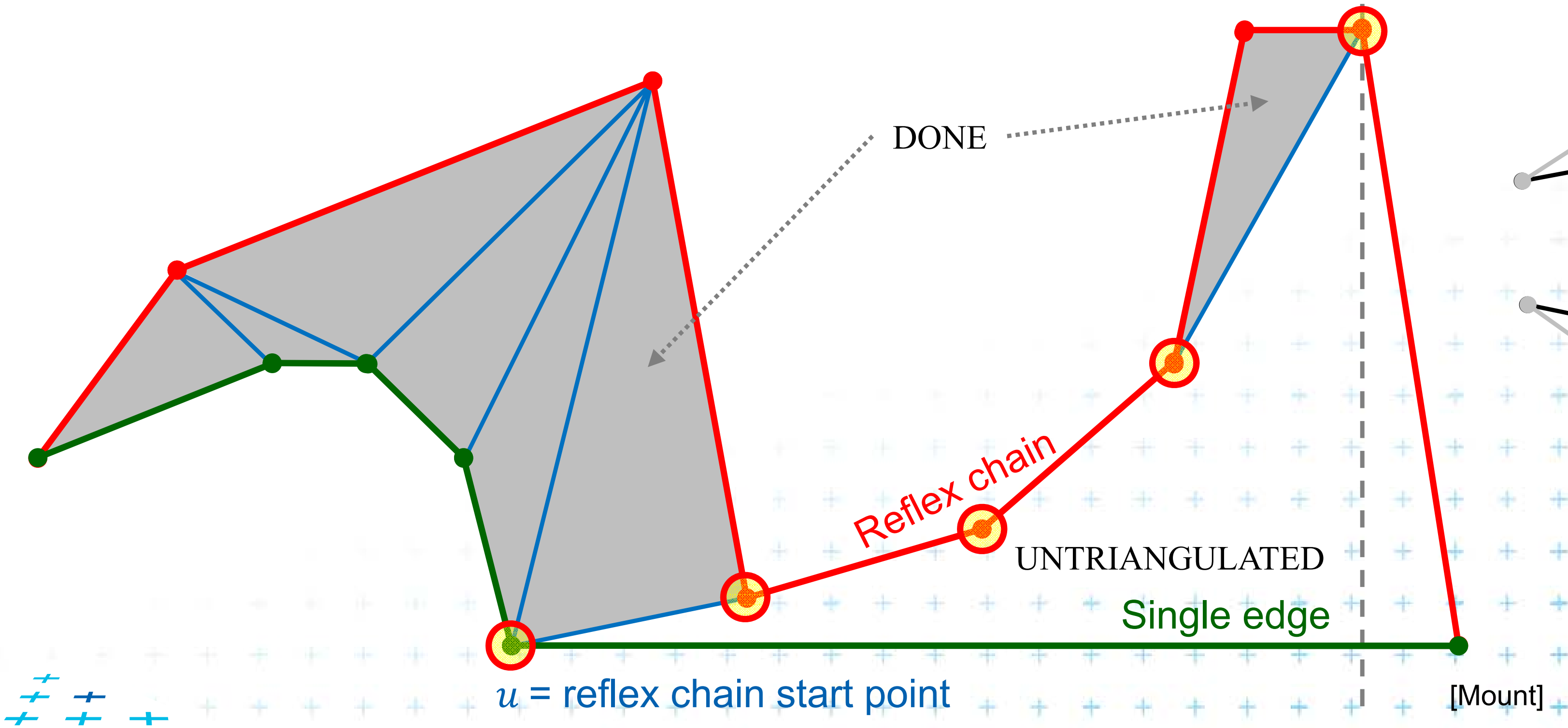
Untriangulated region is bounded by a reflex chain

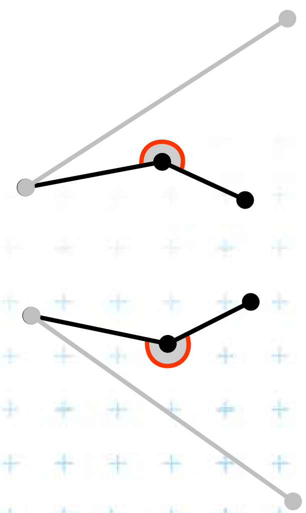
= a sequence of reflex vertices along the not-triangulated part of the polygon

– in the alg. is stored in stack

Reflex vertex

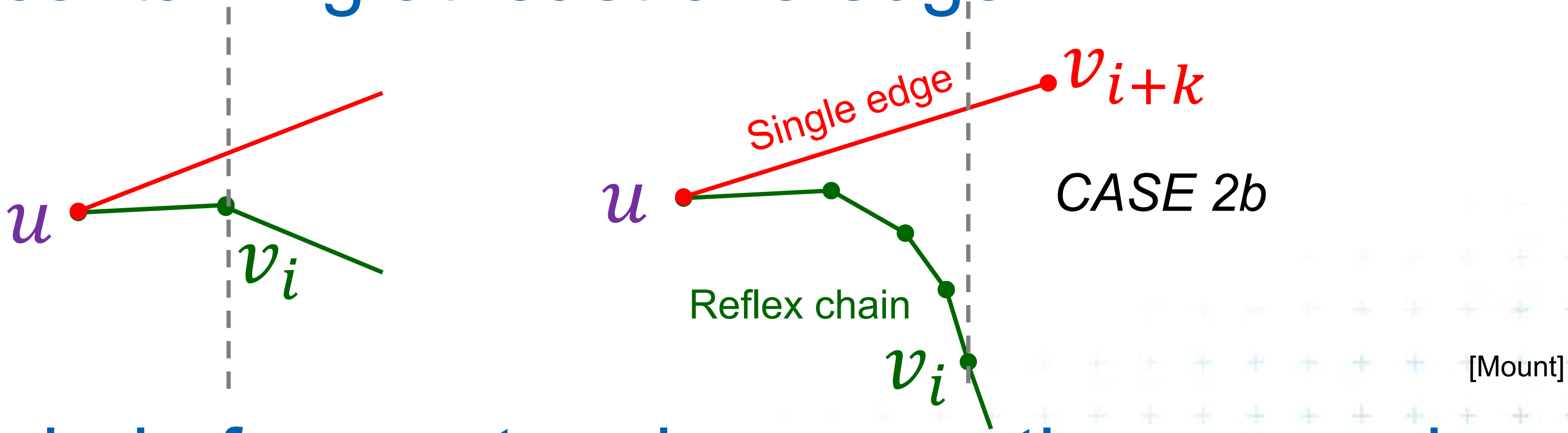
interior angle $\geq \pi$





Main invariant of untriangulated region left from SL

- Let v_i, $i \geq 2$ be the vertex just being processed
- The untriangulated region left of v_i consists of two x-monotone chains (upper and lower) each containing at least one edge



- If the chain from v_i to u has more than one edge
  – these edges form a reflex chain
  – the other chain consist of single edge from u to vertex v_{i+k} right of v_i

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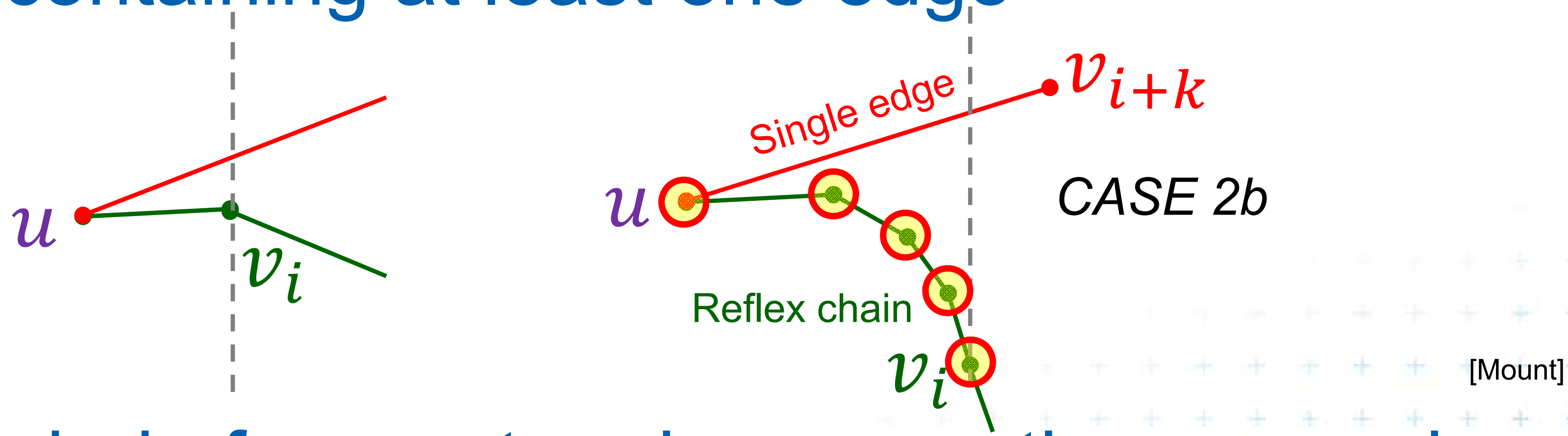

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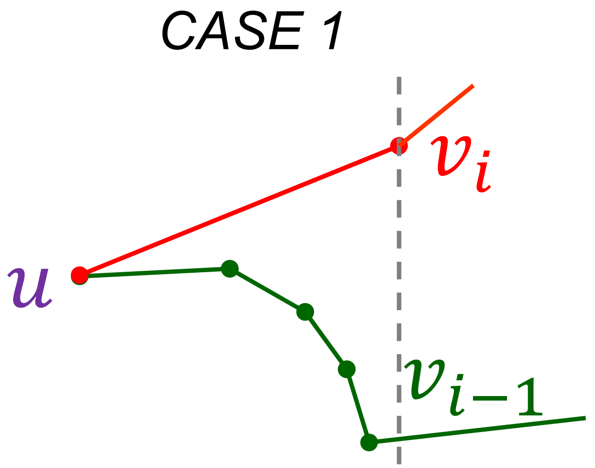
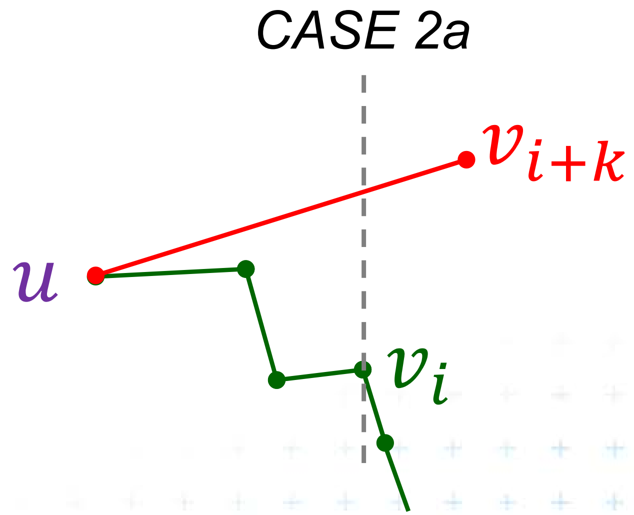
  from u to vertex v_{i+k} right of v_i

The remaining regions are triangulated

- Elsewhere, it would have been triangulated in this step





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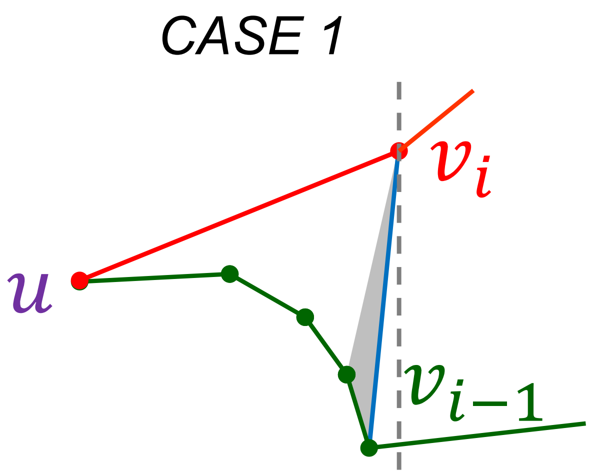

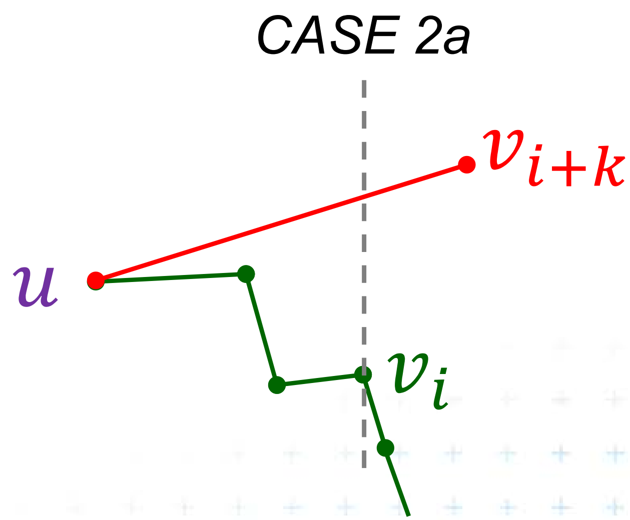

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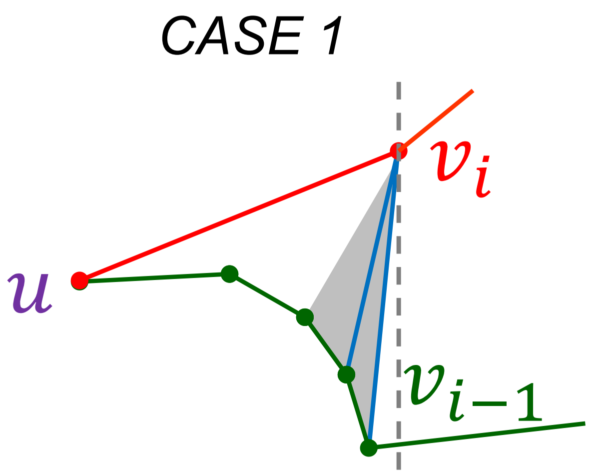

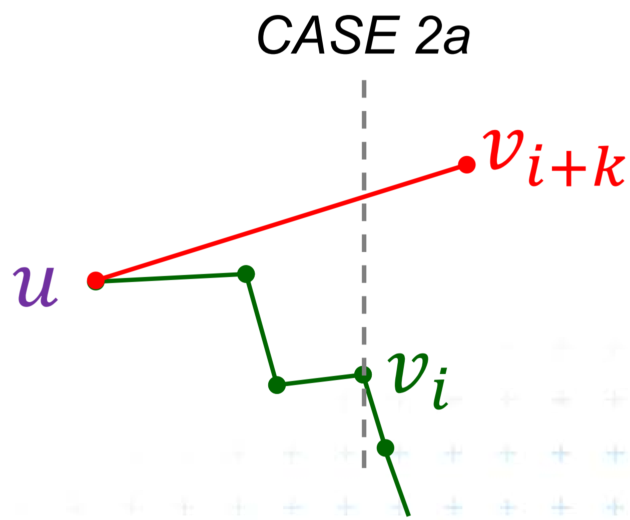

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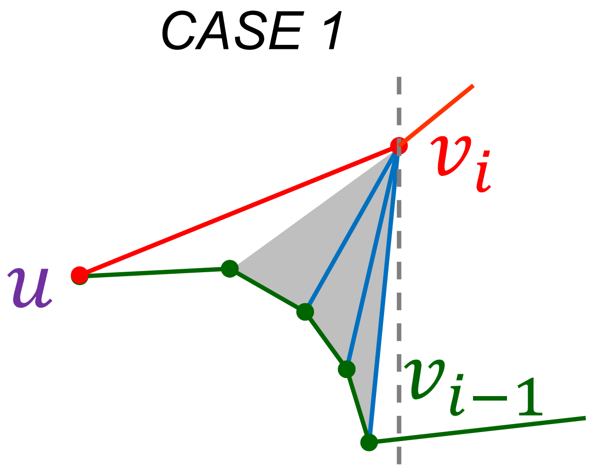

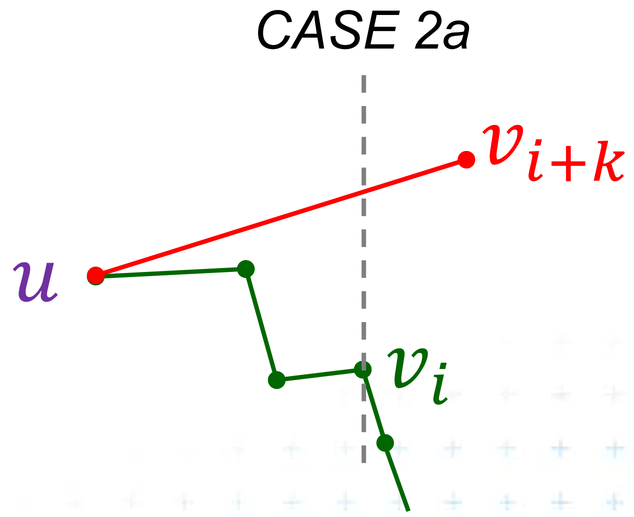

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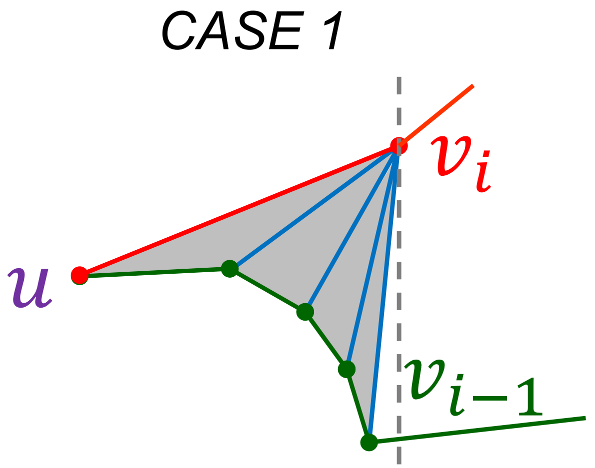

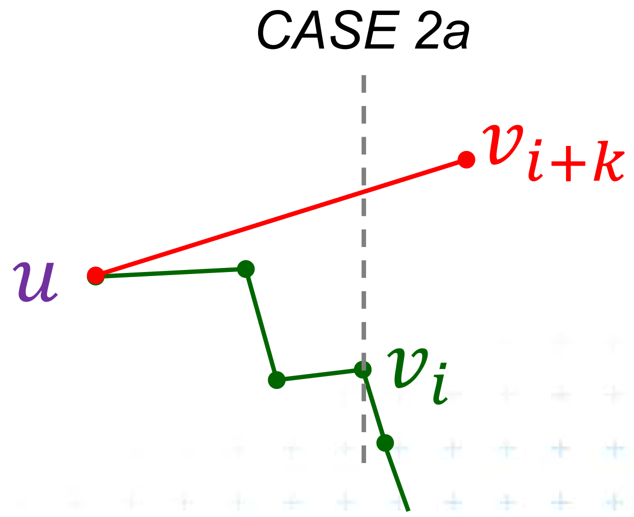

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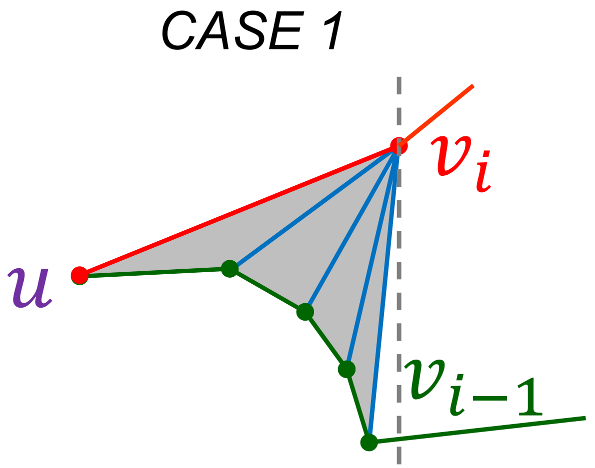

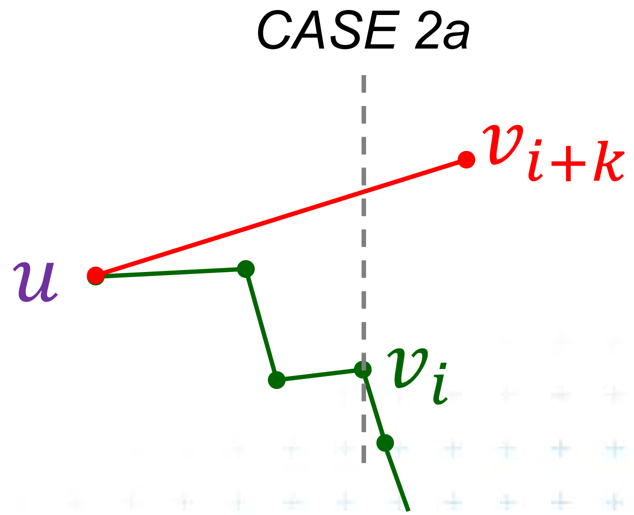

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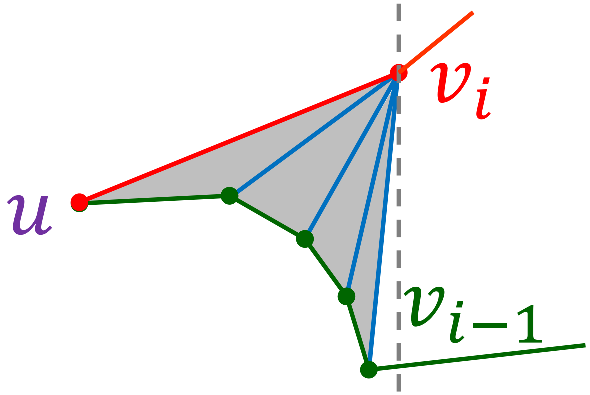
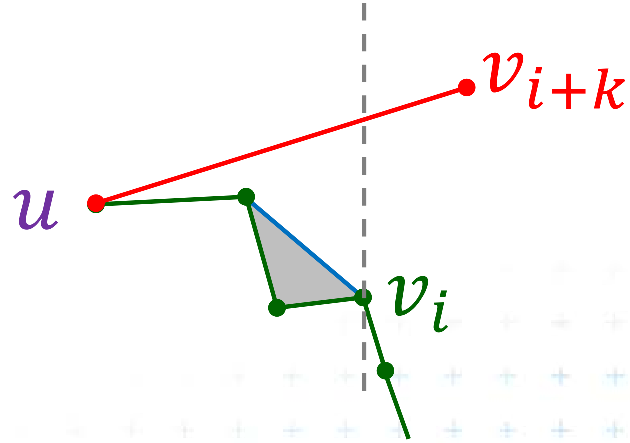
CASE 1



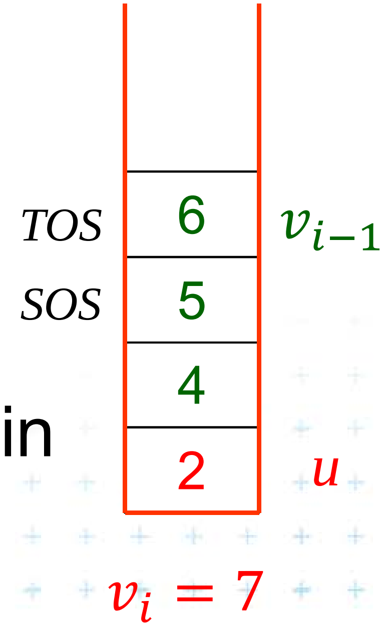
CASE 2a



Triangulation algorithm

Data structures

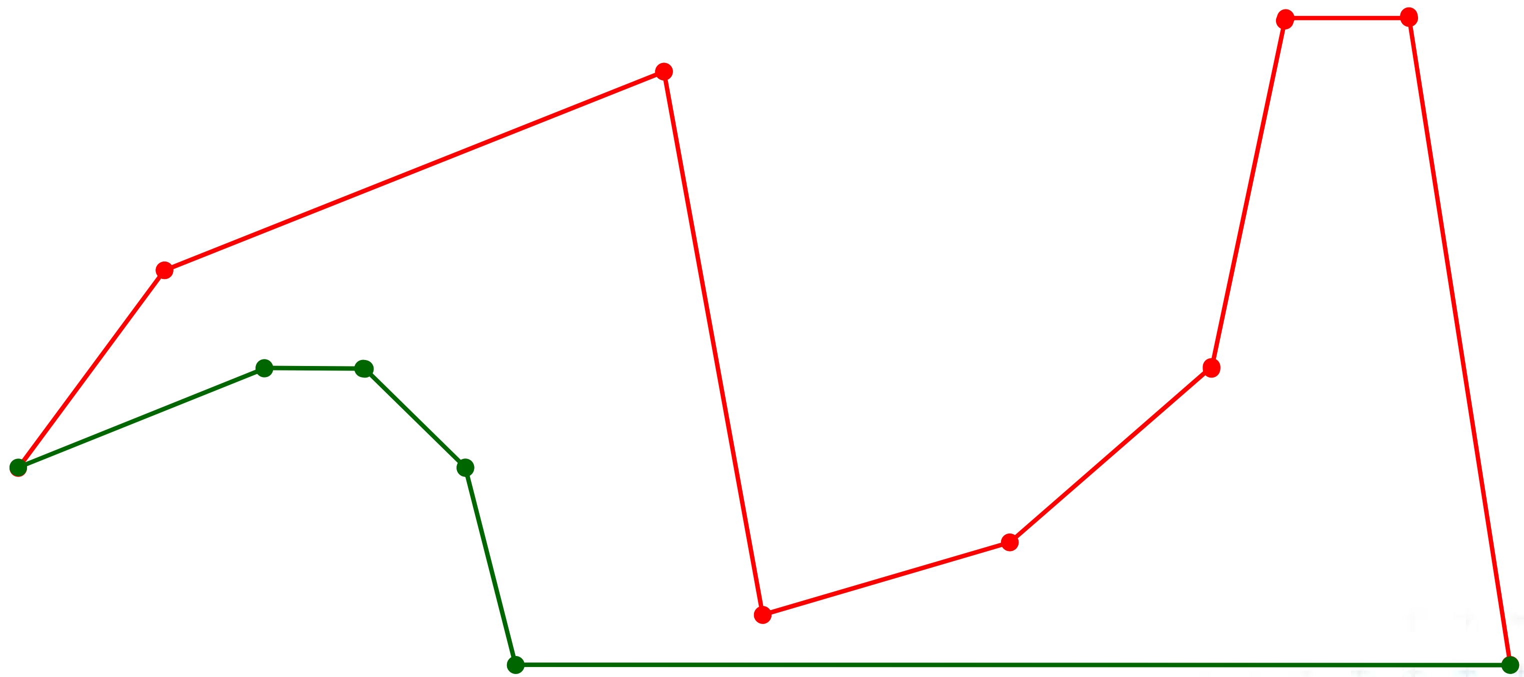
- Event queue with merged upper and lower chain
- Status
  - Current vertex v_i (sweep line position i)
  - Reflex vertices chain in the stack
  - Upper/lower chain flag
    all vertices except u are from the same chain
    u is from the opposite chain

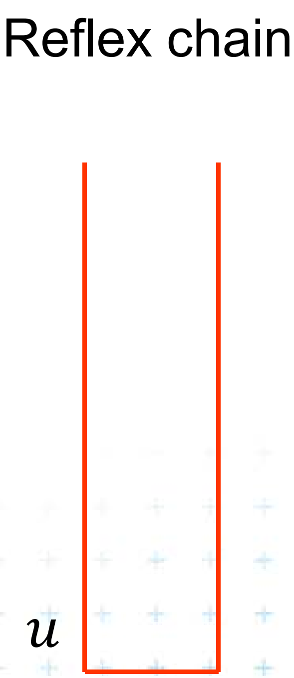


Orientation test

– reflex(TOS, SOS, v_i)

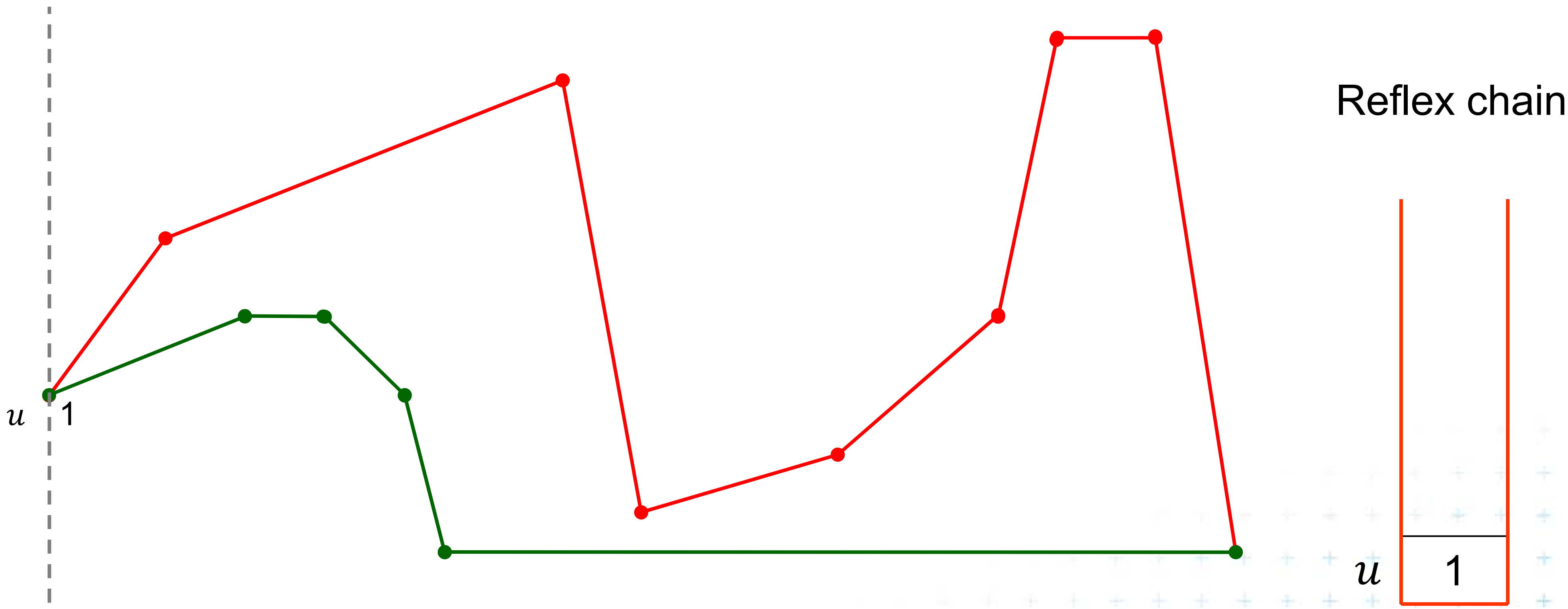
Monotone polygon triangulation algorithm





[Mount]

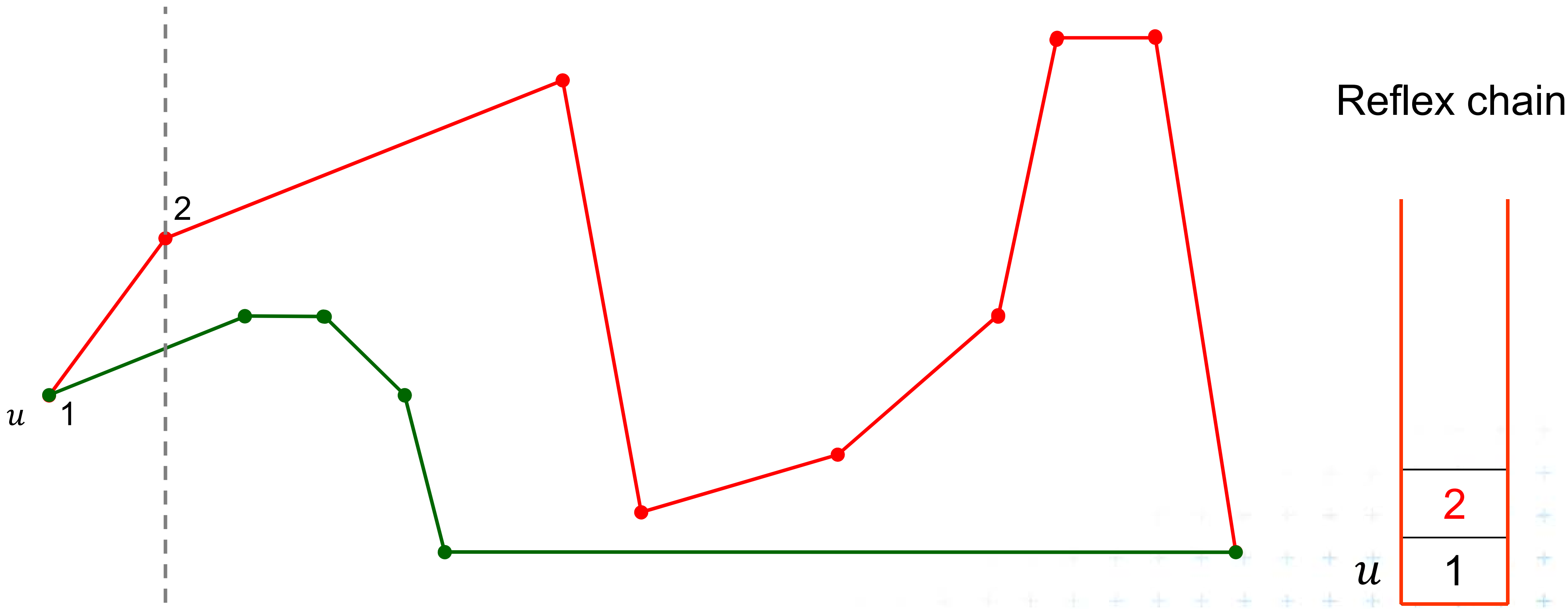
Monotone polygon triangulation algorithm



Start – set reflex chain start u (bottom of stack)

[Mount]

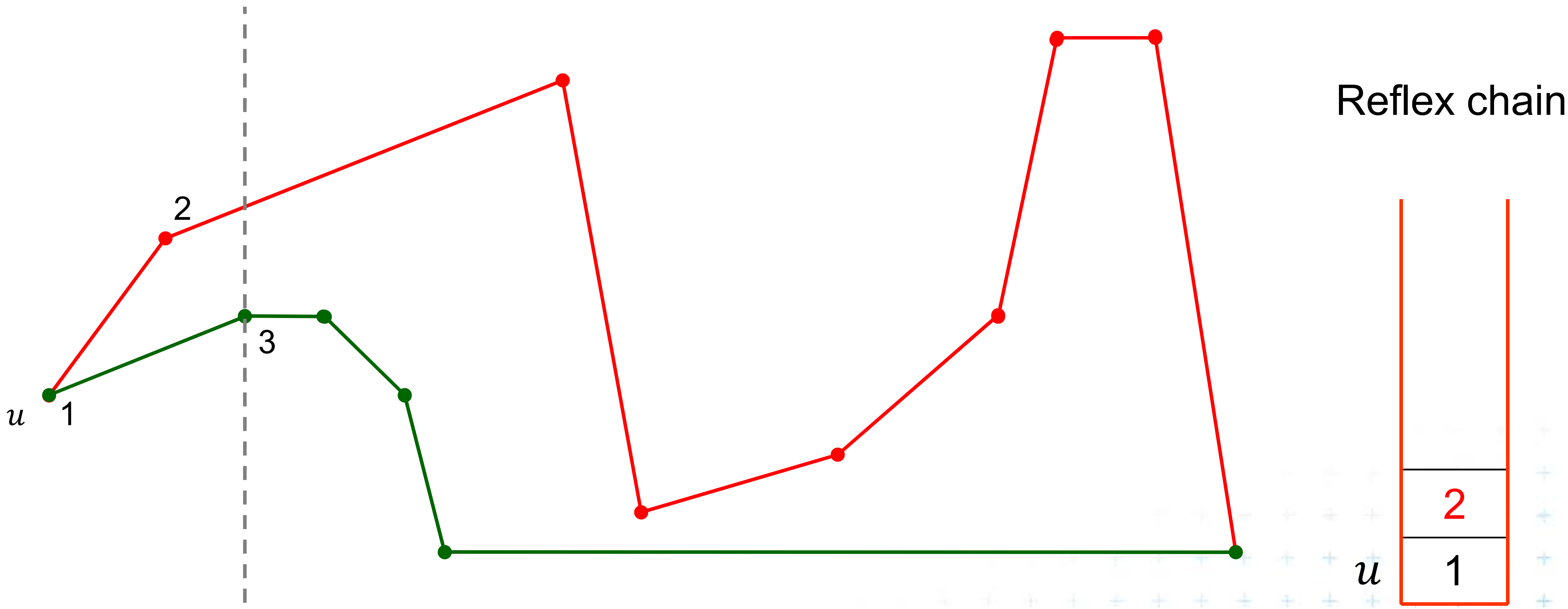
Monotone polygon triangulation algorithm



Start – set trivial reflex chain end (top of stack)

[Mount]

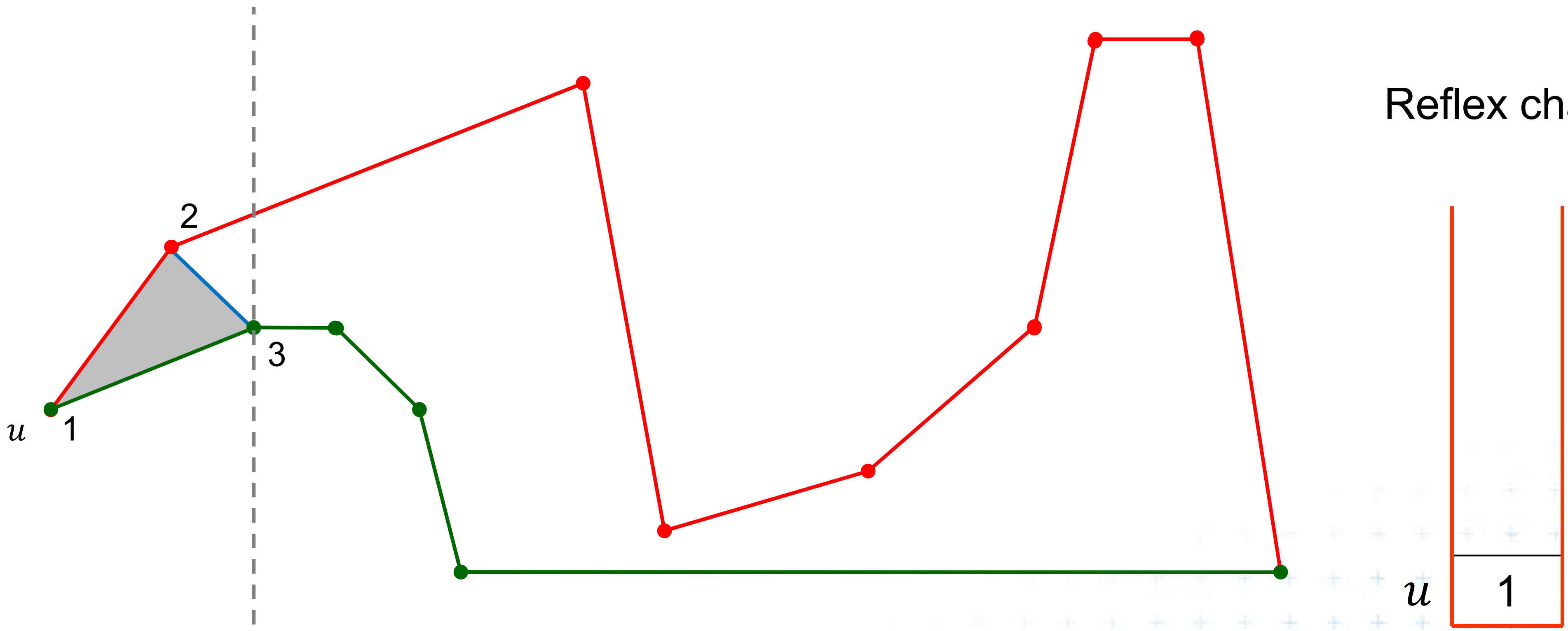
Monotone polygon triangulation algorithm



Case 1 – point v_i on opposite chain from v_{i-1}

[Mount]

Monotone polygon triangulation algorithm

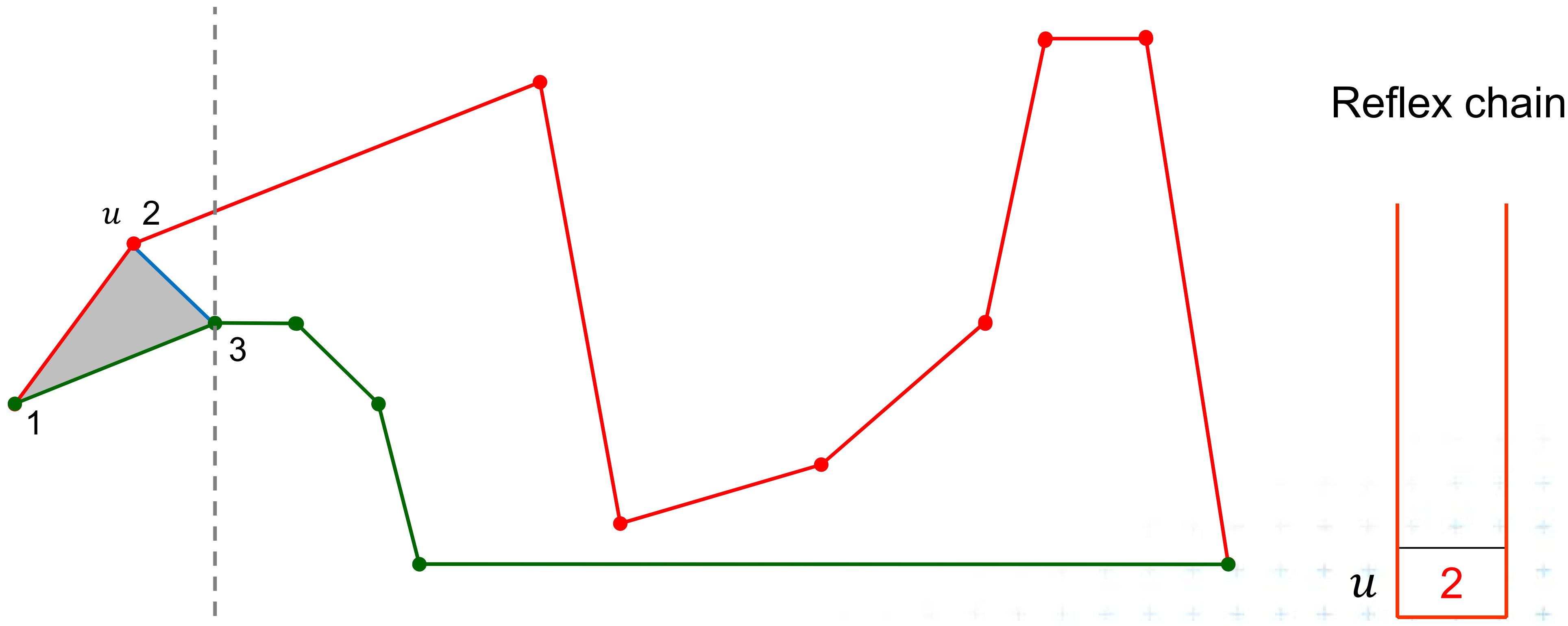


Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

[Mount]

Monotone polygon triangulation algorithm



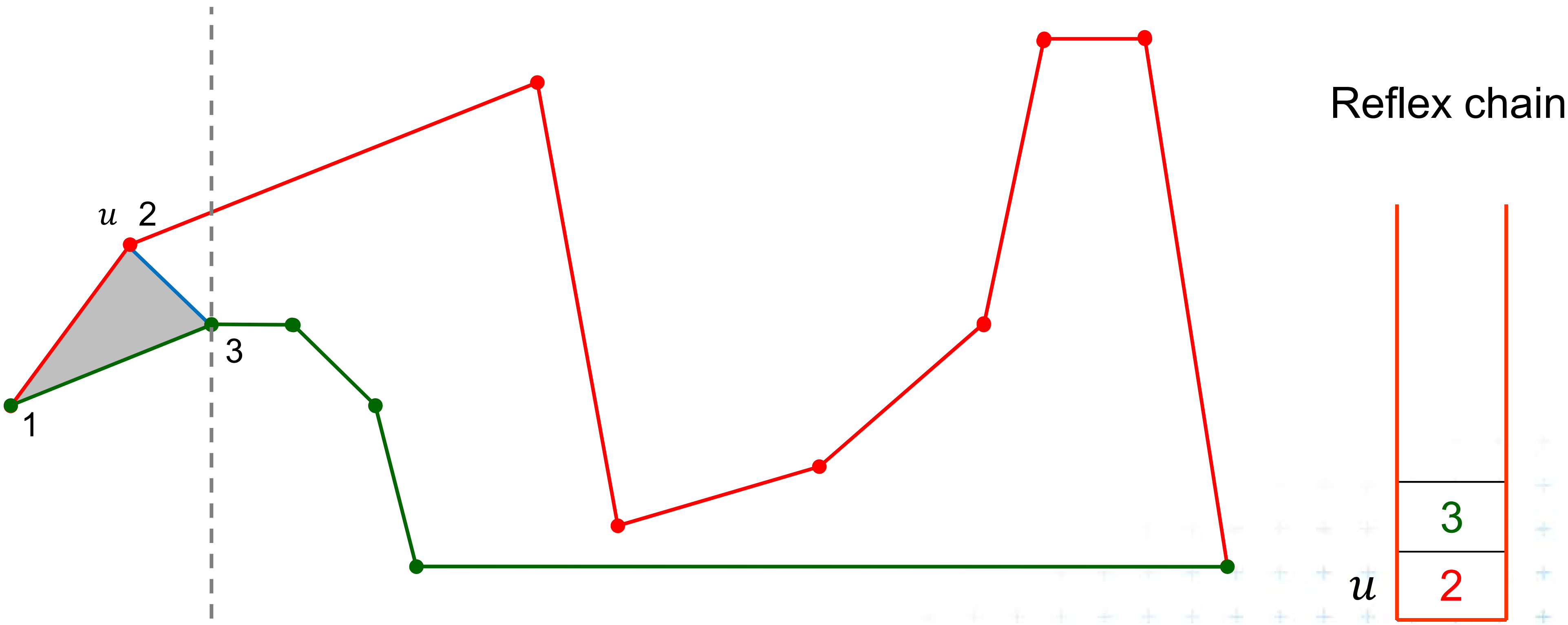
Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

Set trivial reflex chain $v_i v_{i-1}$: New u: pop(), push(v_{i-1}), push(v_i)

[Mount]

Monotone polygon triangulation algorithm



Case 1 – point v_i on opposite chain from v_{i-1}

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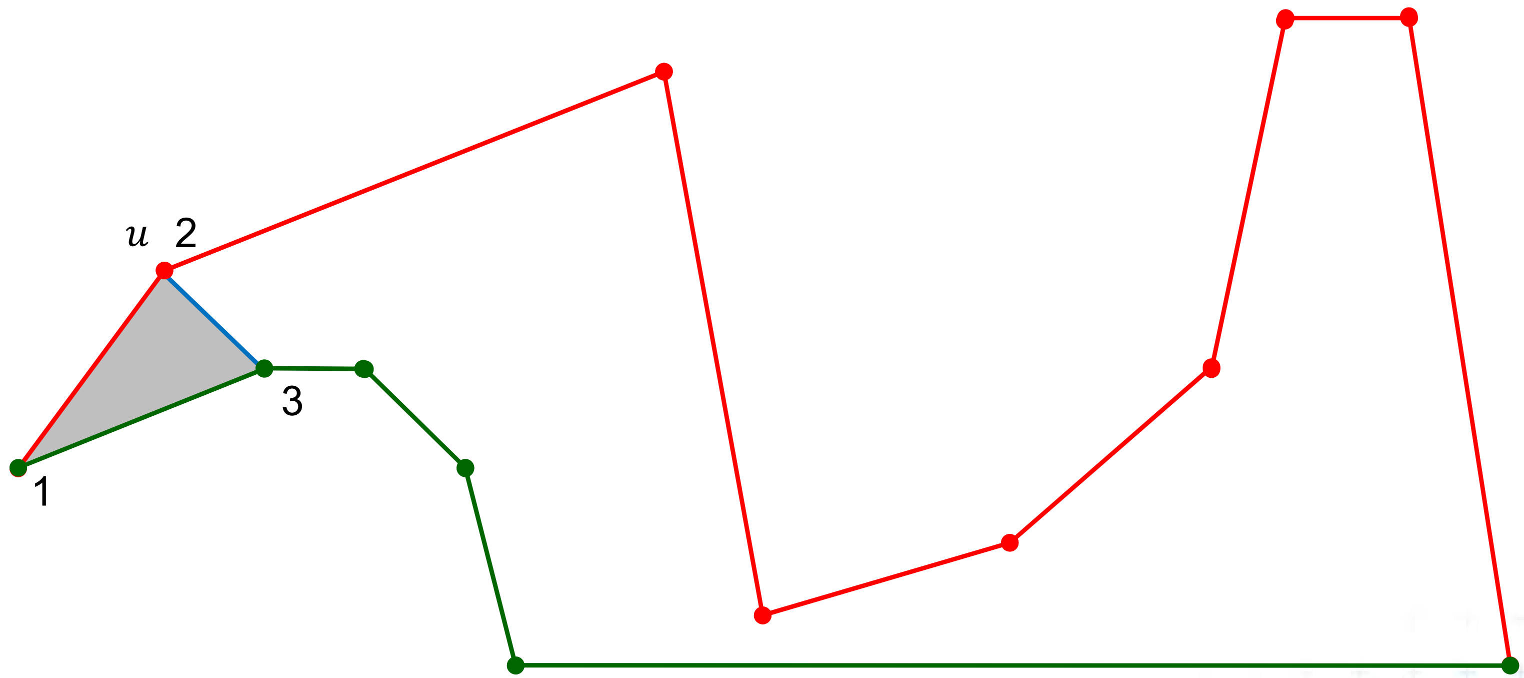
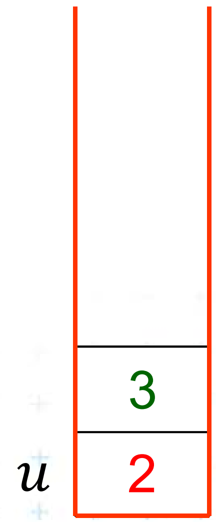
Set trivial reflex chain $v_i v_{i-1}$: New u: pop(), push(v_{i-1}), push(v_i)

[Mount]

Monotone polygon triangulation algorithm

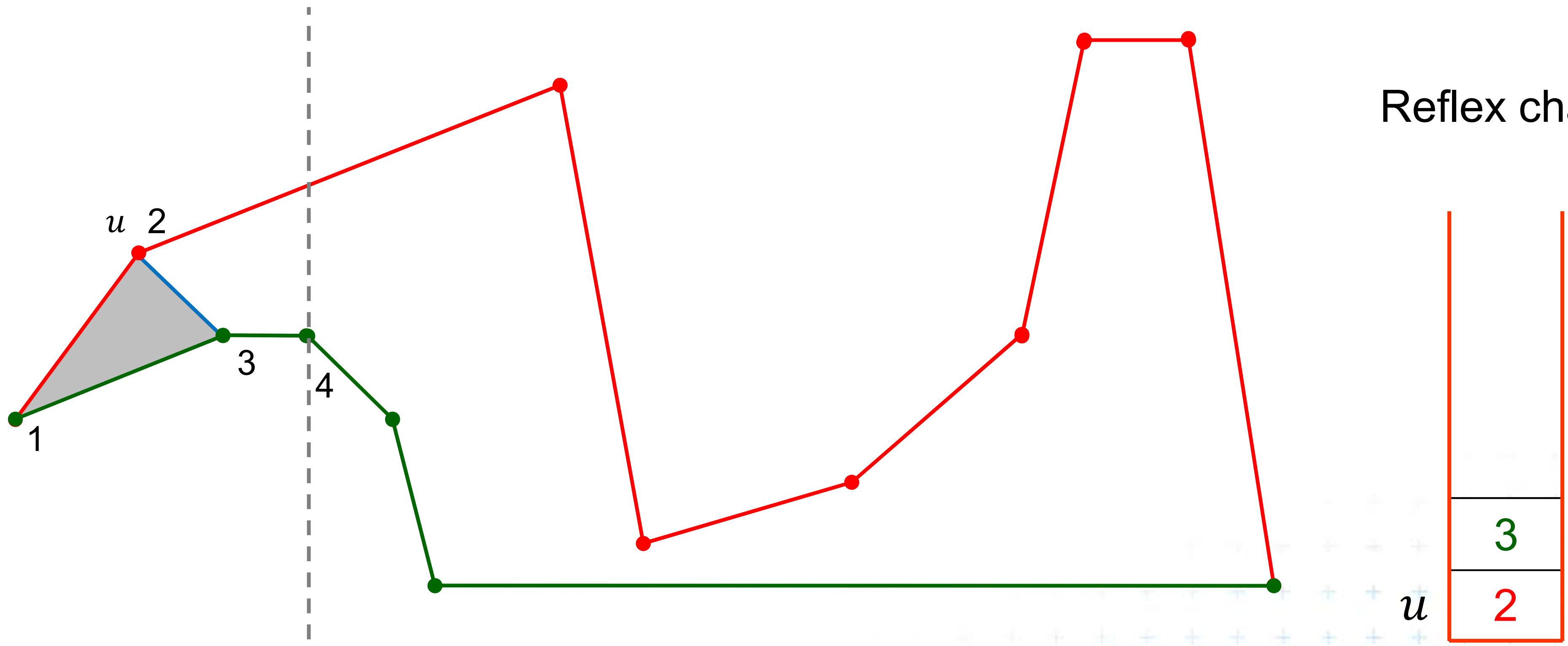


Reflex chain



[Mount]

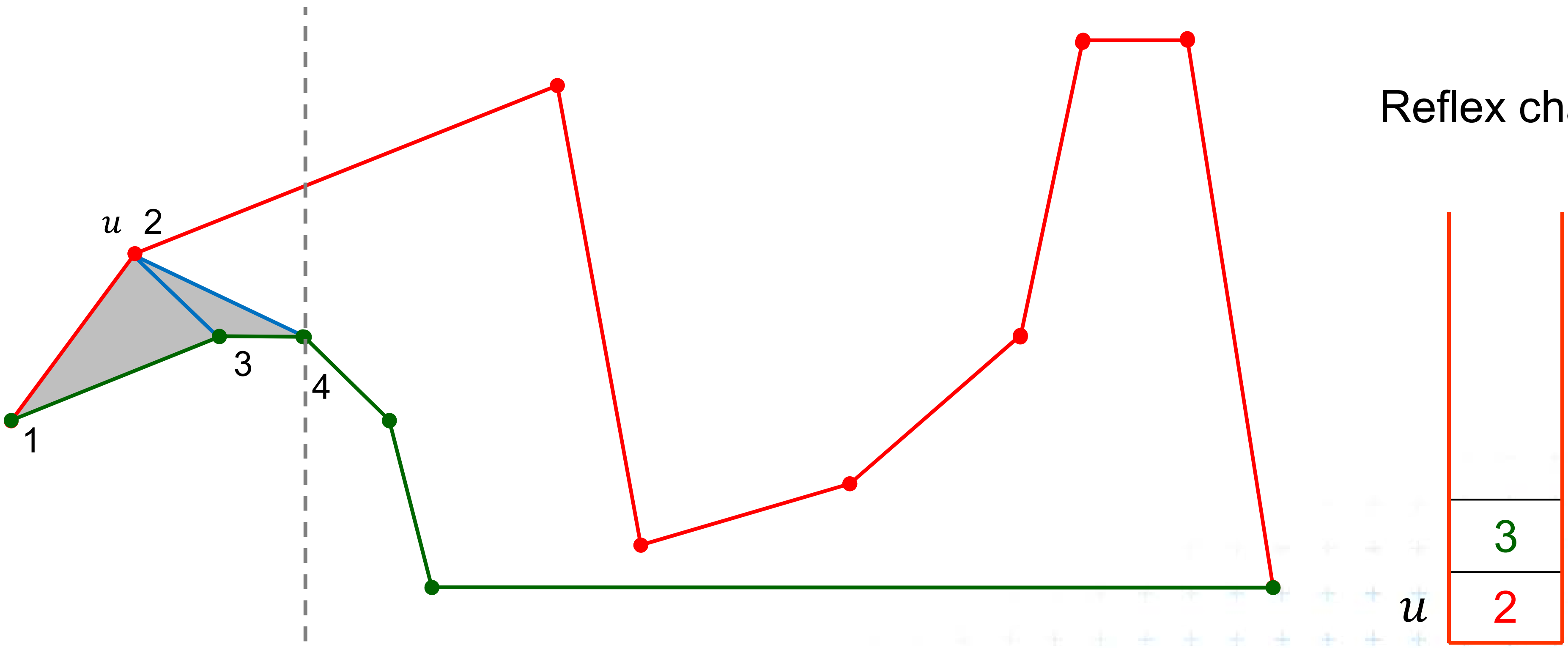
Monotone polygon triangulation algorithm



Case 2a – point v_i on the same chain as non-reflex v_{i-1}

[Mount]

Monotone polygon triangulation algorithm

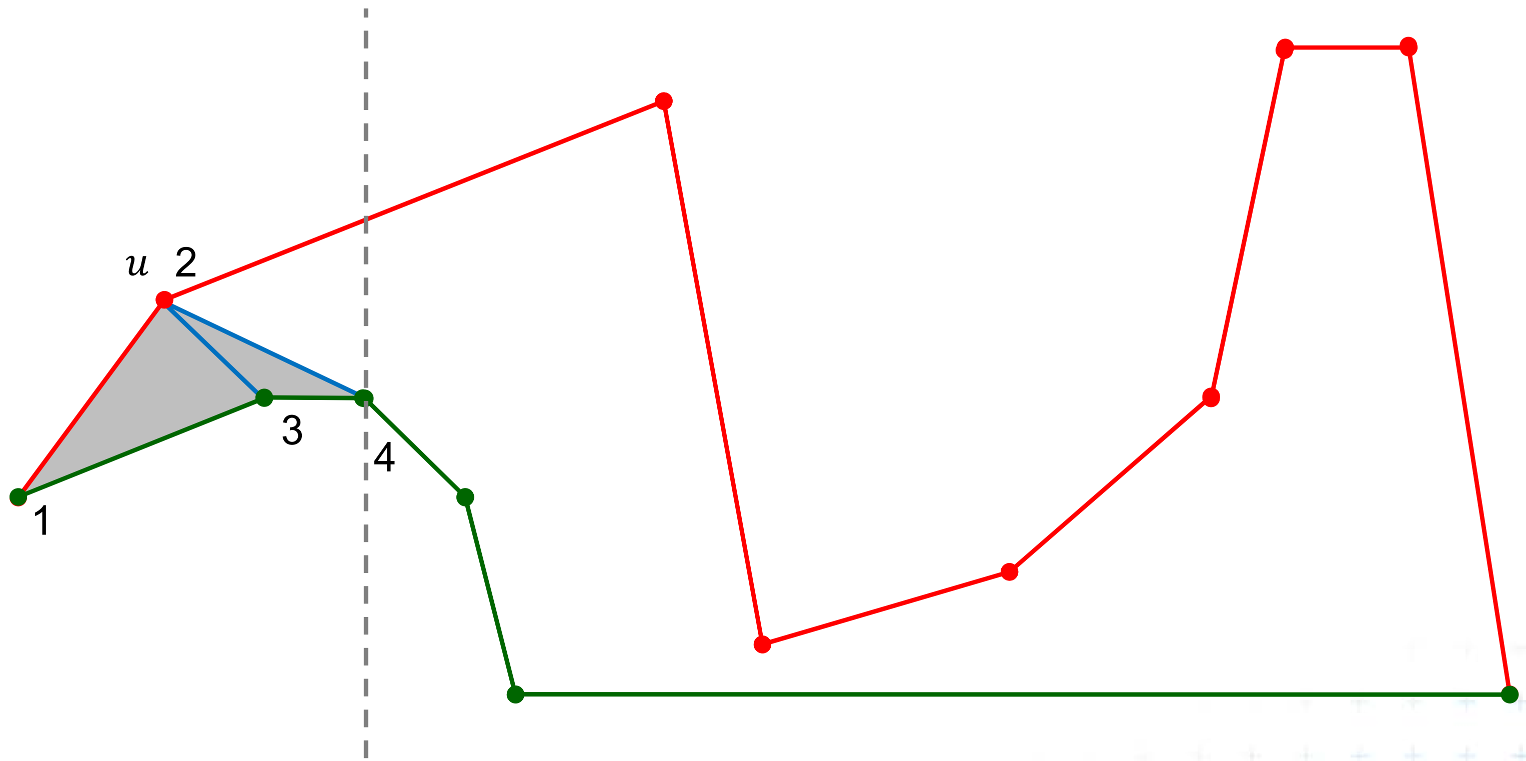


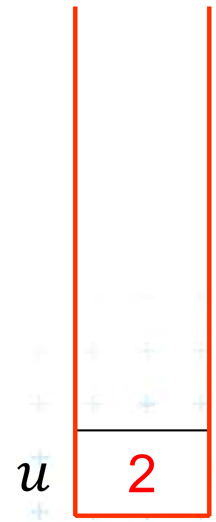
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

[Mount]

Monotone polygon triangulation algorithm



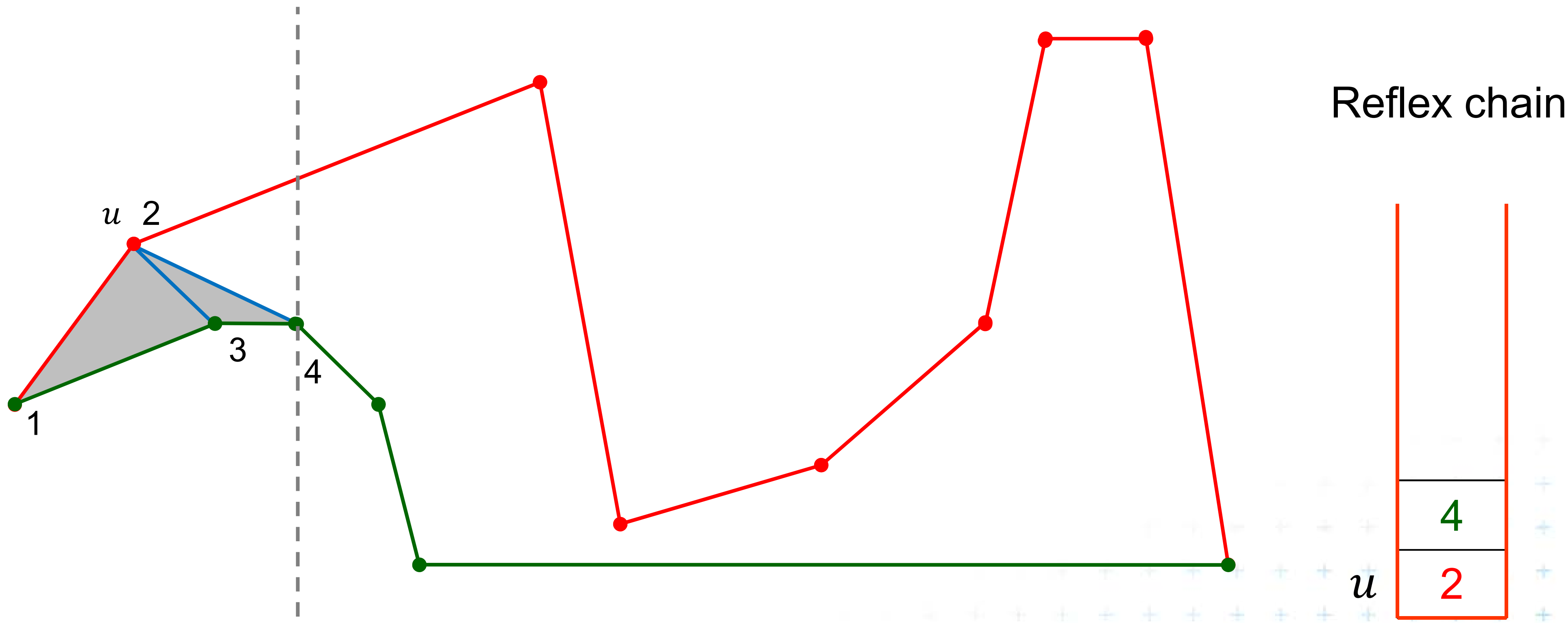


Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

[Mount]

Monotone polygon triangulation algorithm



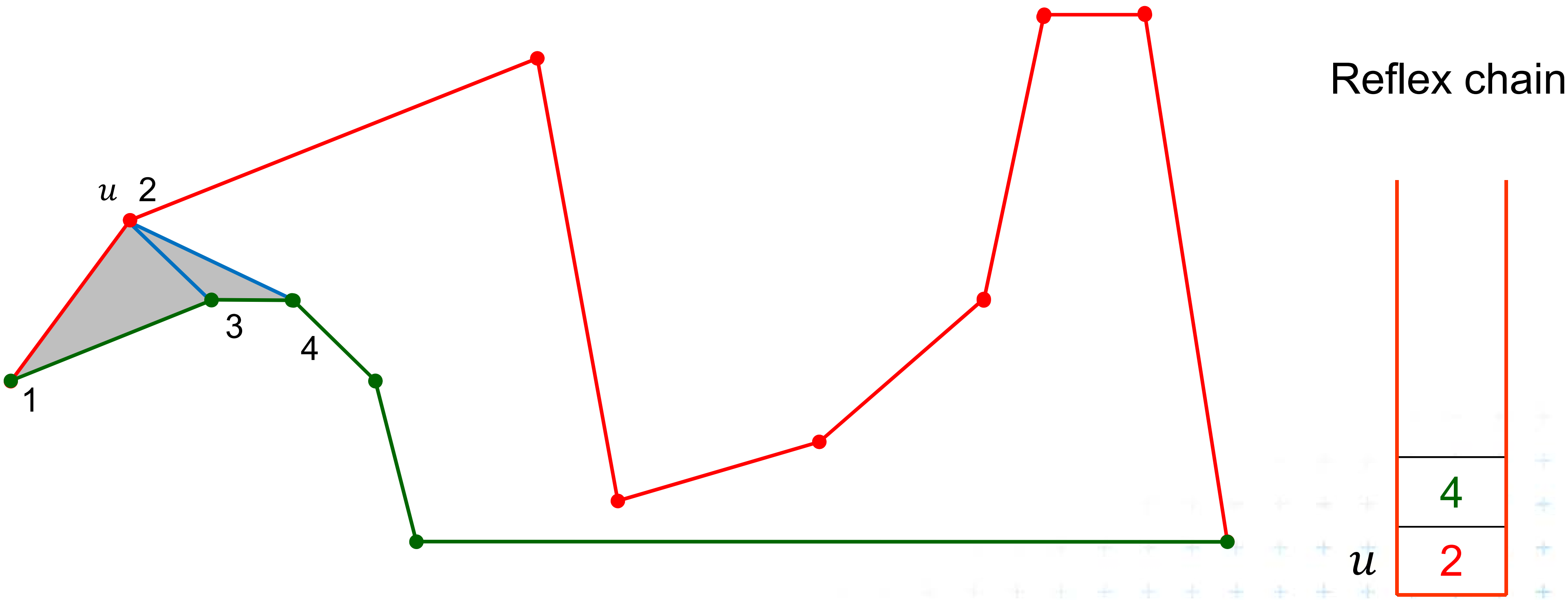
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

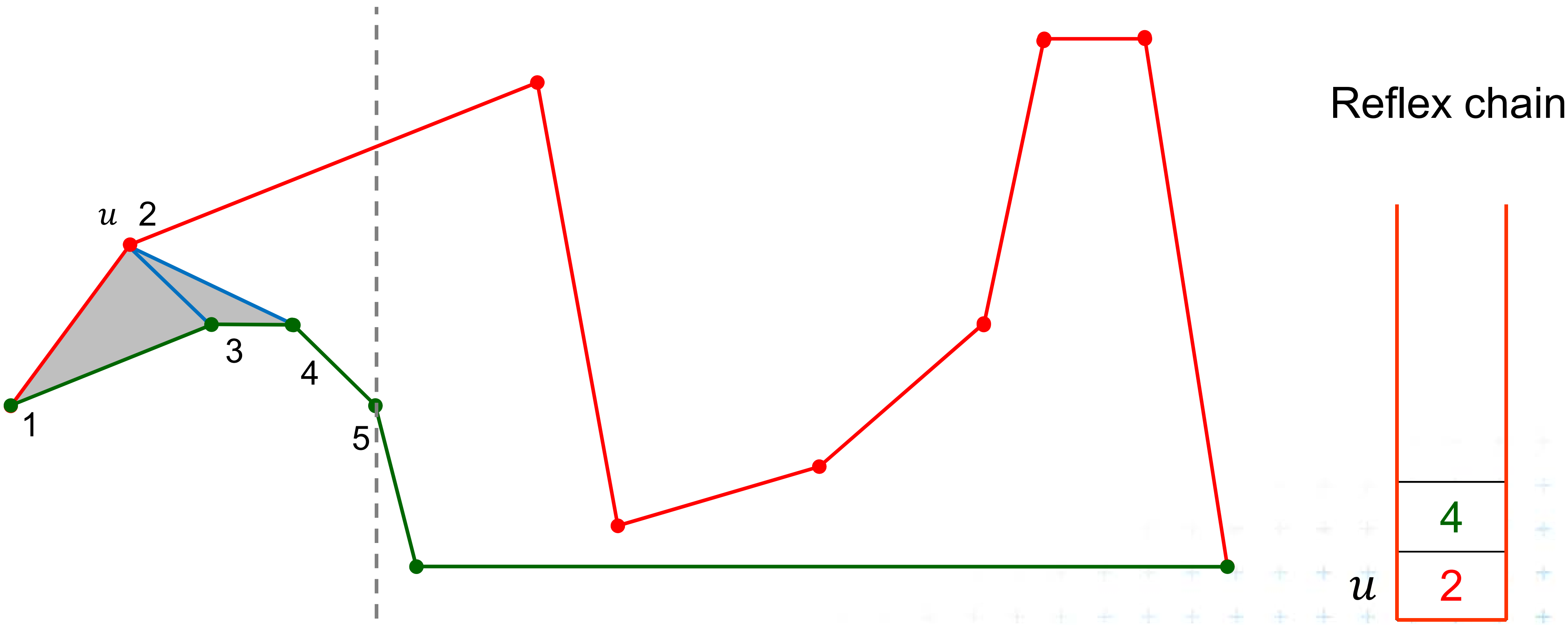
[Mount]

Monotone polygon triangulation algorithm



[Mount]

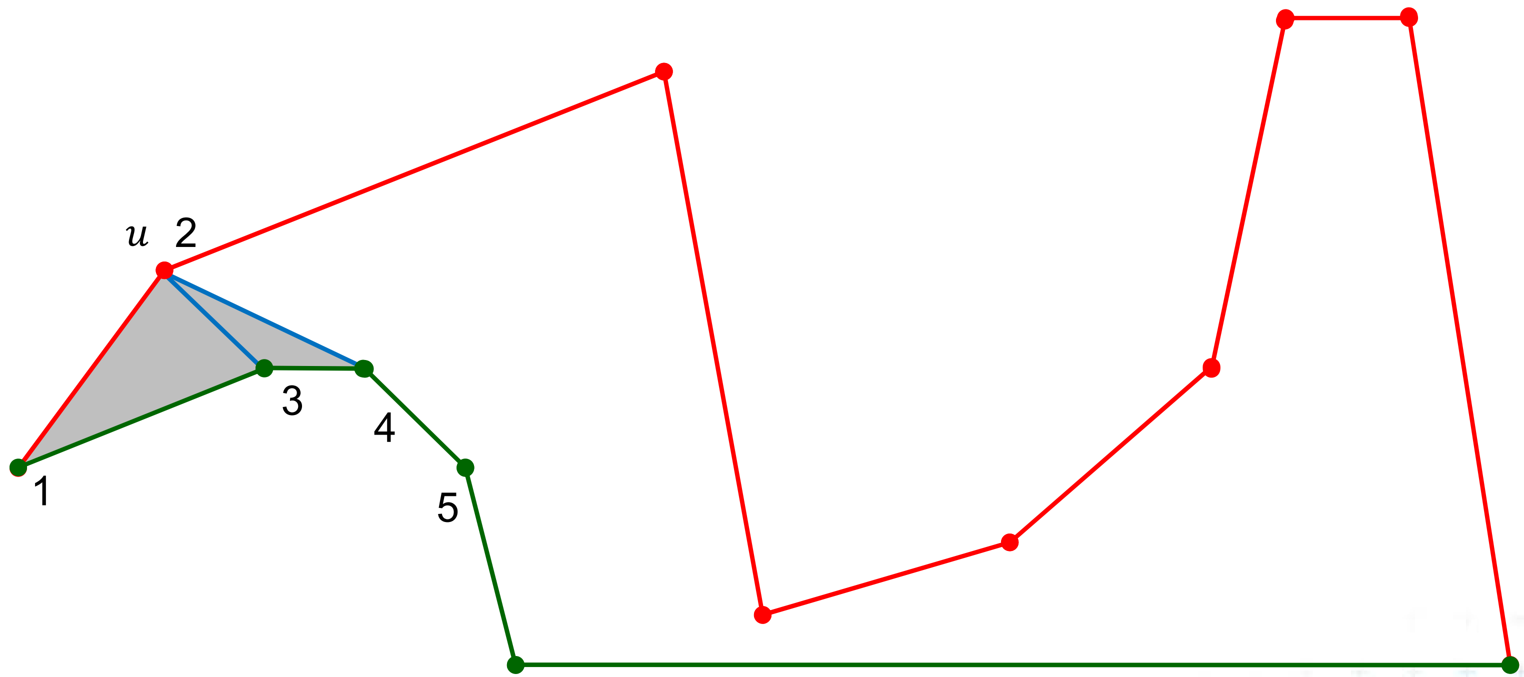
Monotone polygon triangulation algorithm

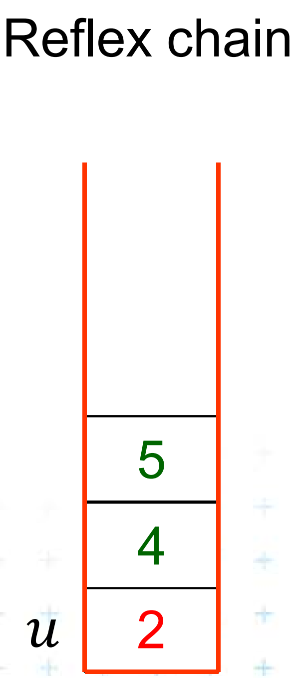


Case 2b – point v_i on the same chain as reflex v_{i-1}

[Mount]

Monotone polygon triangulation algorithm



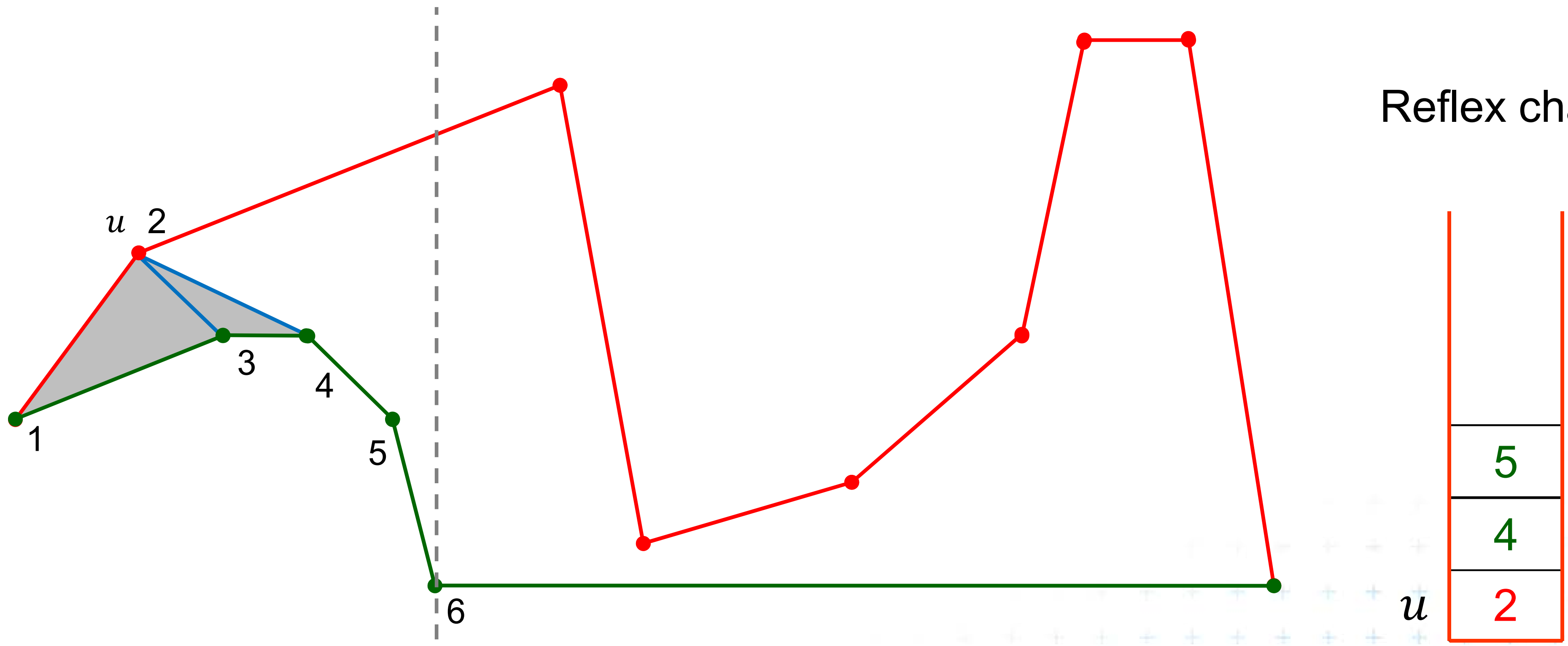


Case 2b – point v_i on the same chain as reflex v_{i-1}

Push point v_i to the reflex chain stack

[Mount]

Monotone polygon triangulation algorithm

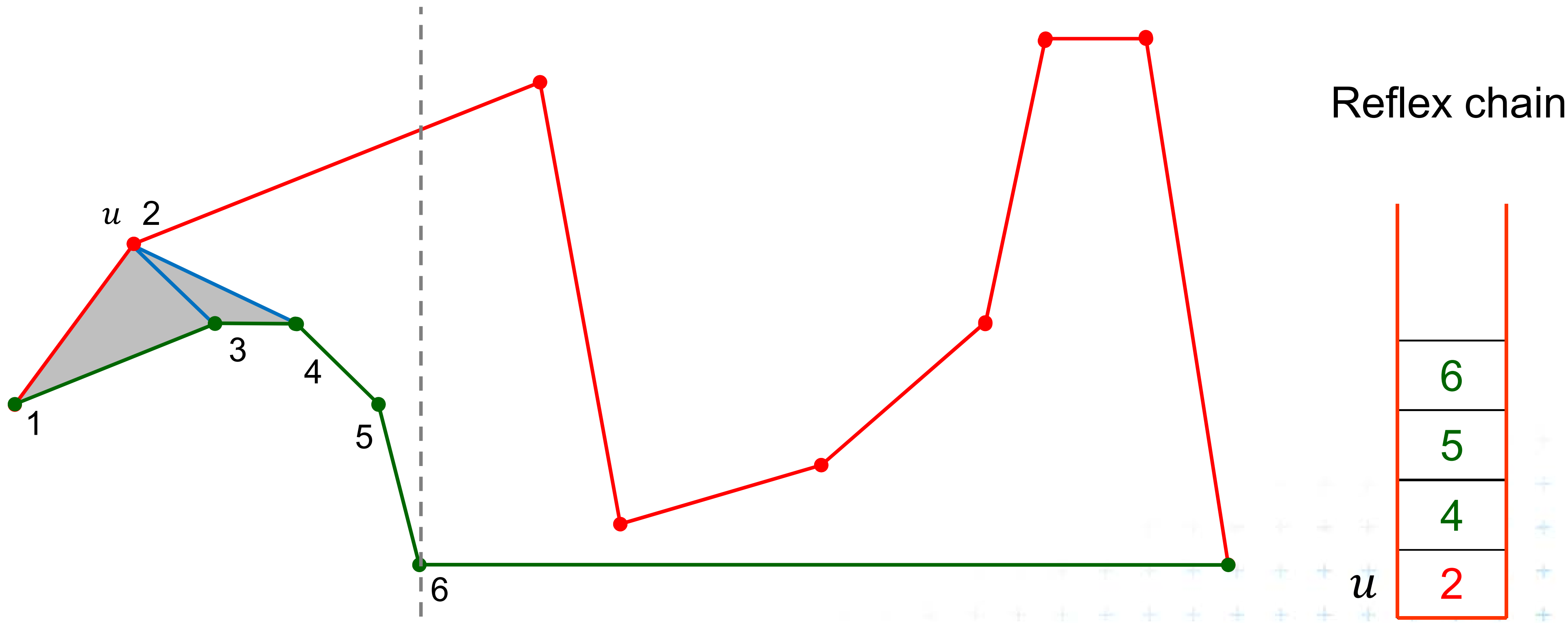


Case 2b – point v_i on the same chain as reflex v_{i-1}

Push point v_i to the reflex chain stack

[Mount]

Monotone polygon triangulation algorithm

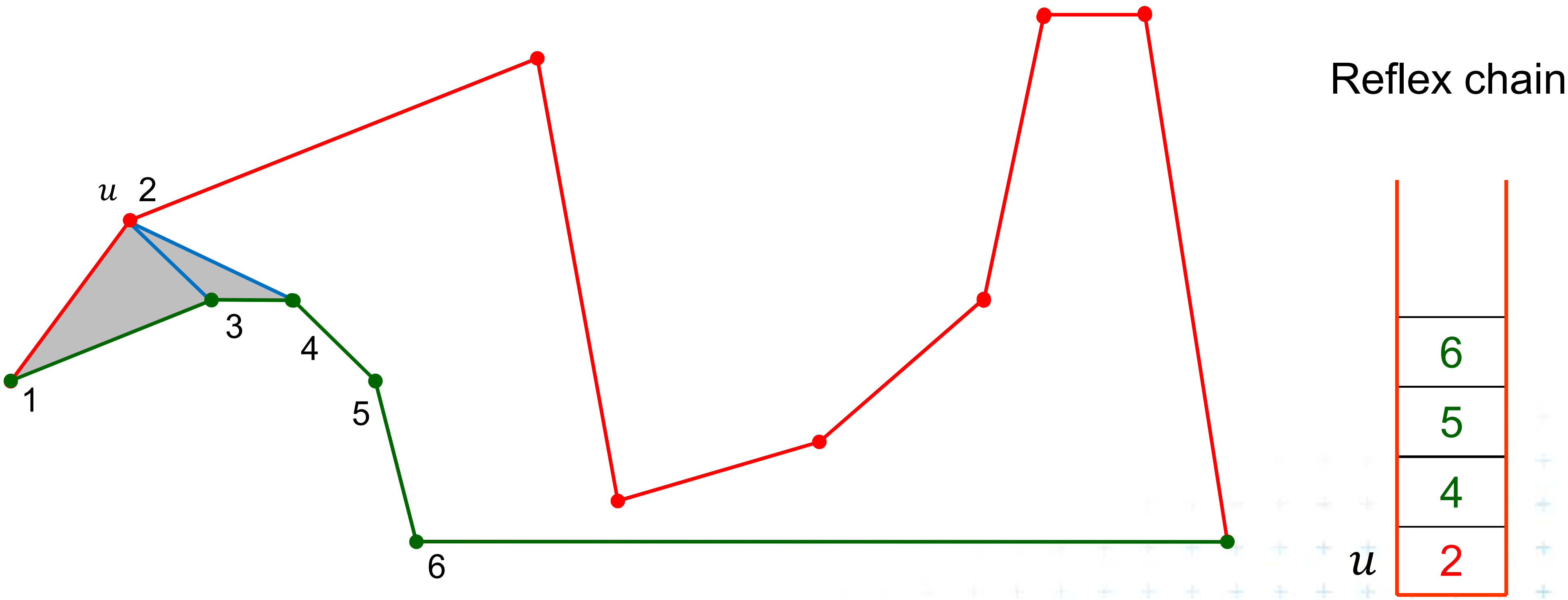


Case 2b – point v_i on the same chain as reflex v_{i-1}

Push point v_i to the reflex chain stack

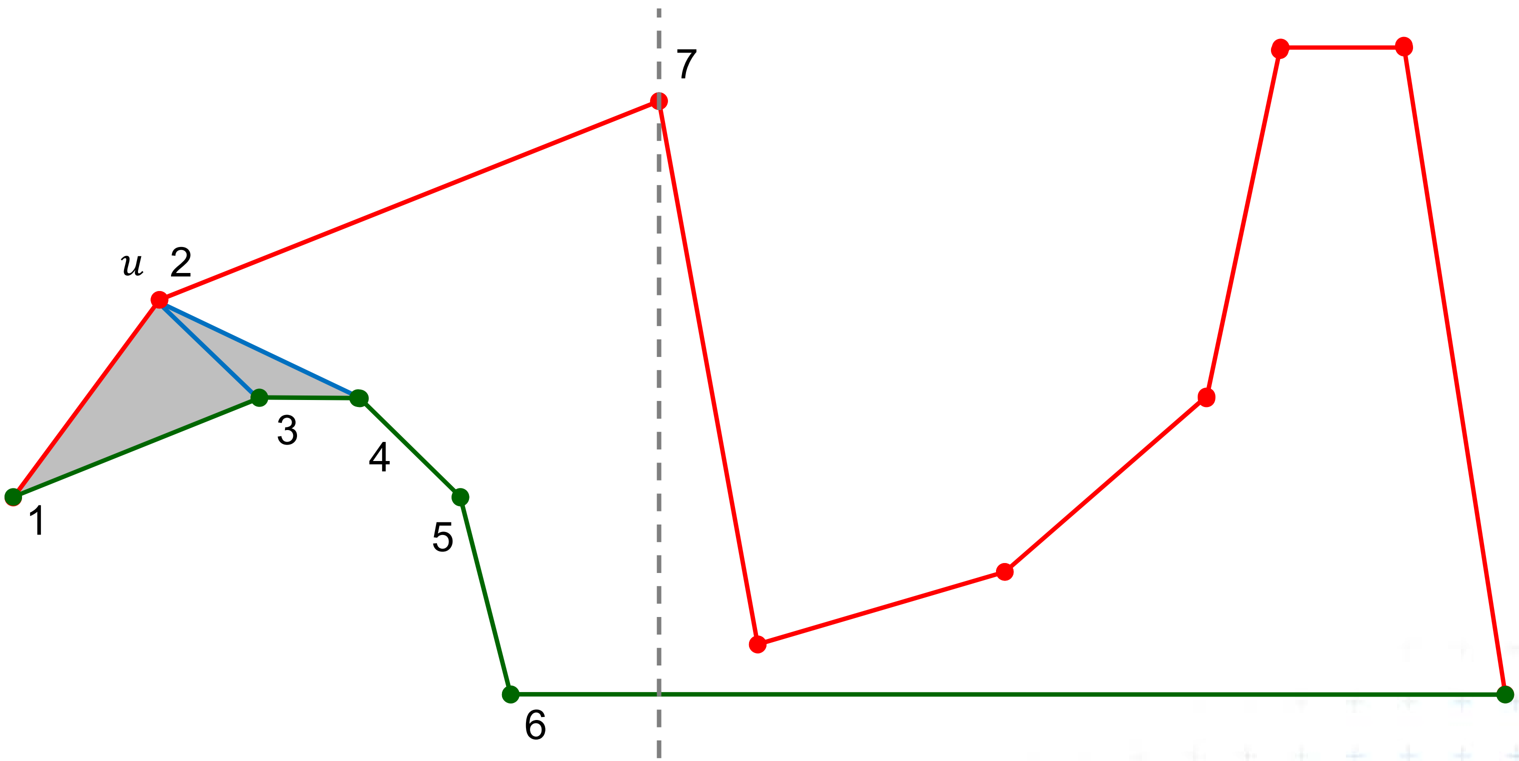
[Mount]

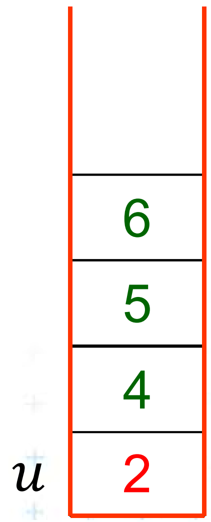
Monotone polygon triangulation algorithm



[Mount]

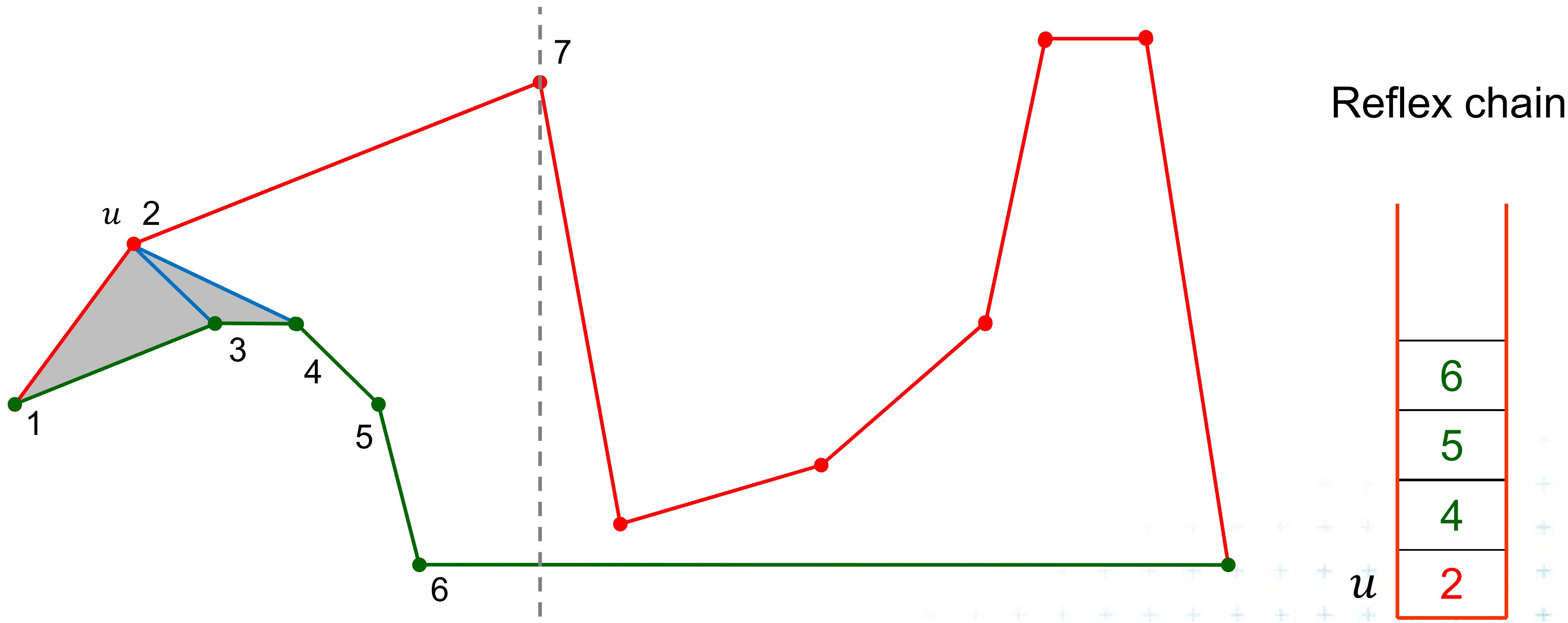
Monotone polygon triangulation algorithm





[Mount]

Monotone polygon triangulation algorithm

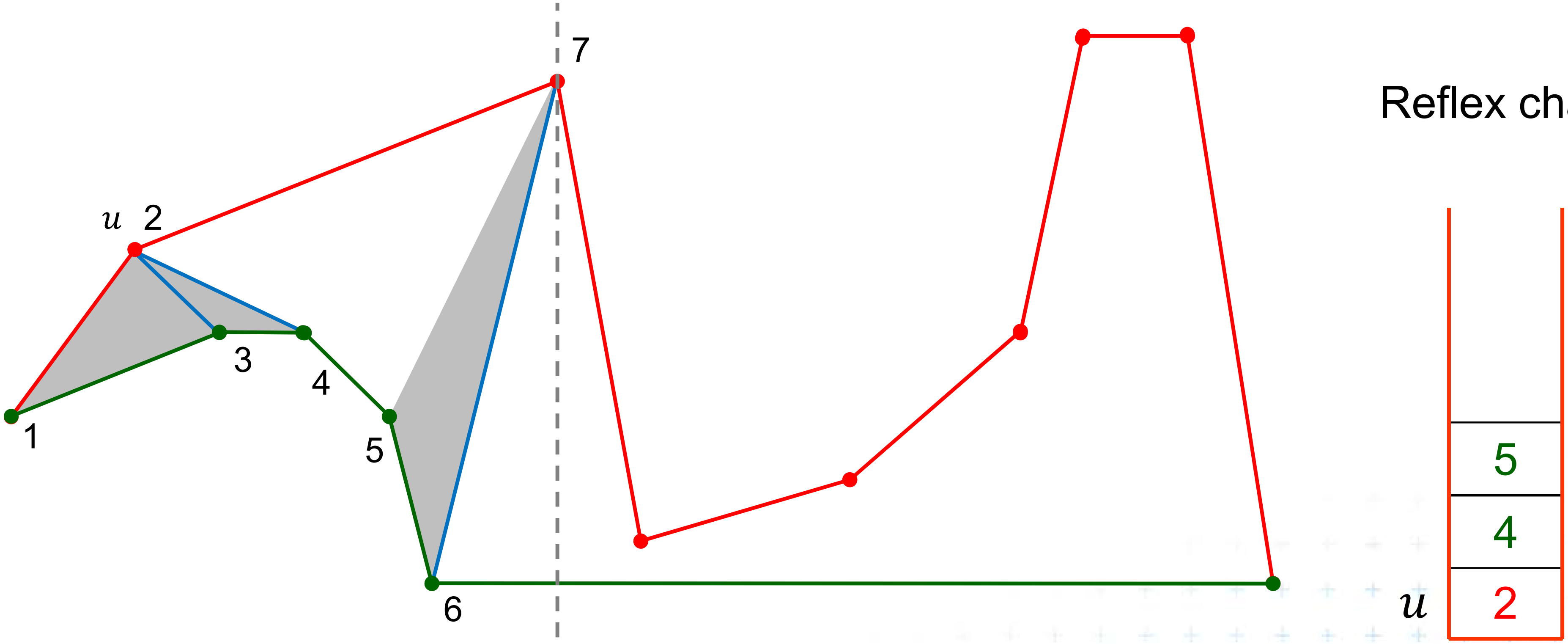


Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

[Mount]

Monotone polygon triangulation algorithm

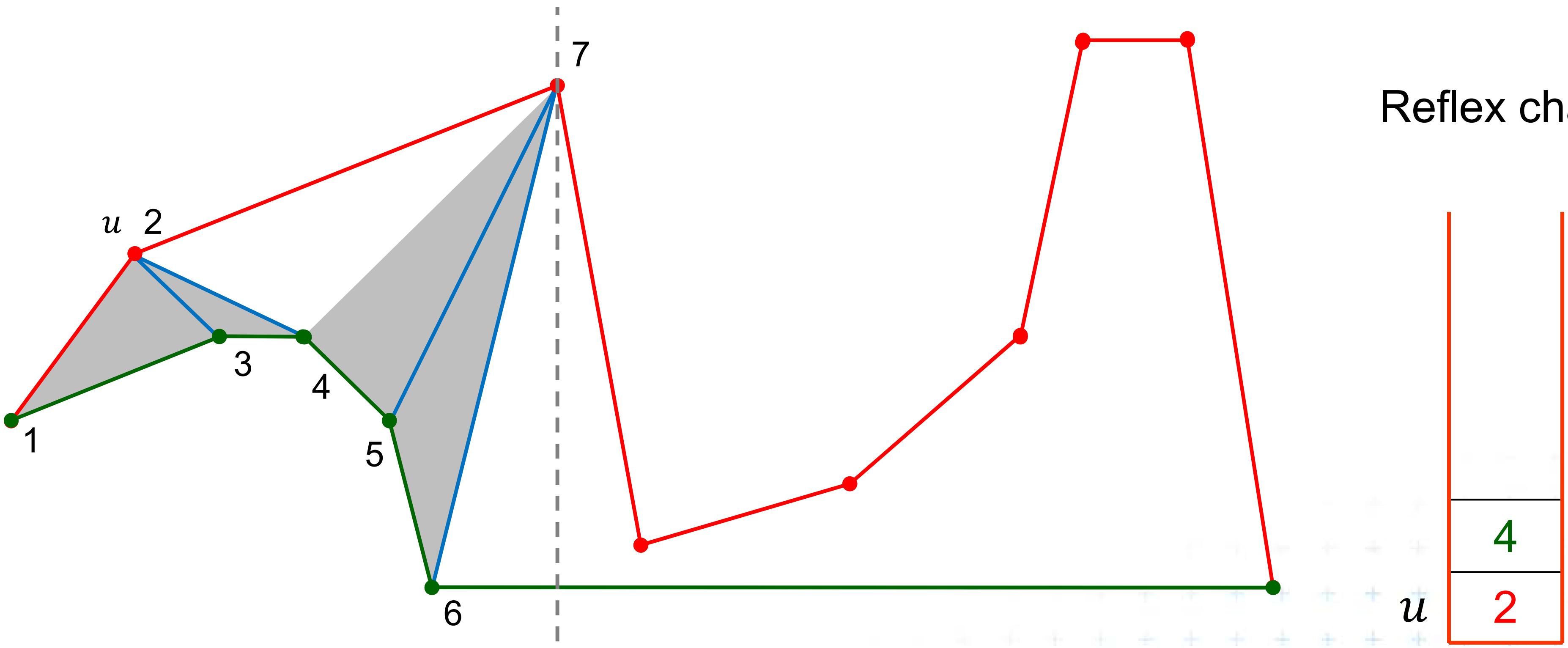


Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

[Mount]

Monotone polygon triangulation algorithm

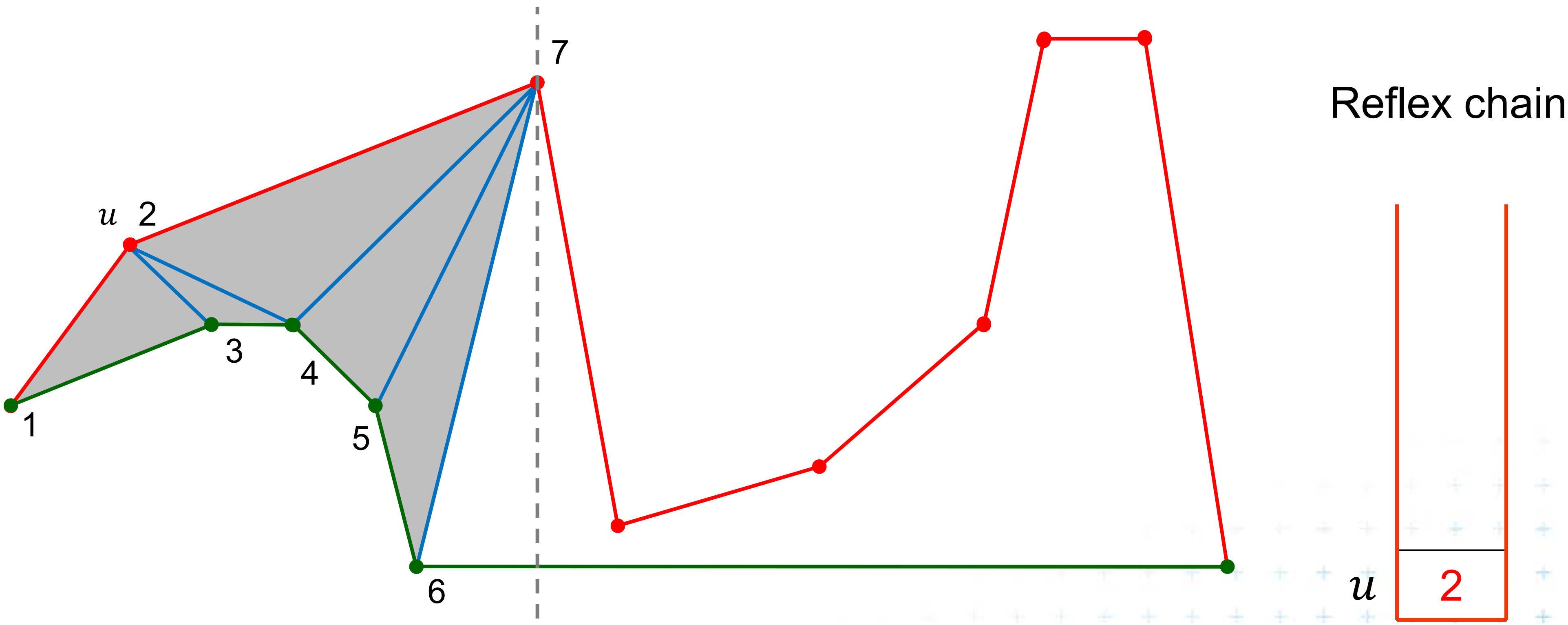


Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

[Mount]

Monotone polygon triangulation algorithm

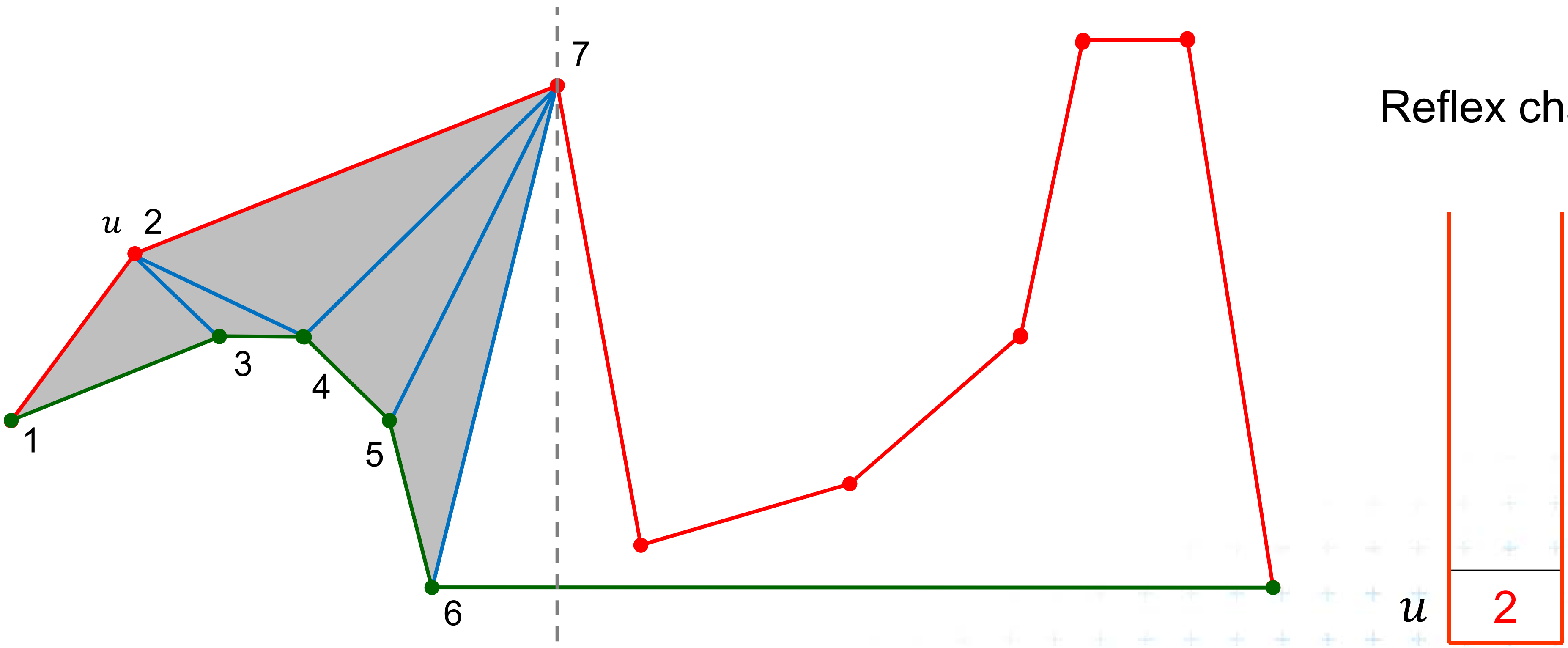


Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

[Mount]

Monotone polygon triangulation algorithm



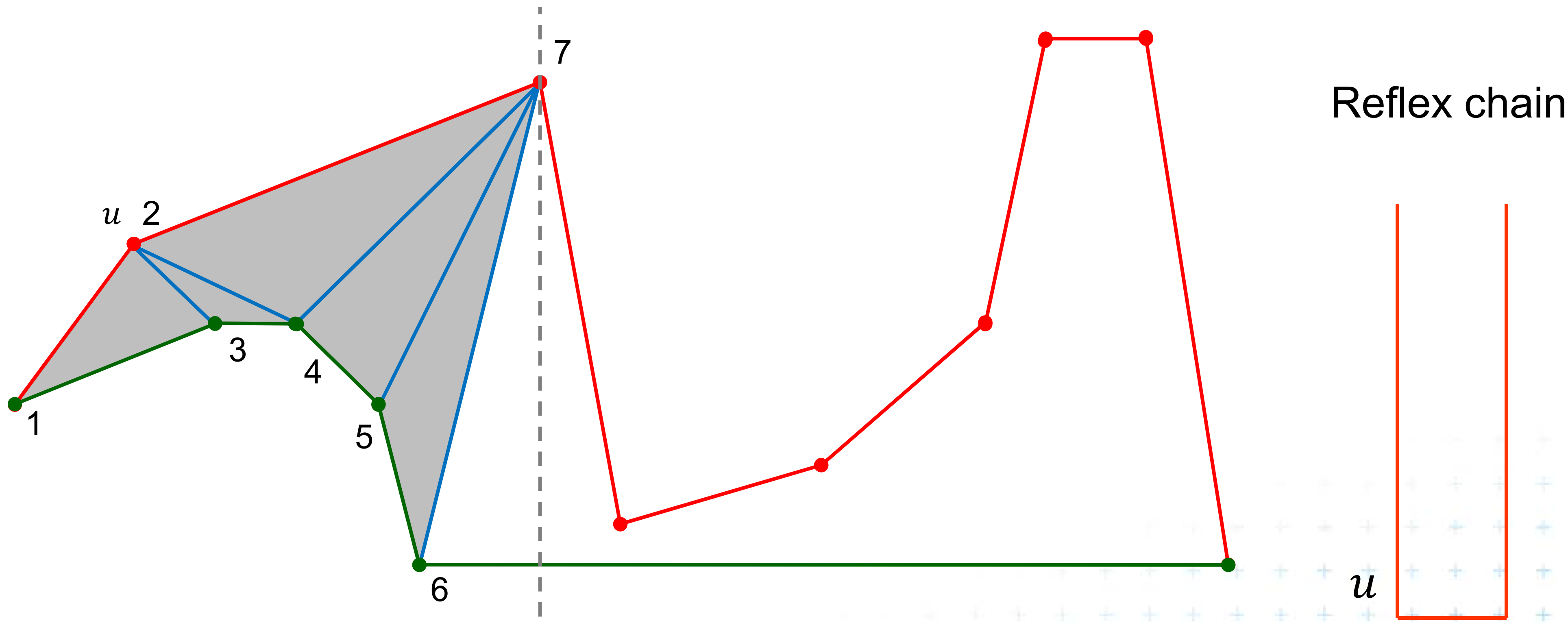
Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

Set trivial reflex chain $v_i v_{i-1}$: New u: pop(), push(v_{i-1}), push(v_i)

[Mount]

Monotone polygon triangulation algorithm



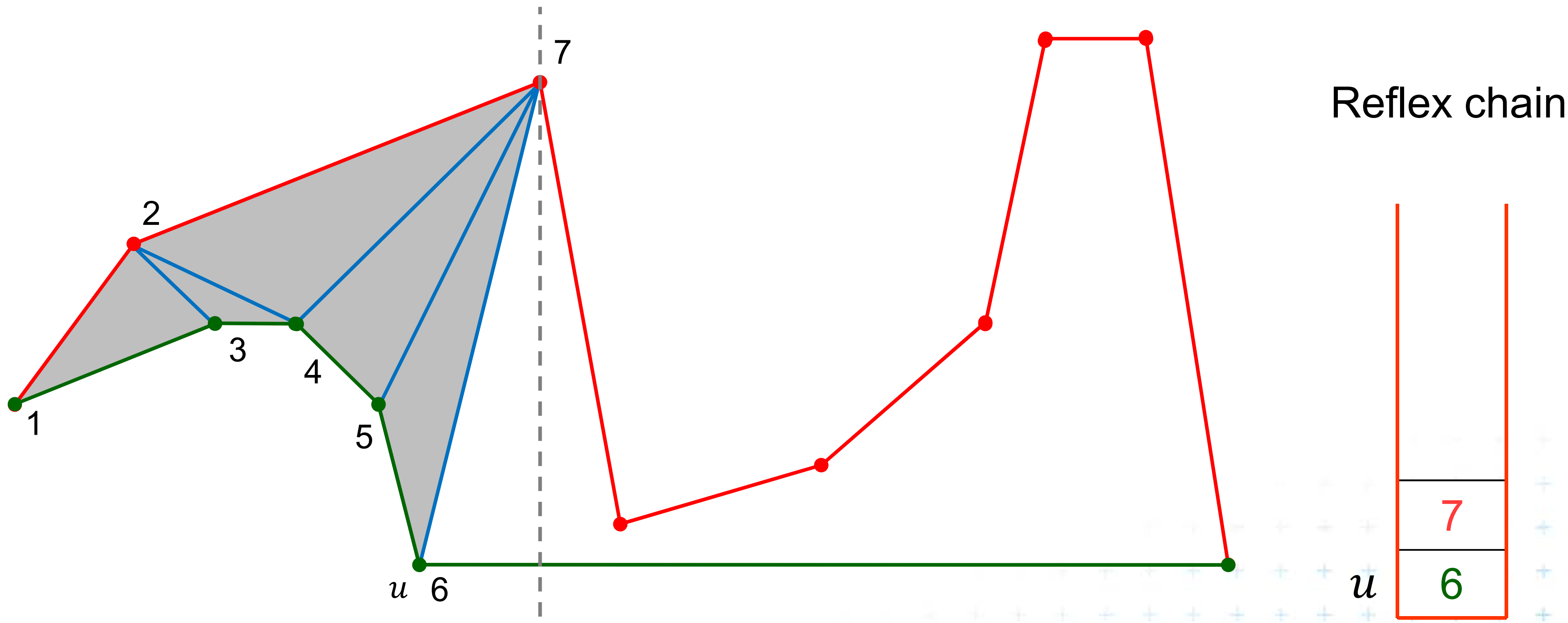
Case 1 – point v_i on opposite chain from v_{i-1}

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

Set trivial reflex chain $v_i v_{i-1}$: New u: pop(), push(v_{i-1}), push(v_i)

[Mount]

Monotone polygon triangulation algorithm



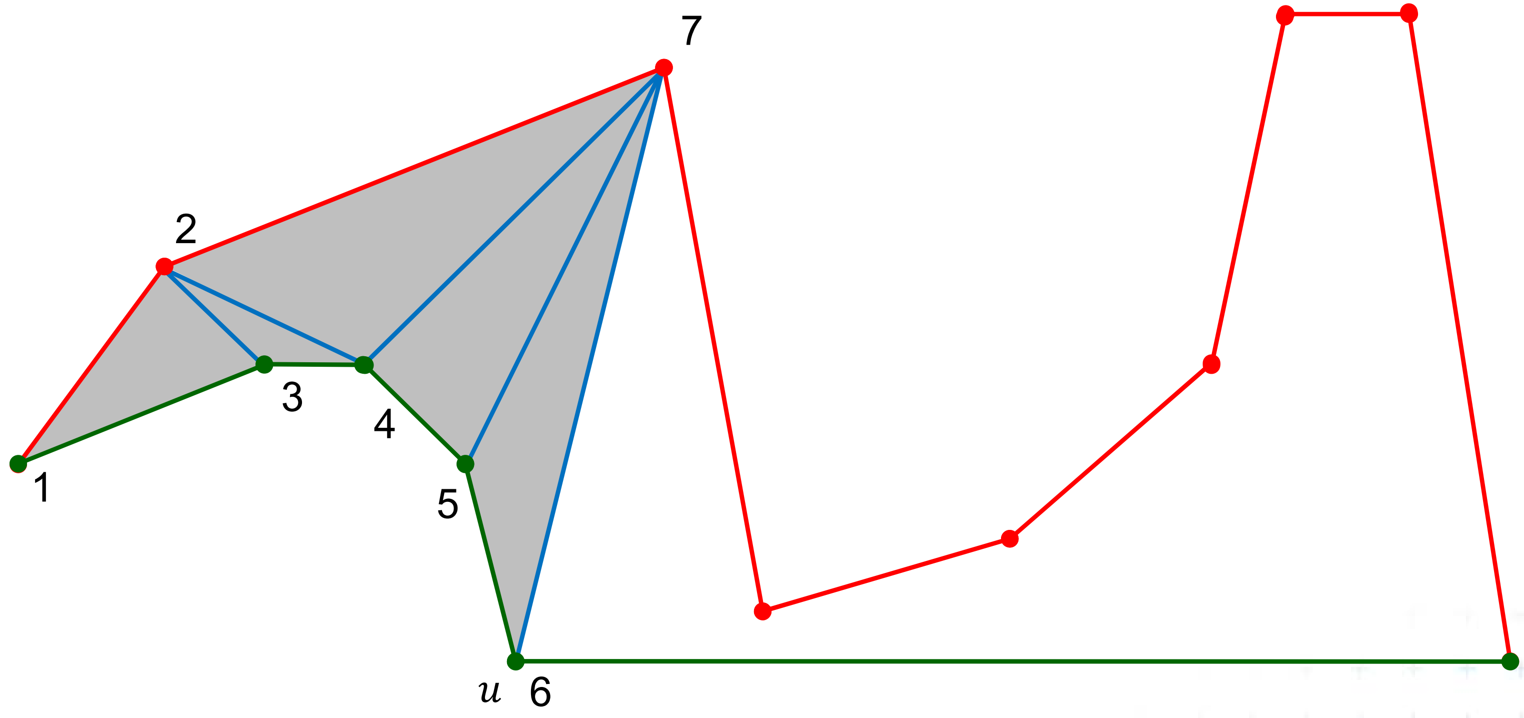
Case 1 – point v_i on opposite chain from v_{i-1}

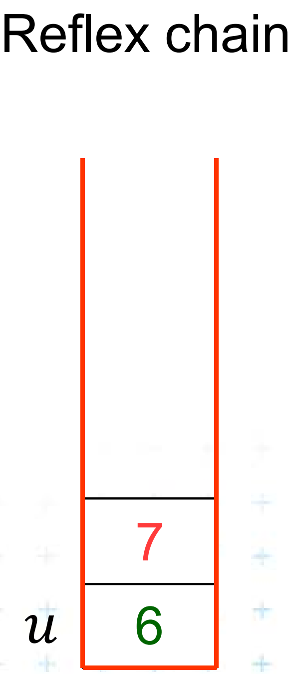
Add diagonal(s) from v_i to all points on reflex chain stack – pop()

Set trivial reflex chain $v_i v_{i-1}$: New u: pop(), push(v_{i-1}), push(v_i)

[Mount]

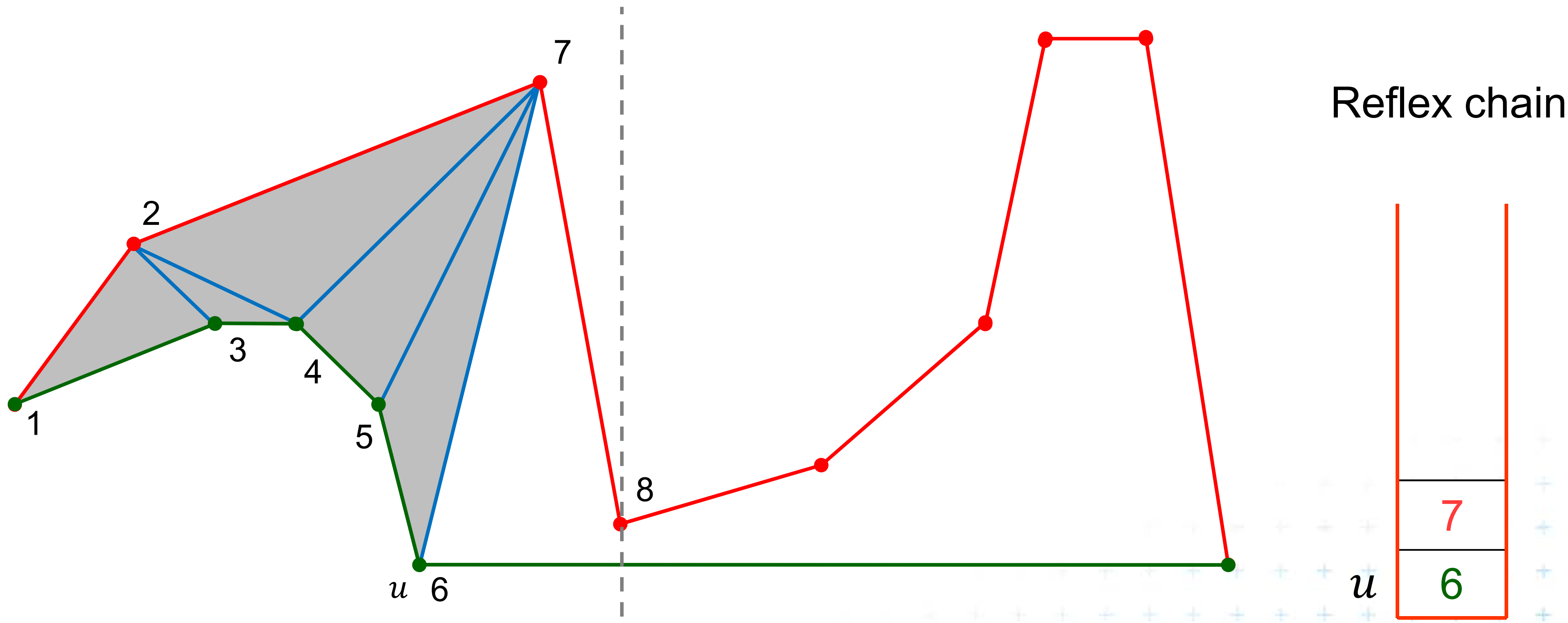
Monotone polygon triangulation algorithm





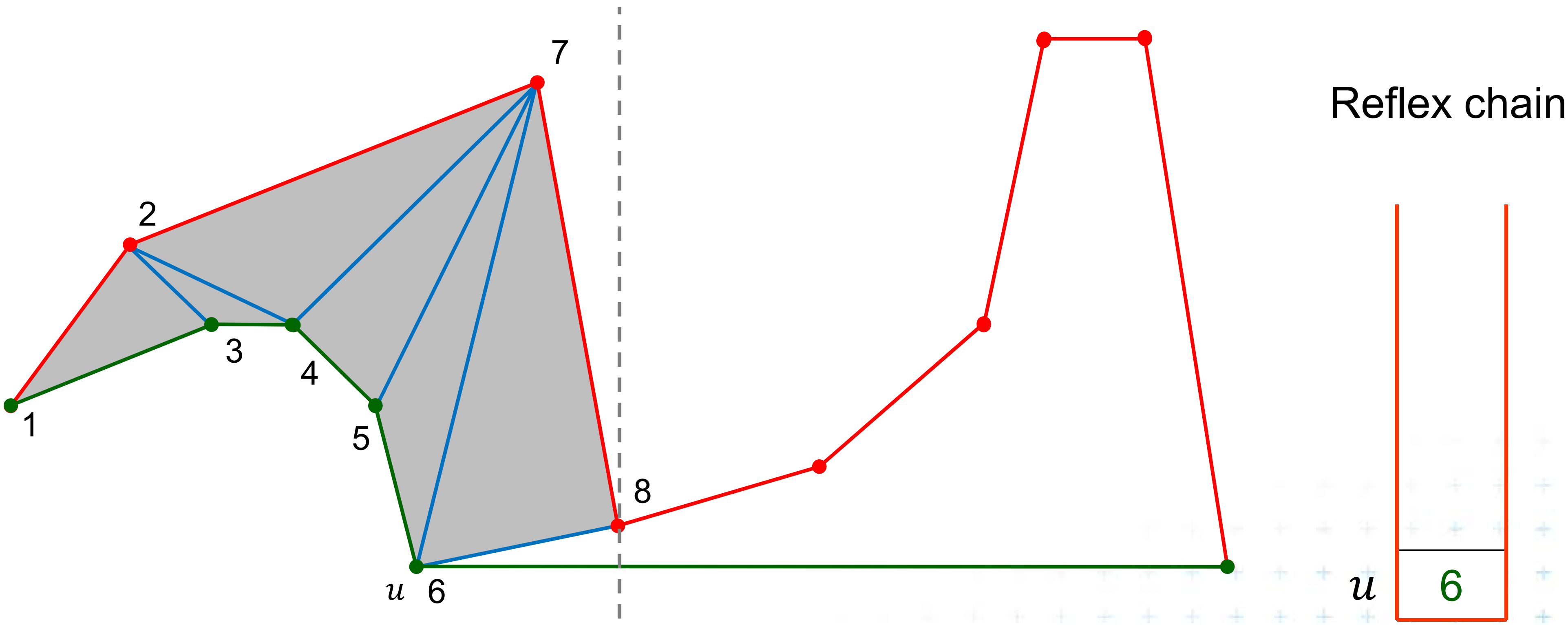
[Mount]

Monotone polygon triangulation algorithm



[Mount]

Monotone polygon triangulation algorithm

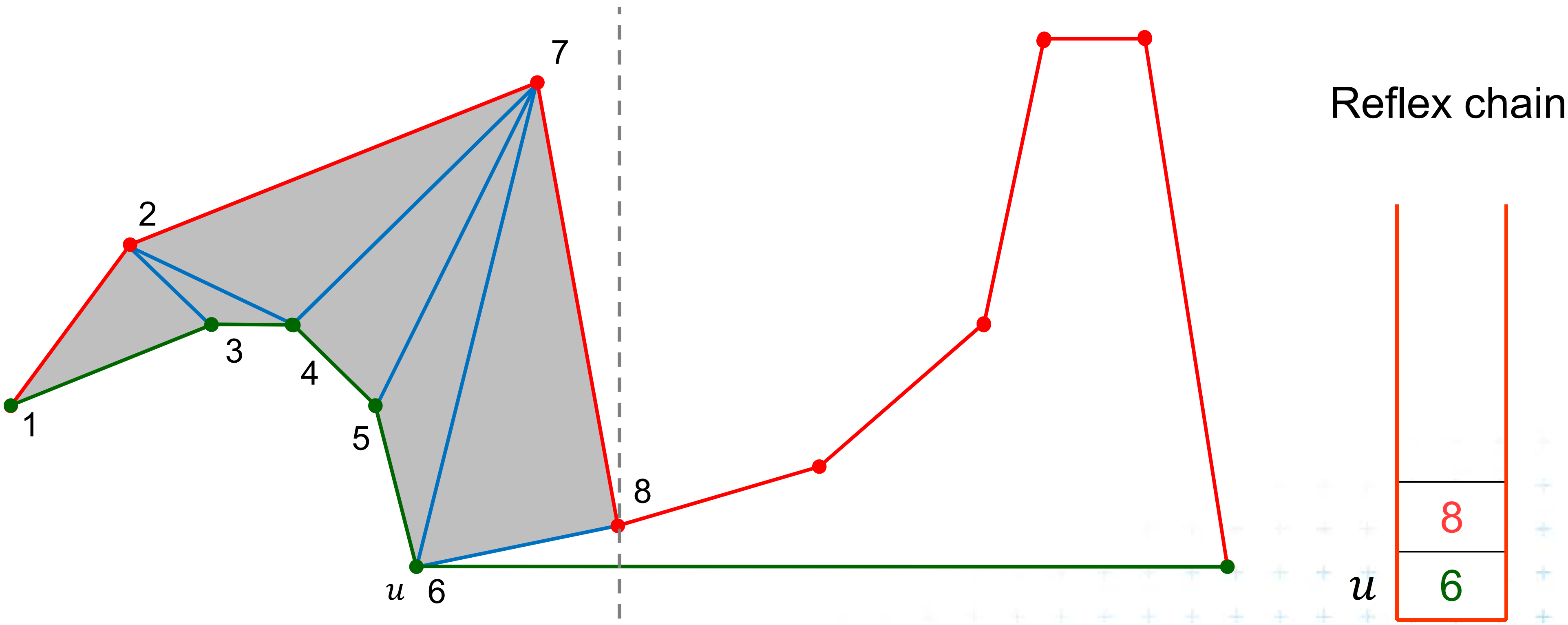


Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

[Mount]

Monotone polygon triangulation algorithm



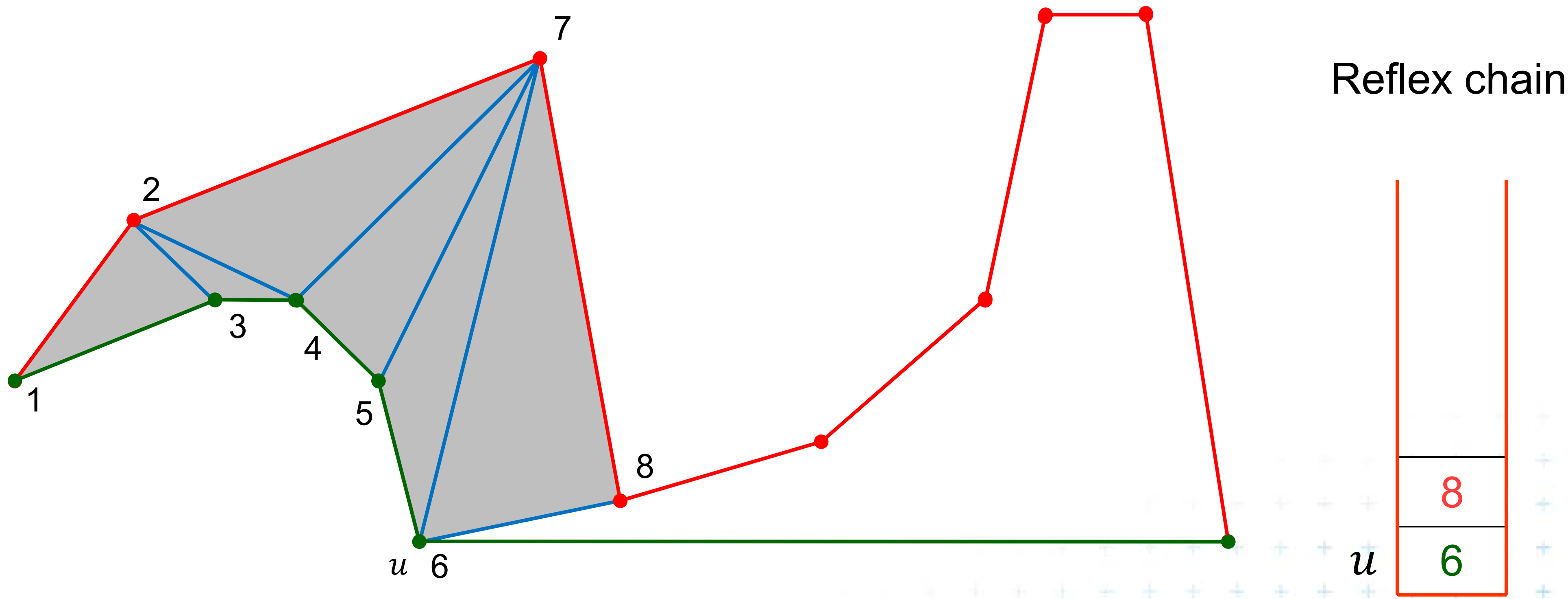
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

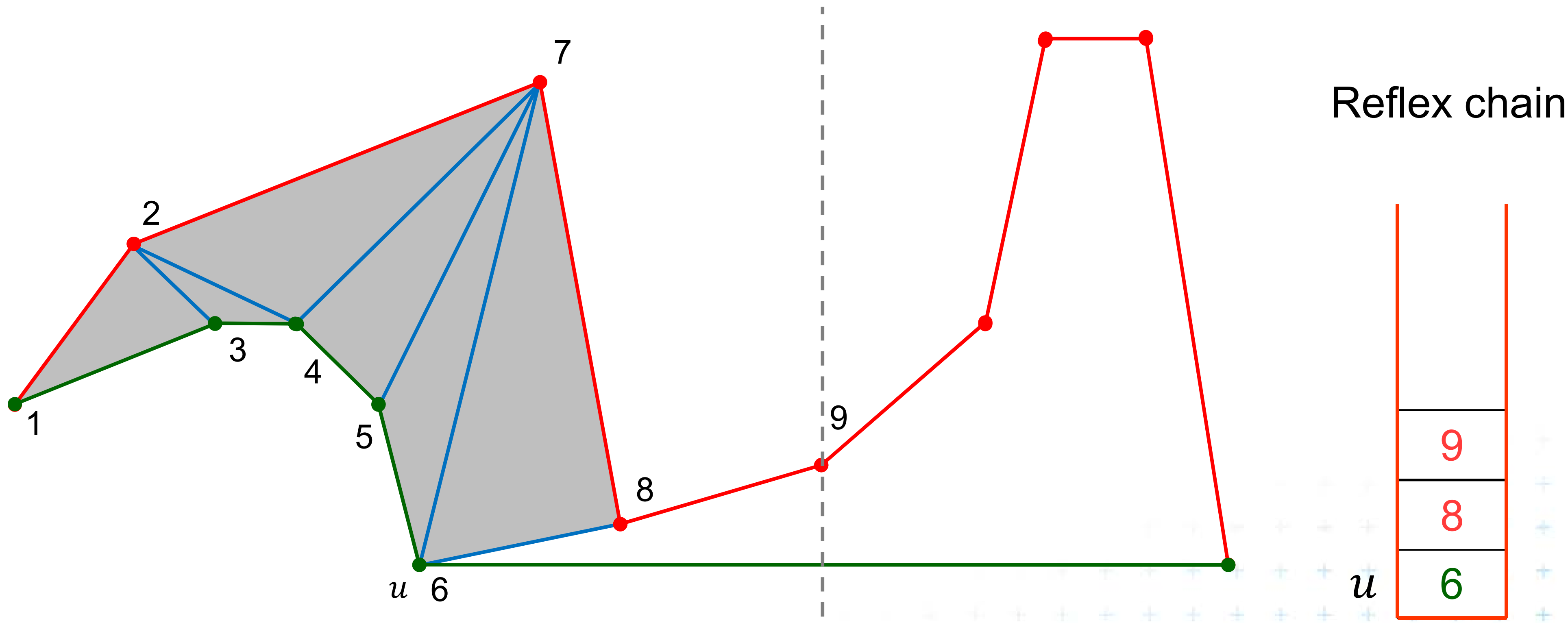
[Mount]

Monotone polygon triangulation algorithm



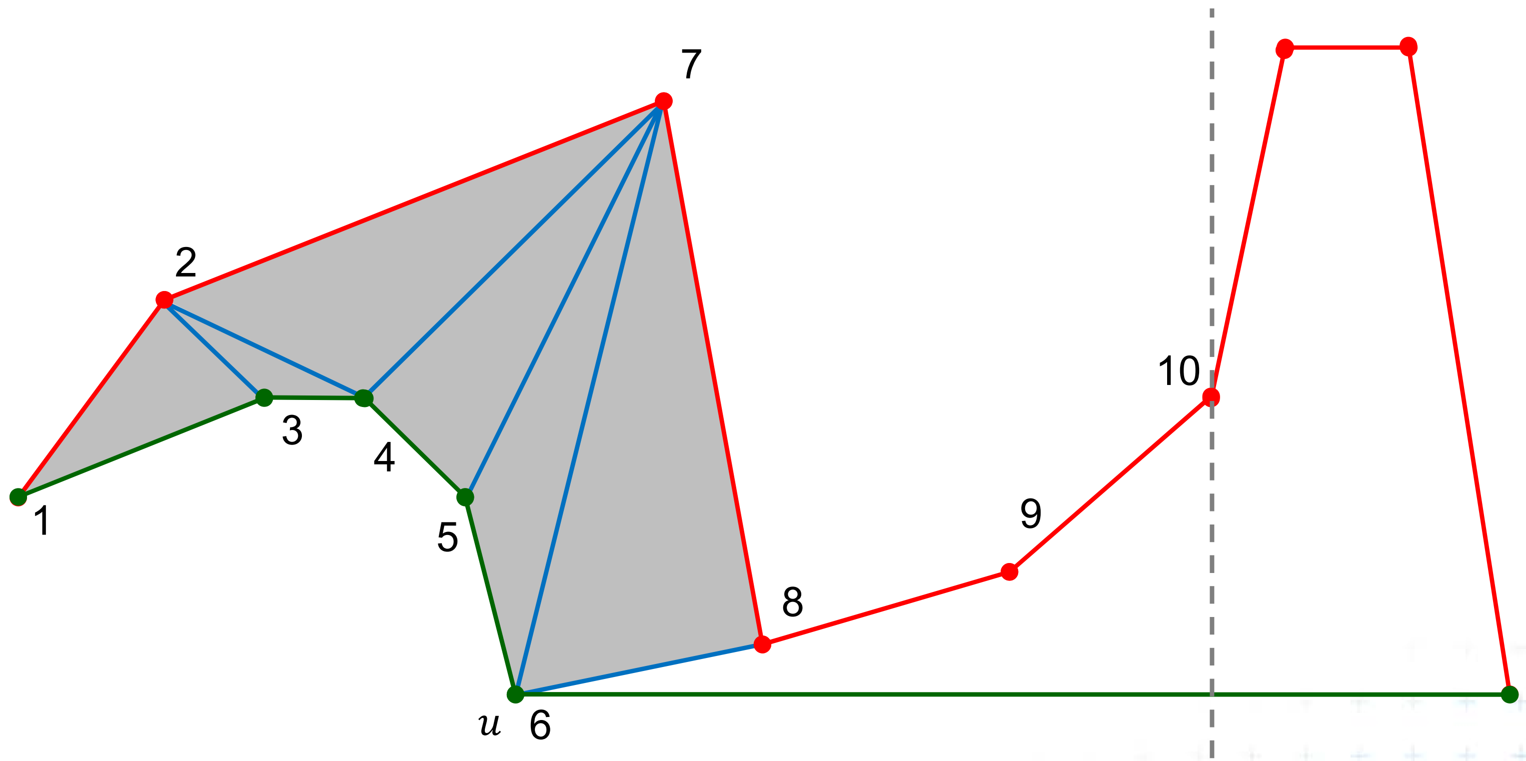
[Mount]

Monotone polygon triangulation algorithm

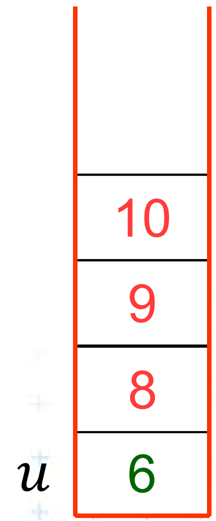


[Mount]

Monotone polygon triangulation algorithm

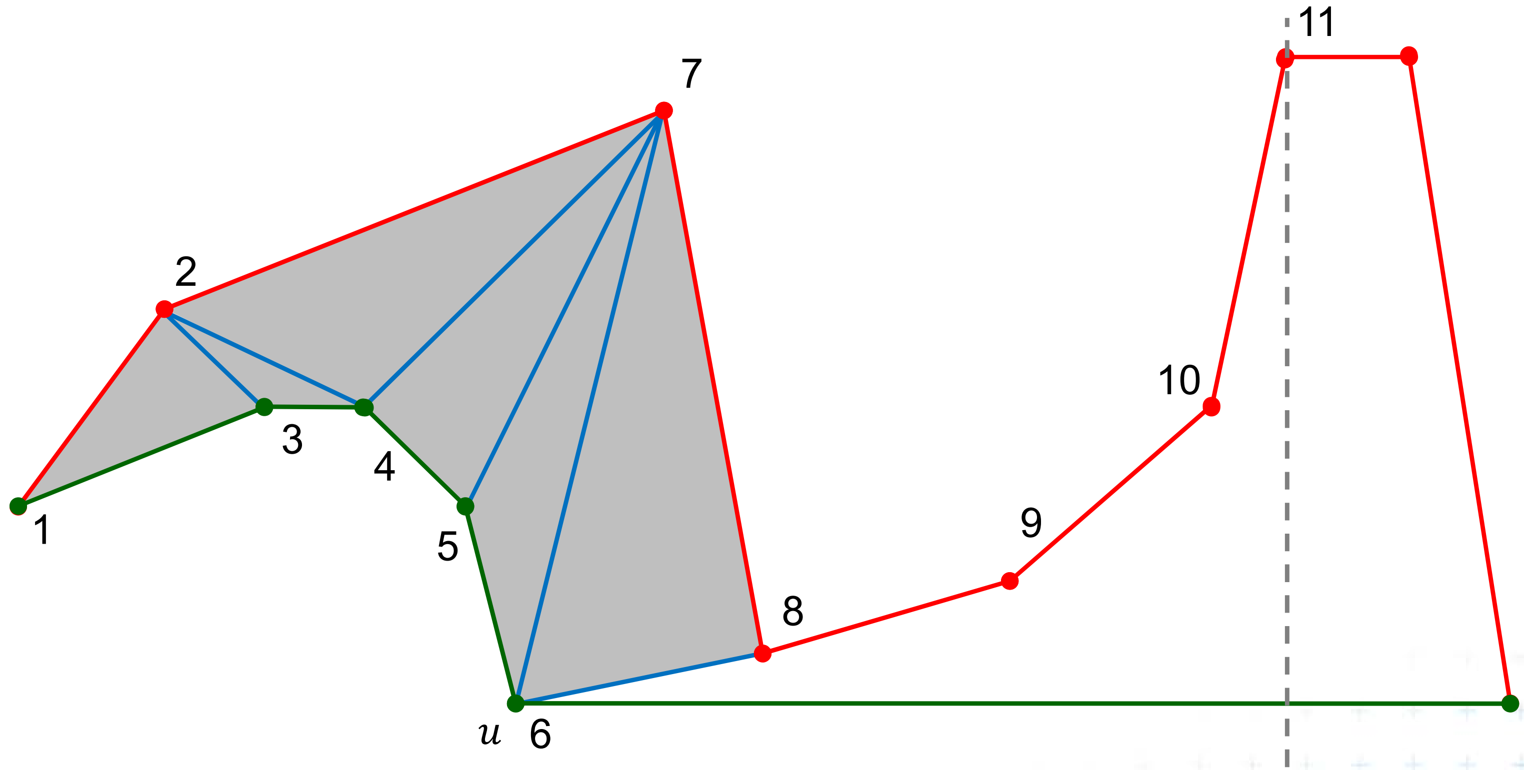


Reflex chain

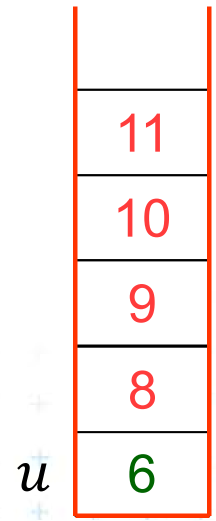


[Mount]

Monotone polygon triangulation algorithm

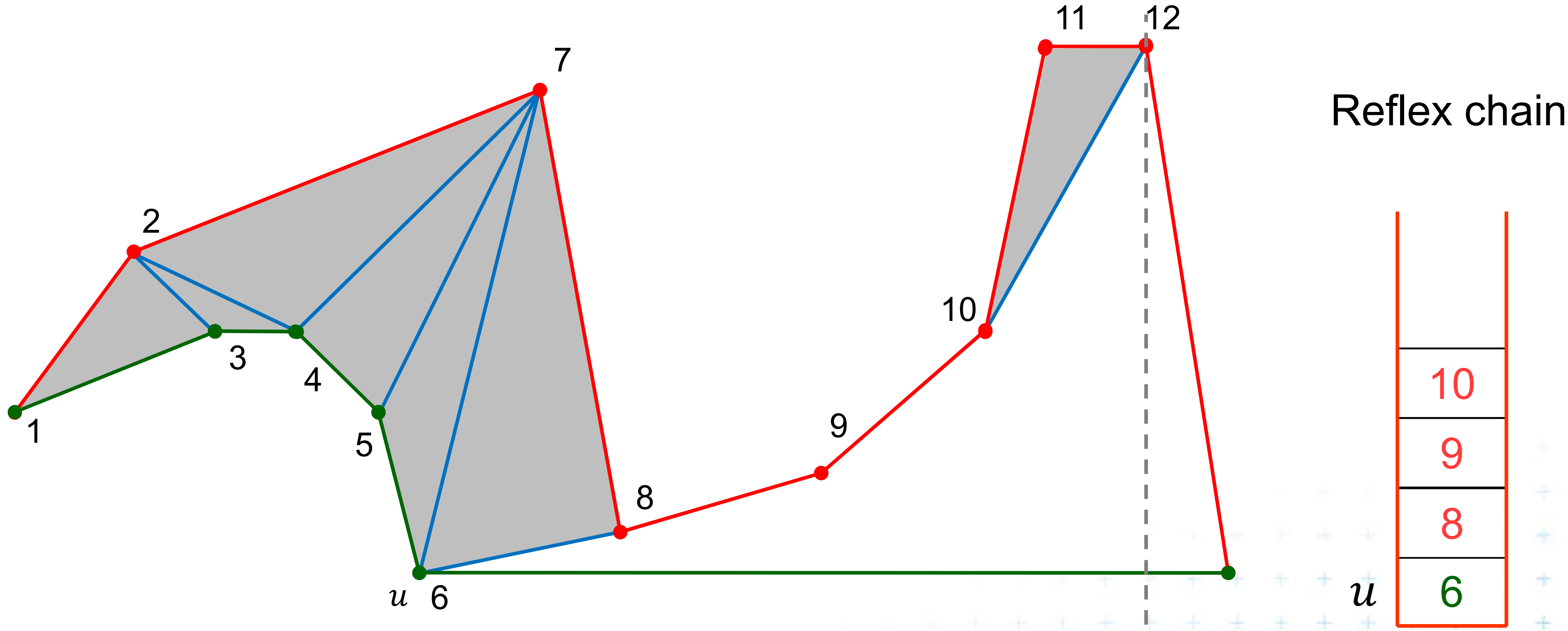


Reflex chain



[Mount]

Monotone polygon triangulation algorithm

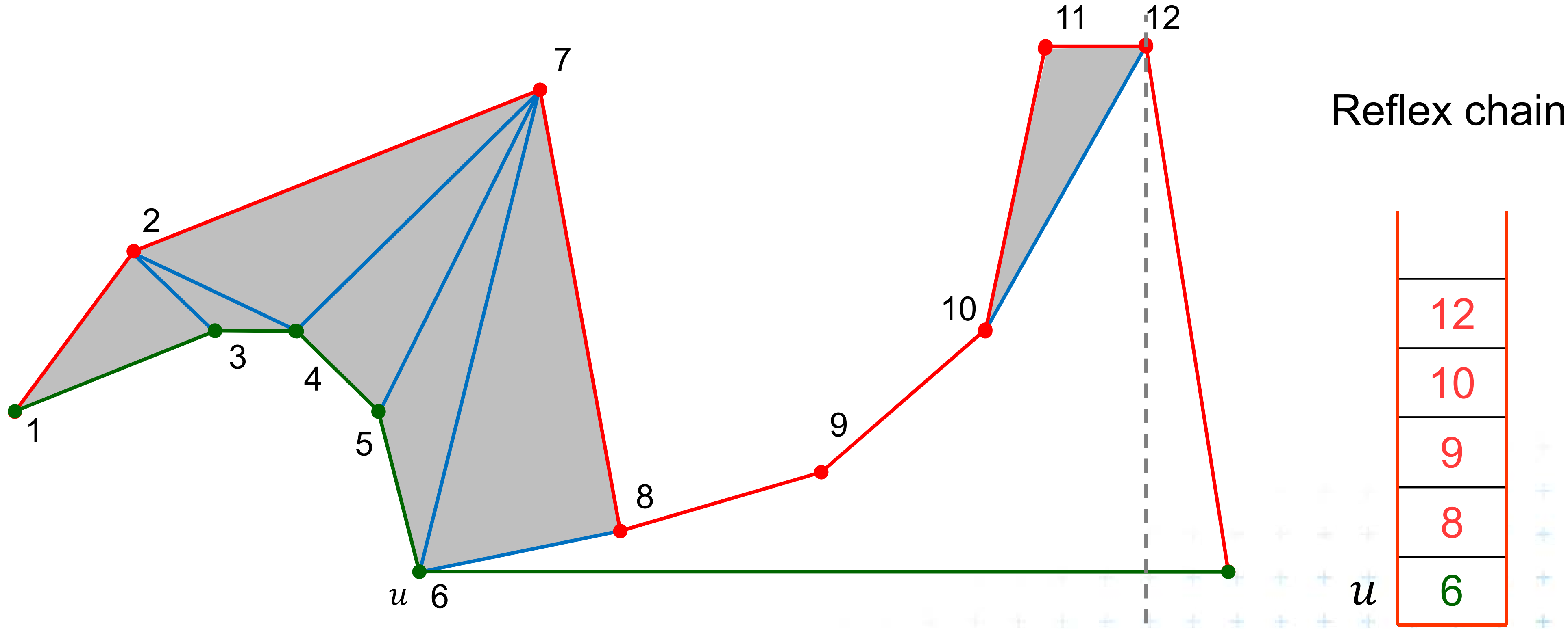


Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

[Mount]

Monotone polygon triangulation algorithm



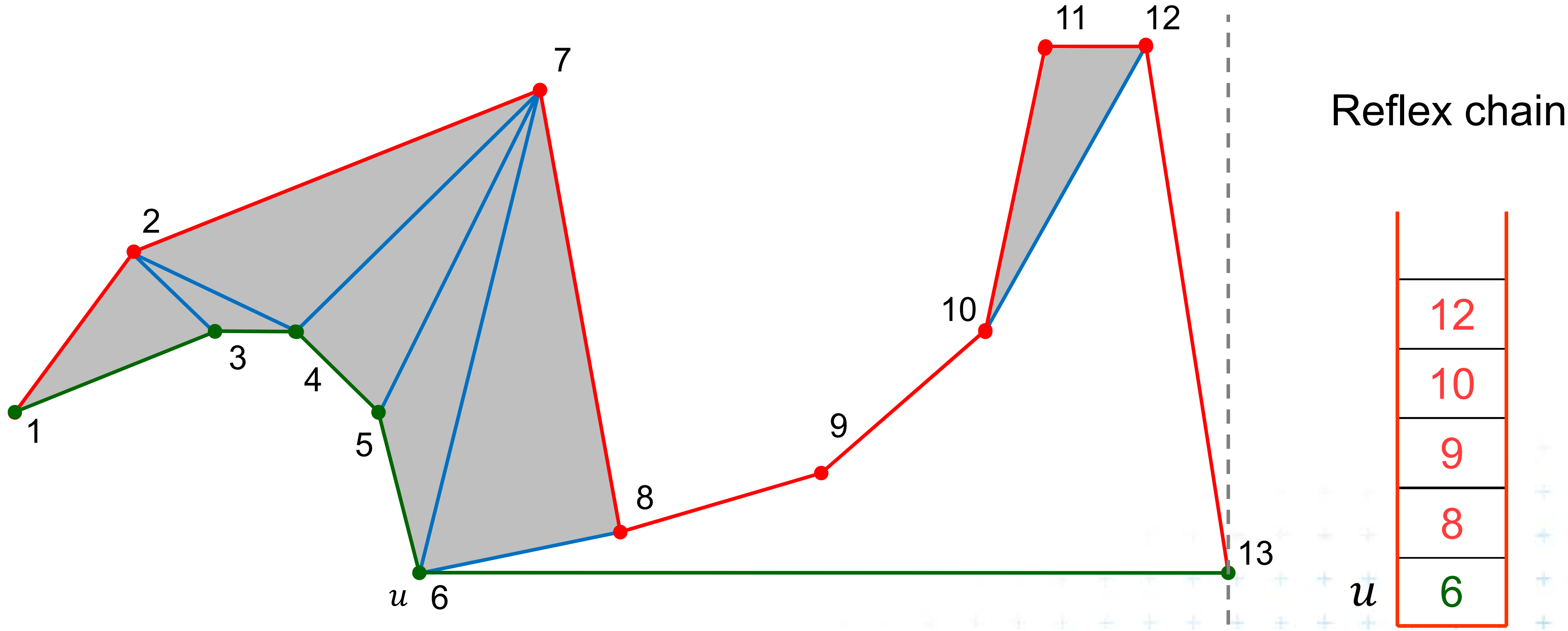
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



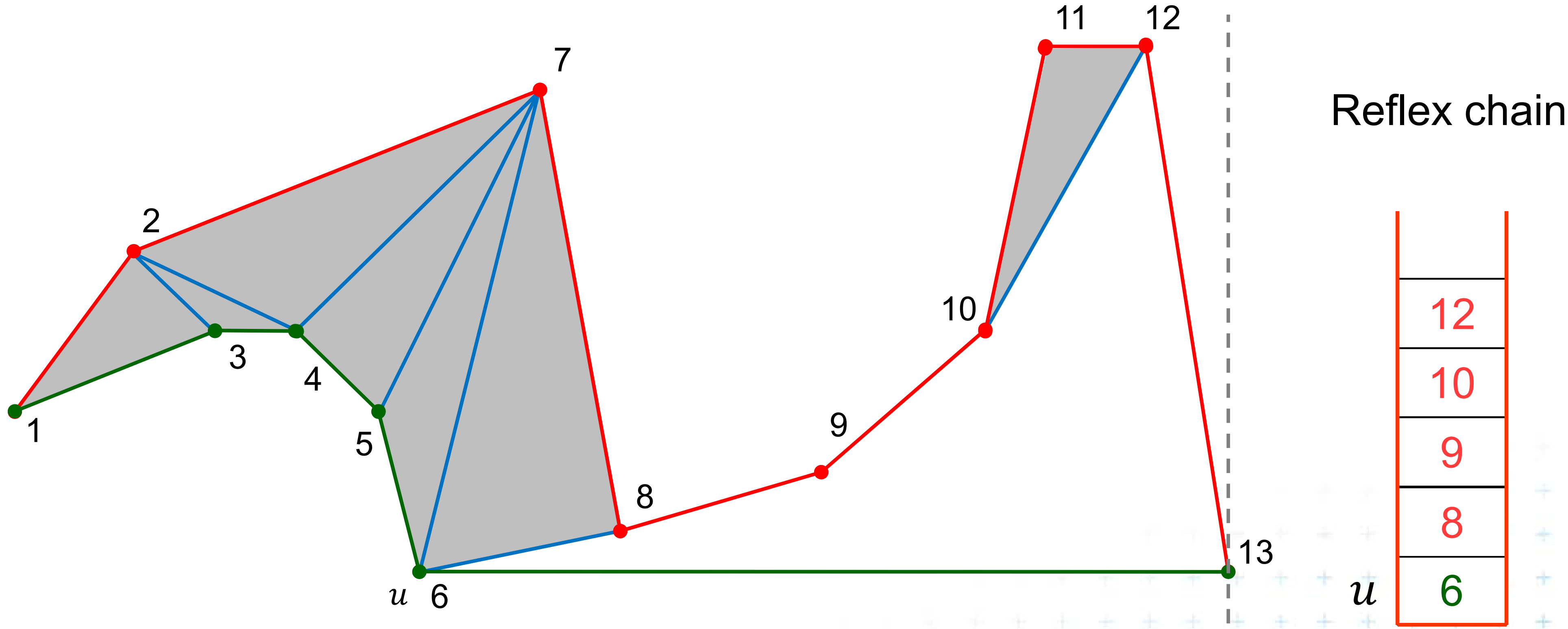
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



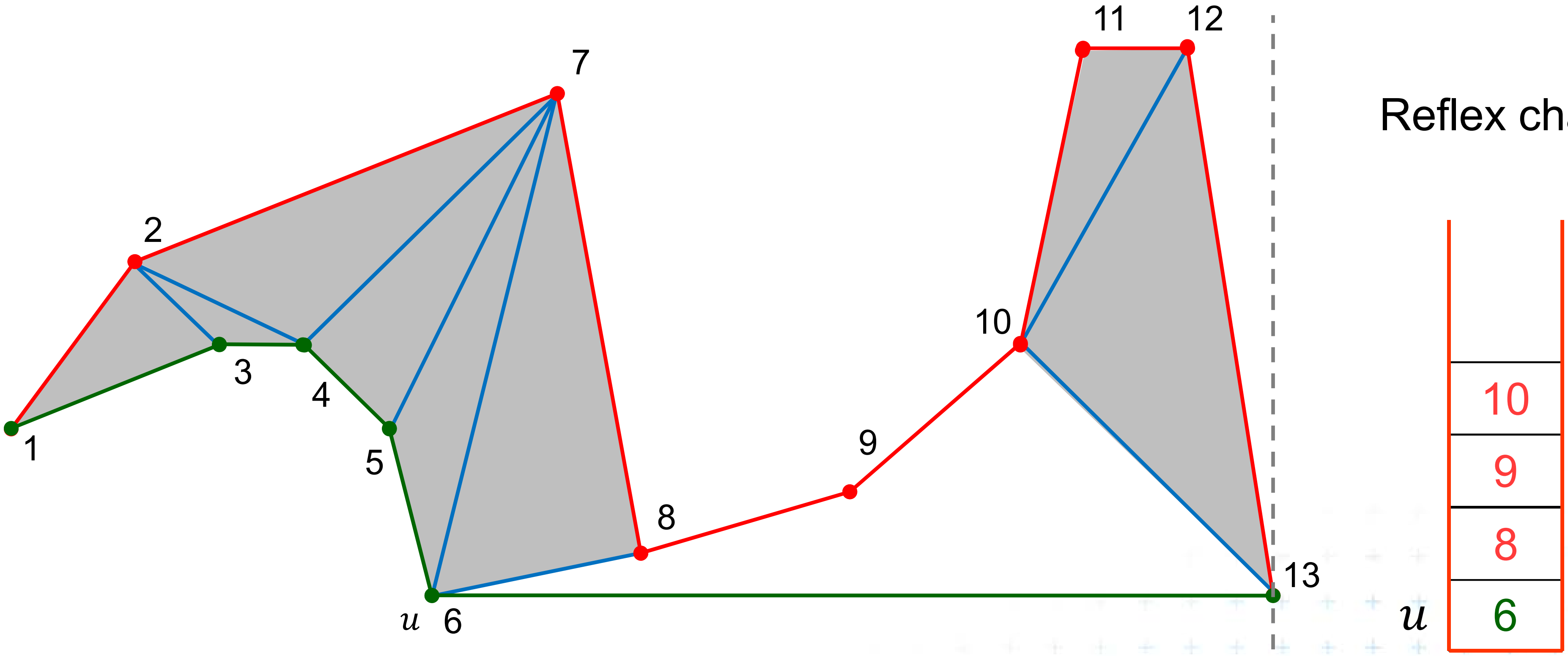
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



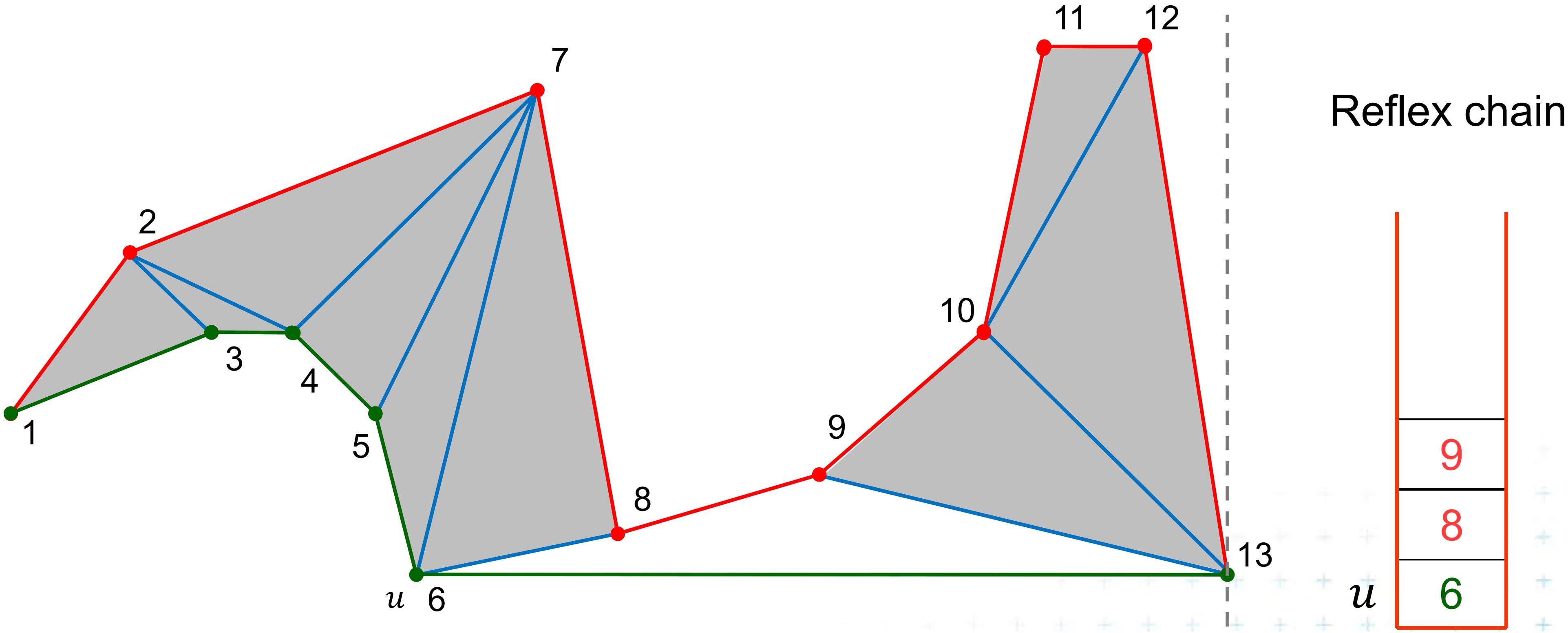
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



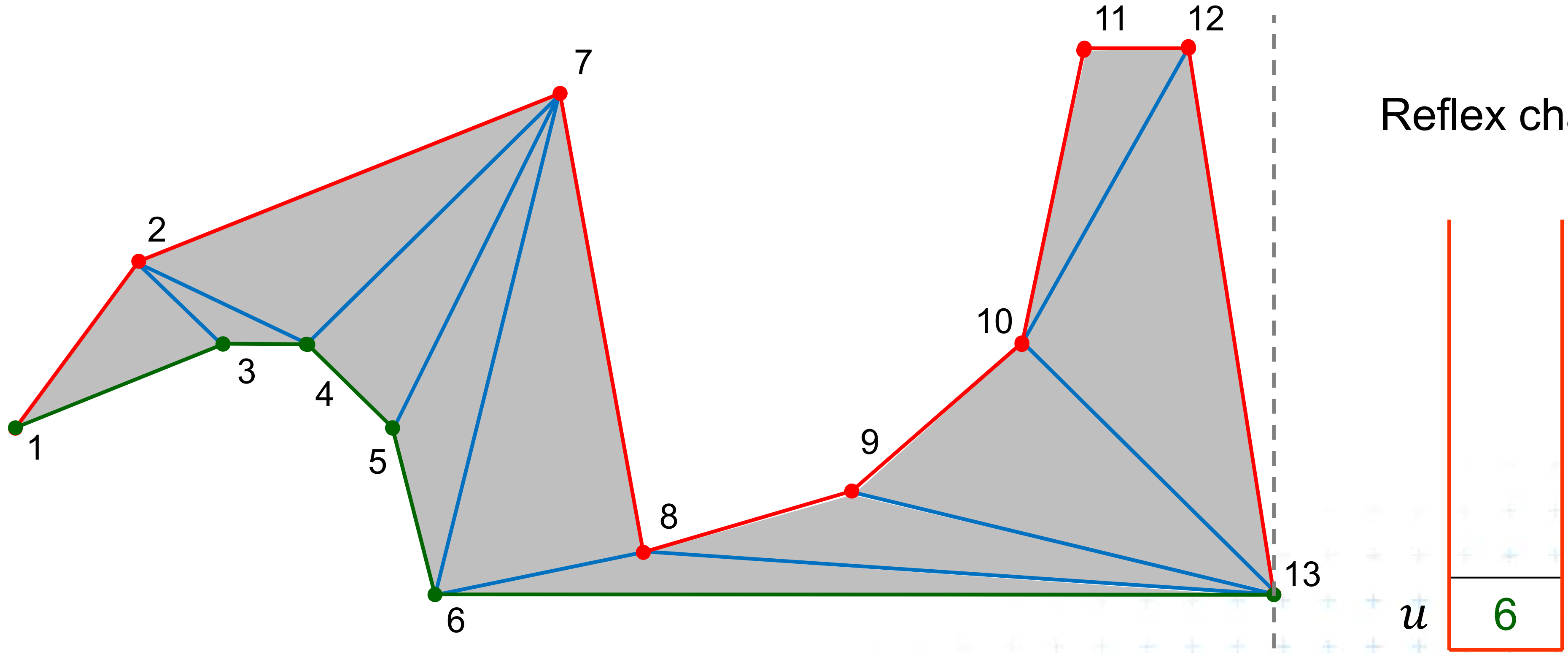
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



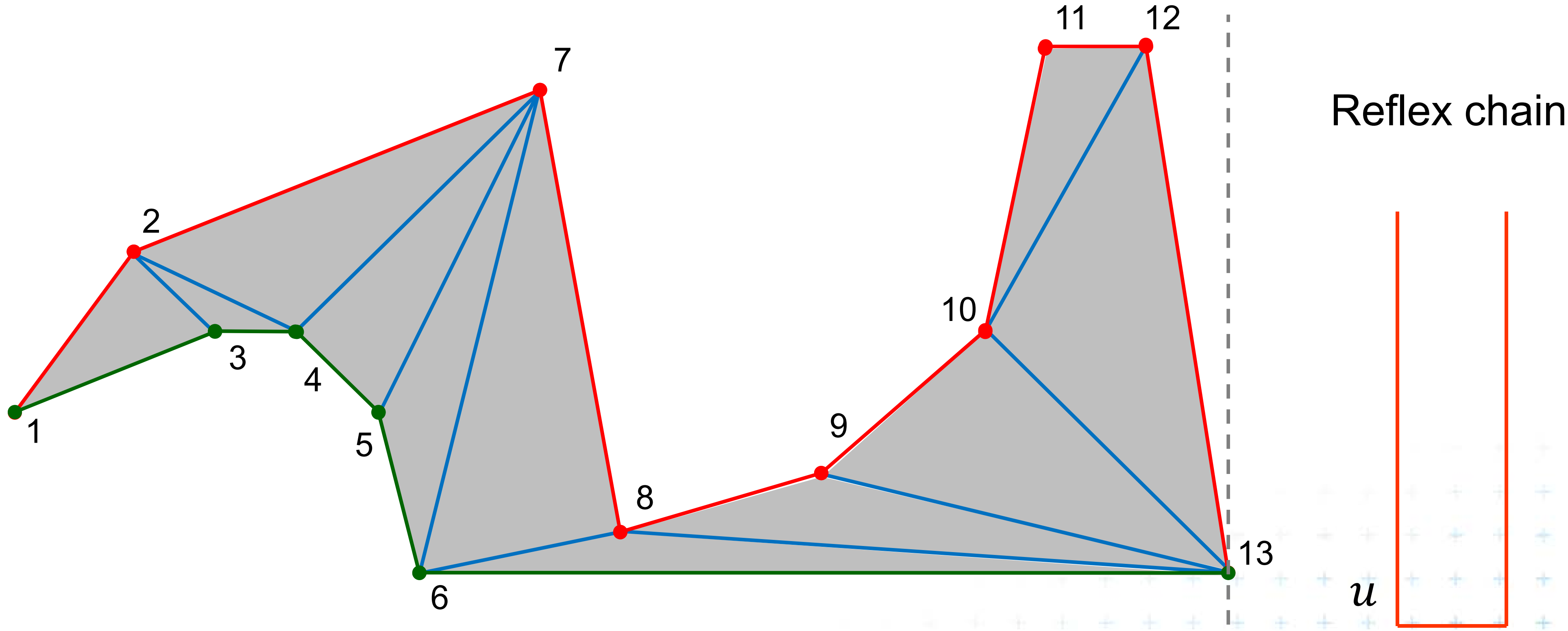
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



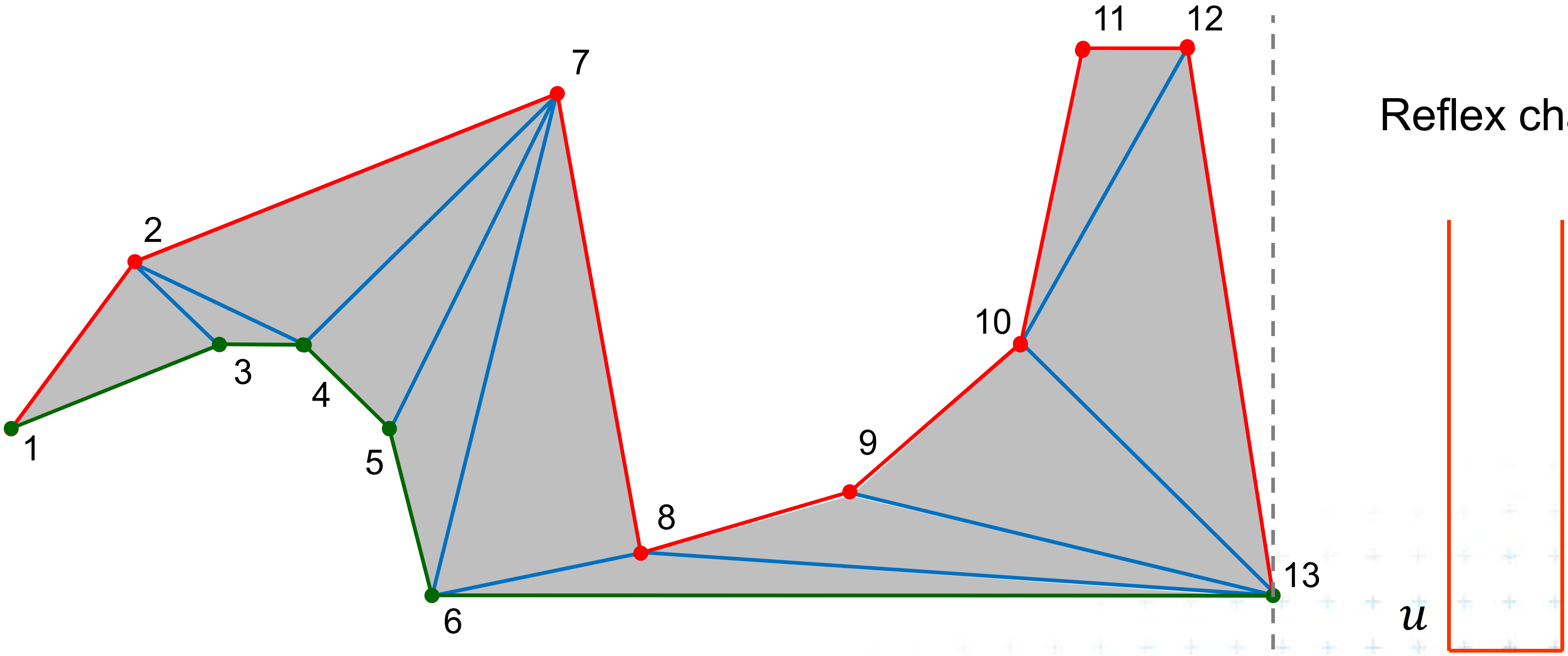
Case 2a – point v_i on the same chain as non-reflex v_{i-1}

Add diagonal(s) from v_i to visible points on reflex chain – pop()

Leave the last visible. Add v_i to reflex chain stack – push(v_i)

[Mount]

Monotone polygon triangulation algorithm



Case 1 – point v_i on opposite chain from v_{i-1}

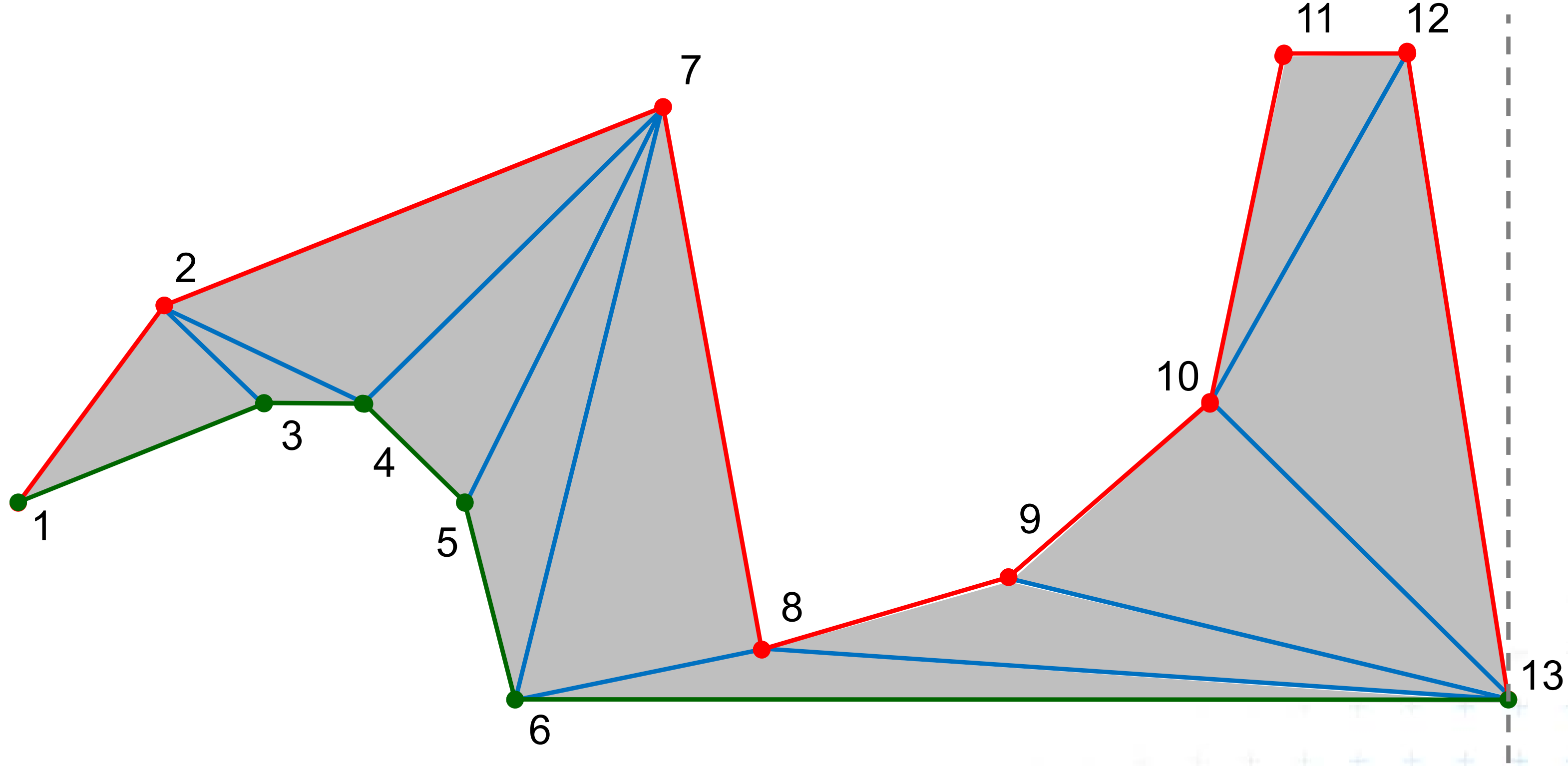
Would do the same from 13

Add diagonal(s) from v_i to all points on reflex chain stack – pop()

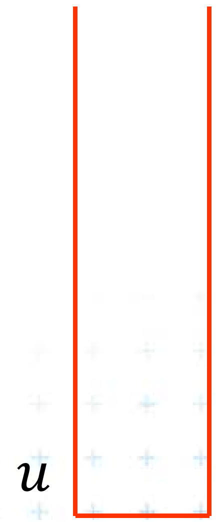
Set trivial reflex chain $v_i v_{i-1}$: New u: pop(), push(v_{i-1}), push(v_i)

[Mount]

Monotone polygon triangulation algorithm



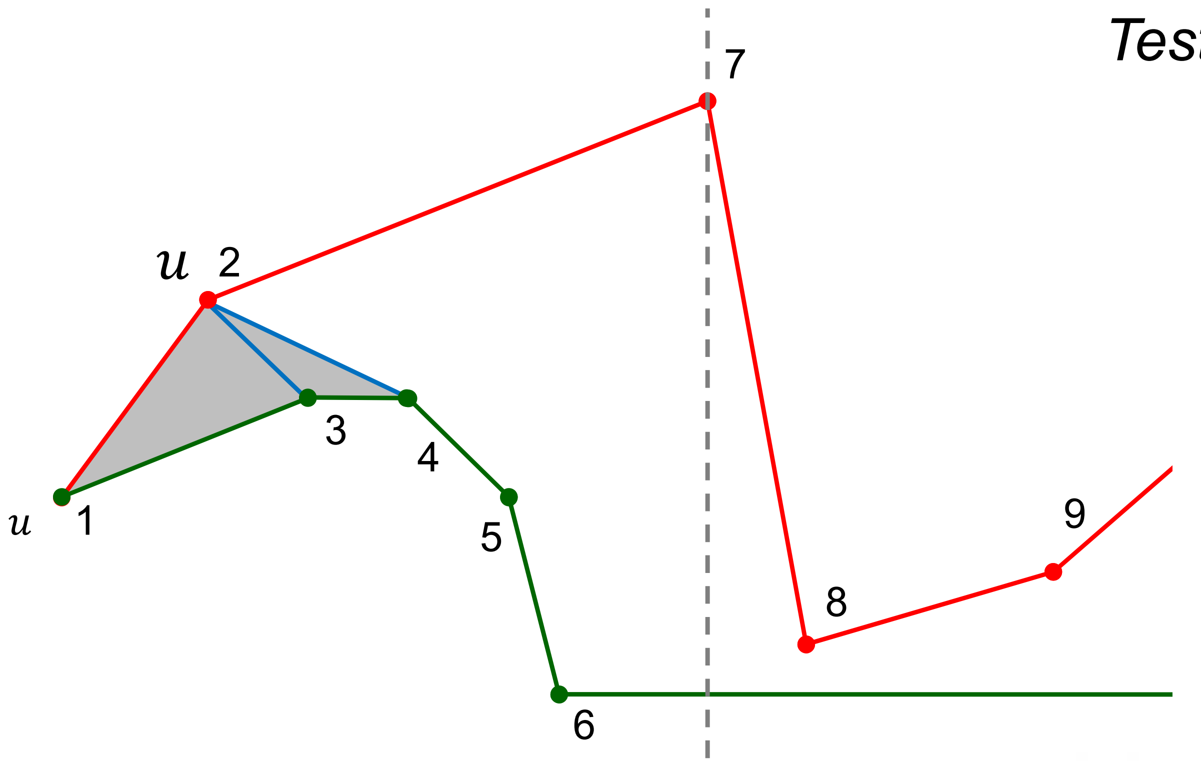
Reflex chain



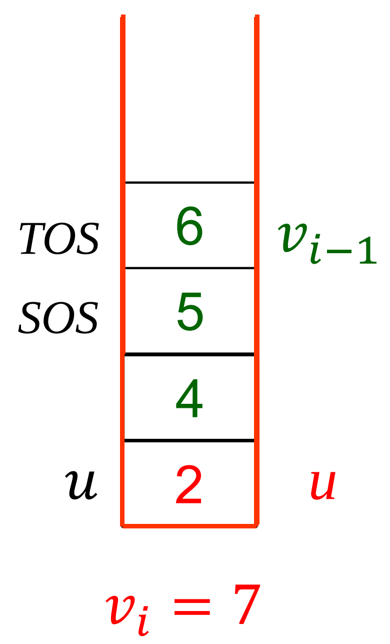
The end

[Mount]

Monotone polygon triangulation algorithm



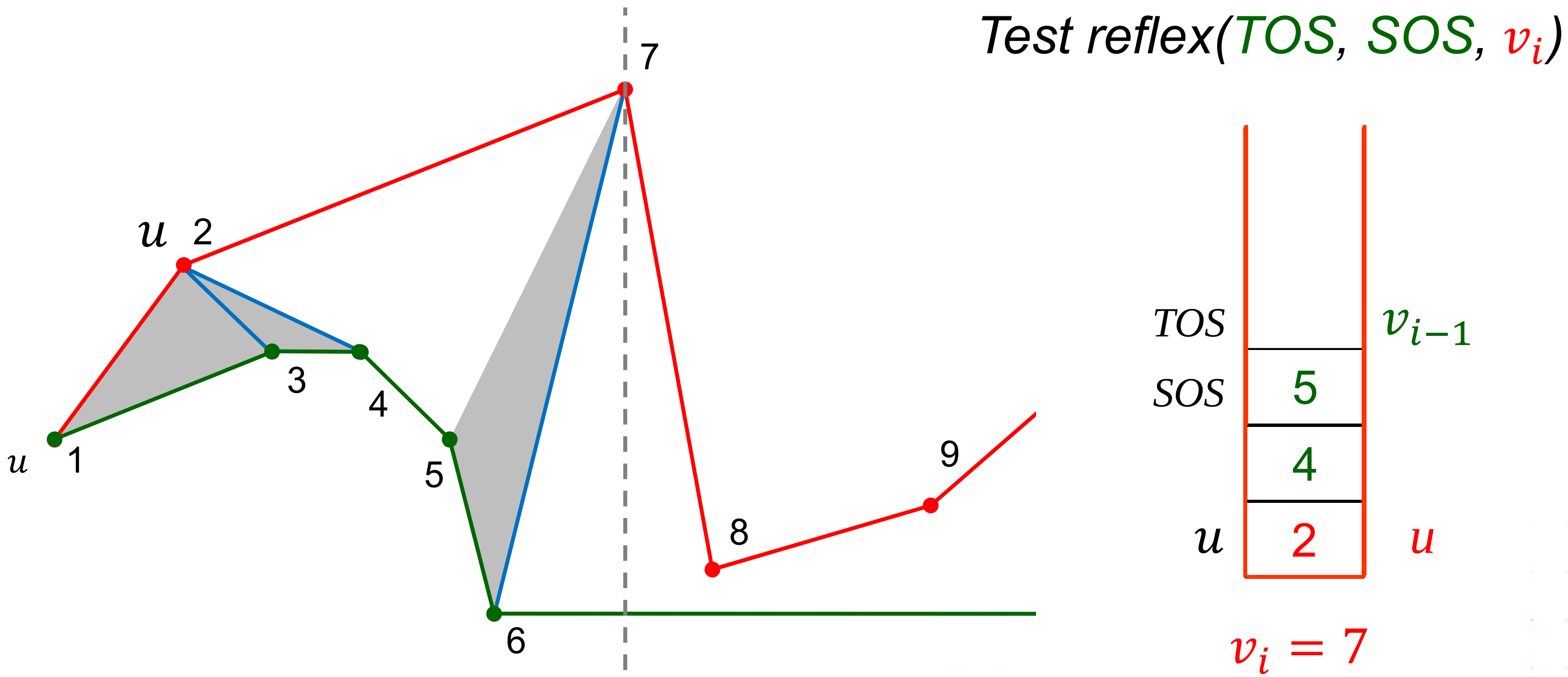
Test reflex(TOS, SOS, v_i)



- Left vertex of the last added opposite diagonal is u
- Vertices between u and v_i are waiting in the stack

[Mount]

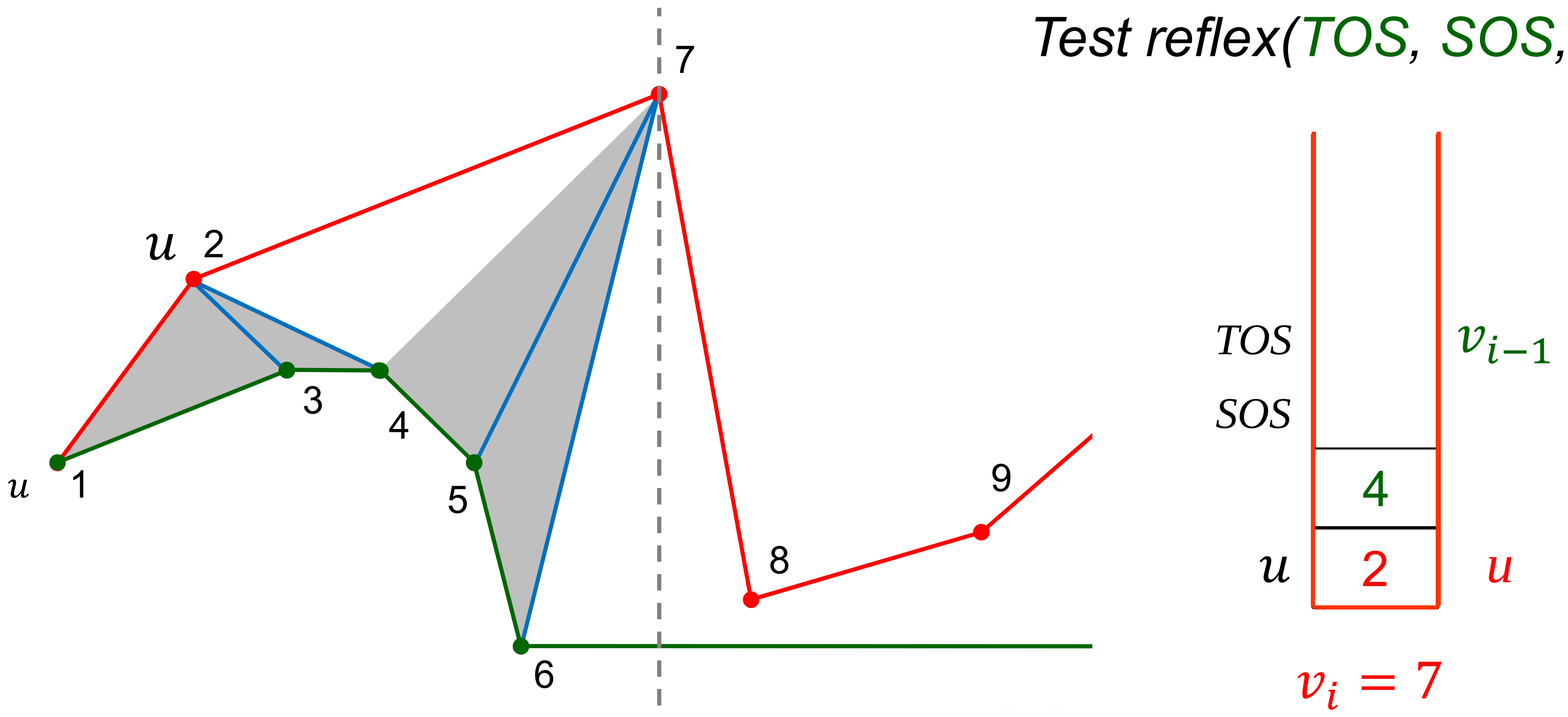
Monotone polygon triangulation algorithm



- Left vertex of the last added opposite diagonal is u
- Vertices between u and v_i are waiting in the stack

[Mount]

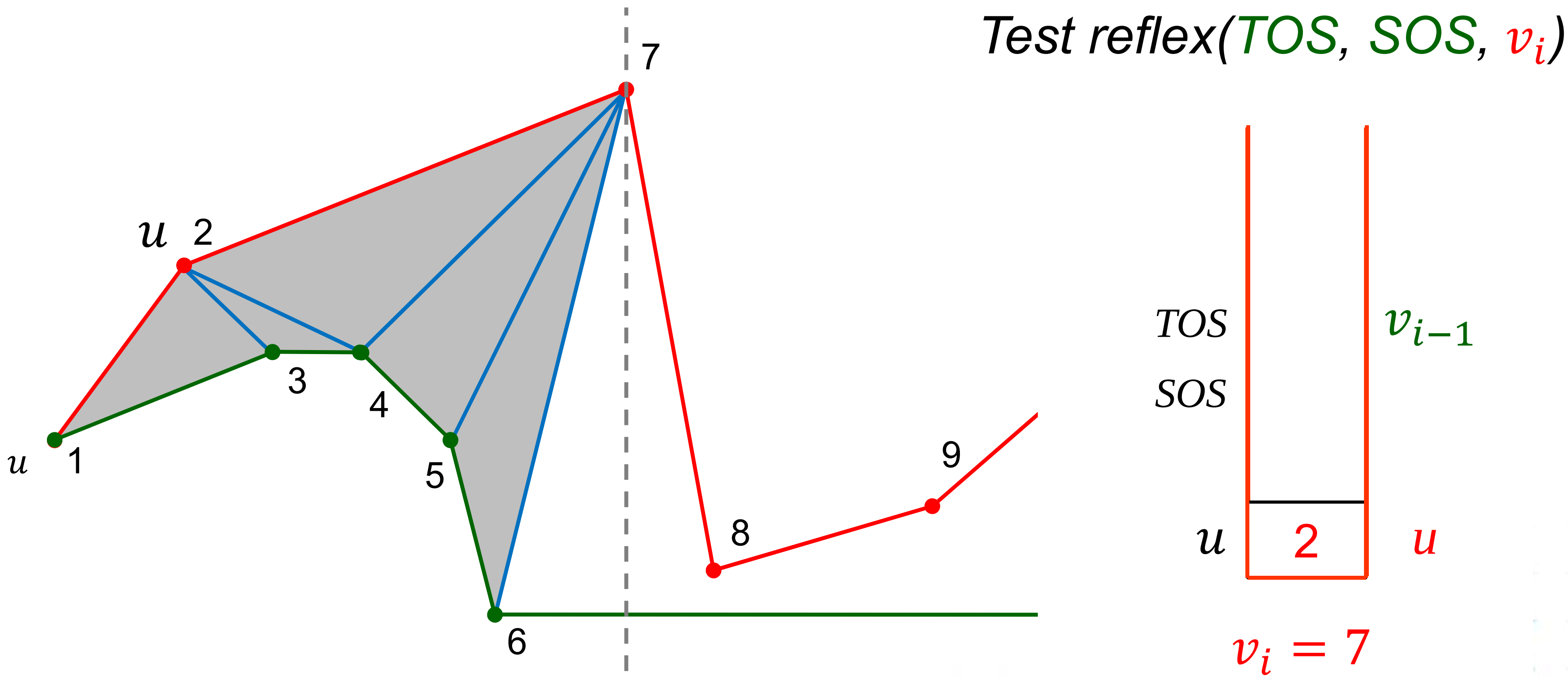
Monotone polygon triangulation algorithm



- Left vertex of the last added opposite diagonal is u
- Vertices between u and v_i are waiting in the stack

[Mount]

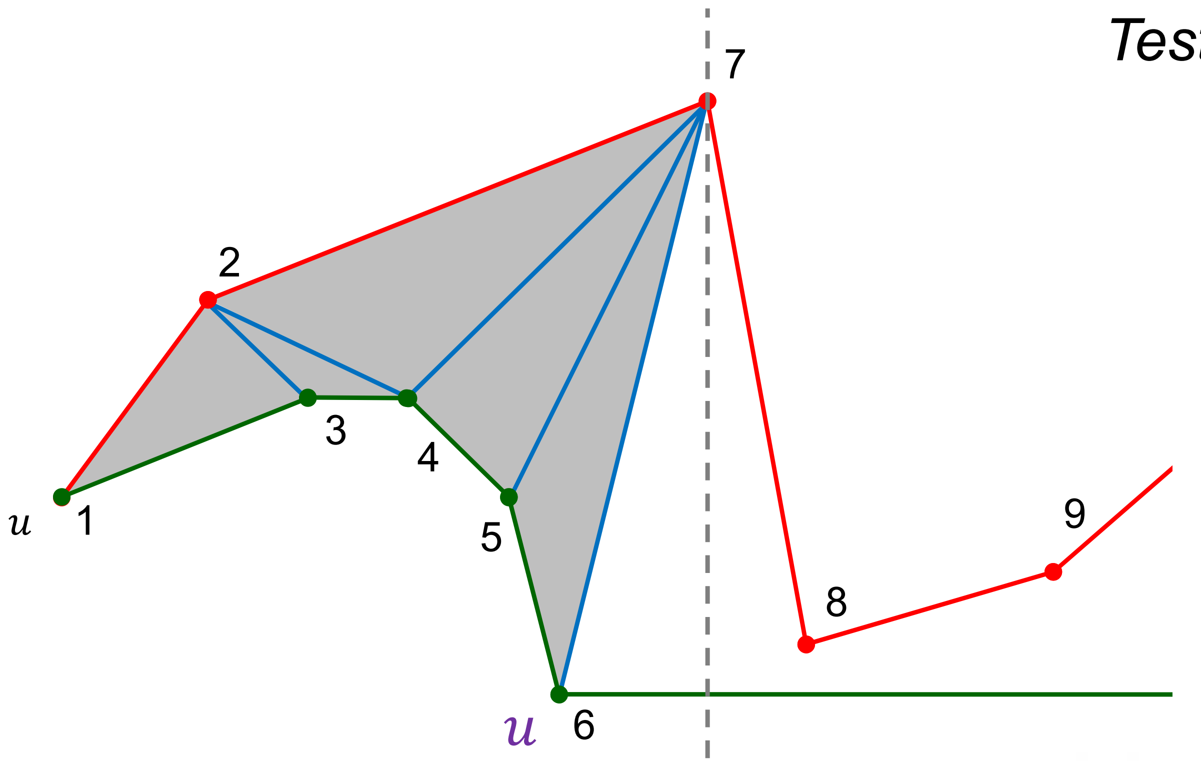
Monotone polygon triangulation algorithm

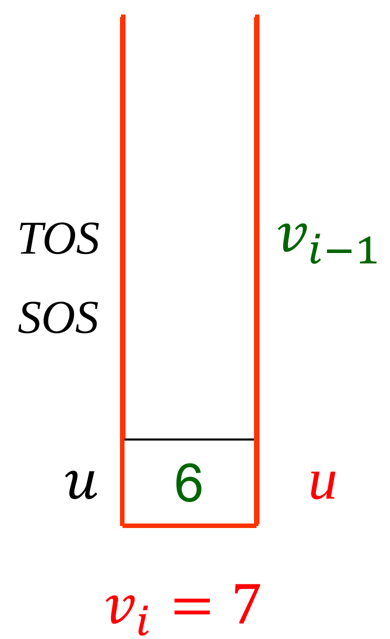


- Left vertex of the last added opposite diagonal is u
- Vertices between u and v_i are waiting in the stack

[Mount]

Monotone polygon triangulation algorithm





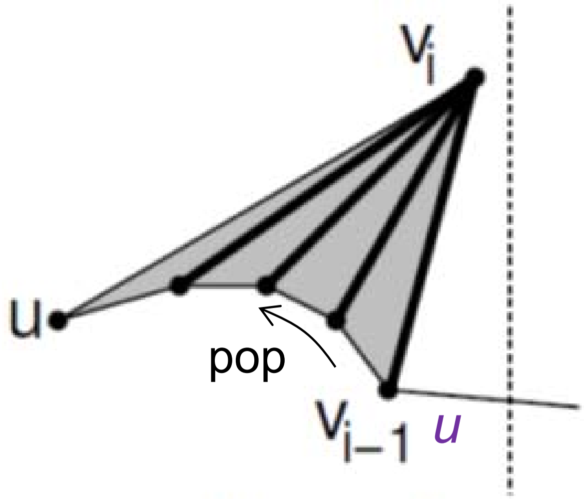
- Left vertex of the last added opposite diagonal is u
- Vertices between u and v_i are waiting in the stack

[Mount]

Triangulation cases for v_i (vertex being just processed)

Case 1: v_i lies on the opposite chain than v_{i-1}

- Add diagonals from next(u) to v_{i-1} (empty the stack-pop)
- Set u = v_{i-1}. Last diagonal (invariant) is $v_{i-1}v_i$
- push u = v_{i-1} and v_i to stack



Case 1

u updated

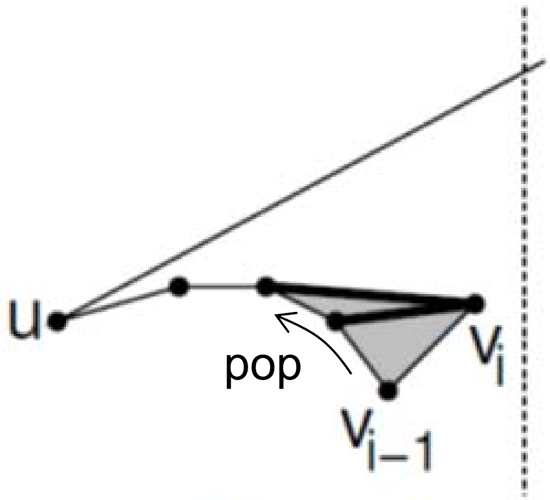
[Mount]

~~u unchanged~~

Triangulation cases for v_i (vertex being just processed)

Case 2a: v_i is on the same chain as v_{i-1}

- walk back, adding diagonals joining v_i to prior vertices until the angle becomes > 180° or u is reached – pop
- push v_i to stack



Case 2a

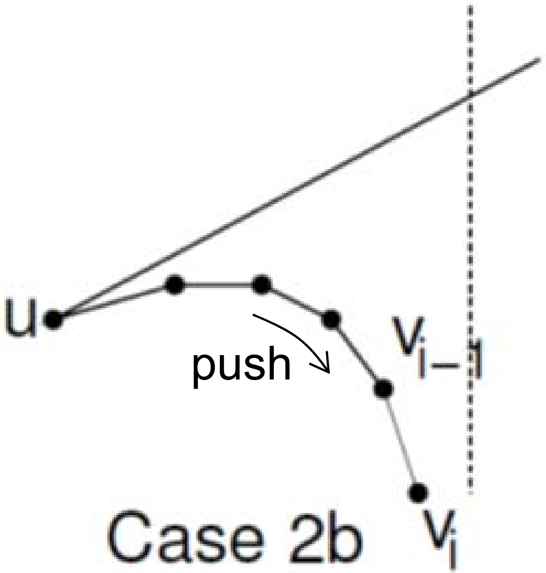
u unchanged

[Mount]

Triangulation cases for v_i (vertex being just processed)

Case 2b: v_i is on the same chain as v_{i-1}

– push v_i to stack



u unchanged

[Mount]

Analysis

Polygon with n vertices has $n-3$ diagonals

$\Rightarrow O(n)$ total time

Algorithm

sorted list of vertices through merging - $O(n)$

stack operations – max n times $O(1)$ - $O(n)$

orientation test - v_i and top two entries

- $O(1)$ per diagonal - $O(n)$

(add diagonal or push)

Simple polygon triangulation

- Simple polygon can be triangulated in 2 steps:
  1. Partition the polygon into x-monotone pieces
  2. Triangulate all monotone pieces

(we discuss the steps in the reversed order)

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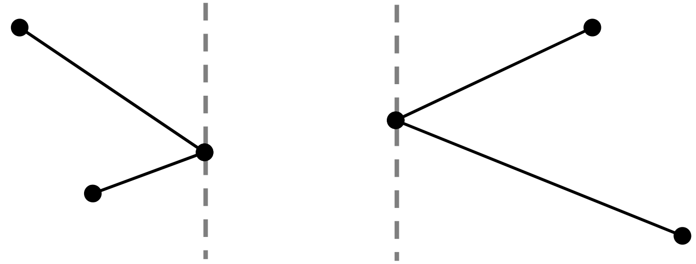
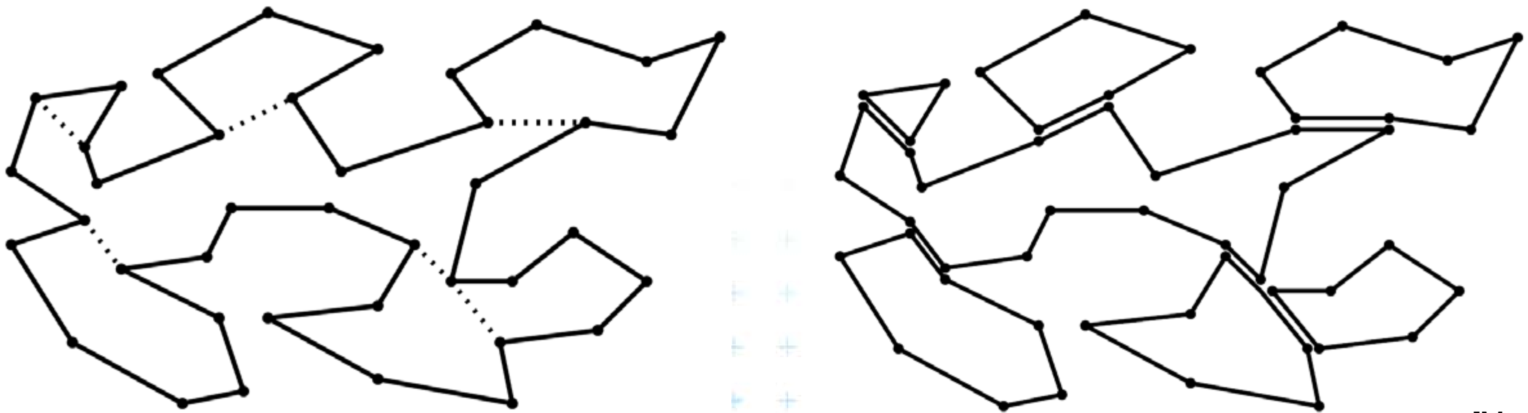
  (we discuss the steps in the reversed order)

1. Polygon subdivision into monotone pieces

- X-monotonicity breaks the polygon in vertices with edges directed both left or both right



- The monotone polygons parts are separated by the splitting diagonals (joining vertex and helper)



[Mount]

Splitting diagonals

Monotone decomposition

Data structures for subdivision

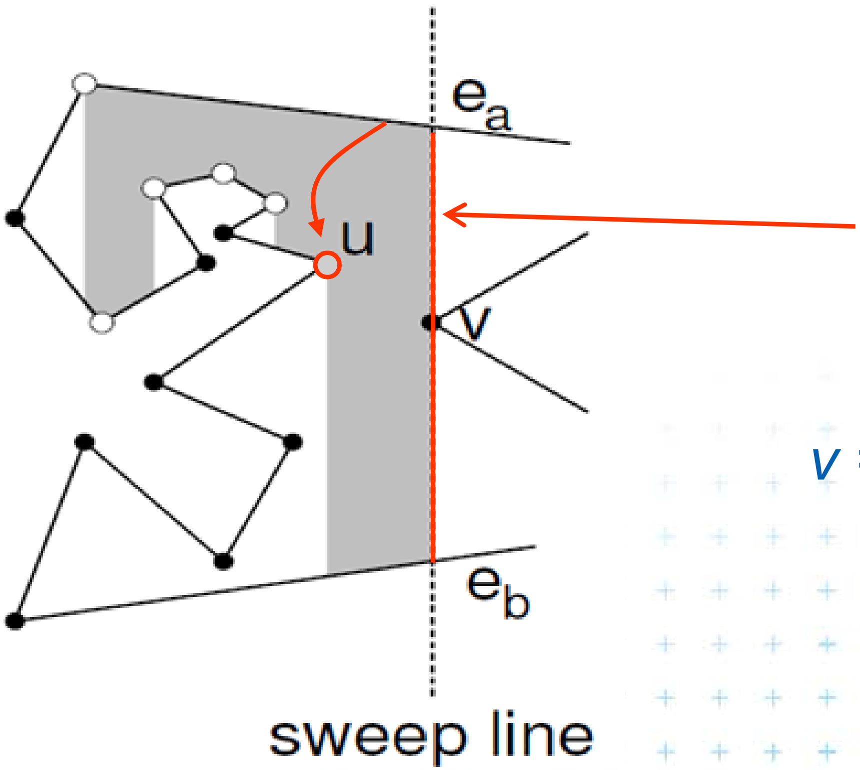
- Events
  - Endpoints of edges, known from the beginning
  - Can be stored in sorted list – no priority queue
- Sweep status
  - List of edges intersecting the sweep line (top to bottom)
  - Stored in O(log n) time dictionary (such as balanced tree)
- Event processing
  - Six event types based on local structure of edges around vertex *v*

Helper – definition

helper(e_a)

= the rightmost vertically visible processed vertex u - *on or* below edge e_a on polygonal chain between edges e_a & e_b

is visible to every point along the sweep line between e_a & e_b



all these vertices

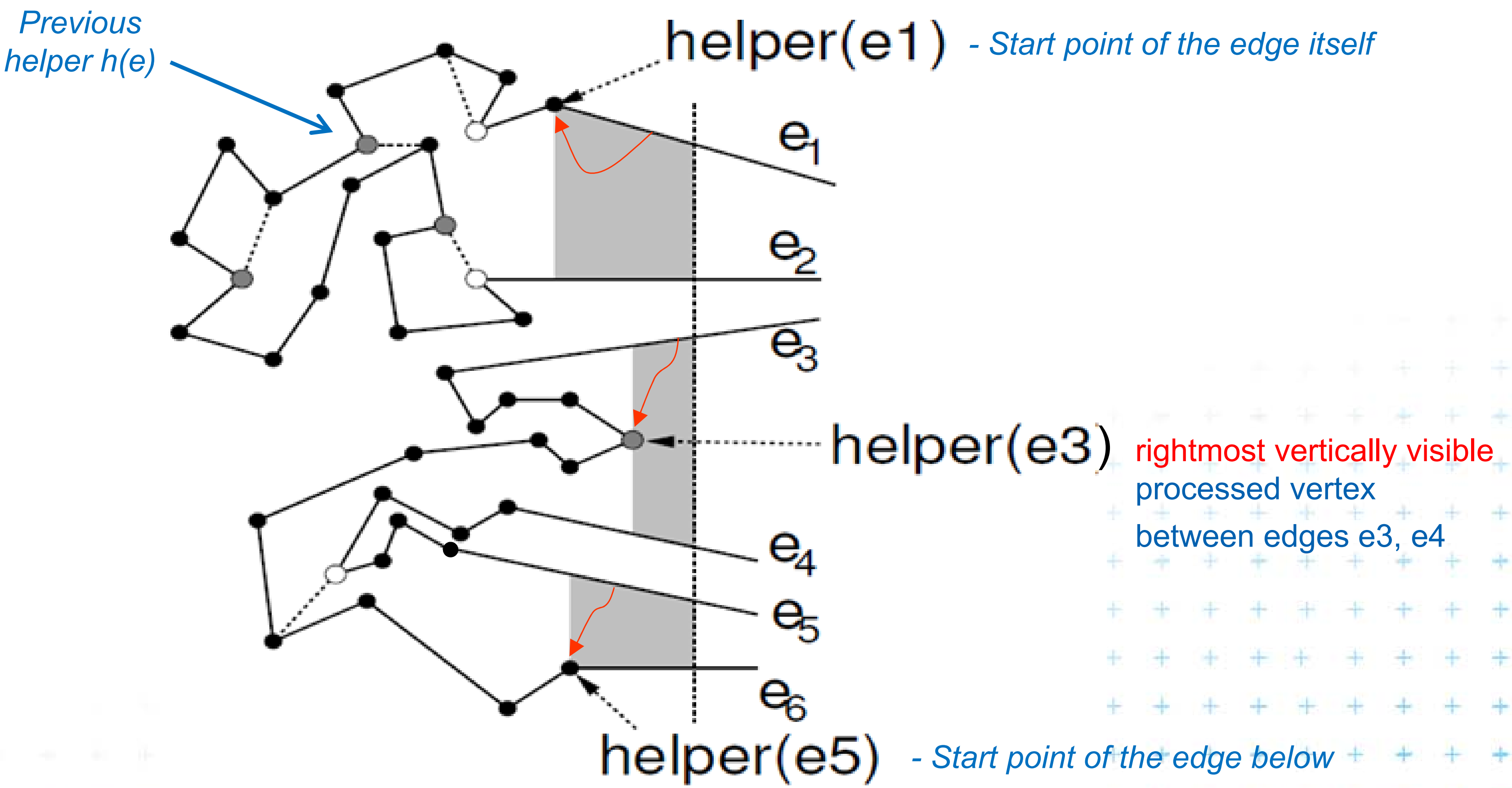
see u = helper(e_a)

v = current vertex
(sweep line stop)

Helper

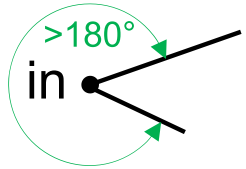
helper(e_a)

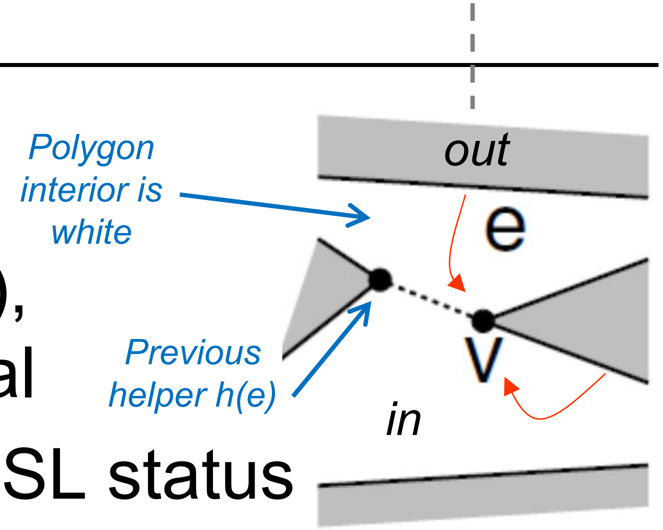
is defined only for edges intersected by the sweep line



Six event types of vertex *v*

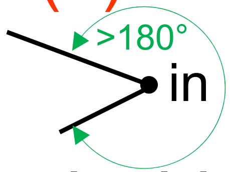
1. Split vertex

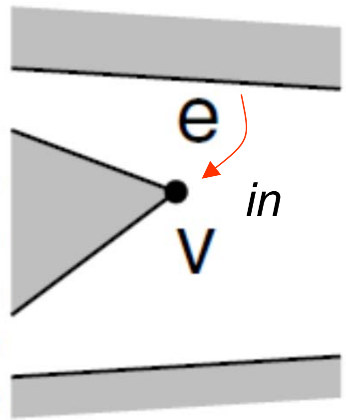




- Find edge *e* above *v* (along the SL), connect *v* with helper(e) by diagonal
- Add 2 new edges incident to *v* into SL status
- Set new helper(e) = *v*, e = lower edge of these two

2. Merge vertex





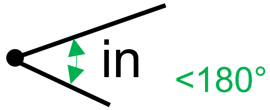
[Mount]

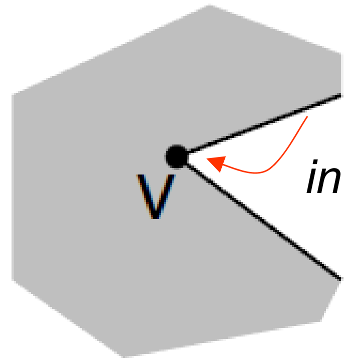
- Find two edges incident with *v* in SL status
- Delete both from SL status
- Let *e* is edge immediately above *v*
- Make helper(e) = *v*

(Interior angle >180° for both – split & merge vertices)

Six event types of vertex *v*

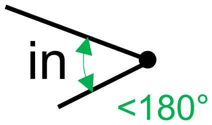
3. Start vertex

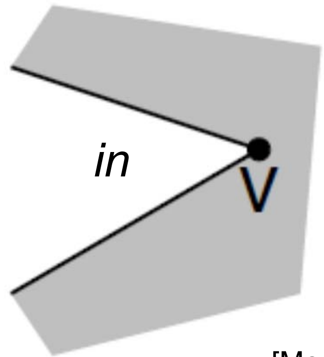




– Both incident edges lie right from *v*
– But interior angle <180°
– Insert both edges to SL status
– Set helper(upper edge) = *v*

4. End vertex



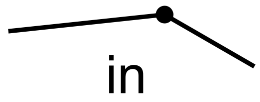


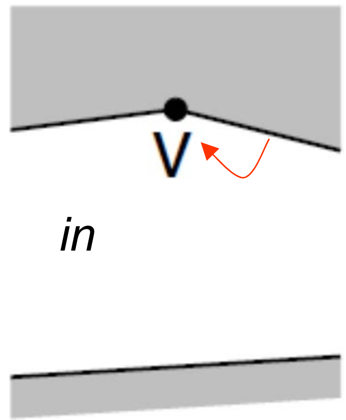
[Mount]

– Both incident edges lie left from *v*
– But interior angle <180°
– Delete both edges from SL status
– No helper set – we are out of the polygon

Six event types of vertex *v*

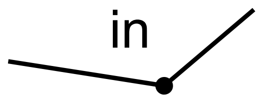
5. Upper chain-vertex

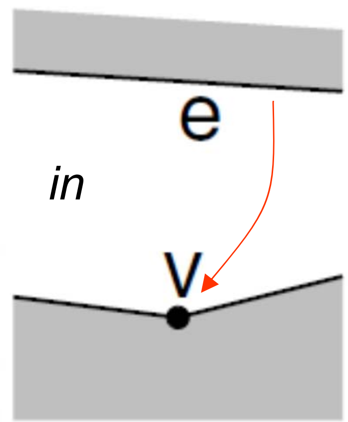




- one side is to the left, one side to the right, interior is below
- replace the left edge with the right edge in SL status
- Make *v* helper of the new (upper) edge

6. Lower chain-vertex





- one side is to the left, one side to the right, interior is above
- replace the left edge with the right edge in SL status
- Make *v* helper of the edge *e above*

[Mount]

Polygon subdivision complexity

- Simple polygon with *n* vertices can be partitioned into x-monotone polygons in
  - O(*n* log *n*) time sort
  - O(*n* log *n*) time (n steps of SL, log n search each)
  - O(*n*) storage

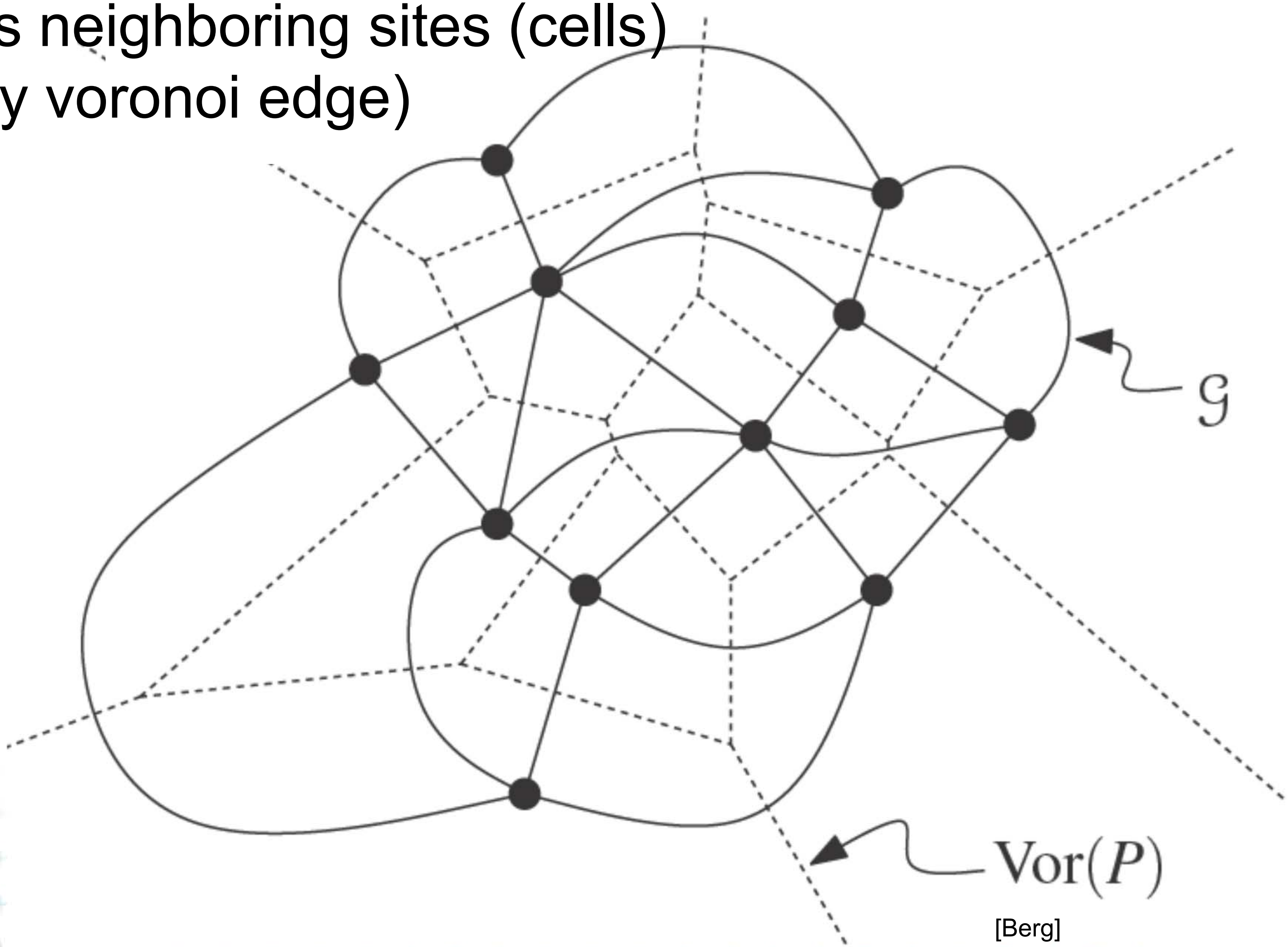
- Complete simple polygon triangulation
  - O(*n* log *n*) time for partitioning into monotone polygons
  - O(*n*) time for triangulation
  - O(*n*) storage

Delaunay triangulation

Dual graph G for a Voronoi diagram

Graph G: Node for each Voronoi-diagram site p ~ VD cell $V(p)$

Arc connects neighboring sites (cells)
(arc for every voronoi edge)



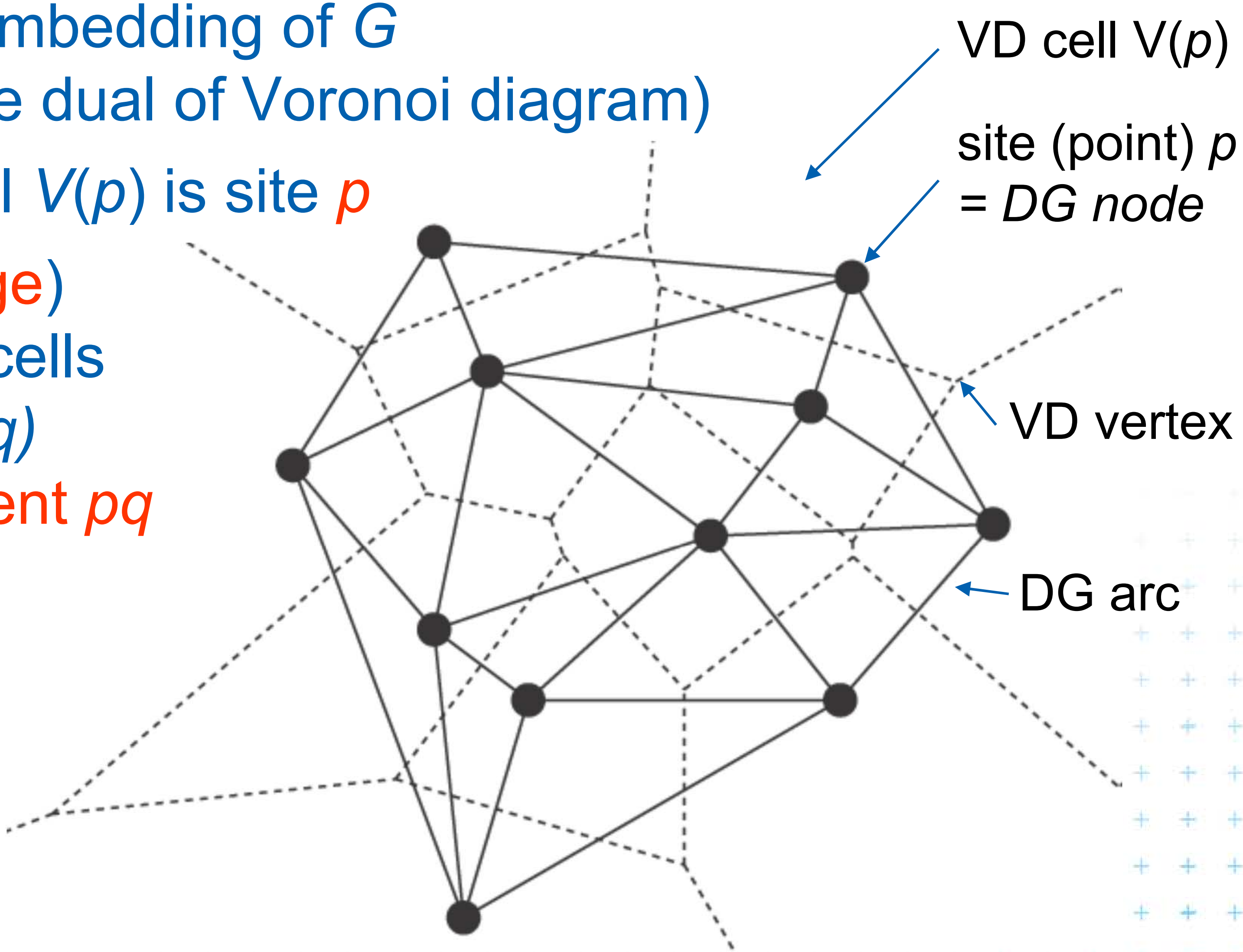
[Berg]

Delaunay graph *DG*(*P*)

[*Борис Николаевич Делоне*]

= straight line embedding of *G*
(straight-line dual of Voronoi diagram)

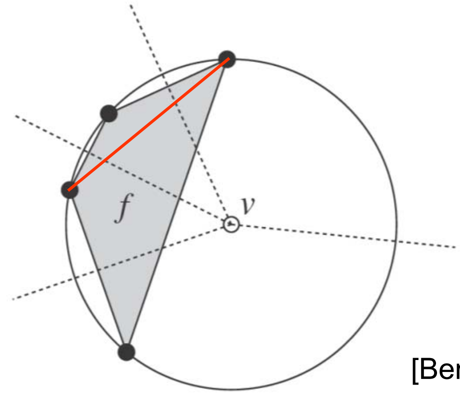
- Node for cell *V*(*p*) is site *p*
- Arc (DG edge) connecting cells *V(p) and V(q)* is the segment *pq*



[Berg]

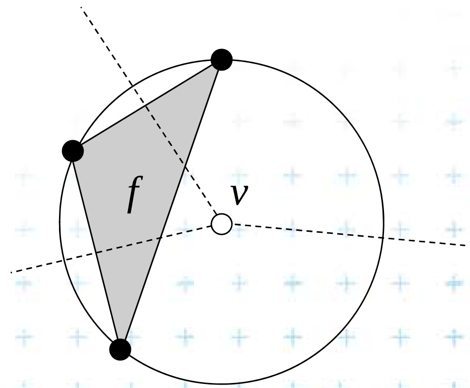
Delaunay graph and Delaunay triangulation

- Delaunay *graph DG(P)* has convex polygonal faces (with number of vertices ≥3, equal to the degree of Voronoi vertex)
  – Triangulate faces with more vertices *DG(P)* sites not in general position such triangulation is not unique

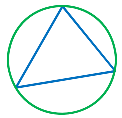


[Berg]

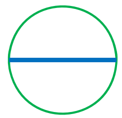
- Delaunay *triangulation DT*(*P*) = Delaunay graph for sites in general position
  – No four sites on a circle
  – Faces are triangles (Voronoi vertices have degree = 3)
  – DT is unique



Delaunay triangulation properties 1/2

Circumcircle property 

- The circumcircle of any triangle in DT is empty (no sites)
  Proof: It's center is the Voronoi vertex
- Three points *a,b,c* are vertices of the same face of *DG*(*P*)
  iff circle through *a,b,c* contains no point of *P* in its interior

Empty circle property and legal edge 

- Two points *a,b* form an edge of *DG*(*P*) – it is a legal edge
  iff ∃ closed disc with *a,b* on its boundary that contains
  no other point of *P* in its interior … disc minimal diameter = dist(a,b)

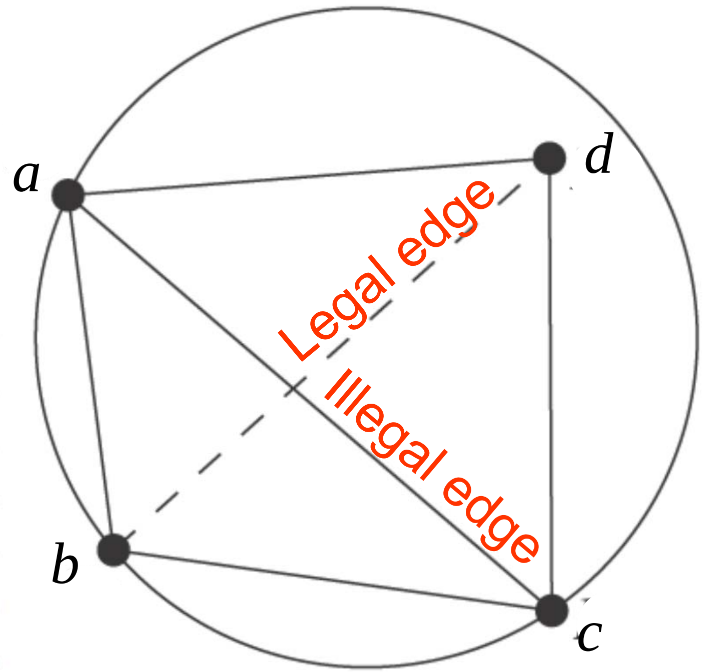
Closest pair property

- The closest pair of points in *P* are neighbors in *DT*(*P*)

Delaunay triangulation properties 2/2

- DT edges do not intersect
- Triangulation *T* is legal, **iff** *T* is a Delaunay triangulation (i.e., if it does not contain illegal edges)
- Edge in DT that was legal before may become illegal if one of the triangles incident to it changes
- In convex quadrilateral *abcd* (*abcd* do not lie on common circle) exactly one of *ac, bd* is an illegal edge and the other edge is legal
  = principle of edge flip operation

Non-convex quad has only one diagonal



[Berg]

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Non-convex quad has only one diagonal

- In convex quadrilateral *abcd* (*abcd* do not lie on common circle) exactly one of *ac, bd*
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  - and the other edge is legal
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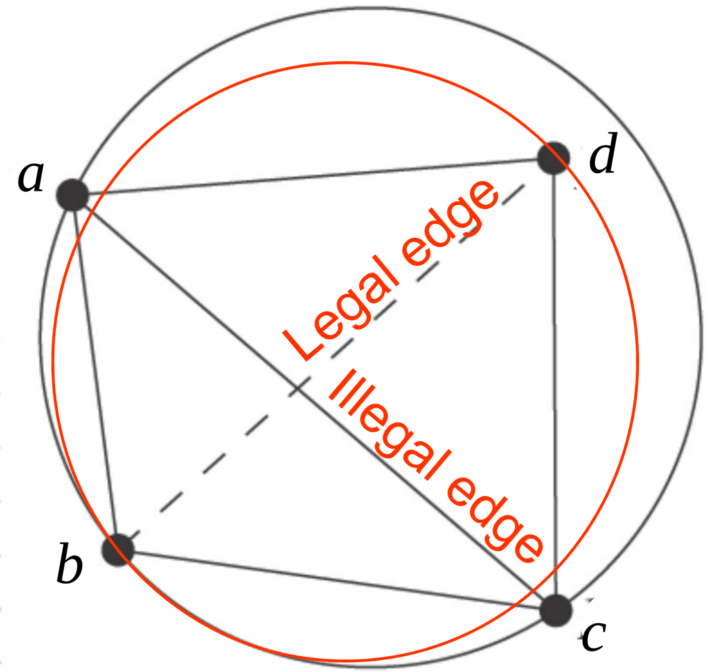


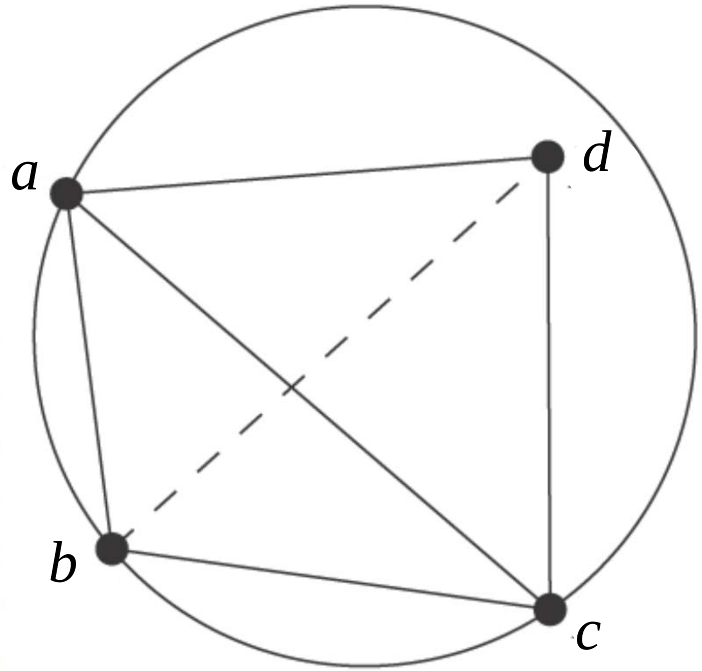
[Berg]

Edge flip operation

Edge flip iflips illegal → legal edge

= a local operation, that increases the angle vector

- Given two adjacent triangles $\triangle abc$ and $\triangle cda$ such that their union forms a convex quadrilateral, the edge flip operation replaces the diagonal *ac* with *bd*.



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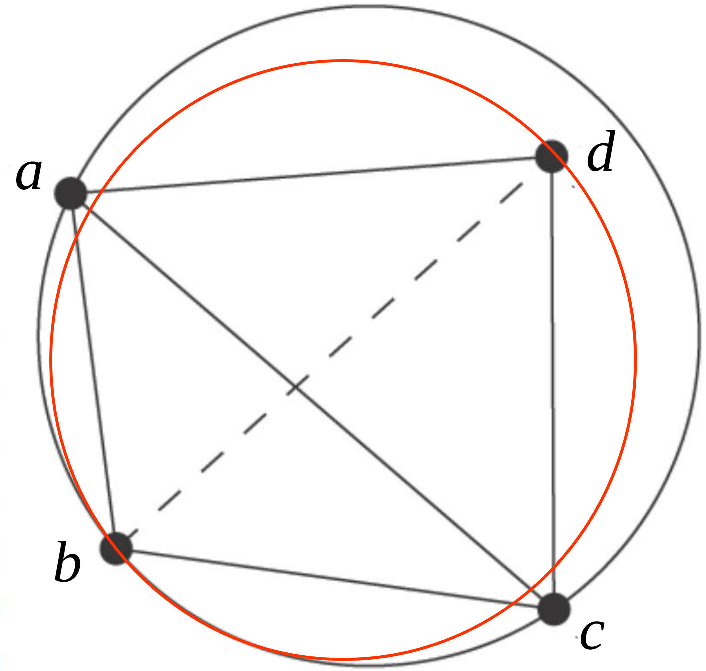

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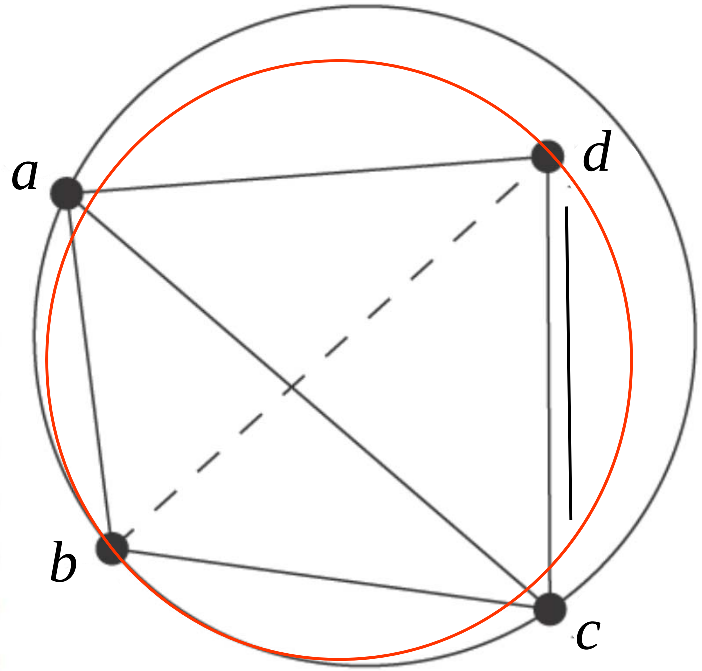

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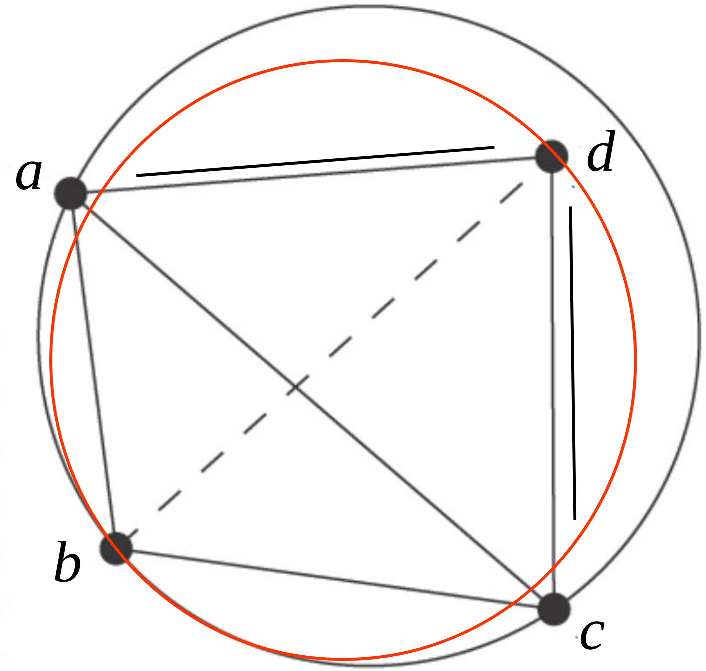


[Berg]

Delaunay triangulation

- Let *T* be a triangulation with *m* triangles (and 3*m angles*)
- Angle-vector

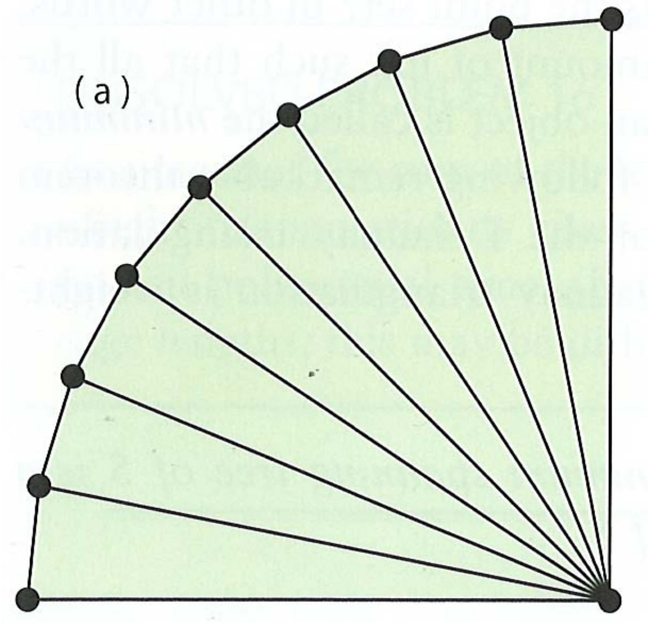
  = non-decreasing ordered sequence $(\alpha_1, \alpha_2, \dots, \alpha_{3m})$ inner angles of triangles, $\alpha_i \leq \alpha_j$, for $i < j$
- In the plane, Delaunay triangulation has the lexicographically largest angle sequence
  - It maximizes the minimal angle (the first angle in angle-vector)
  - It maximizes the second minimal angle, …
  - It maximizes all angles
  - It is an angle sequence optimal triangulation

Delaunay triangulation

- It maximizes the minimal angle
  – The smallest angle in the DT is at least as large as the smallest angle in any other triangulation.
- Minimum spanning tree is a subset of DT min. kostra
- However, the Delaunay triangulation
  – does not necessarily minimize the maximum angle.
  – does not necessarily minimize the length of the edges.

DT and minimal weight triangulation

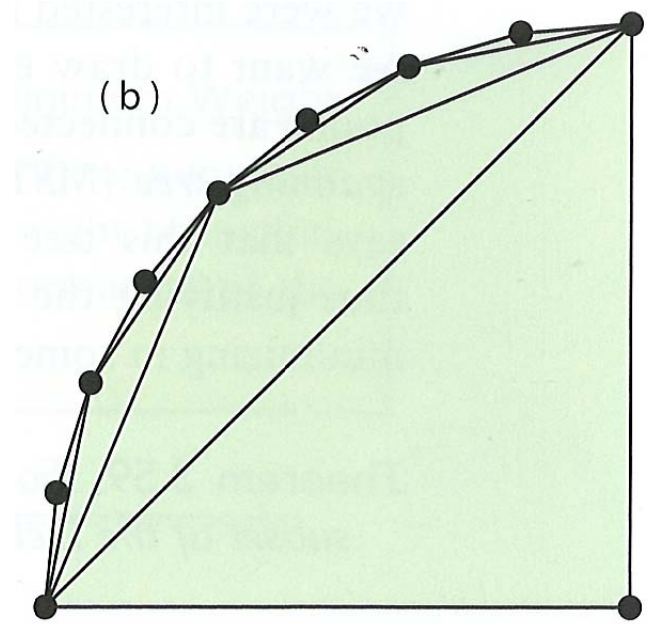
32 points on unit circle + center, $\frac{1}{4}$ shown



Delaunay triangulation

Total weight close to $2\pi + 32$

Connect every 2, 4, 8 + center



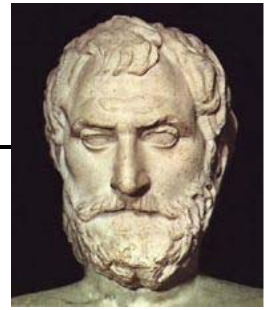
Minimum weight triangulation
(minimum sum of edge lengths)

Total weight far less than $8\pi + 4 > 4x$ around

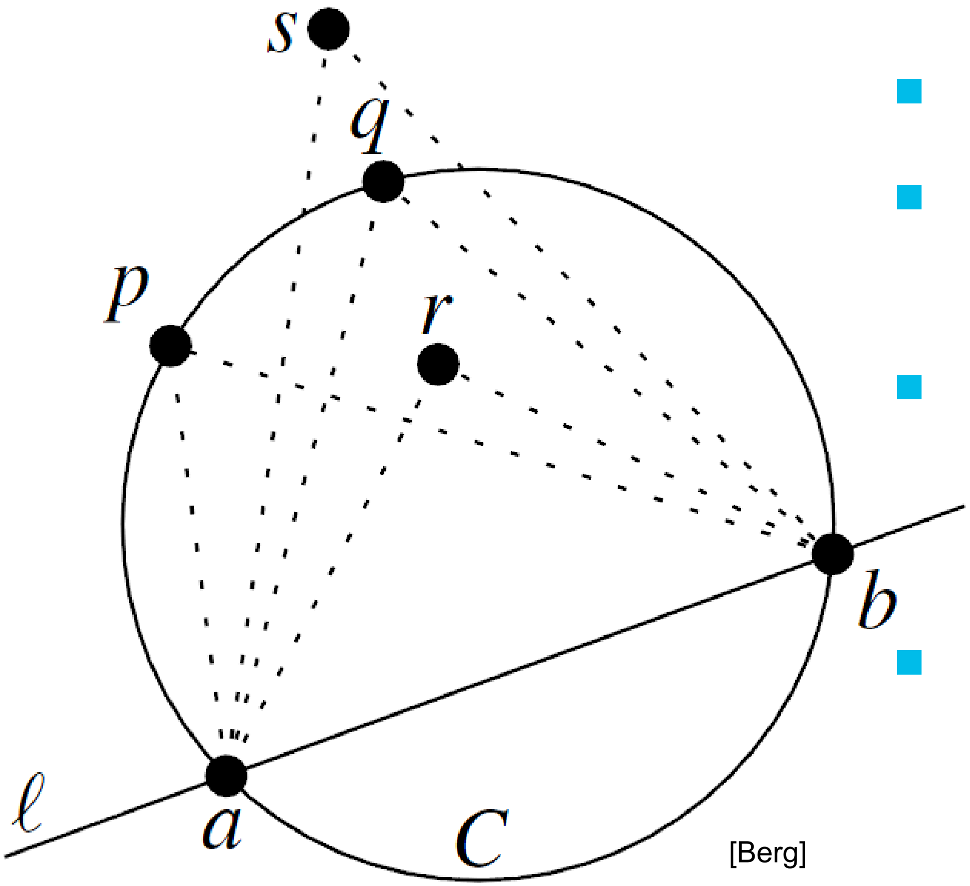
$$2\pi + 32 \approx 38 \quad > \quad 29 \approx 8\pi + 4$$

[Devadoss]

Thales's theorem (624-546 BC)



Respective Central Angle Theorem



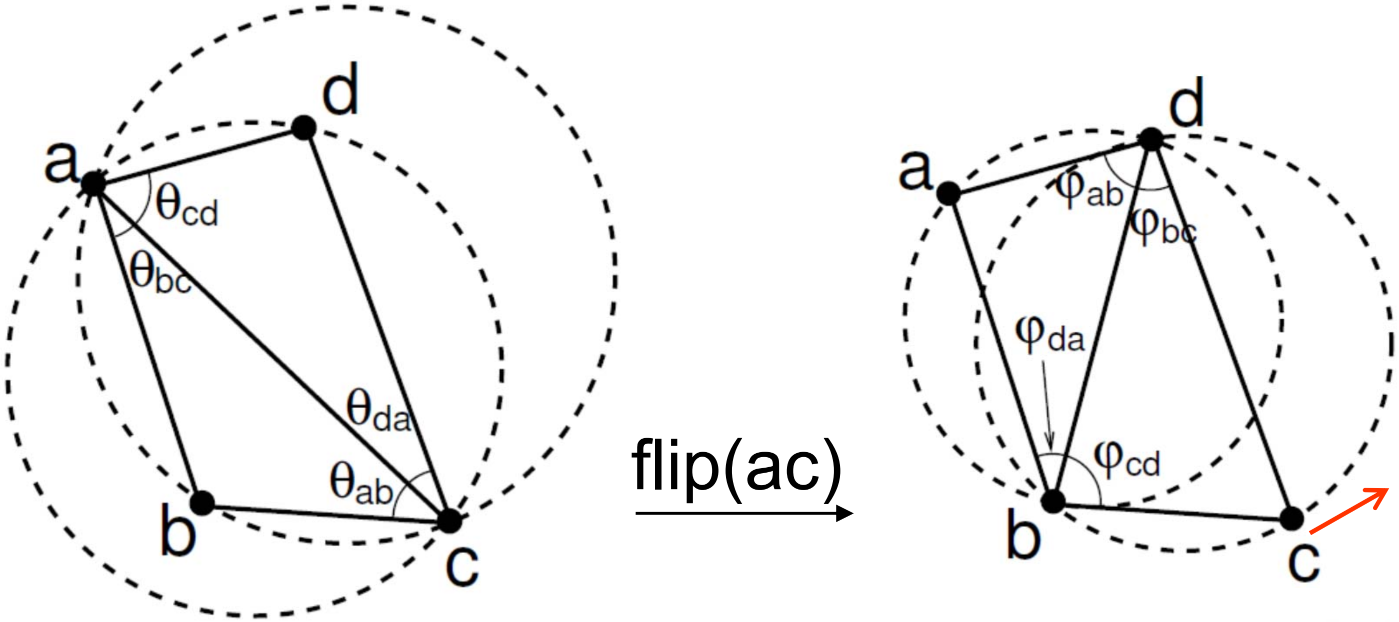
- Let C = circle,
- l =line intersecting C in points a, b
- p, q, r, s = points on the same side of l
  p,q on C , r is *in*, s is out
- Then for the angles holds:
  $\measuredangle arb > \measuredangle apb = \measuredangle aqb > \measuredangle asb$

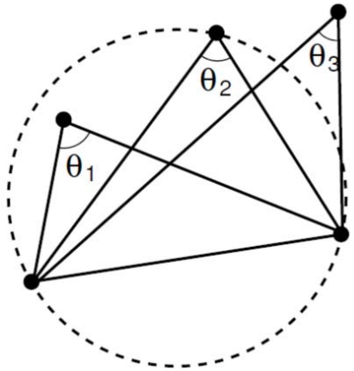
http://www.mathopenref.com/arccentralangletheorem.html

Edge flip of illegal edge and angle vector

- The minimum angle increases after the edge flip

of illegal edge ac > bd





$\theta_1 > \theta_2 > \theta_3$

[Mount]

$|bd| < |ac|$ $\varphi_{ab} > \theta_{ab}$ $\varphi_{bc} > \theta_{bc}$ $\varphi_{cd} > \theta_{cd}$ $\varphi_{da} > \theta_{da}$

=> After limited number of edge flips

- Terminate with lexicographically maximum triangulation
- It satisfies the empty circle condition => Delauney T.

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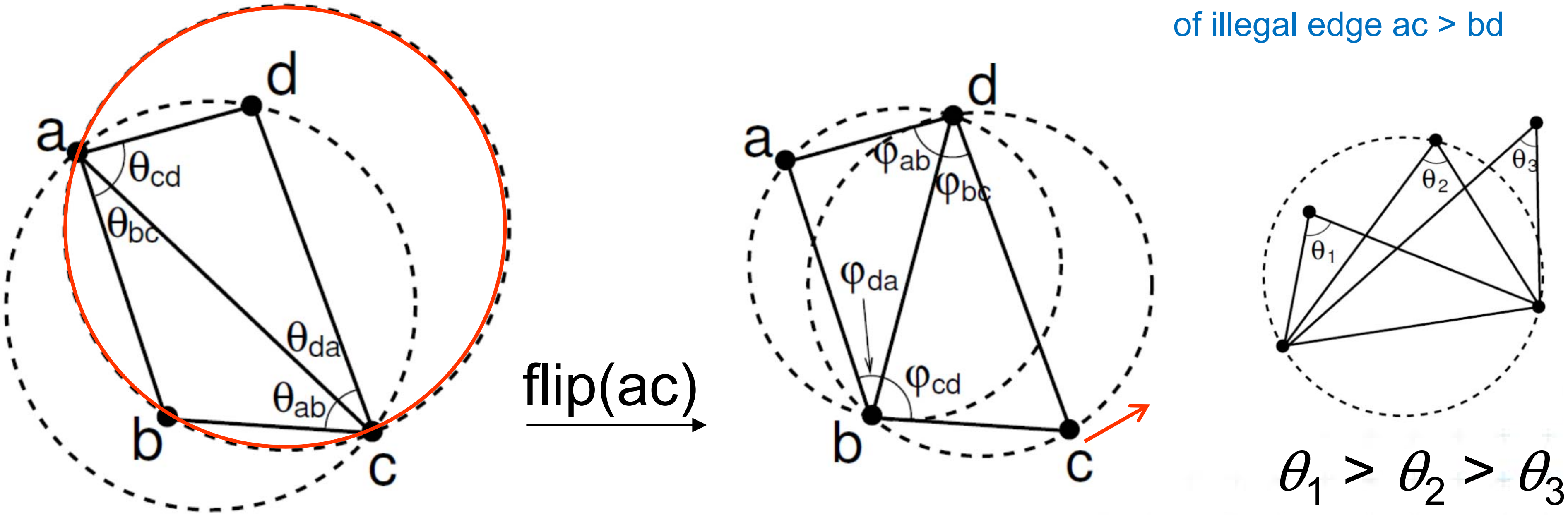

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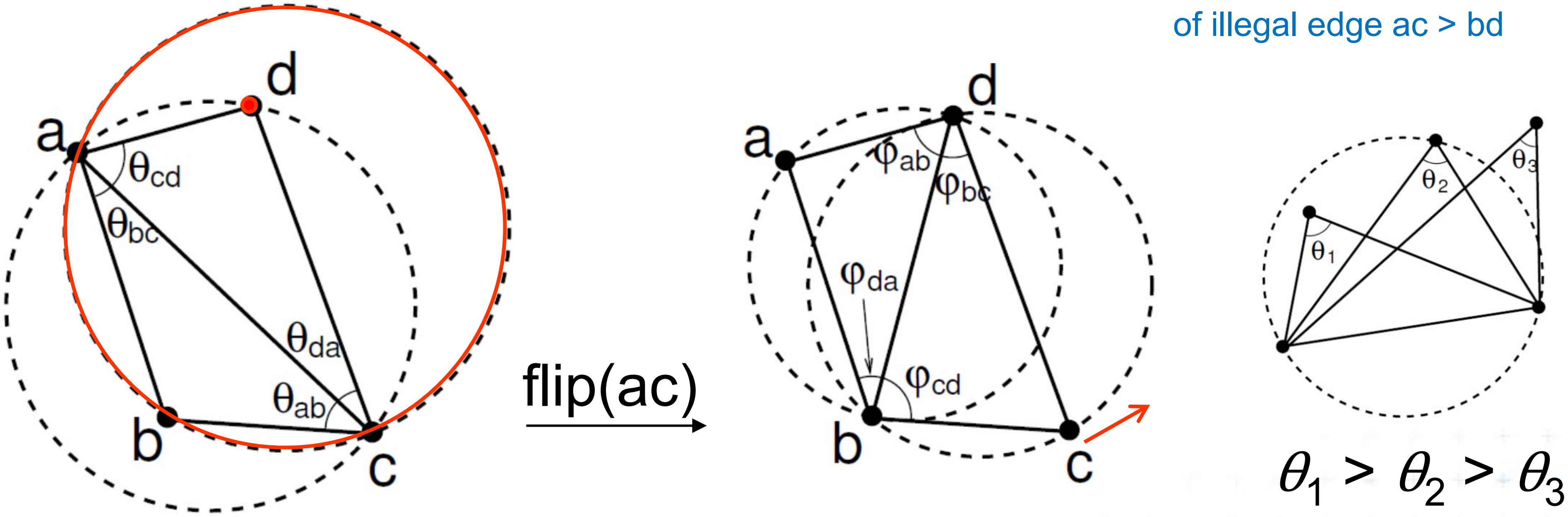

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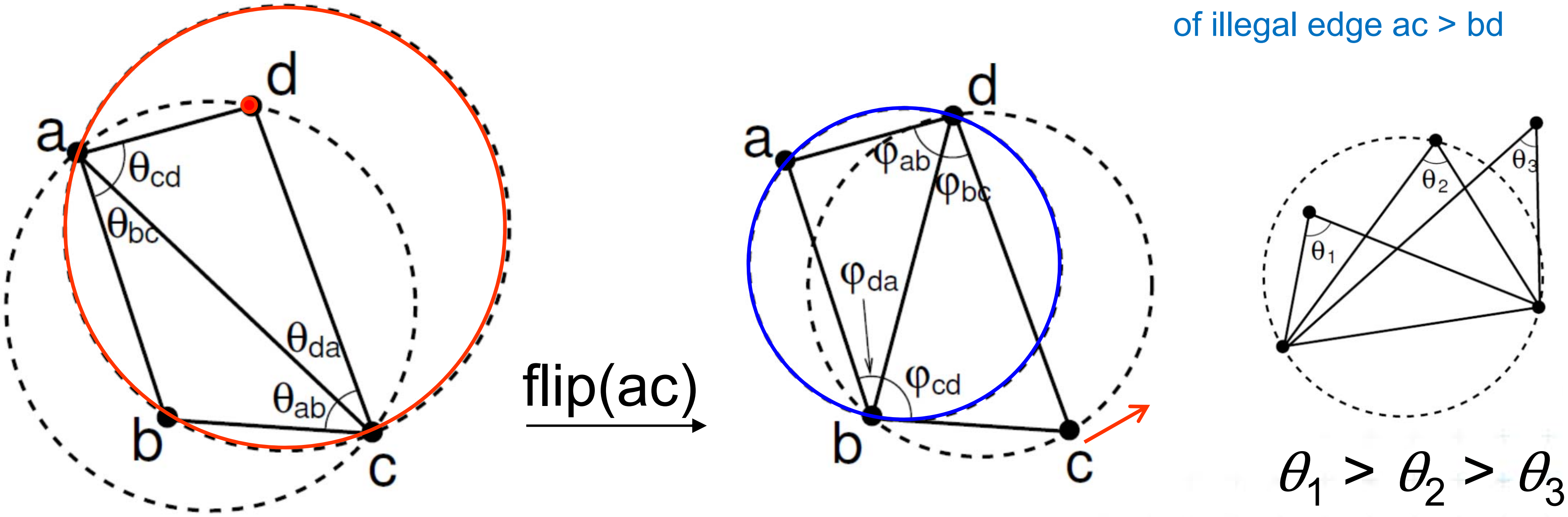

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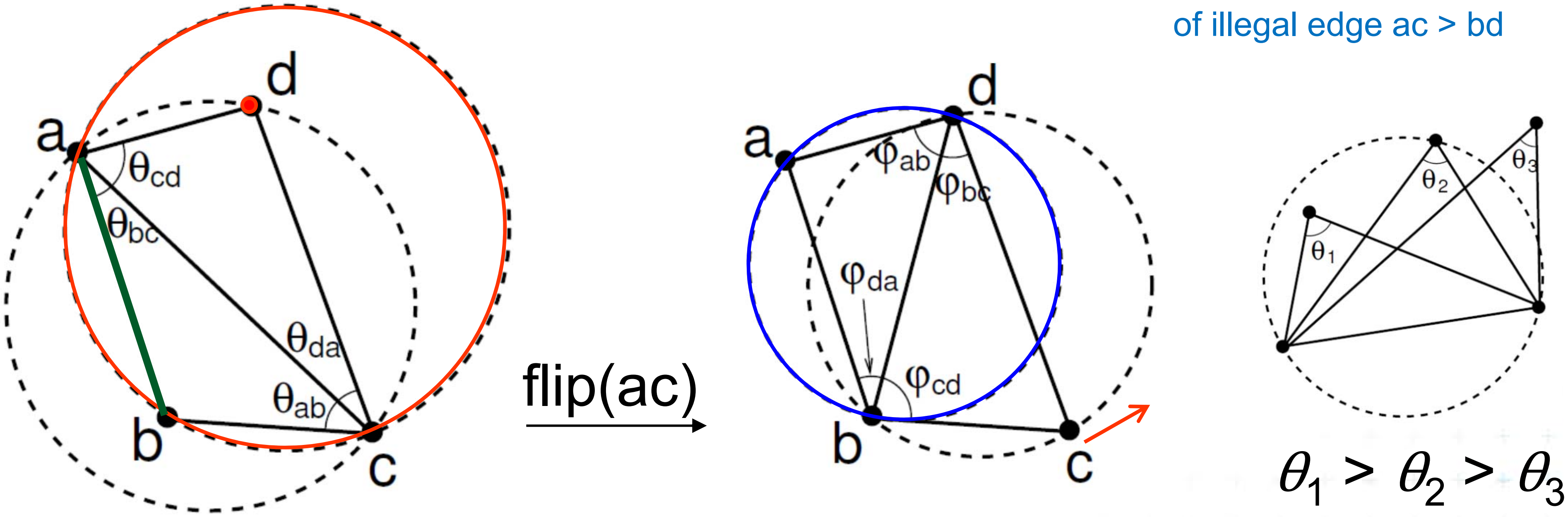

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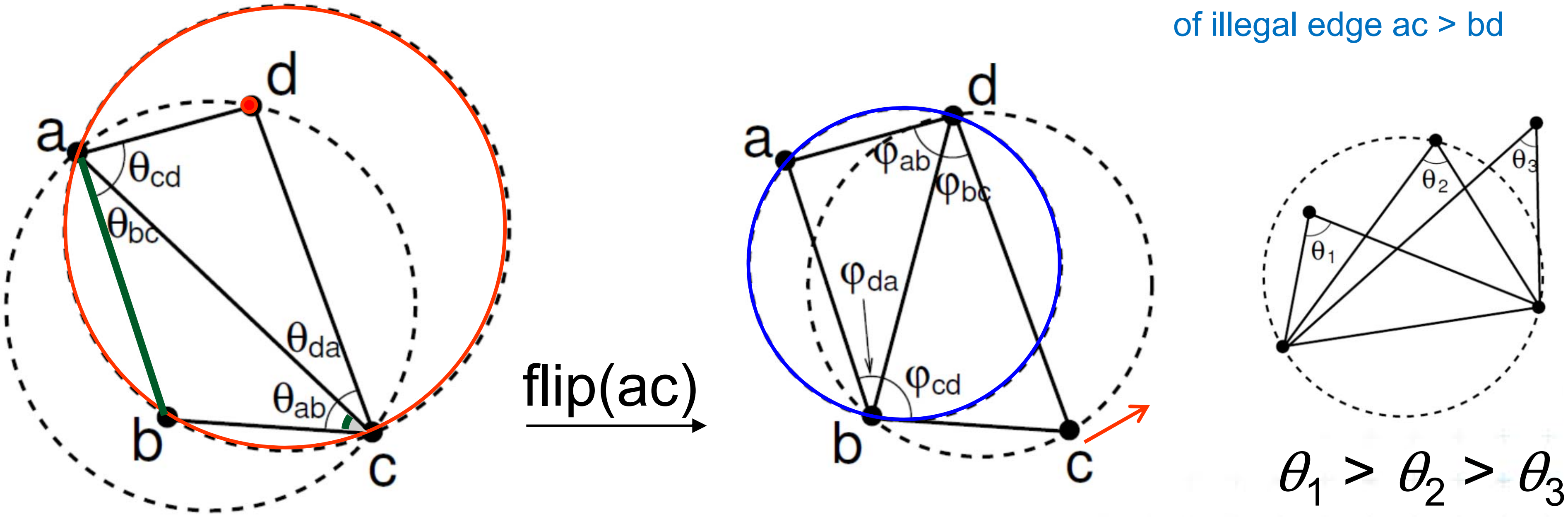

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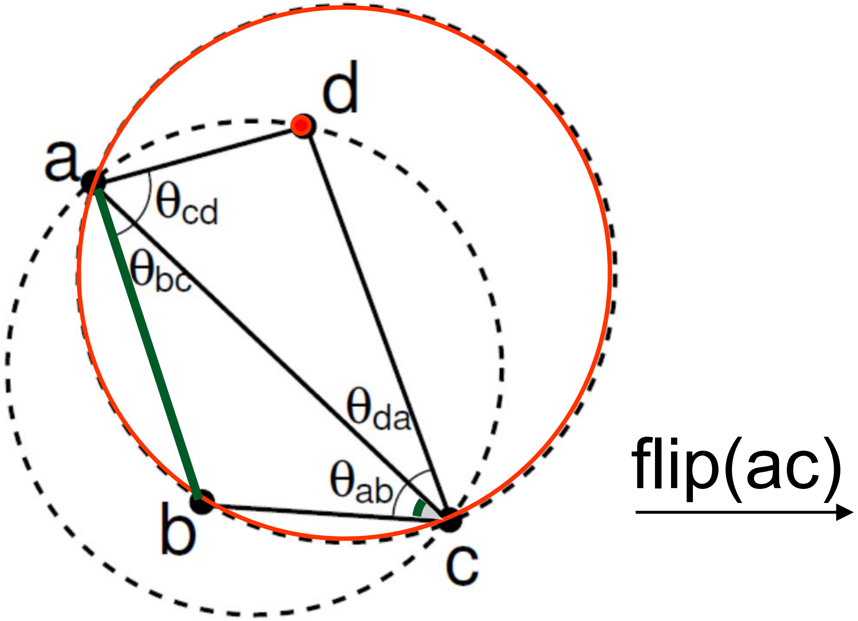

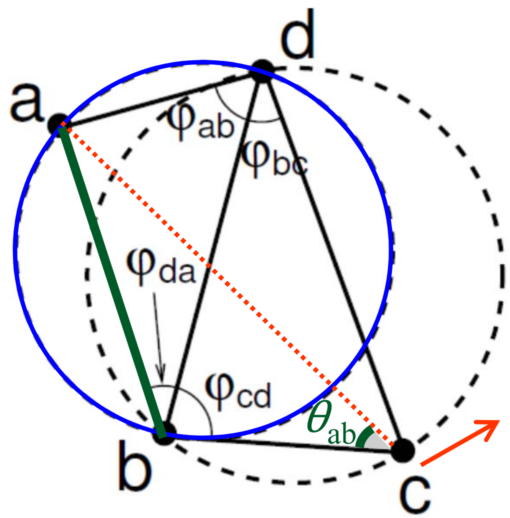

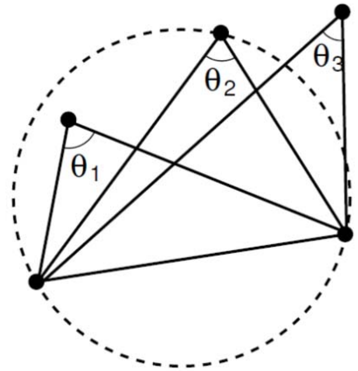

$\theta_1 > \theta_2 > \theta_3$

[Mount]

$|bd| < |ac| \qquad \varphi_{ab} > \theta_{ab} \qquad \varphi_{bc} > \theta_{bc} \qquad \varphi_{cd} > \theta_{cd} \qquad \varphi_{da} > \theta_{da}$

=> After limited number of edge flips

- Terminate with lexicographically maximum triangulation
- It satisfies the empty circle condition => Delauney T.

Edge flip of illegal edge and angle vector

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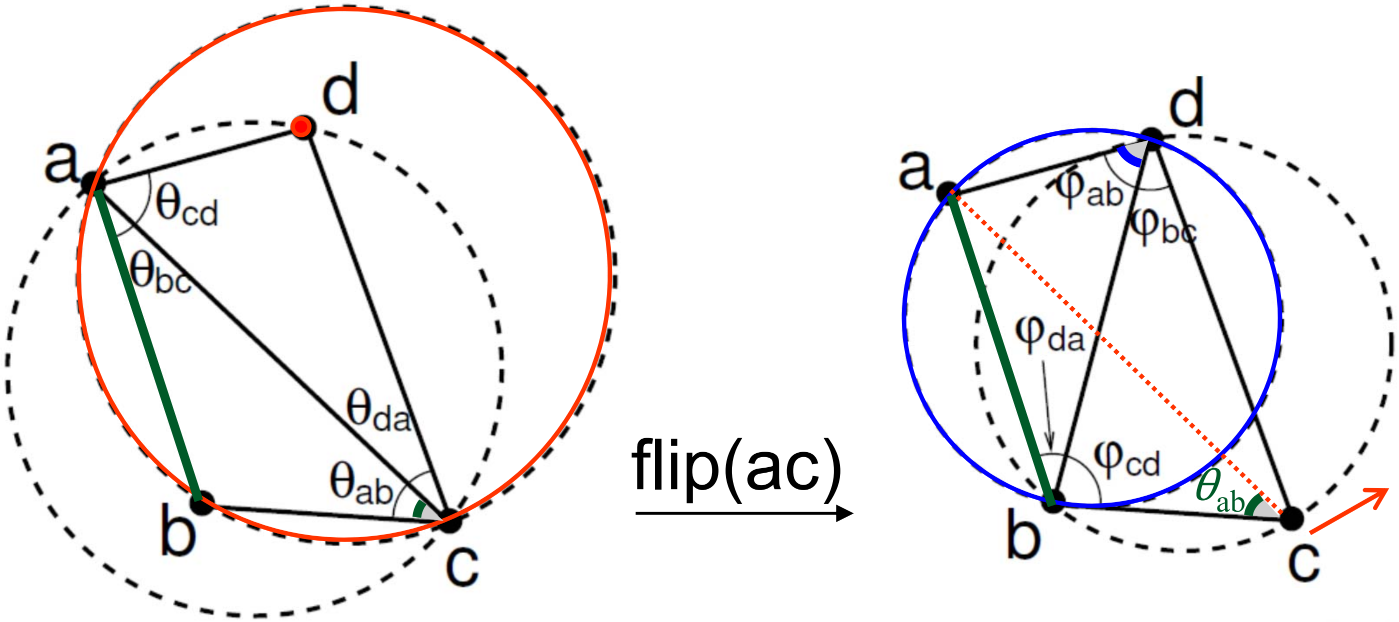

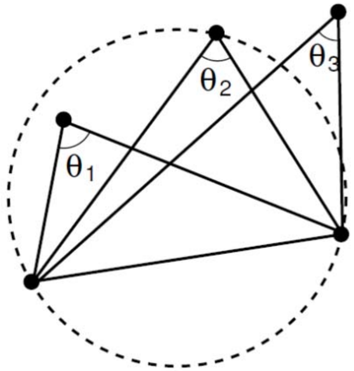

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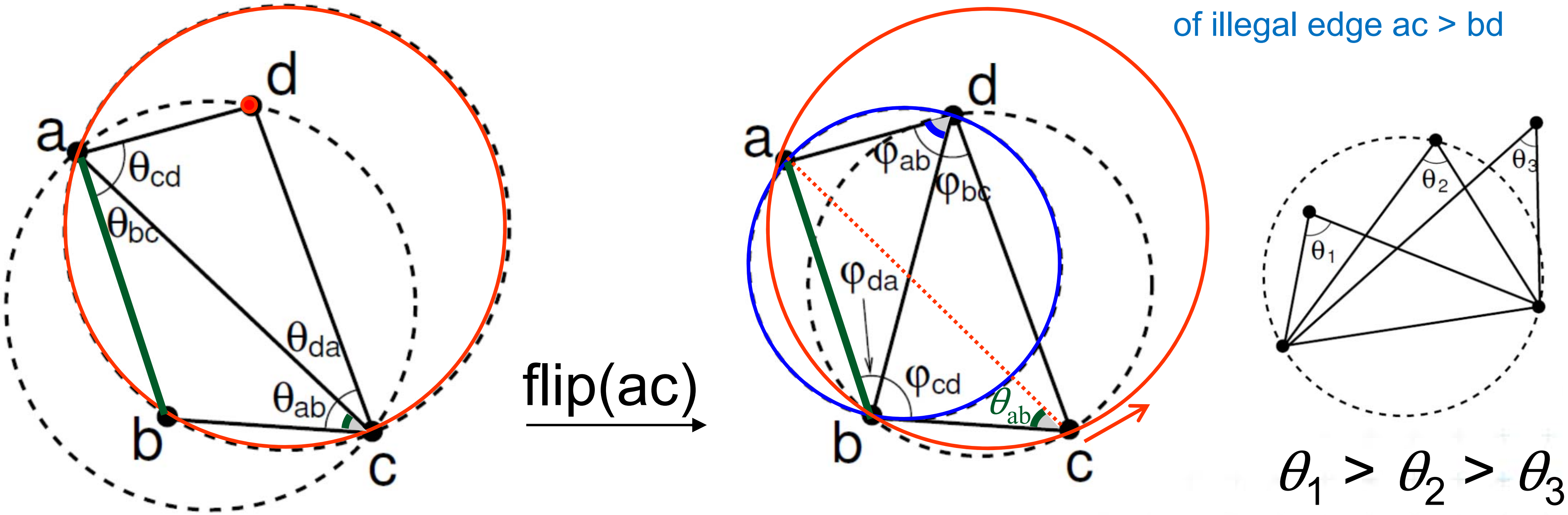

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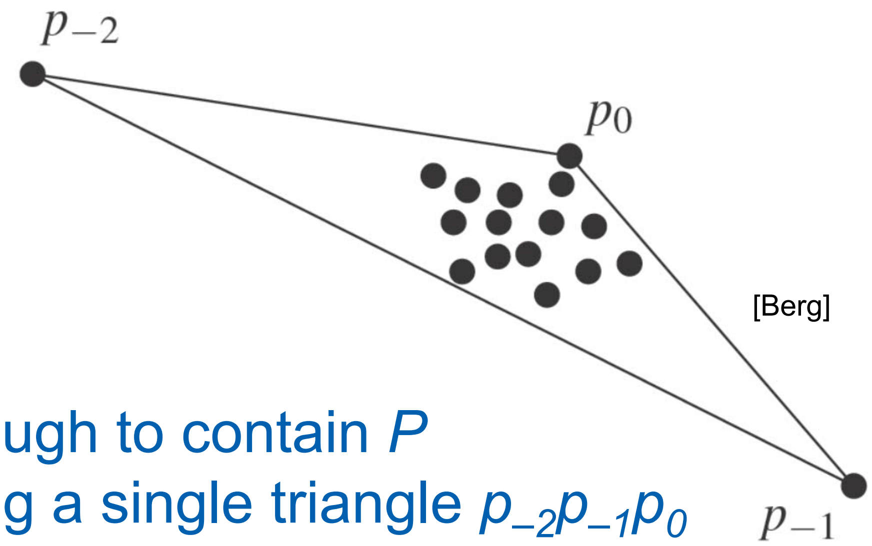
Incremental DT algorithm

Incremental algorithm principle

1. Create a large triangle containing all points (to avoid problems with unbounded cells)
   - must be larger than the largest circle through 3 points
   - will be discarded at the end
2. Insert the points in random order
   - Find triangle with inserted point *p*
   - Add edges to its vertices (these new edges are correct)
   - Check correctness of the old edges (triangles) "around *p*" and legalize (flip) potentially illegal edges
3. Discard the large triangle and incident edges

Incremental algorithm in detail



DelaunayTriangulation(*P*)
Input: Set P of n points in the plane
Output: A Delaunay triangulation T of P

1. Let p_{-2}, p_{-1}, p_0 form a triangle large enough to contain P
2. Initialize T as the triangulation consisting a single triangle $p_{-2}p_{-1}p_0$
3. Compute random permutation $p_1, p_2, \ldots, p_n$ of $P \setminus \{p_0\}$
4. **for** $r = 1$ **to** n **do**
5. $\quad T$ = Insert(p_r , T)
6. Discard p_{-1}, p_{-2} with all incident edges from T
7. **return** T

Incremental algorithm – insertion of a point

Insert(*p*, *T*)

Input: Point *p* being inserted into triangulation *T*

Output: Correct Delaunay triangulation after insertion of *p*

1. Find a triangle *abc* $\in$ *T* containing *p*
2. **if** *p* lies in the interior of *abc* **then**
3. Insert edges *pa, pb, pc* into triangulation *T* (splitting *abc* into 3 triangles *pab, pbc, pca*)
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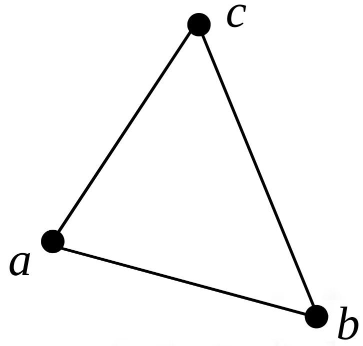

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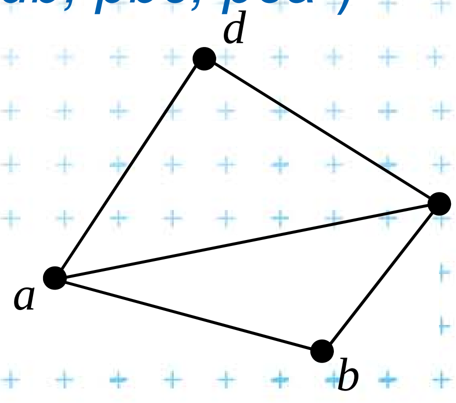

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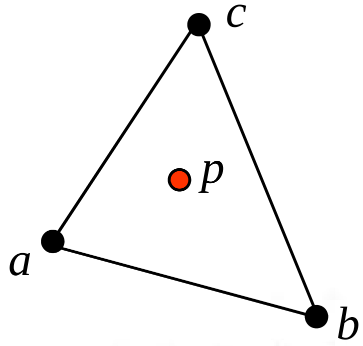

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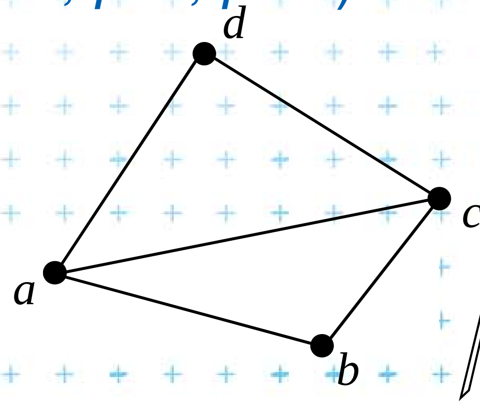

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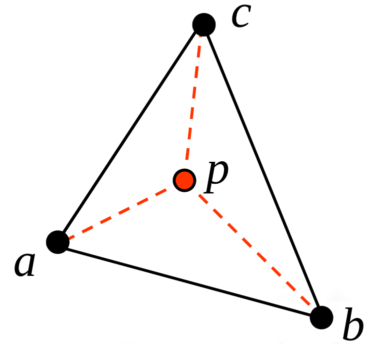

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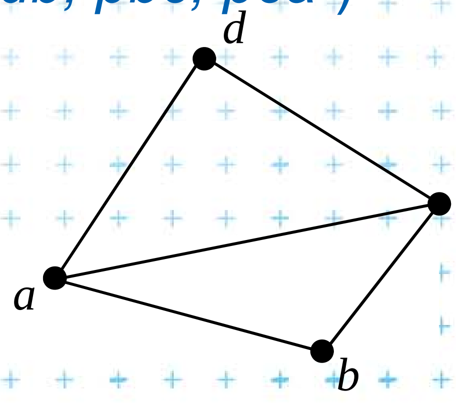

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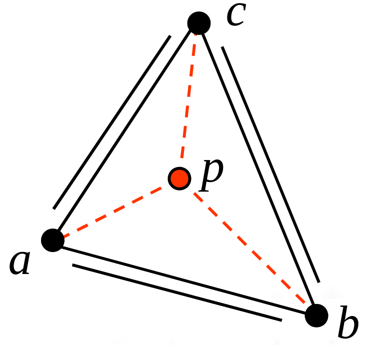

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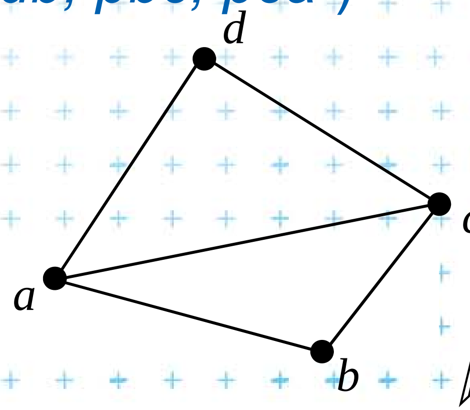

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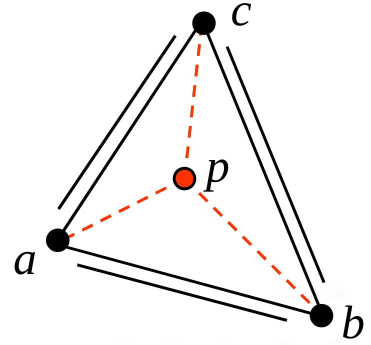

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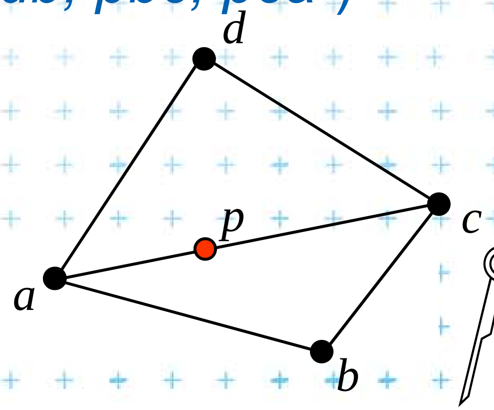

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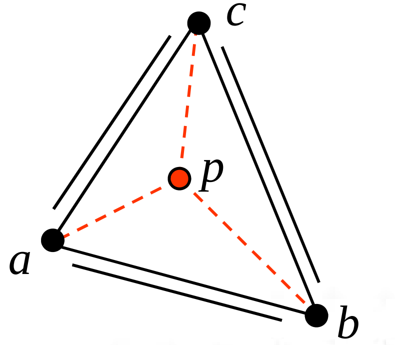

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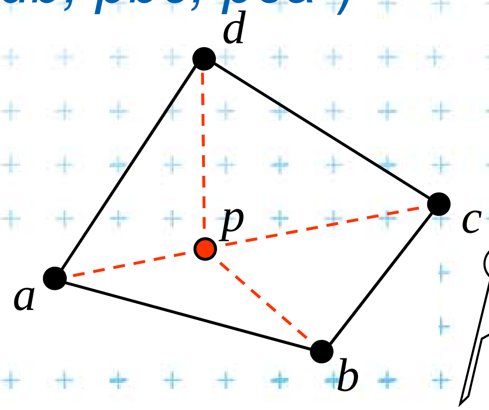

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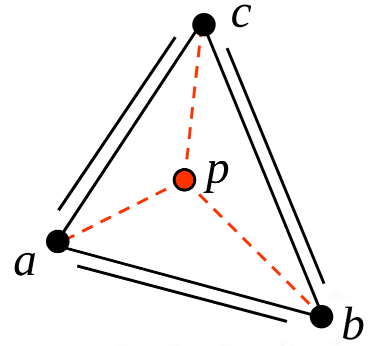

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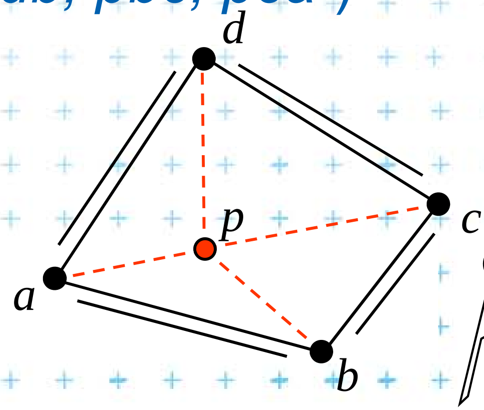

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LegalizeEdge(*p, ab, T*)

Input: Edge *ab* being checked after insertion of point *p* to triangulation *T*

Output: Delaunay triangulation of *p* ∪*T*

1. **if**(*ab* is edge on the exterior face) **return**
2. let *d* be the vertex to the right of edge *ab*
3. if(inCircle(*p, a, b, d*)) // *d* is in the circle around *pab* => *d* is illegal
4. Flip edge *ab* for *pd*
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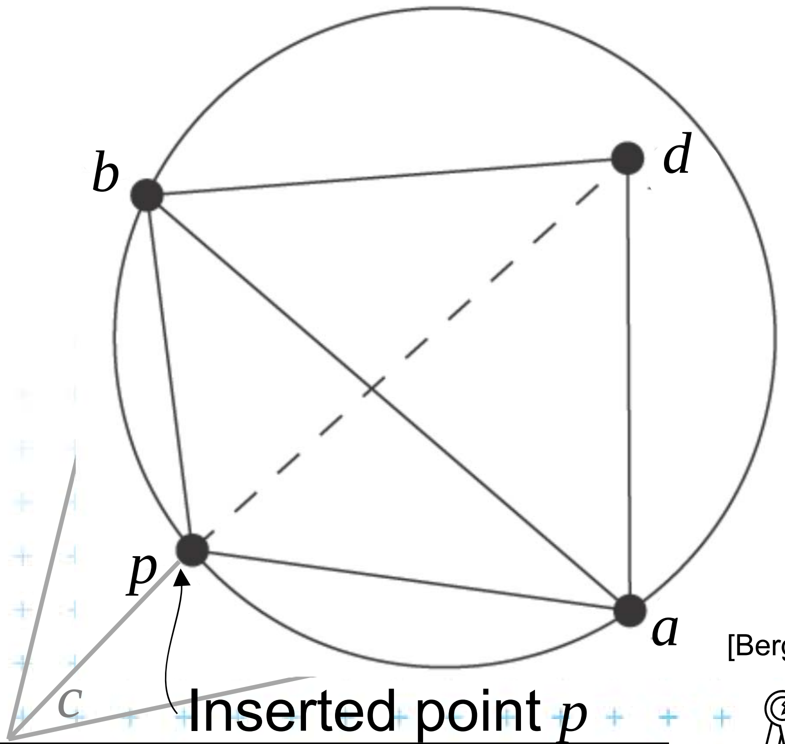

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DCGI

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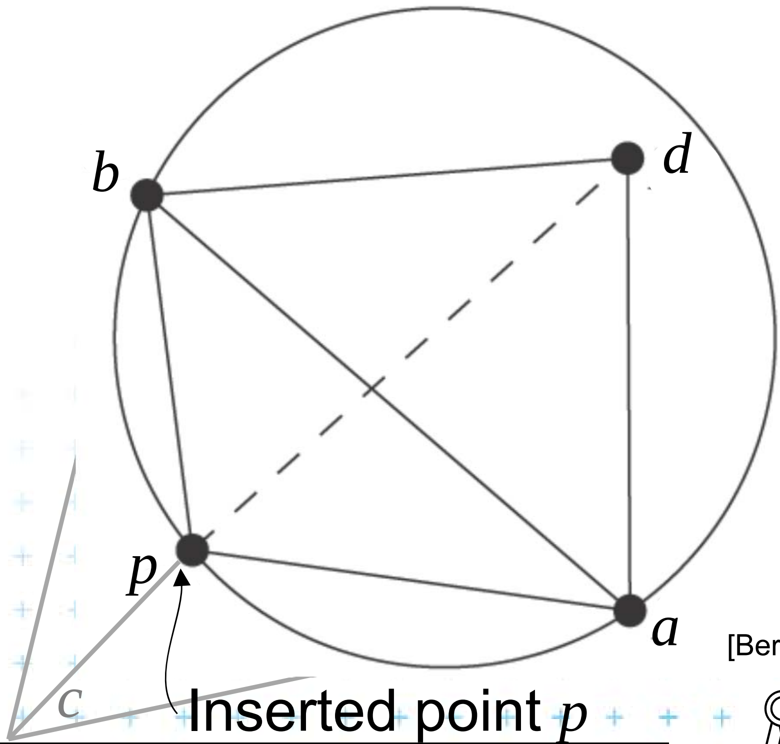

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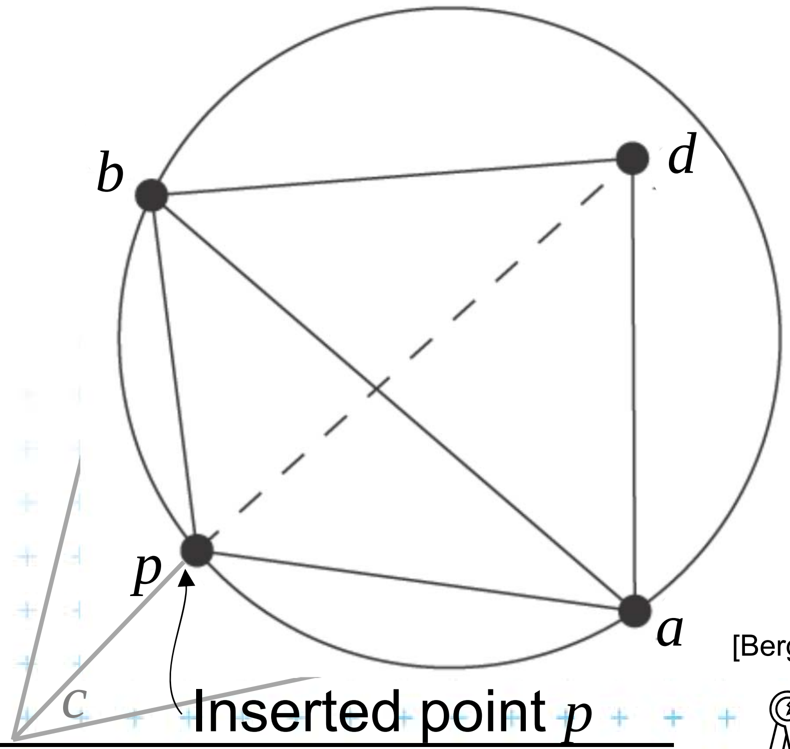

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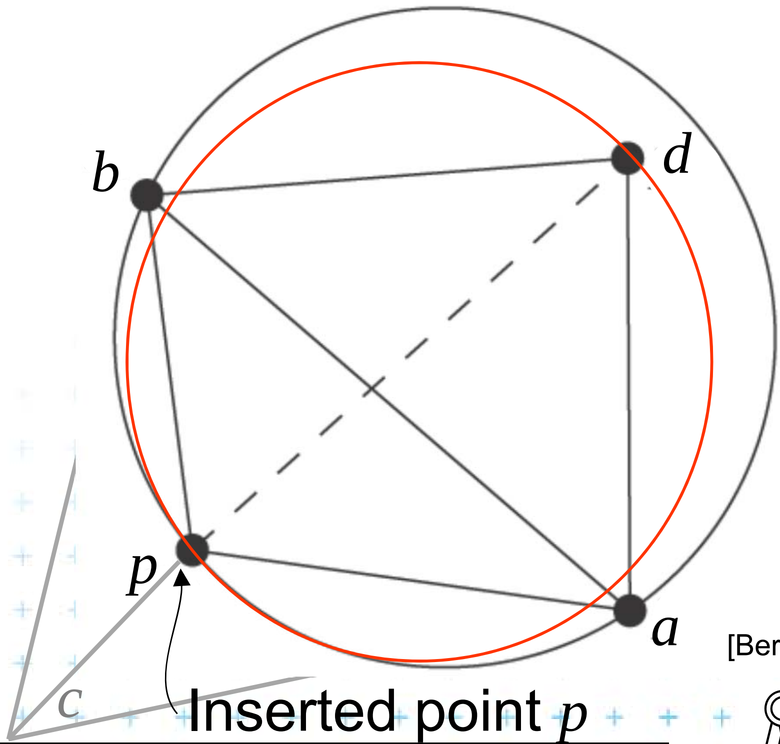

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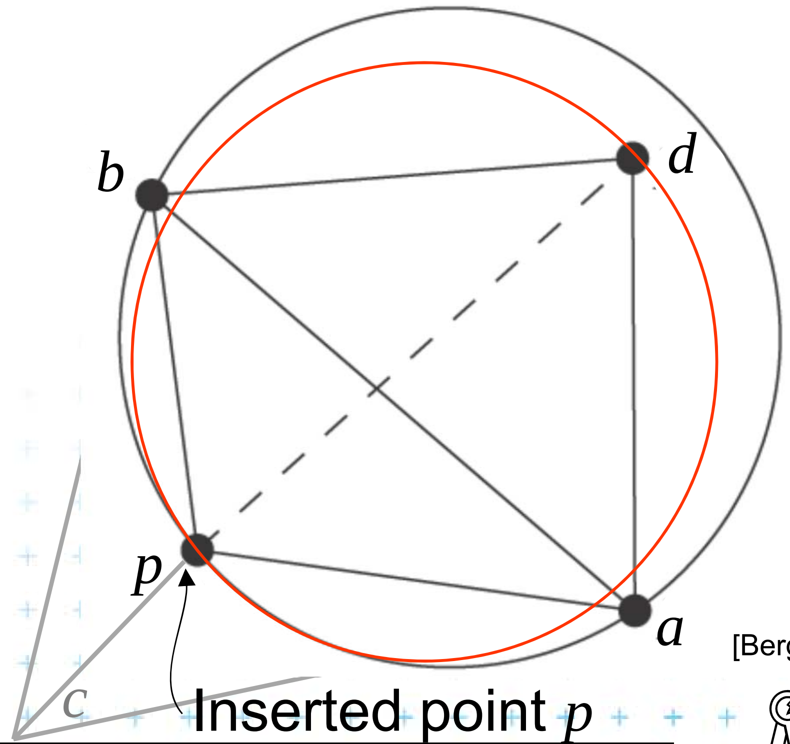

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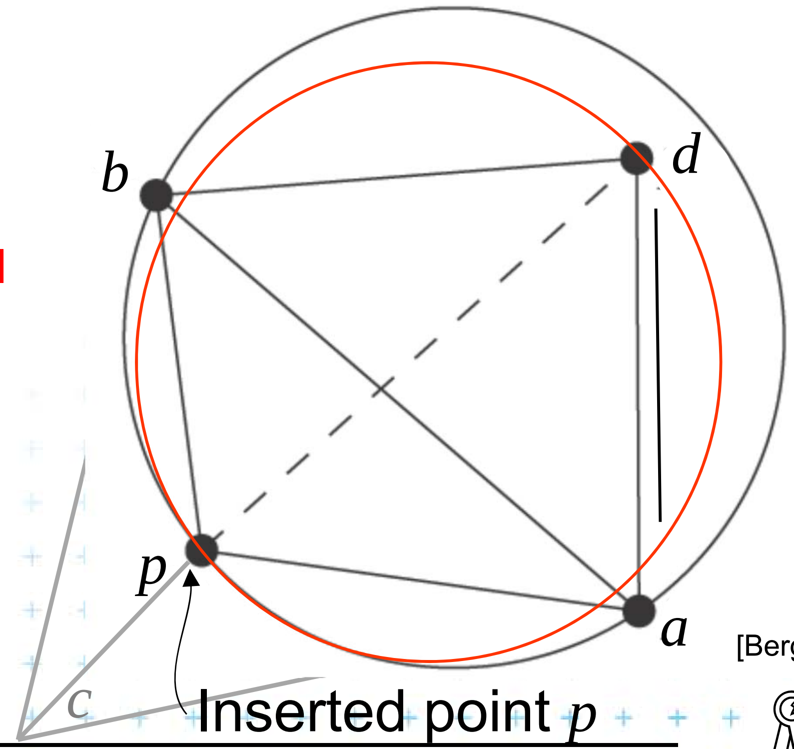

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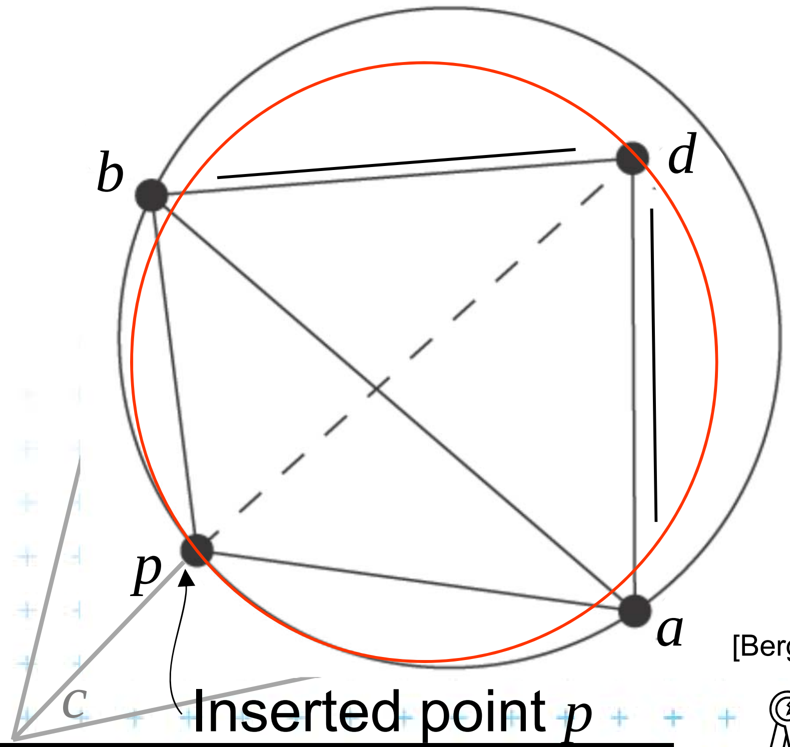

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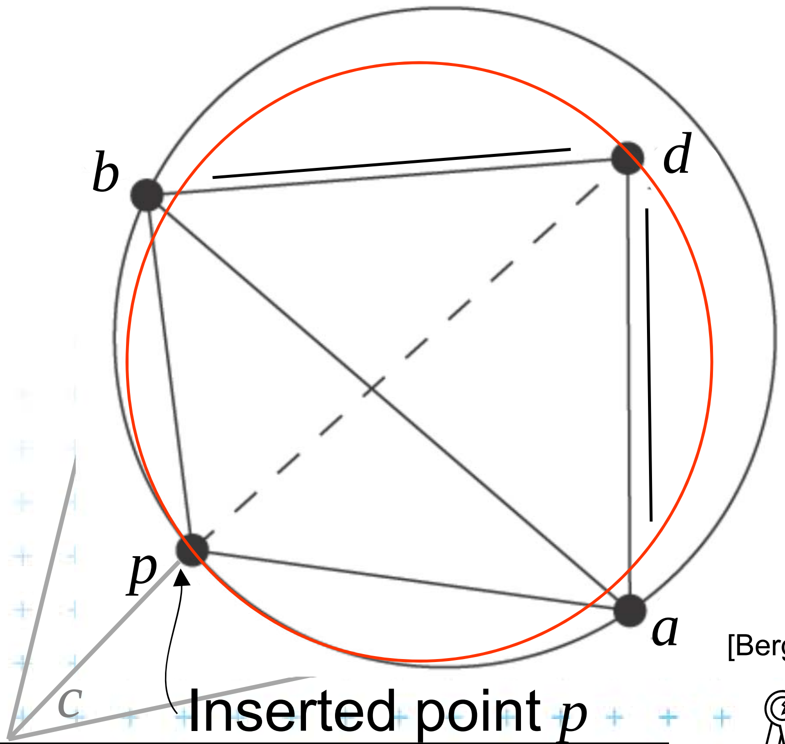
We must check and possibly flip edges *ad, db*

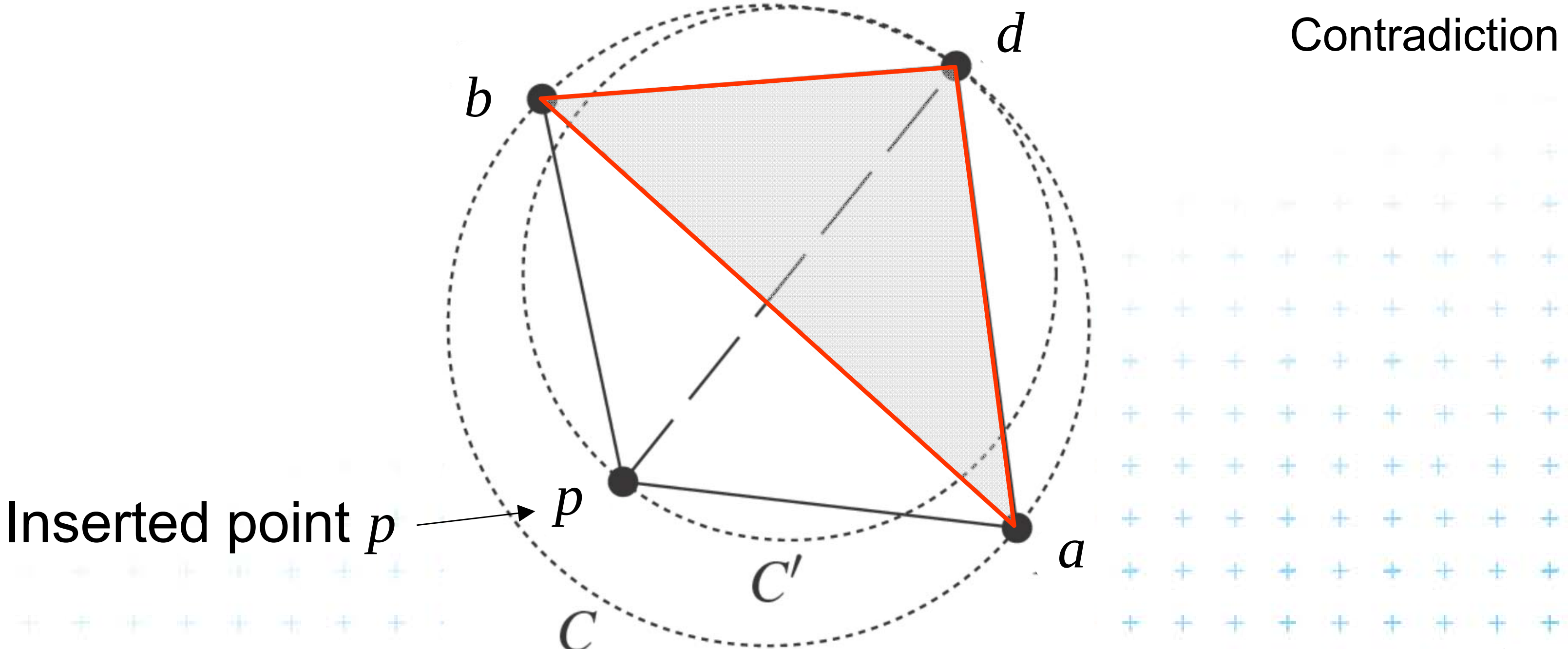
(We must check and possibly flip edges *bc* & *ca* of the triangle *abc* - lines 5,6 in Insert(p, T))



[Berg]

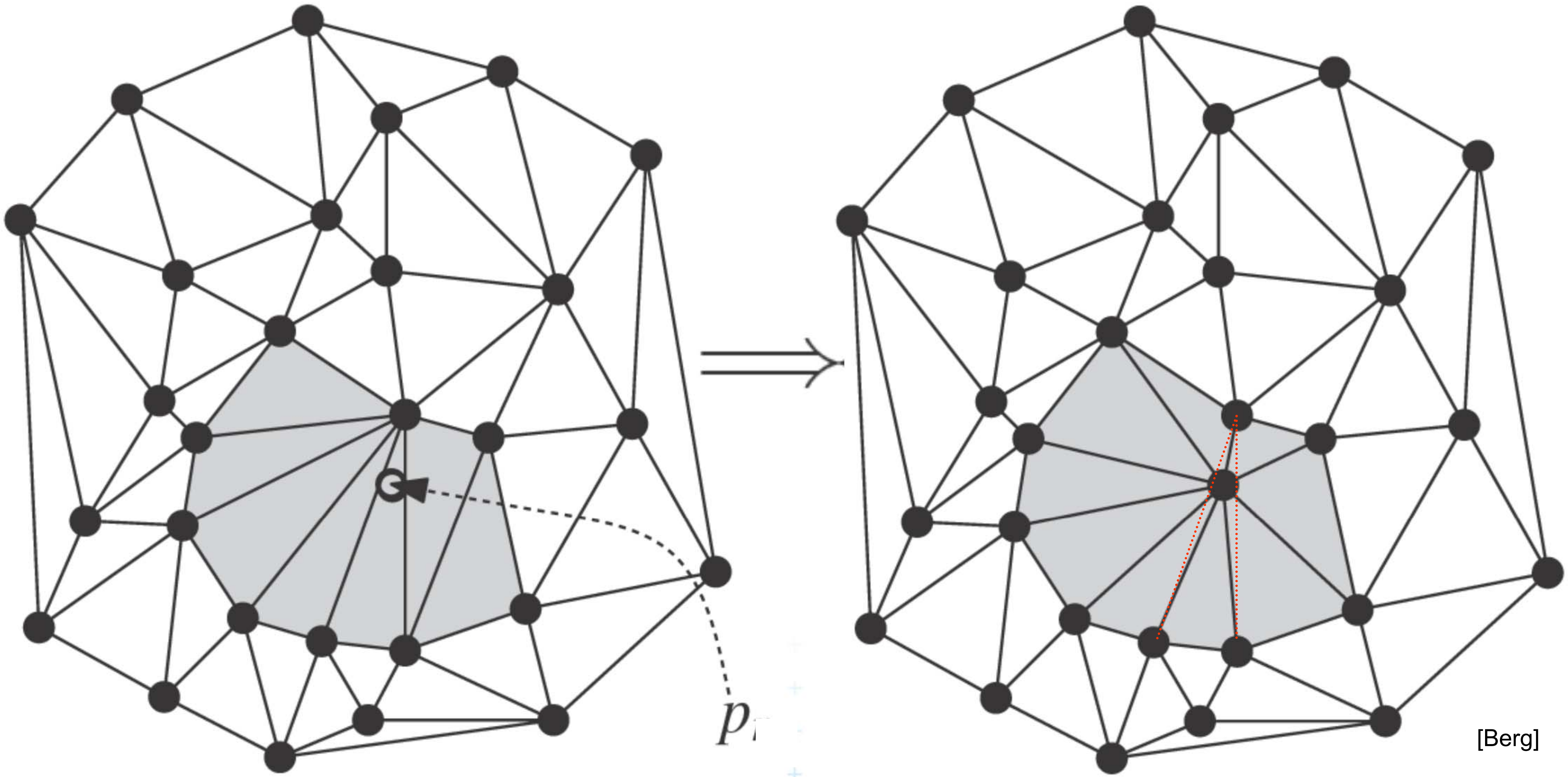
Correctness of edge flip of illegal edge

- Assume point p is in C (it violates DT criteria for adb)
- adb was a triangle of DT => C was an empty circle
- Create circle C' trough point p, C' is inscribed to C, $C' \subset C$
  => C' is also an empty circle ($a, b \notin C$)
  => new edge *pd* is also a Delaunay edge



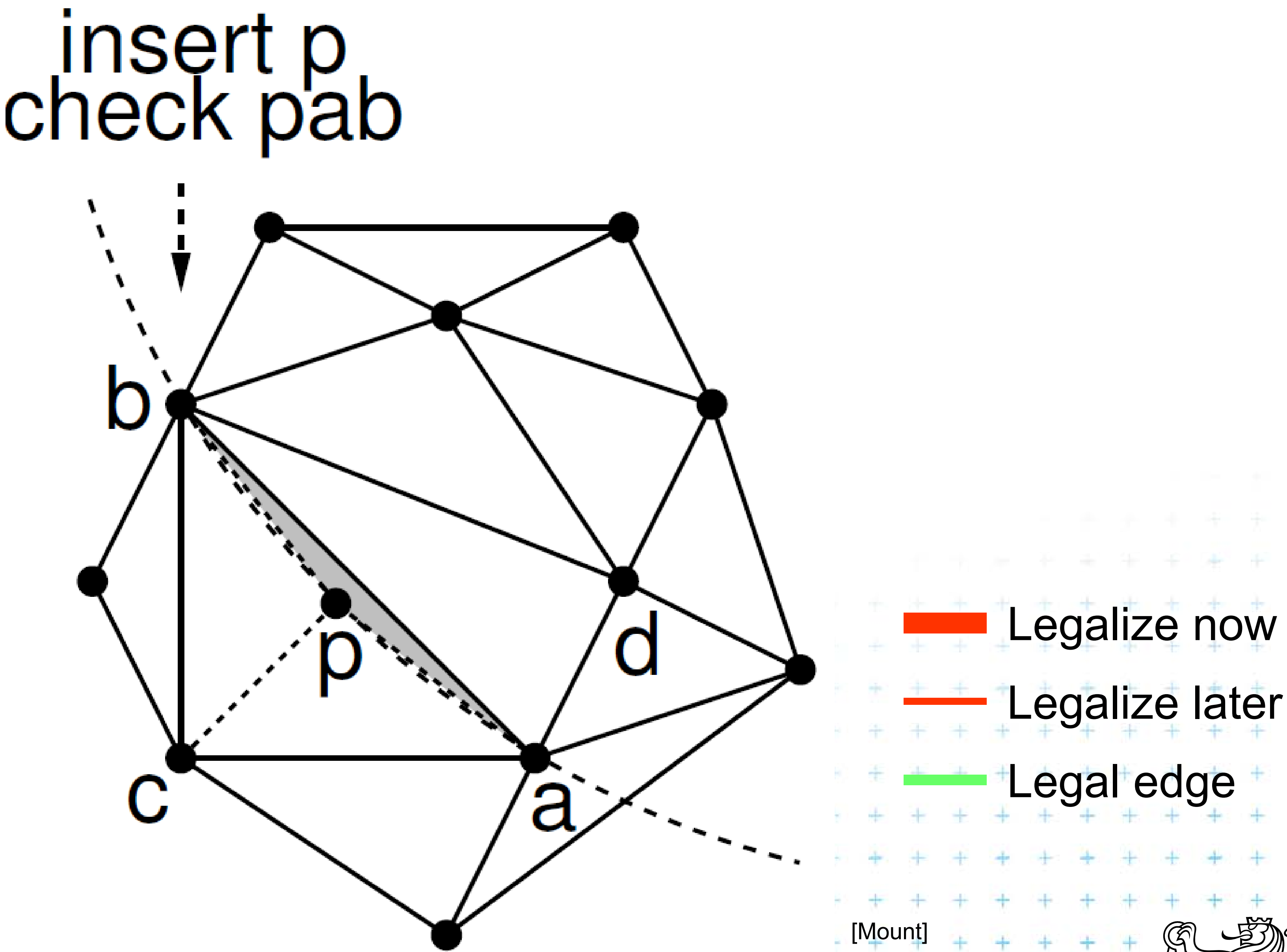
[Berg]

DT- point insert and mesh legalization



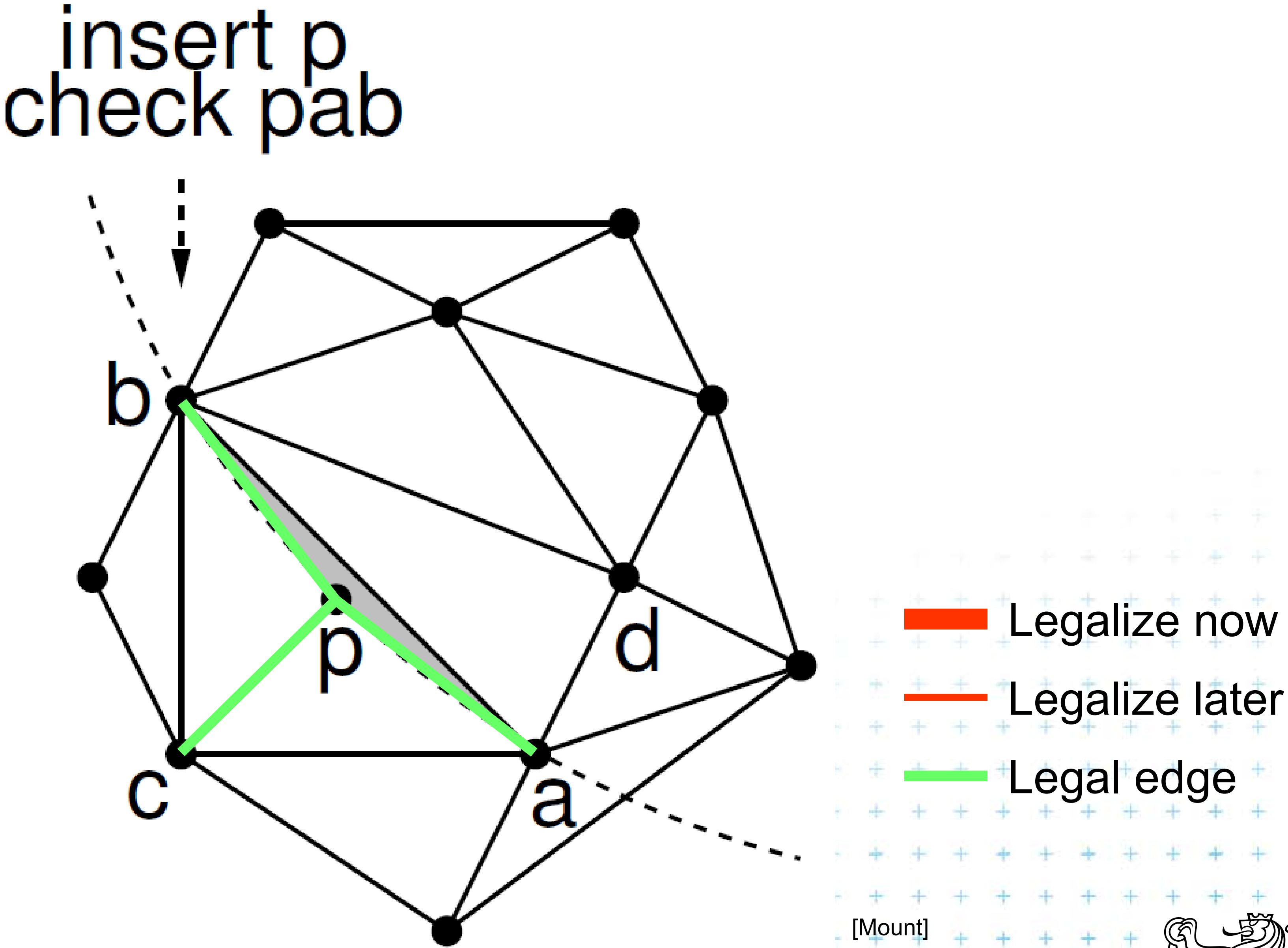
Every new edge created due to insertion of *p* will be incident to *p*

Delaunay triangulation – other point insert



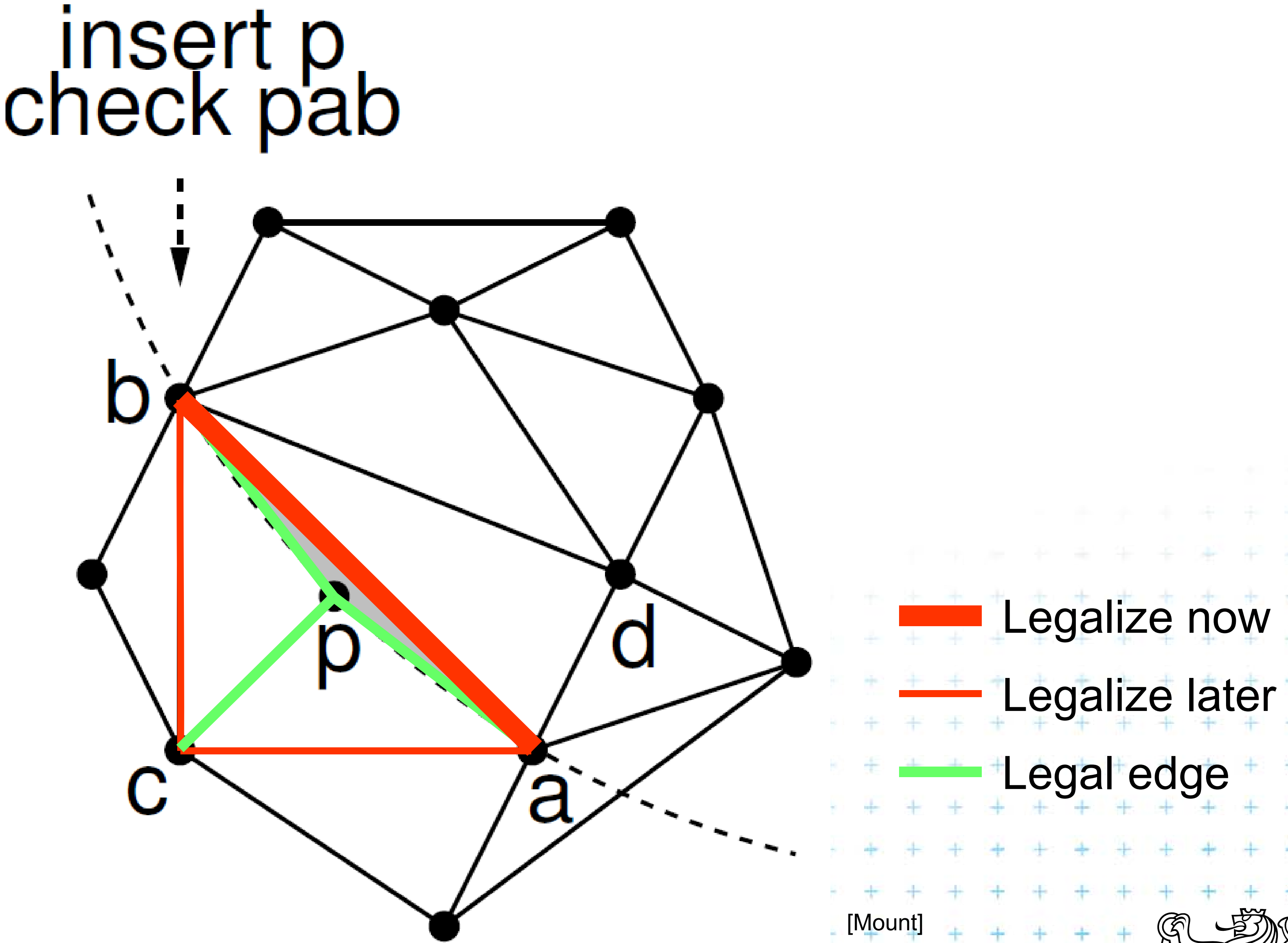
[Mount]

Delaunay triangulation – other point insert



[Mount]

Delaunay triangulation – other point insert



[Mount]

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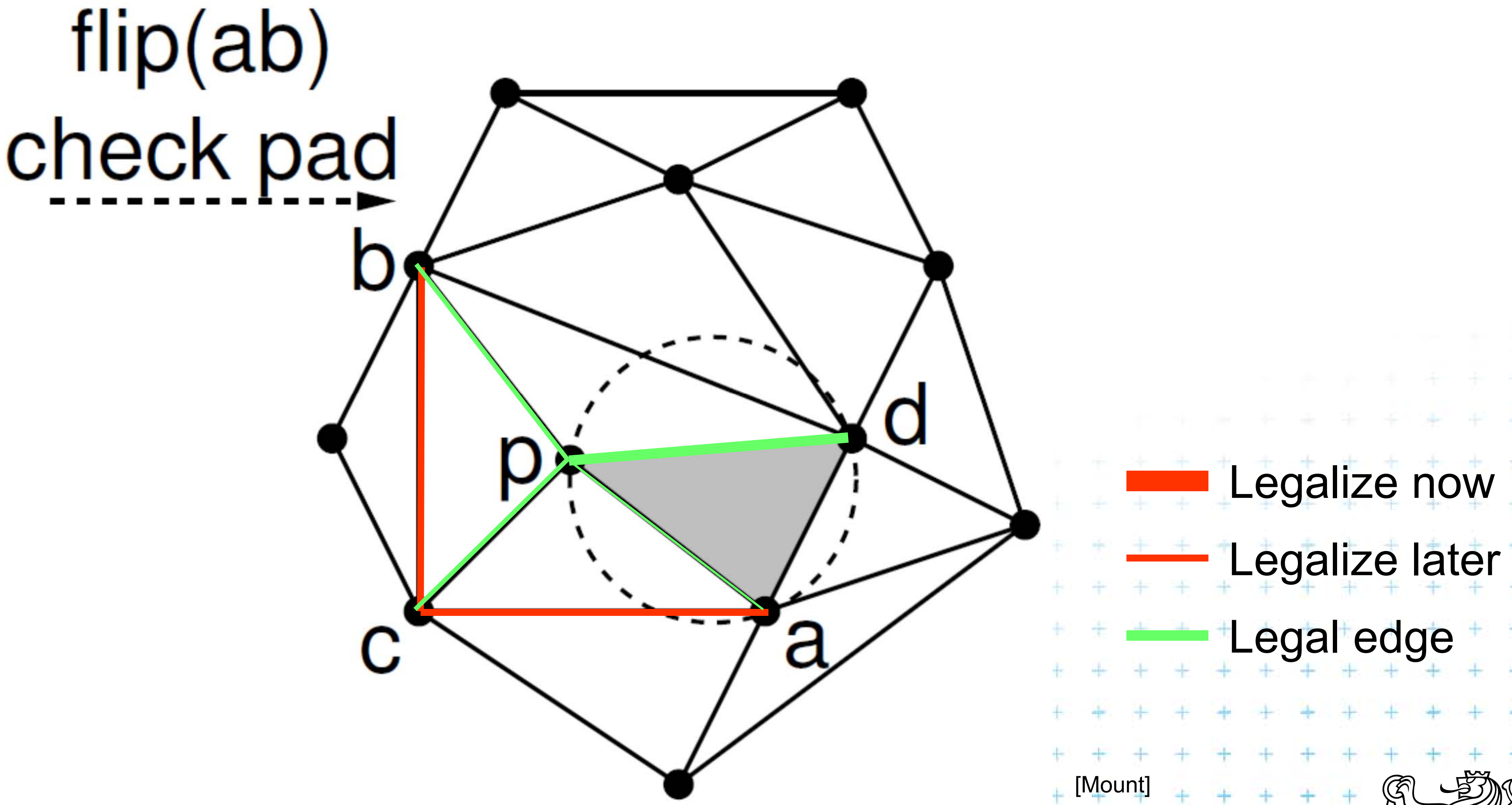

[Mount]

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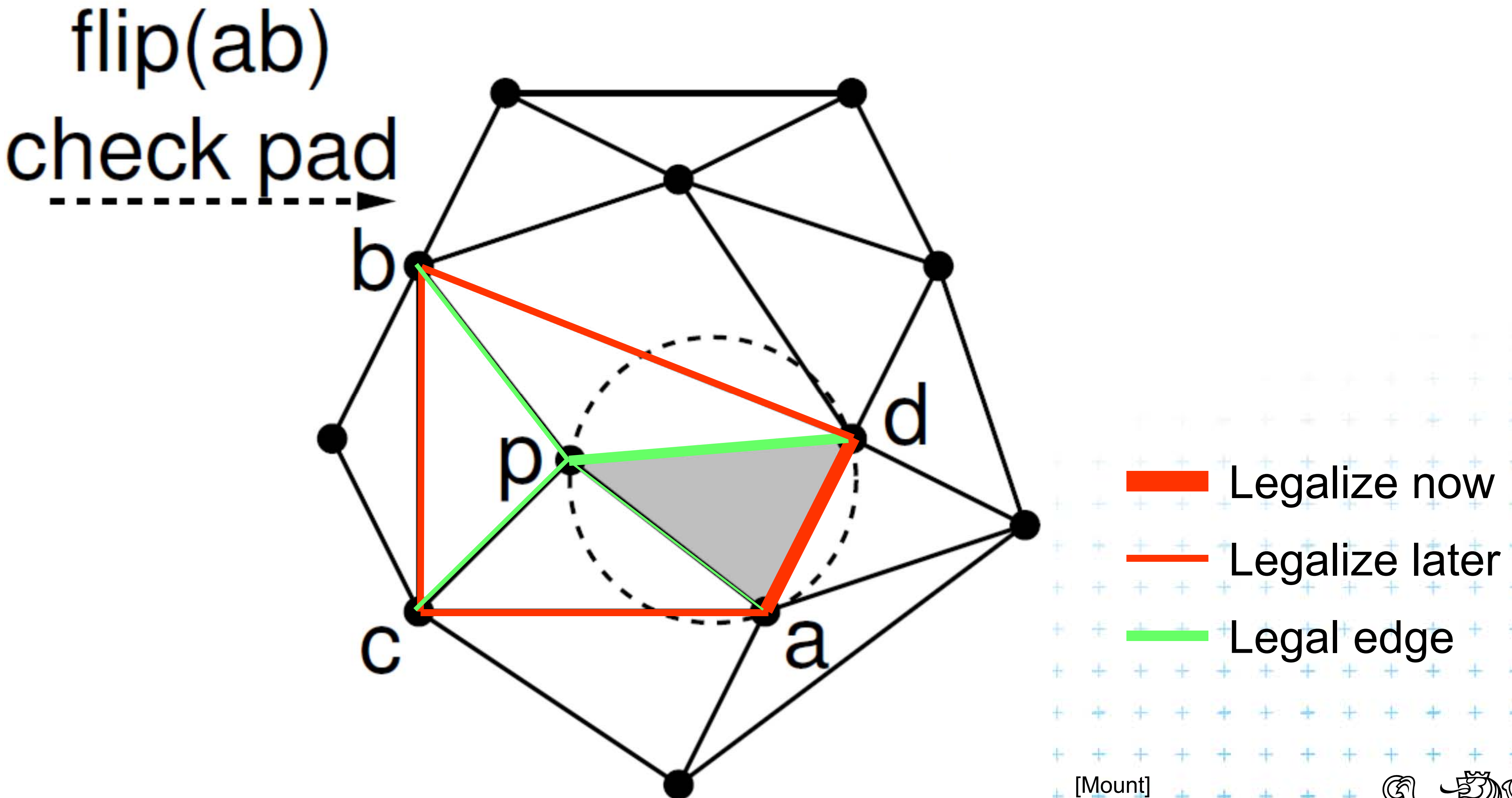
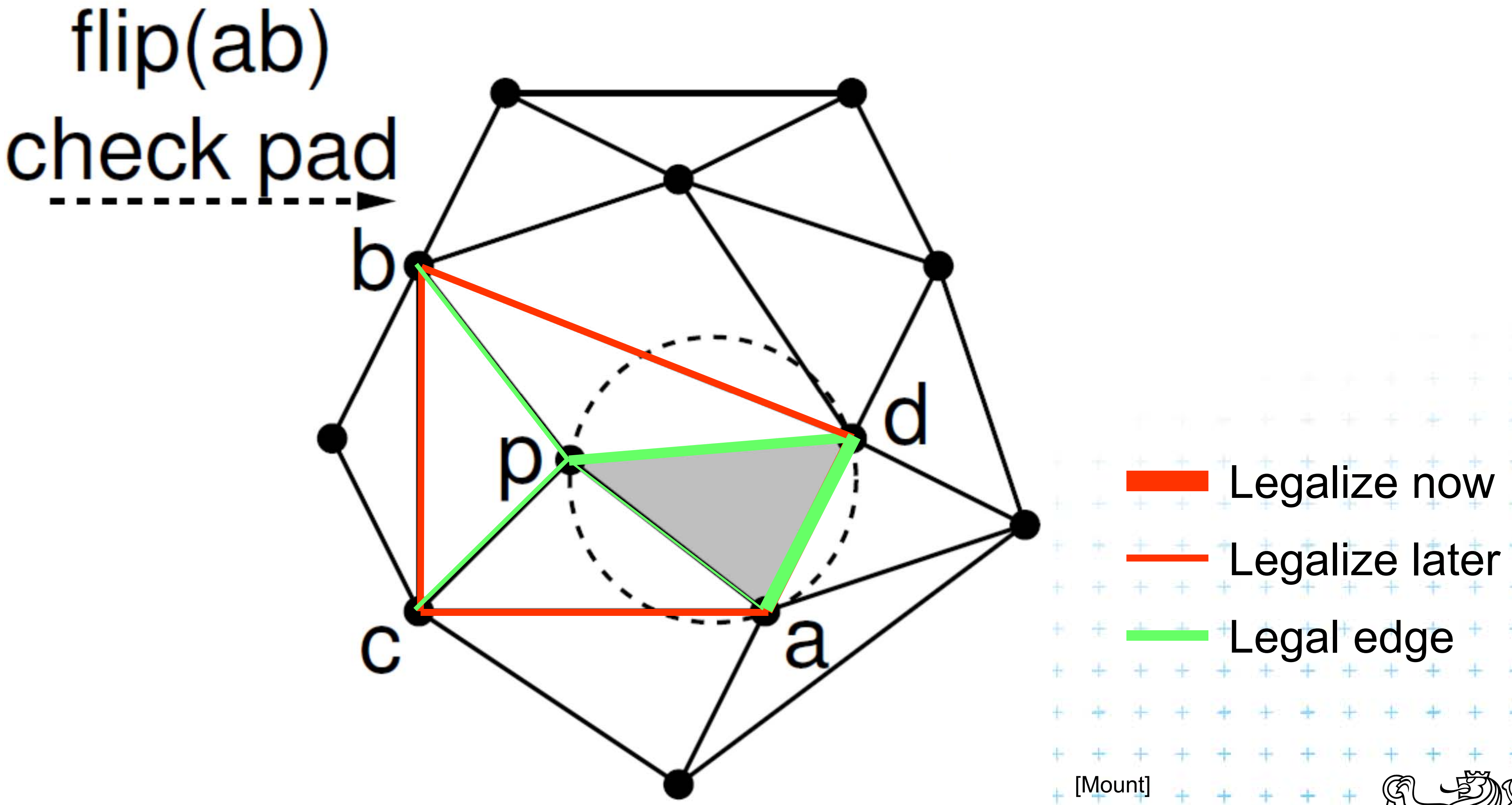


[Mount]

Delaunay triangulation – other point insert



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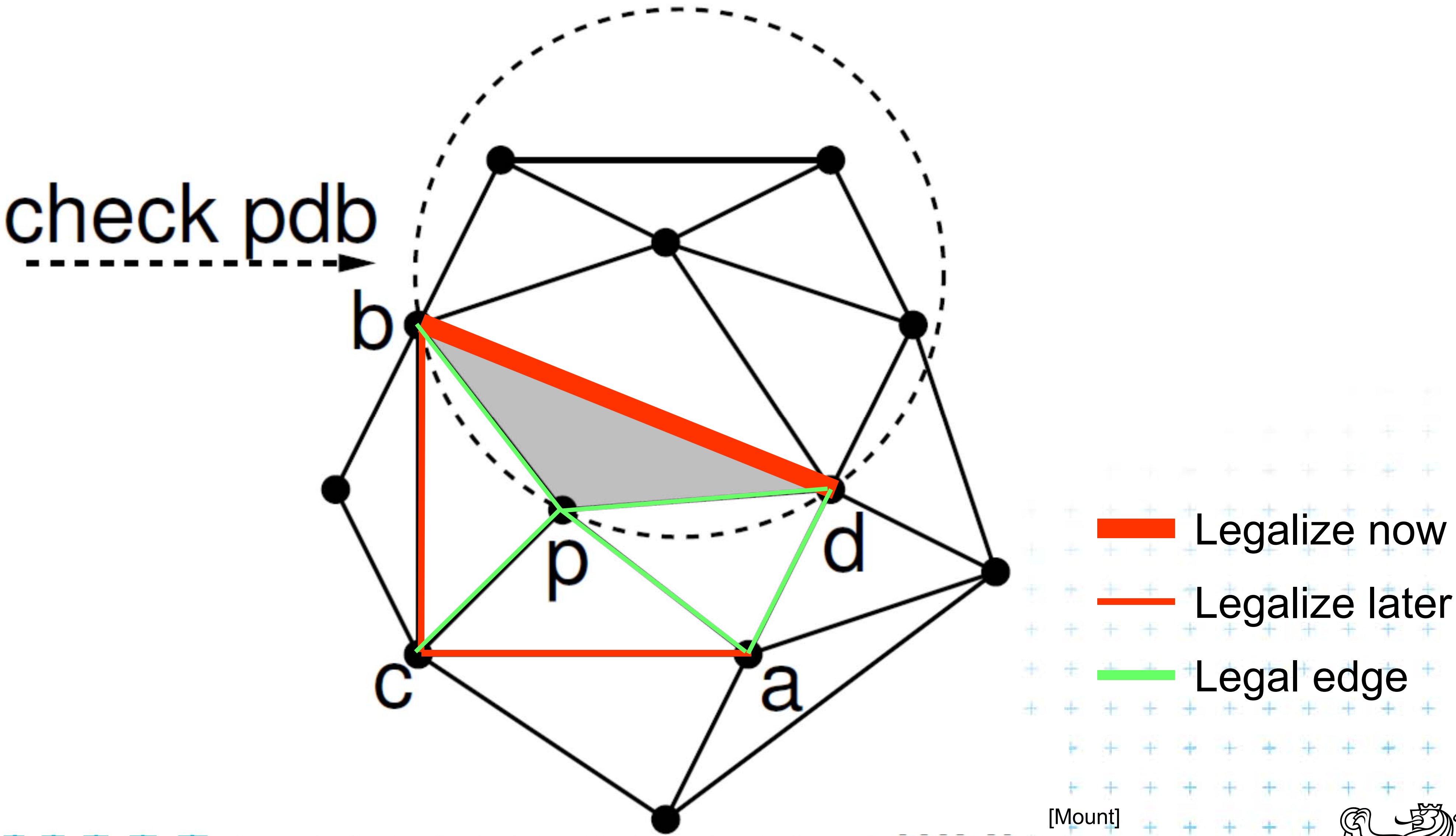
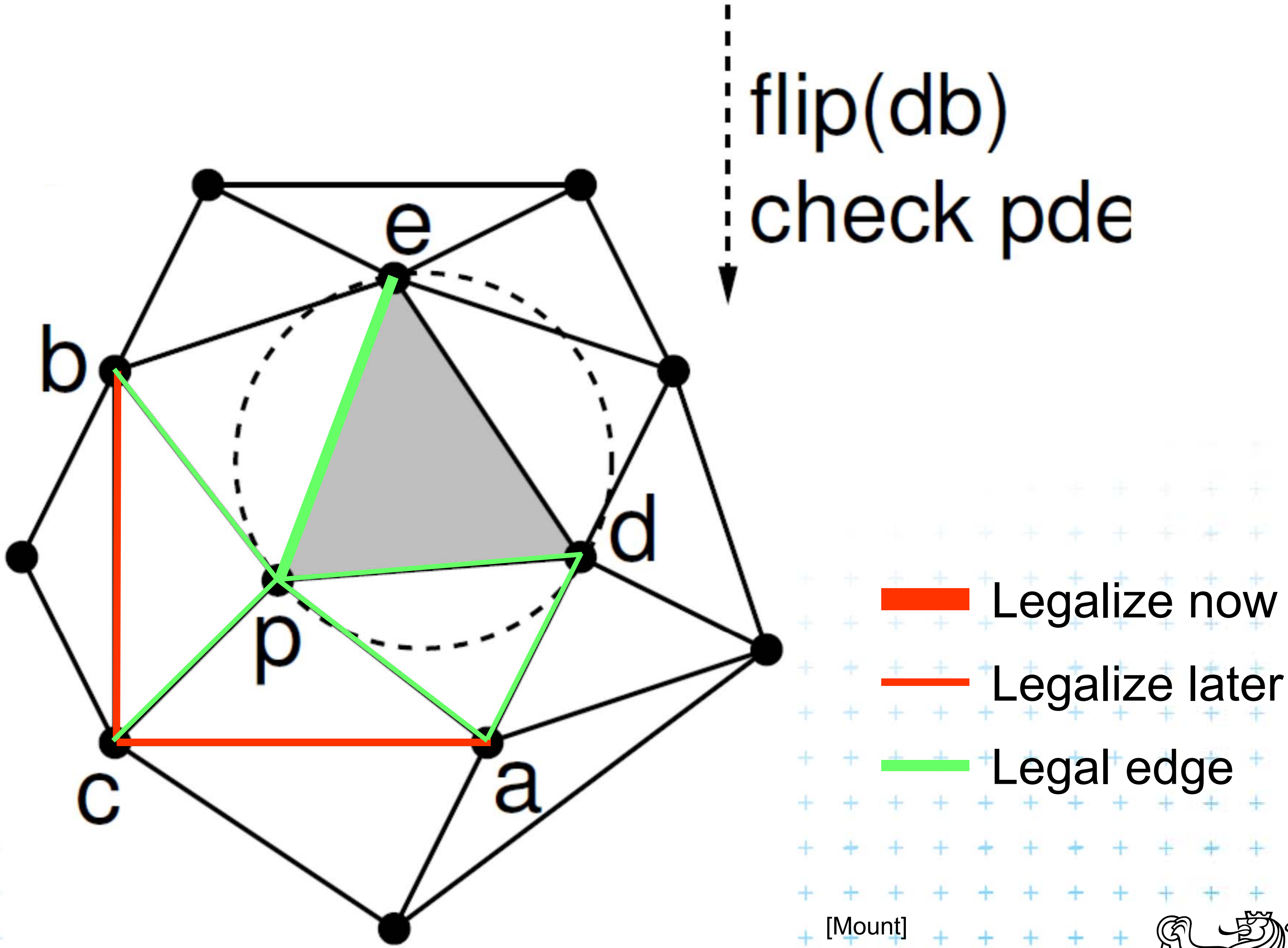


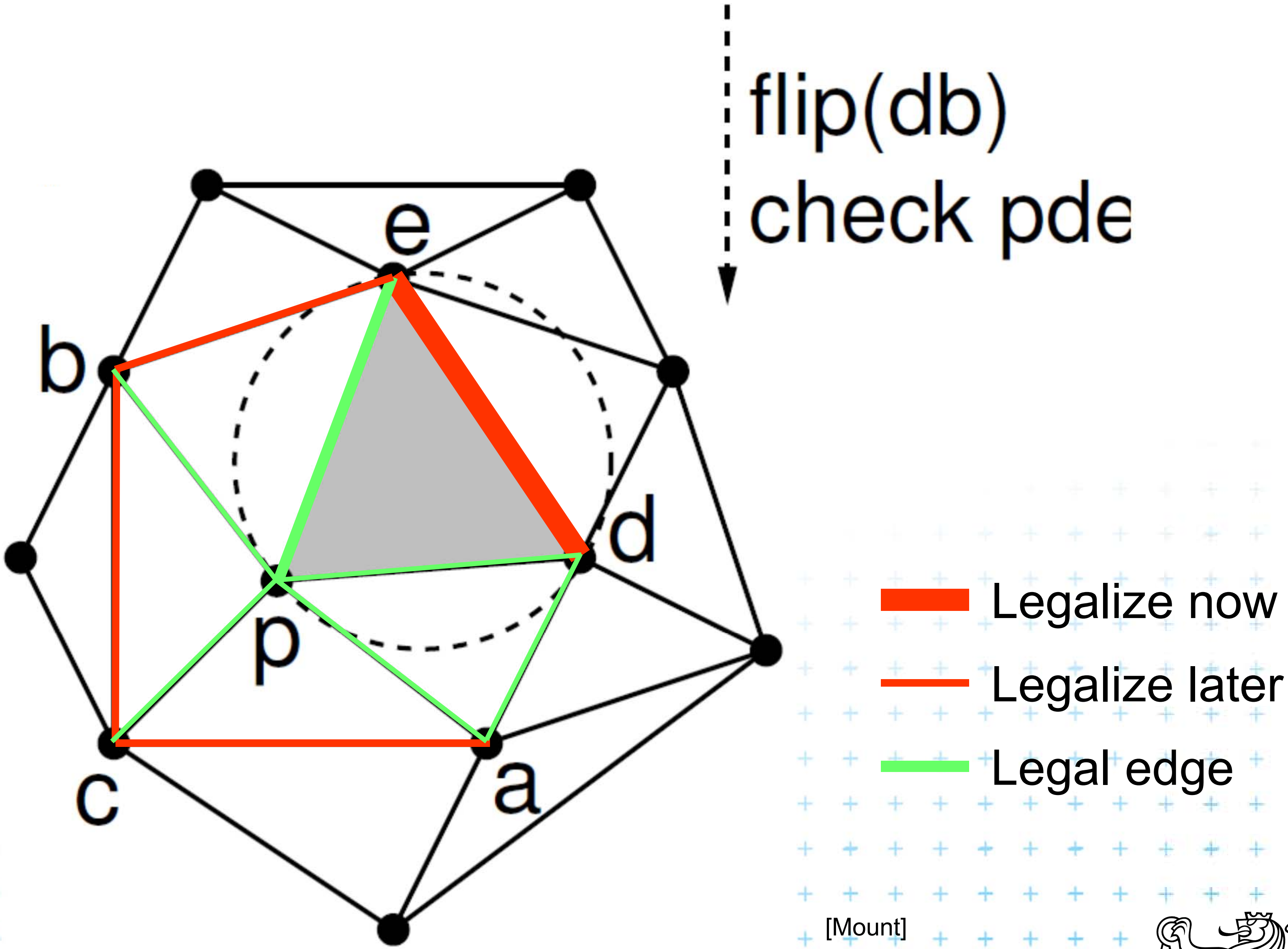
[Mount]

Delaunay triangulation – other point insert



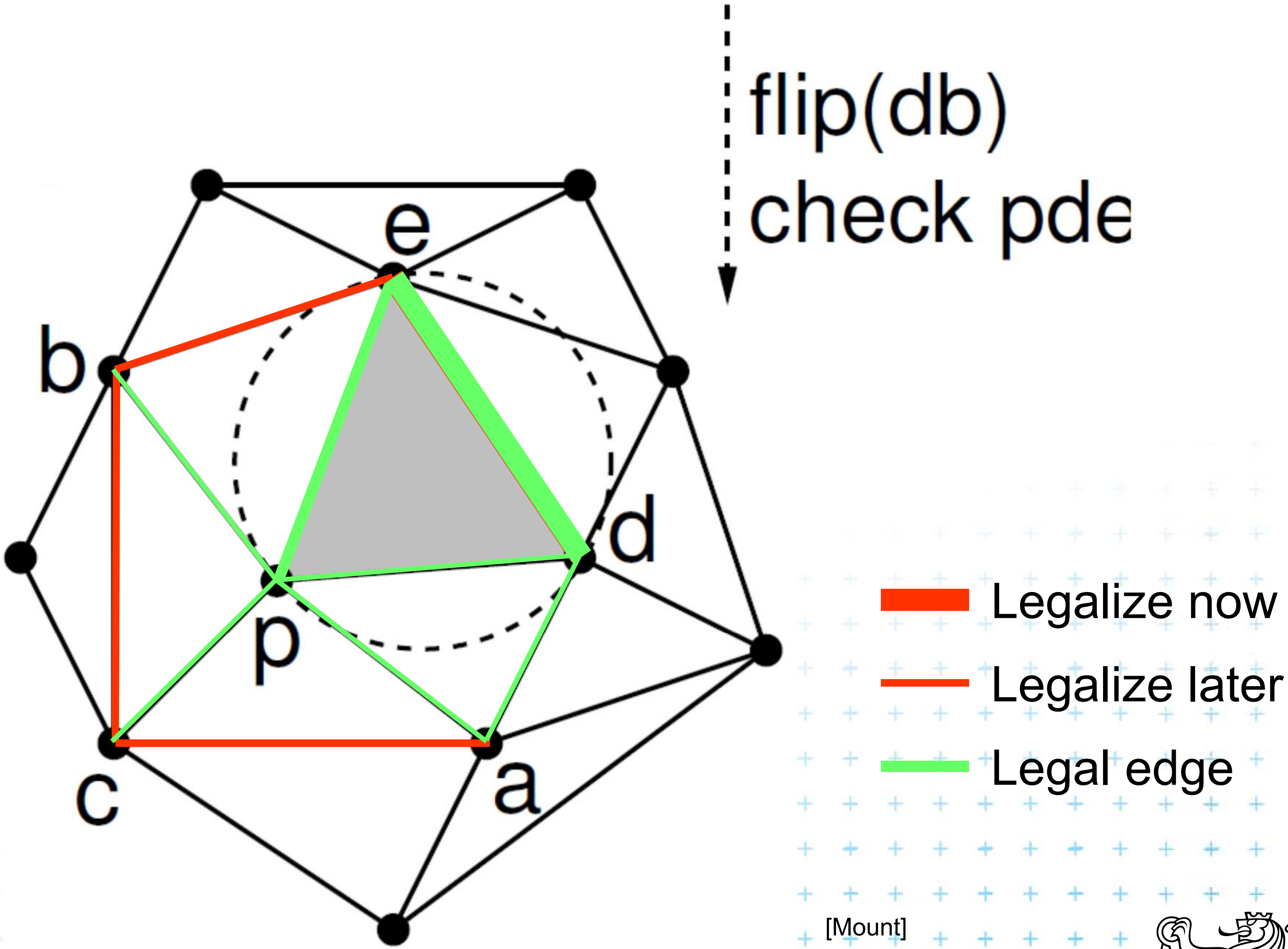
[Mount]

Delaunay triangulation – other point insert



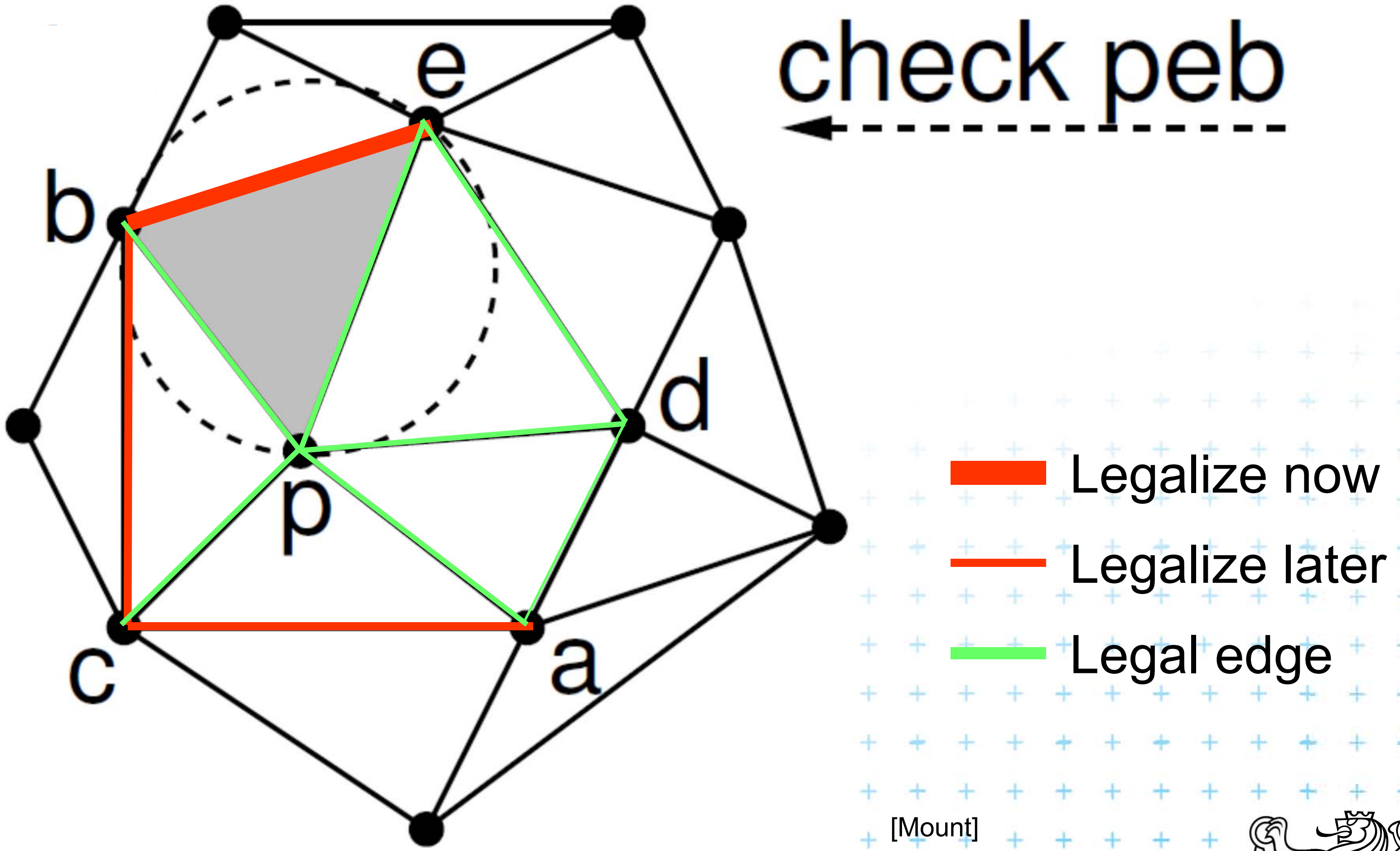
[Mount]

Delaunay triangulation – other point insert



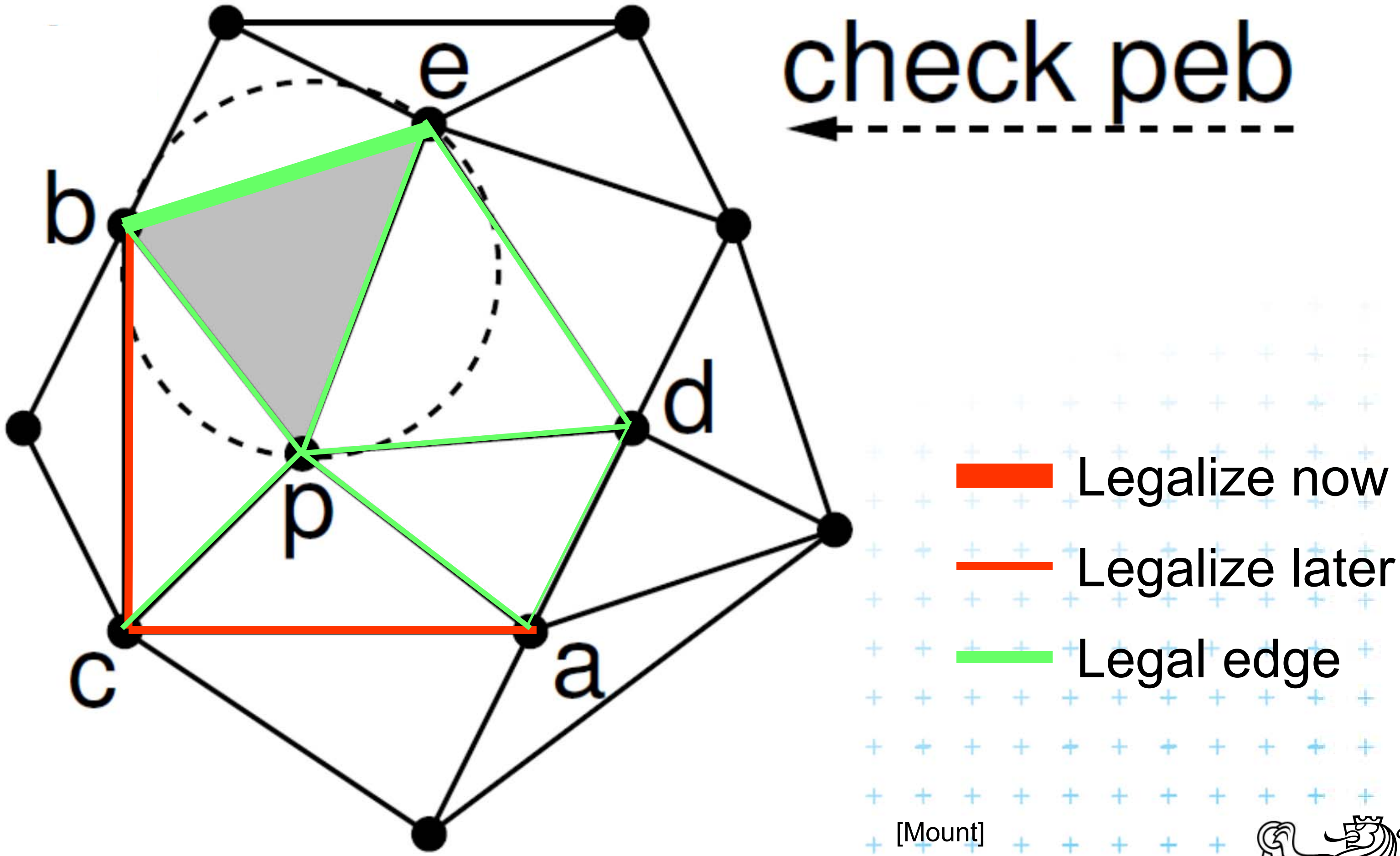
[Mount]

Delaunay triangulation – other point insert



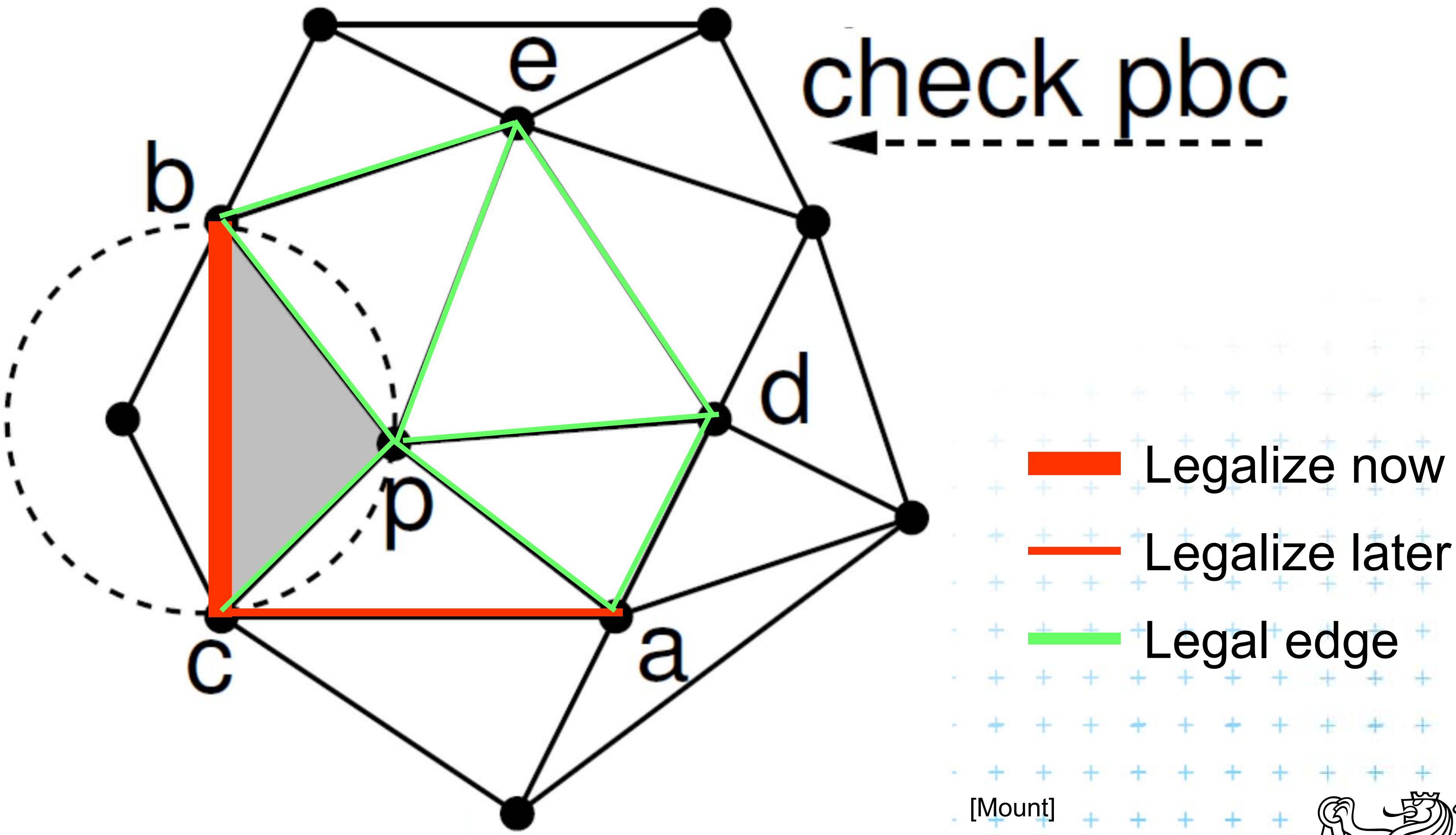
[Mount]

Delaunay triangulation – other point insert



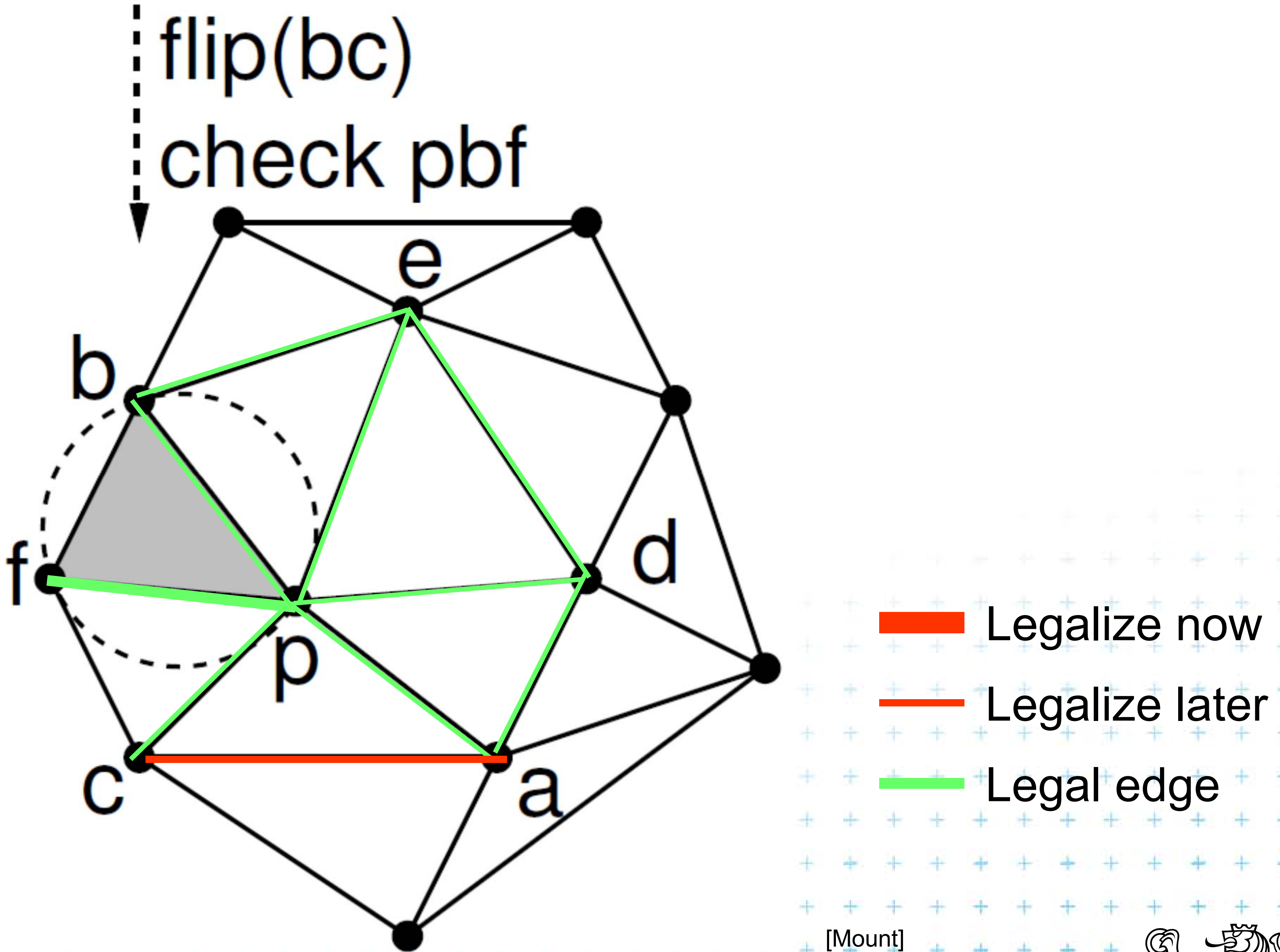
[Mount]

Delaunay triangulation – other point insert



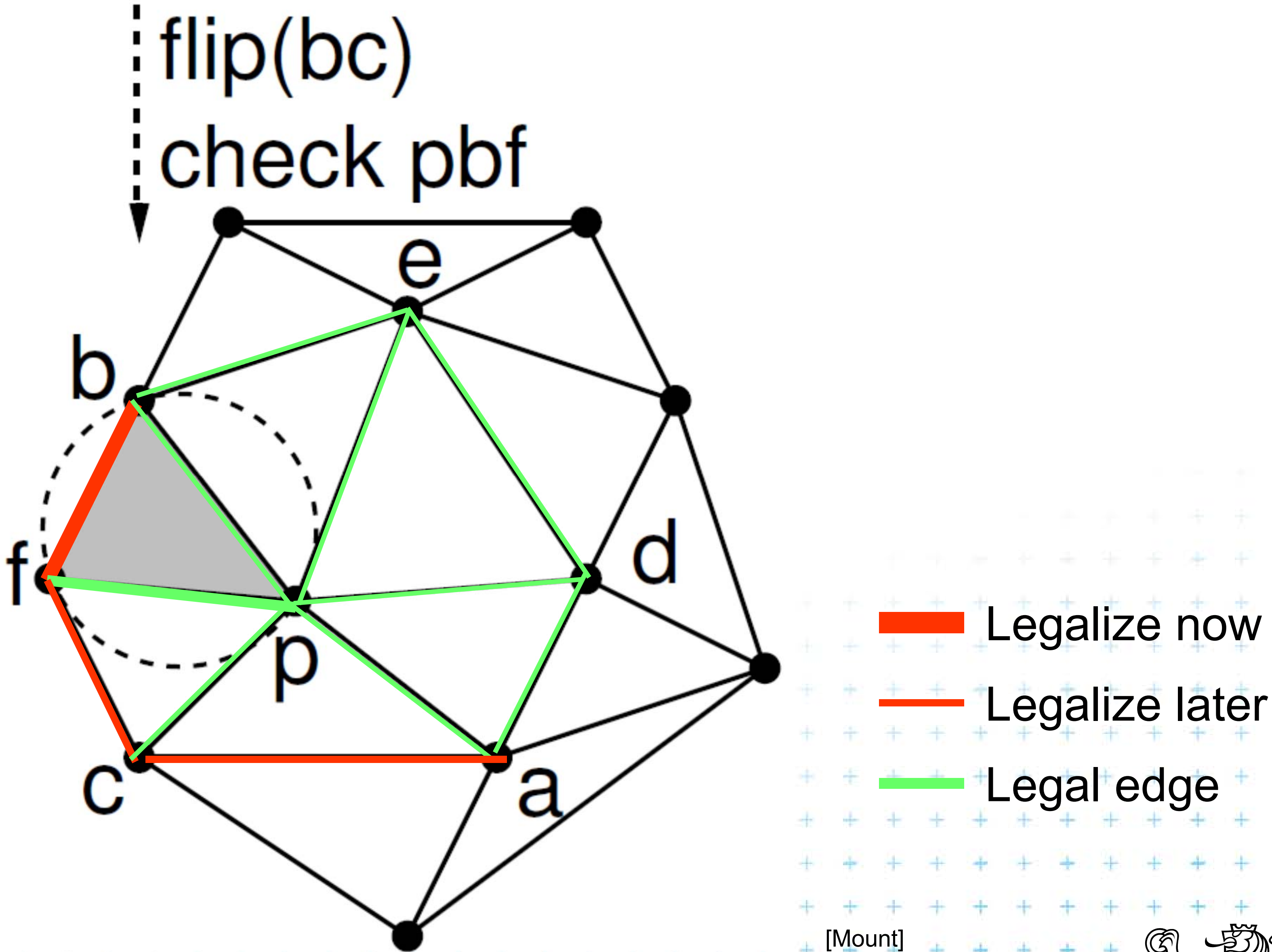
[Mount]

Delaunay triangulation – other point insert



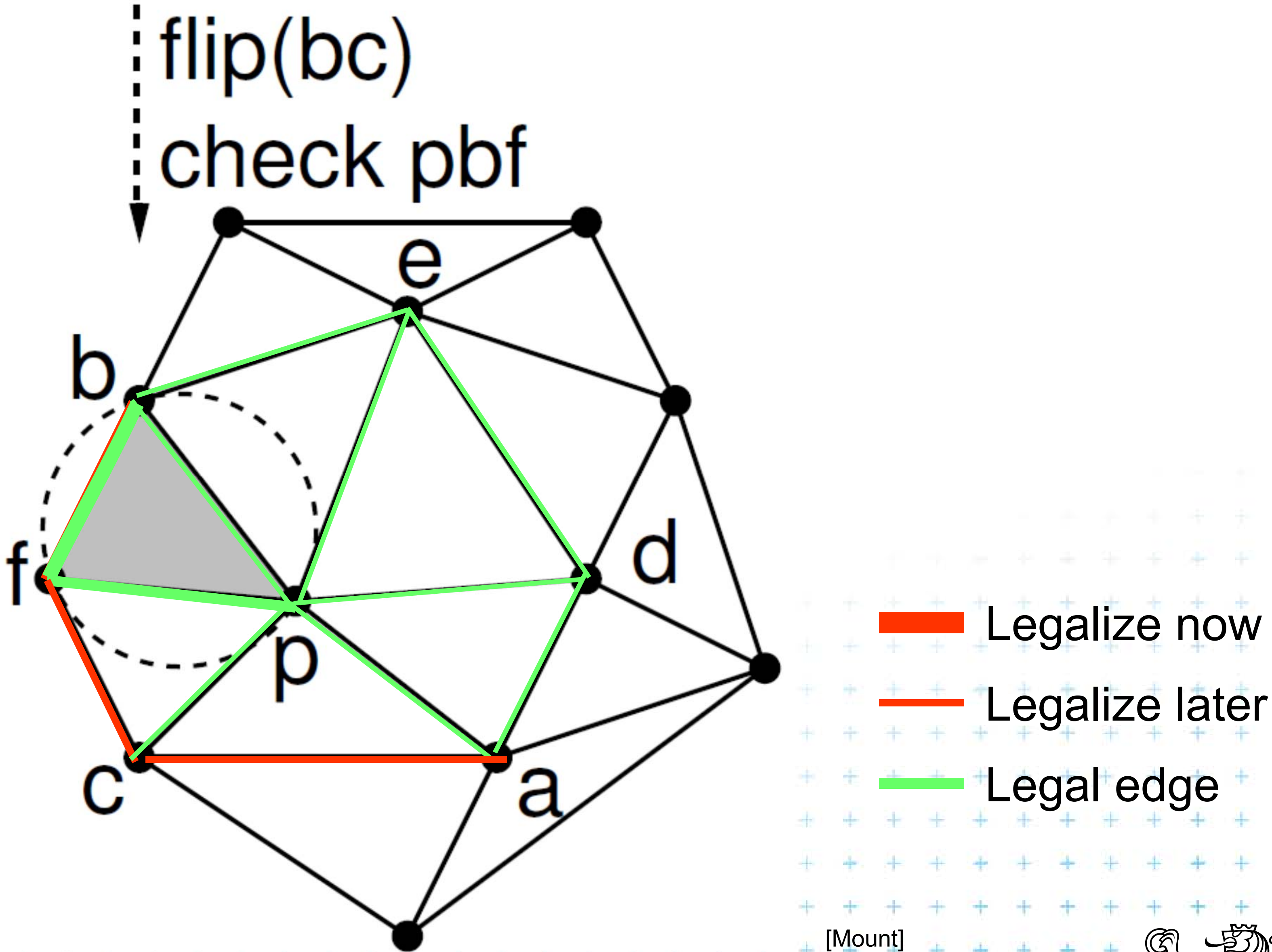
[Mount]

Delaunay triangulation – other point insert



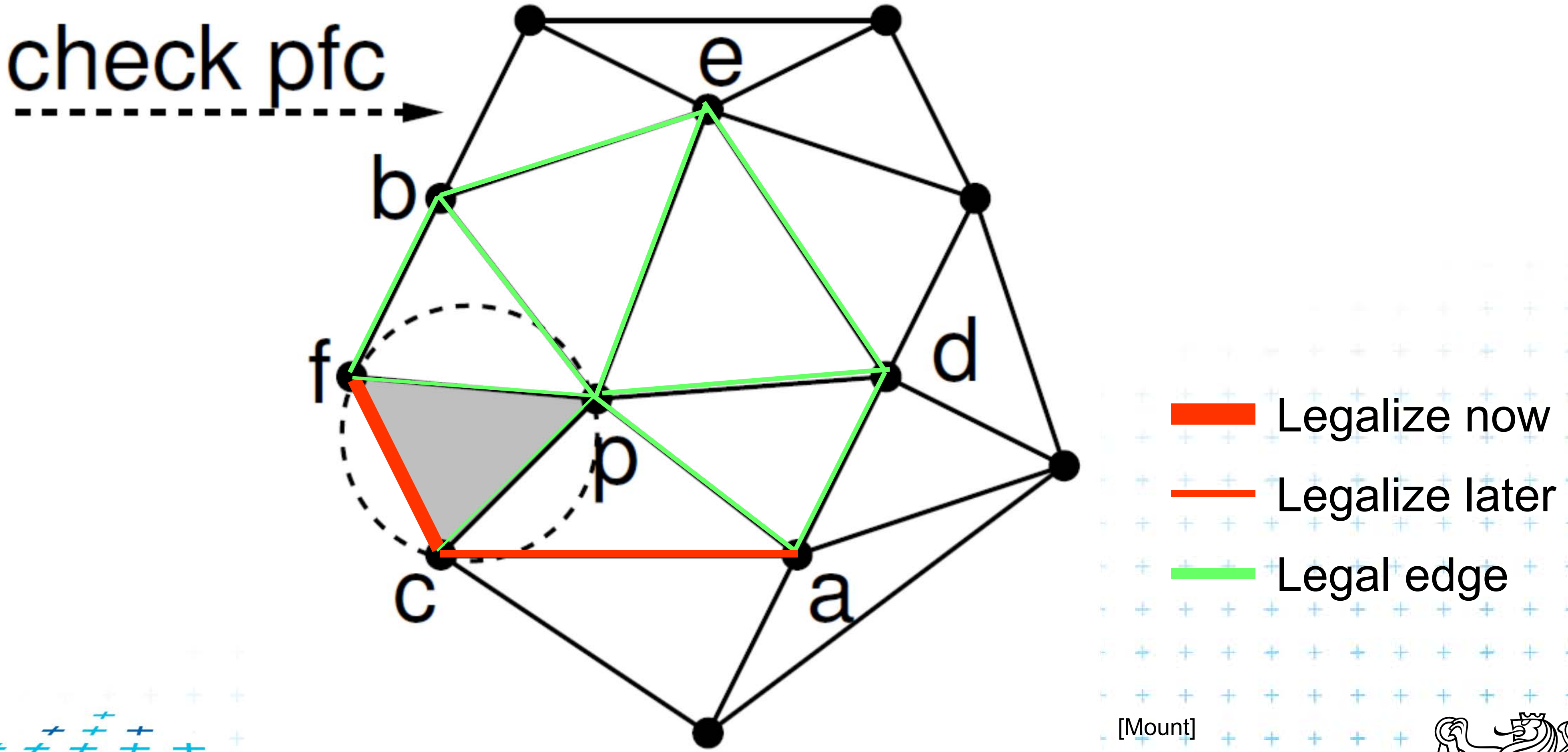
[Mount]

Delaunay triangulation – other point insert



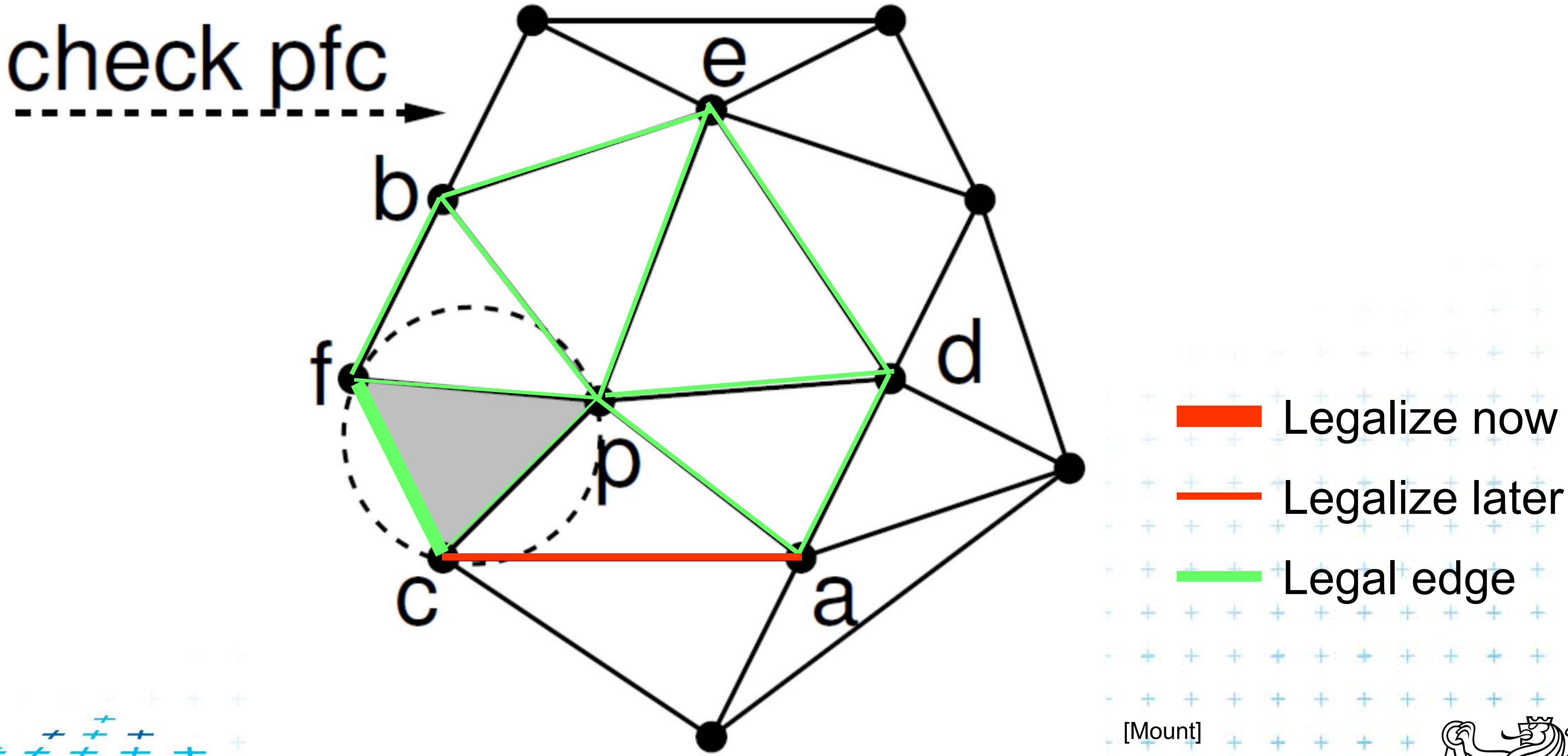
[Mount]

Delaunay triangulation – other point insert



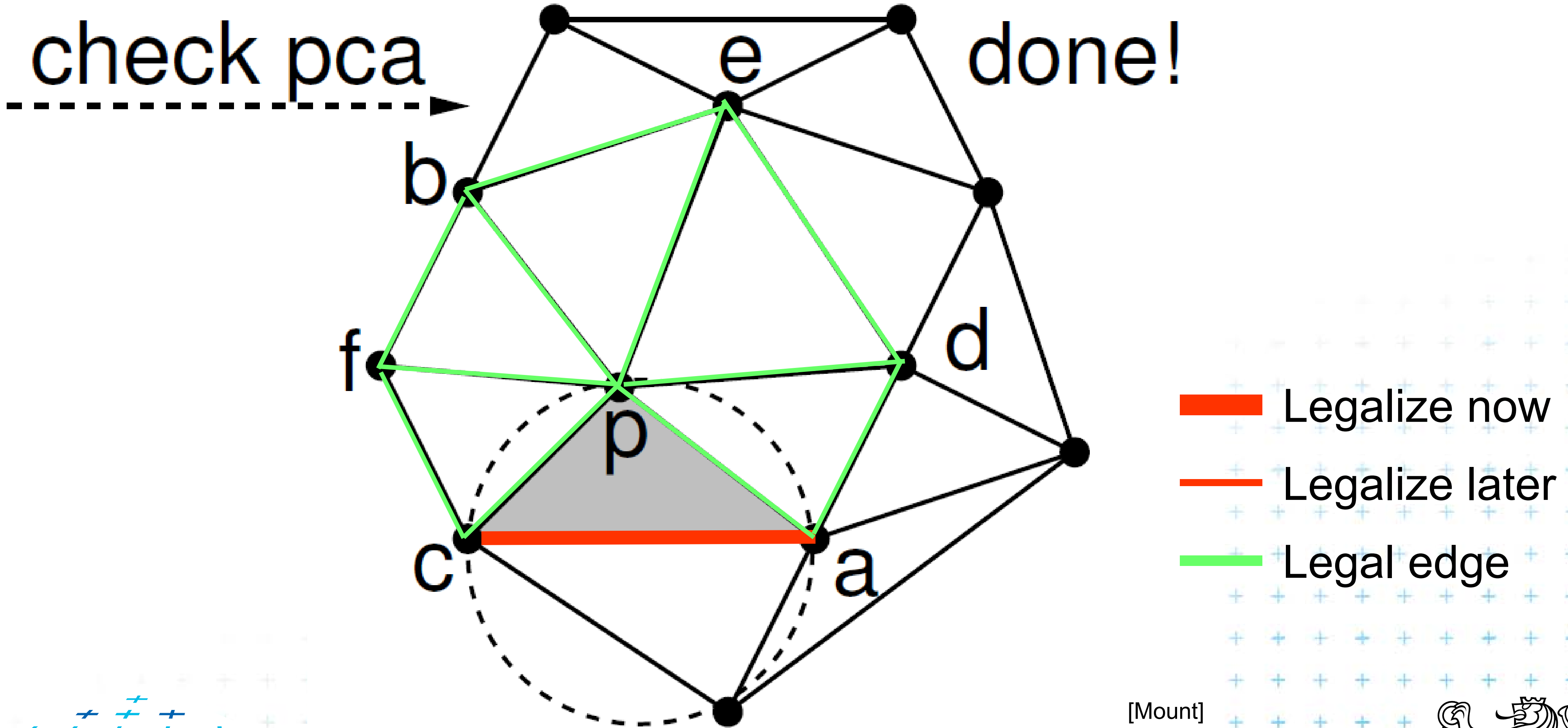
[Mount]

Delaunay triangulation – other point insert



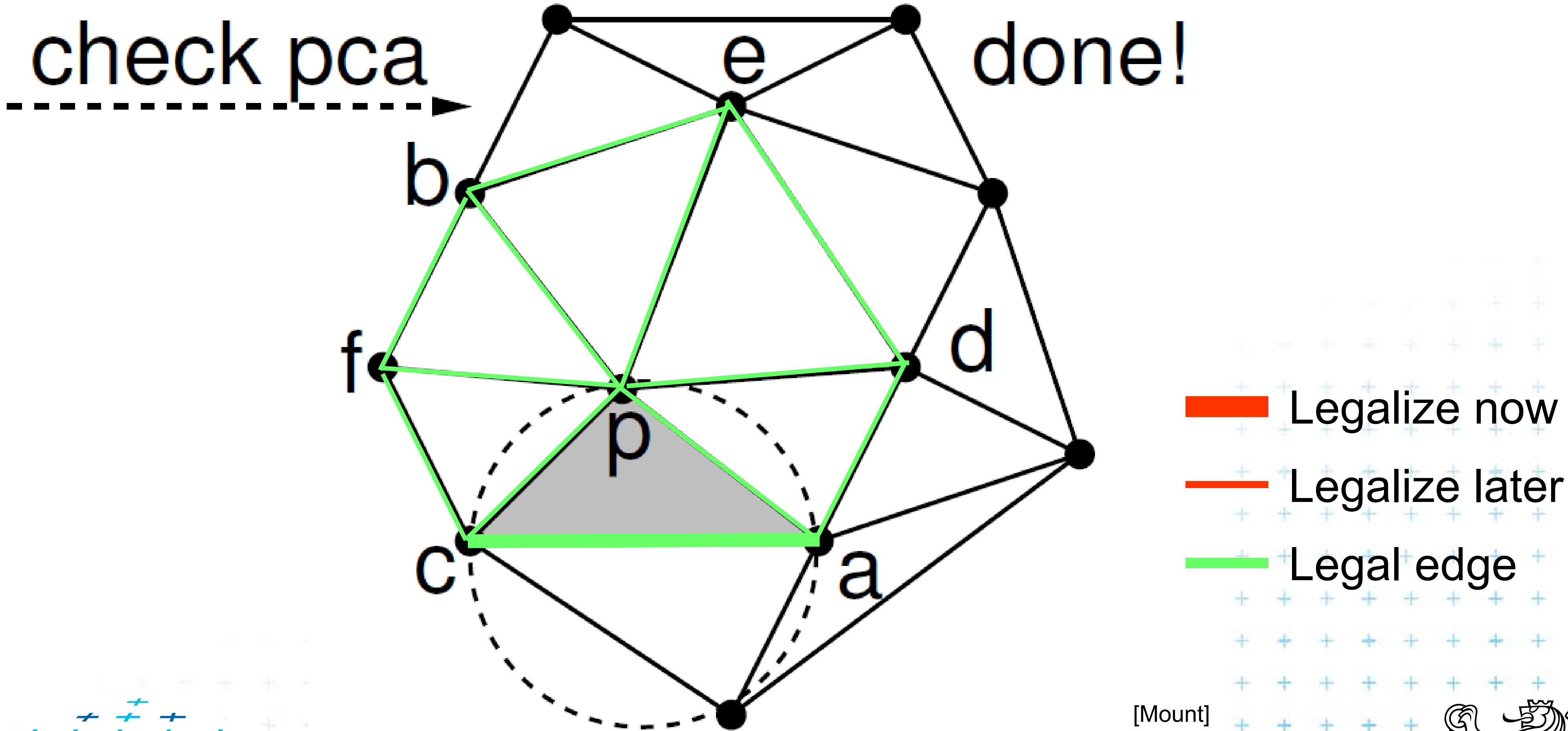
[Mount]

Delaunay triangulation – other point insert



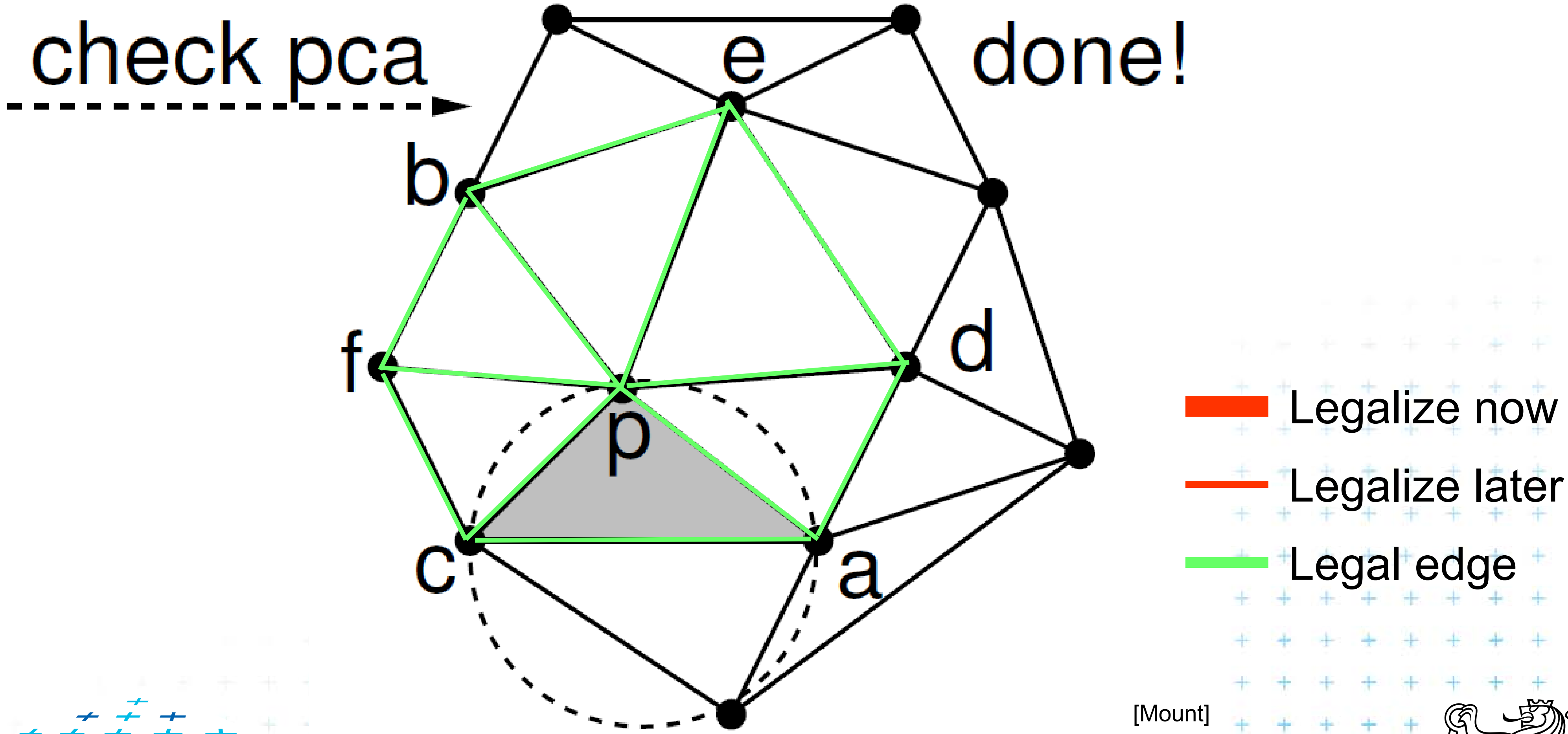
[Mount]

Delaunay triangulation – other point insert



[Mount]

Delaunay triangulation – other point insert



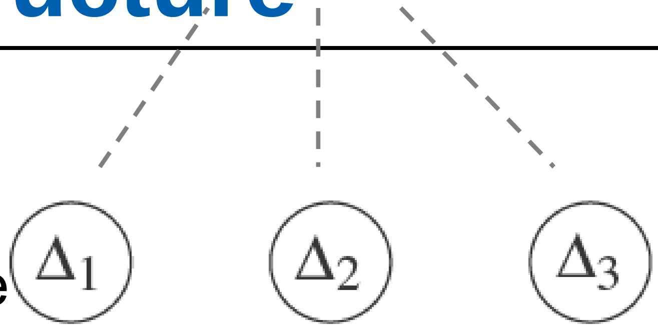
Correctness of the algorithm

- Every new edge (created due to insertion of *p)*
  - is incident to *p*
  - must be legal
    => no need to test them
- Edge can only become illegal if one of its incident triangle changes
  - Algorithm tests any edge that may become illegal
    => the algorithm is correct
- Every edge flip makes the angle-vector larger
  => algorithm can never get into infinite loop

Point location data structure

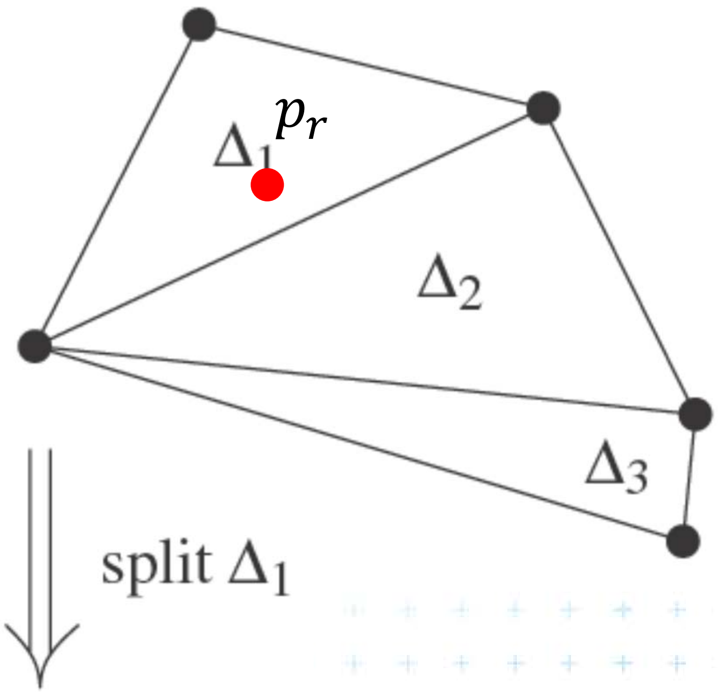
- For finding a triangle $abc \in T$ containing p
  - Leaves for active (current) triangles
  - Internal nodes for destroyed triangles
  - Links to new triangles
- Search p: start in root (initial triangle)
  - In each inner node of T:
    - Check all children (max three)
    - Descend to child containing p

Point location data structure



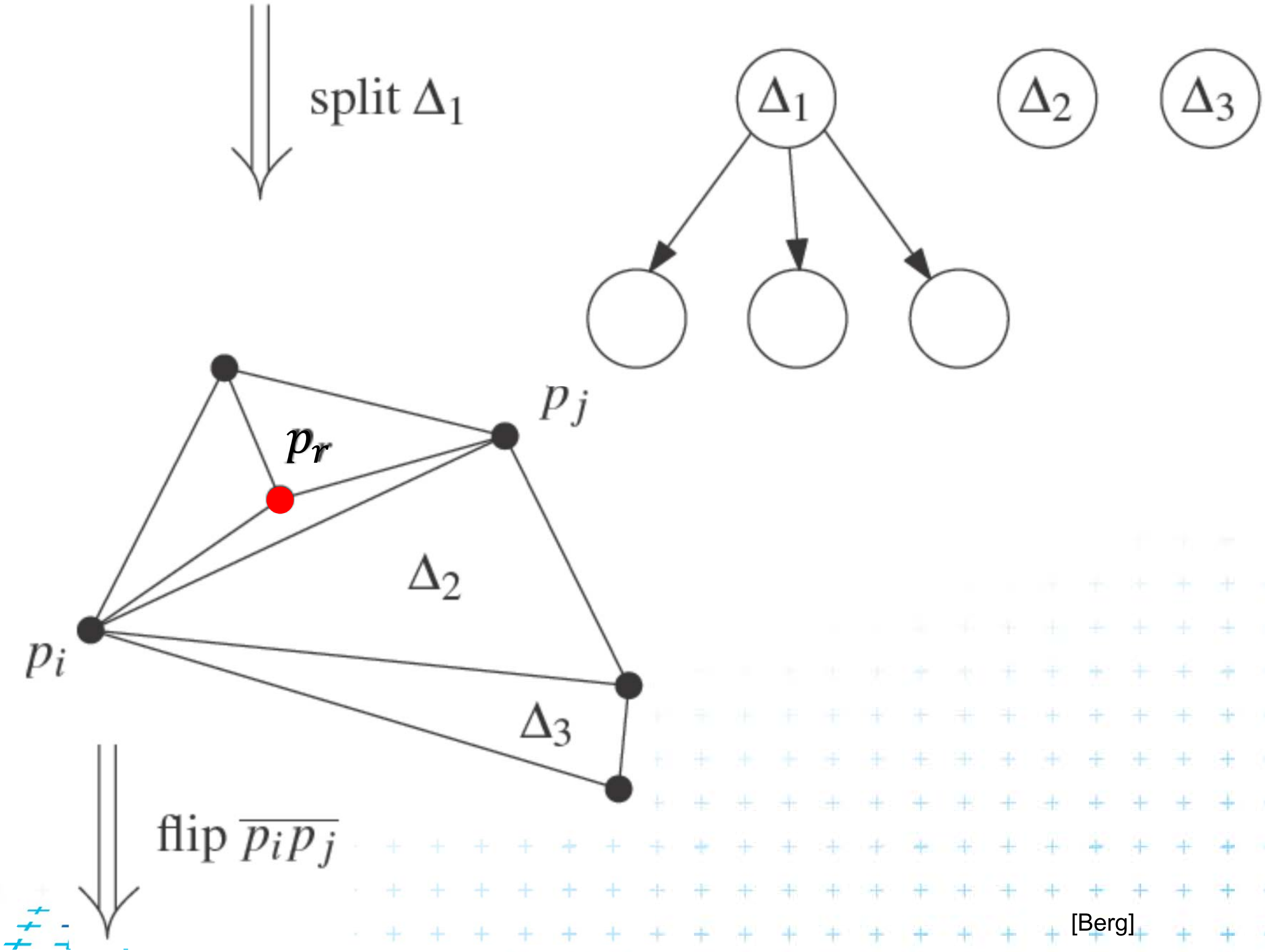
Simplified
- it should also contain the root node of the large triangle

New point p_r inserted to tr. 1



[Berg]

Point location data structure



[Berg]

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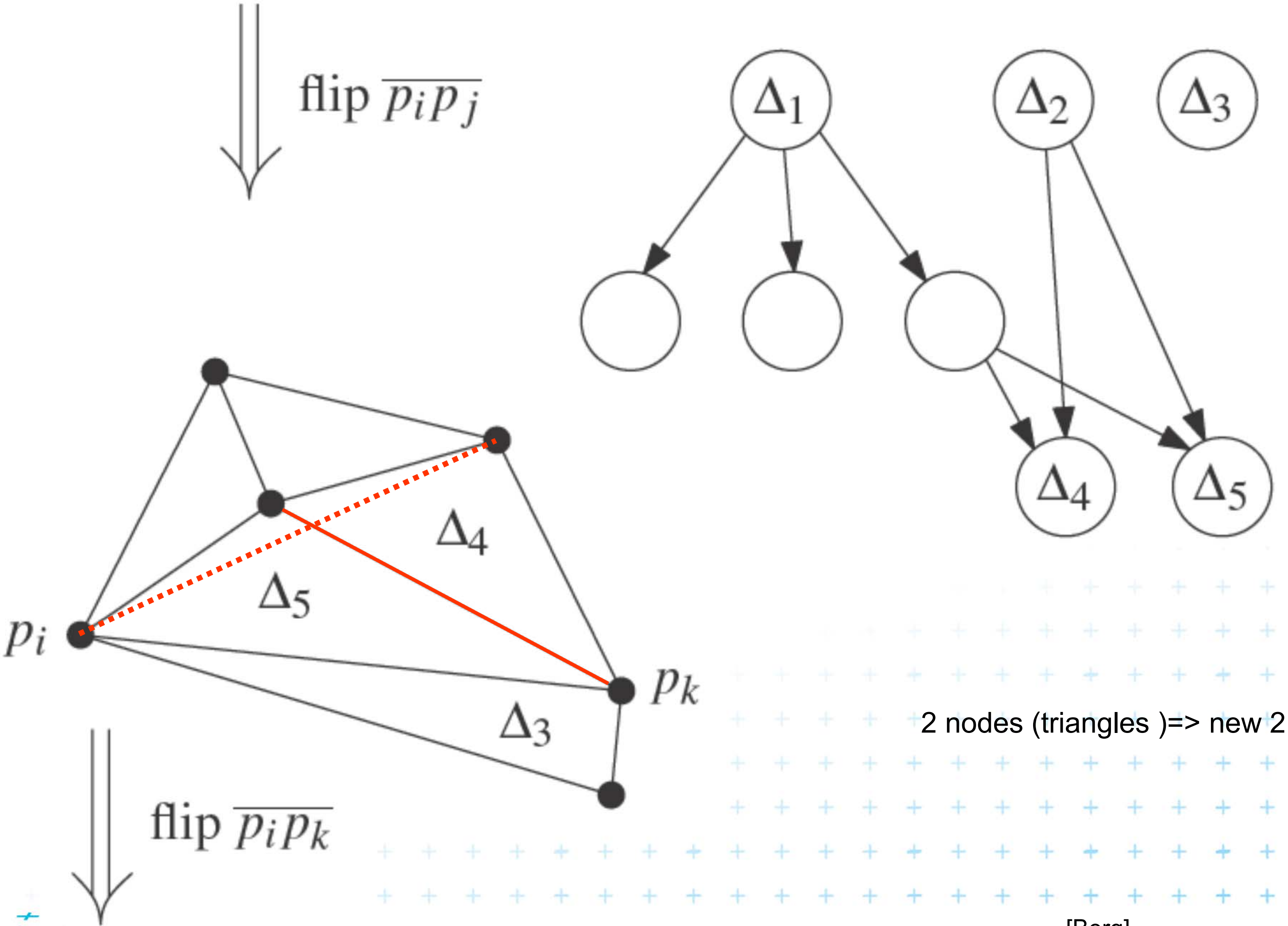
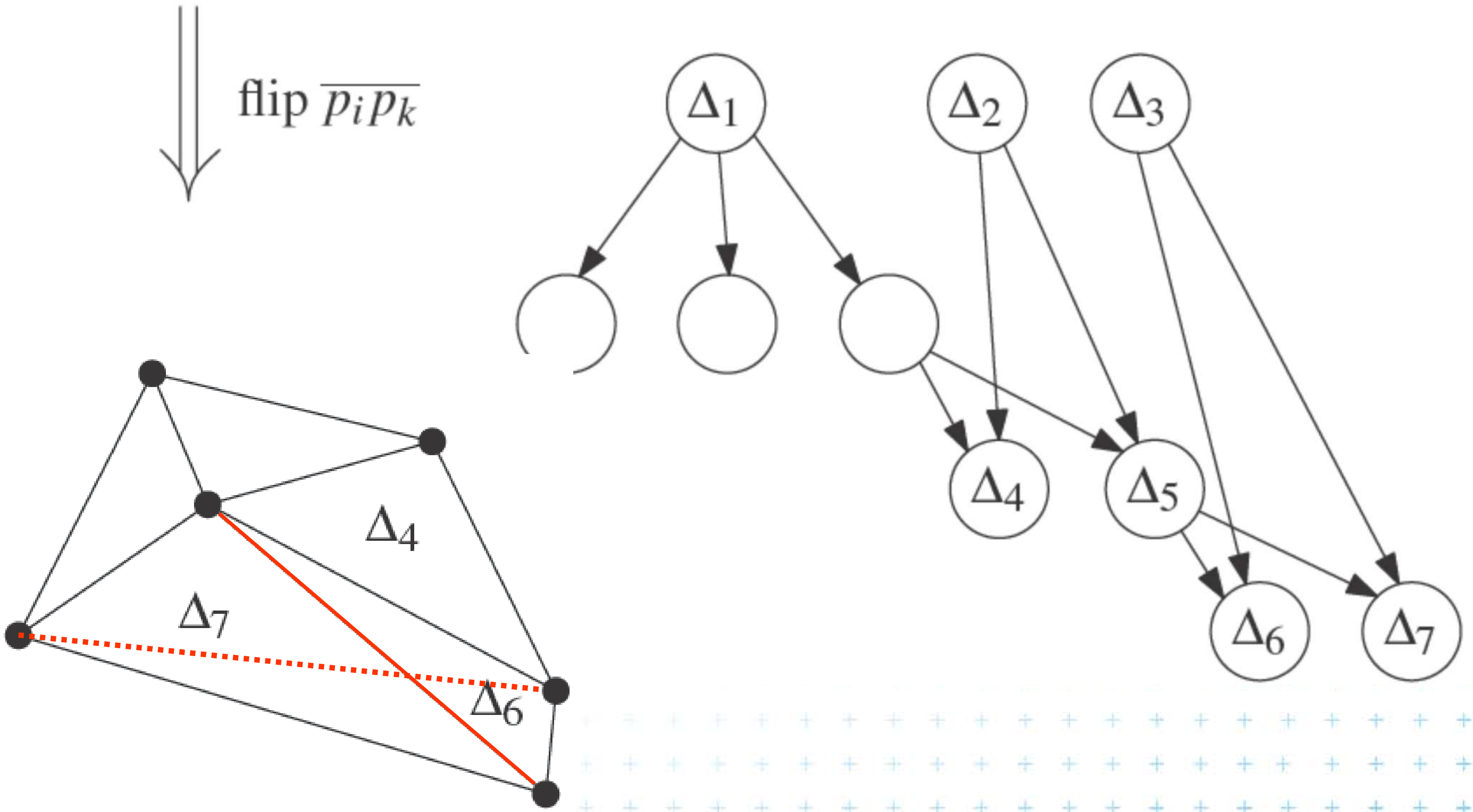


2 nodes (triangles)=> new 2 nodes

[Berg]

Point location data structure

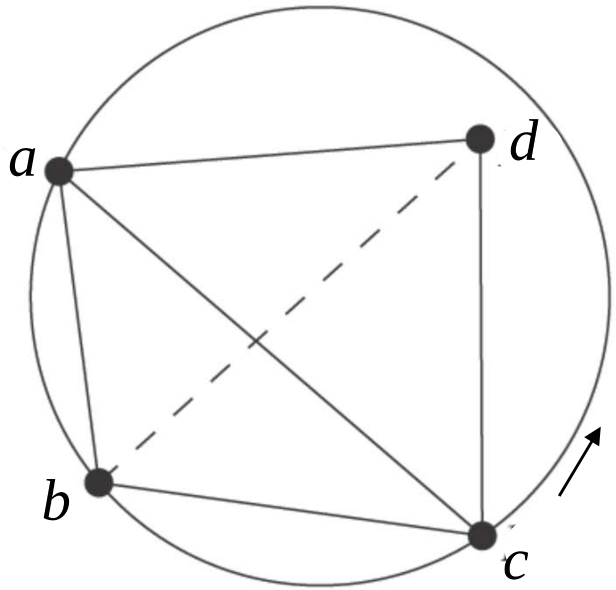


[Berg]

InCircle test

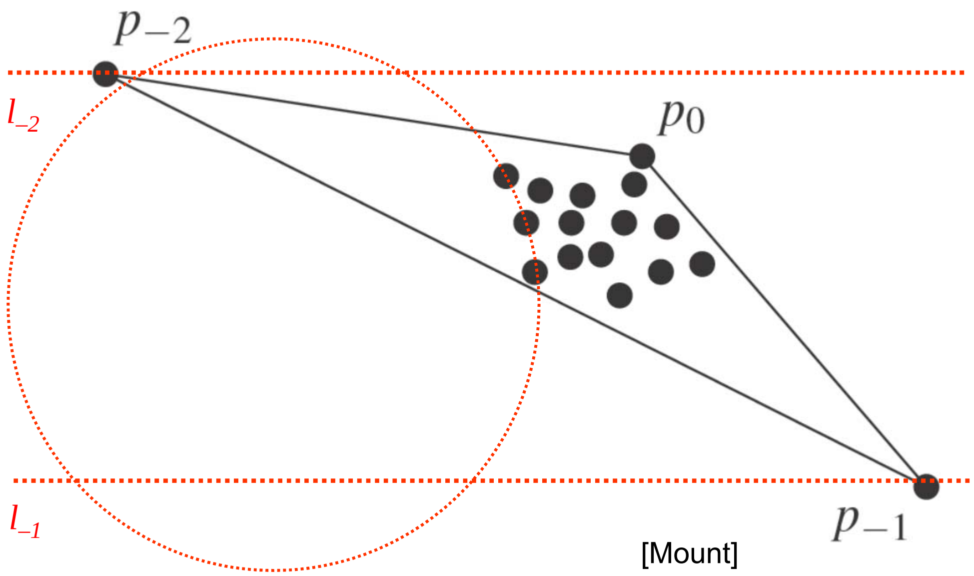
- *a,b,c* are counterclockwise in the plane
- Test, if *d* lies to the left of the oriented circle through *a,b,c*

$$\mathrm{inCircle}(a,b,c,d) = \det \begin{pmatrix} a_x & a_y & a_x^2 + a_y^2 & 1 \\ b_x & b_y & b_x^2 + b_y^2 & 1 \\ c_x & c_y & c_x^2 + c_y^2 & 1 \\ d_x & d_y & d_x^2 + d_y^2 & 1 \end{pmatrix} > 0$$



[Mount]

Creation of the initial triangle



Idea: For given points set *P*:

- Initial triangle $p_{-2}p_{-1}p_0$
  - Must contain all points of *P*
  - Must not be (none of its points) in any circle defined by non-collinear points of *P*
- l_{-2} = horizontal line above *P*
- l_{-1} = horizontal line below *P*
- p_{-2} = lies on l_{-2} as far left that p_{-2} lies outside every circle
- p_{-1} = lies on l_{-1} as far right that p_{-1} lies outside every circle defined by 3 non-collinear points of *P*

Replaced by symbolical tests with this triangle
=> p_{-1} and p_{-2} always out

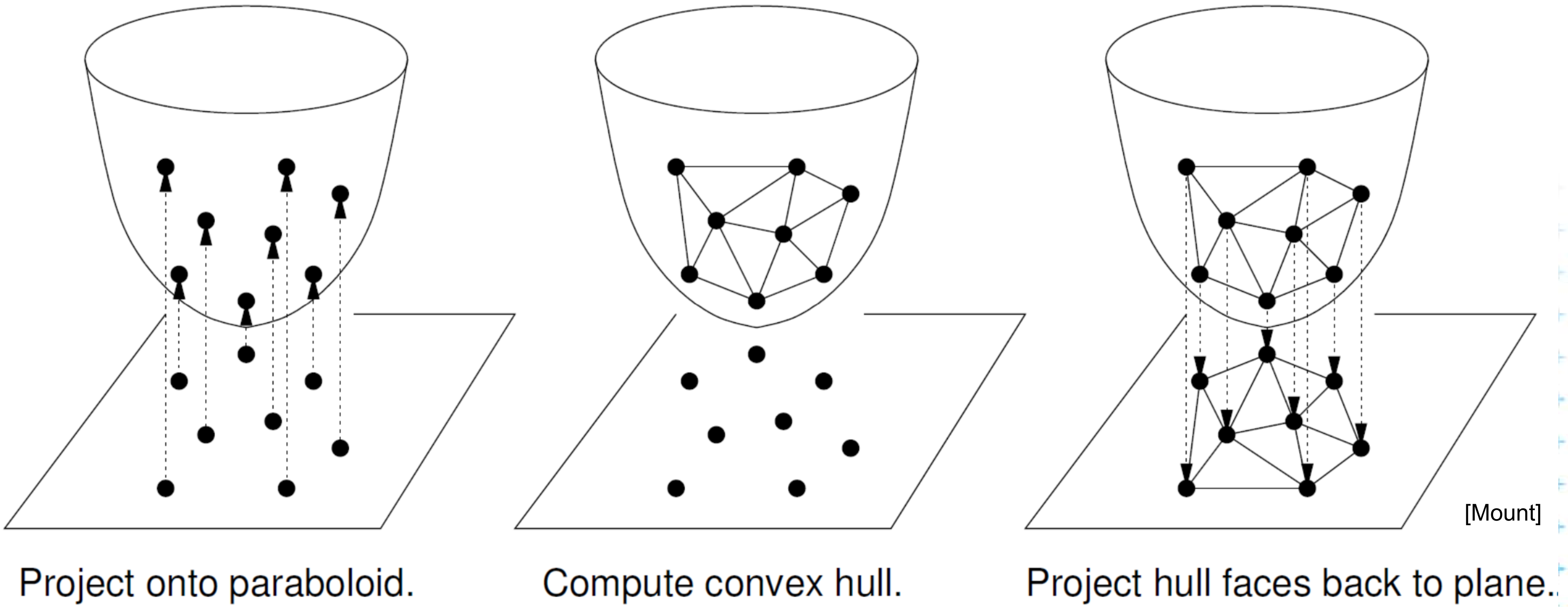
Complexity of incremental DT algorithm

- Delaunay triangulation of a pointset P in the plane can be computed in
  - $O(n \log n)$ expected time
  - using $O(n)$ storage
- For details see [Berg, Section 9.4]

  Idea
  - expected number of created triangles is $9n + 1$
  - expected search $O(\log n)$ in the search structure done n times for n inserted points

Delaunay triangulations and Convex hulls

- Delaunay triangulation in R^d can be computed as part of the convex hull in R^{d+1} (lower CH)
- 2D: Connection is the paraboloid: $z = x^2 + y^2$



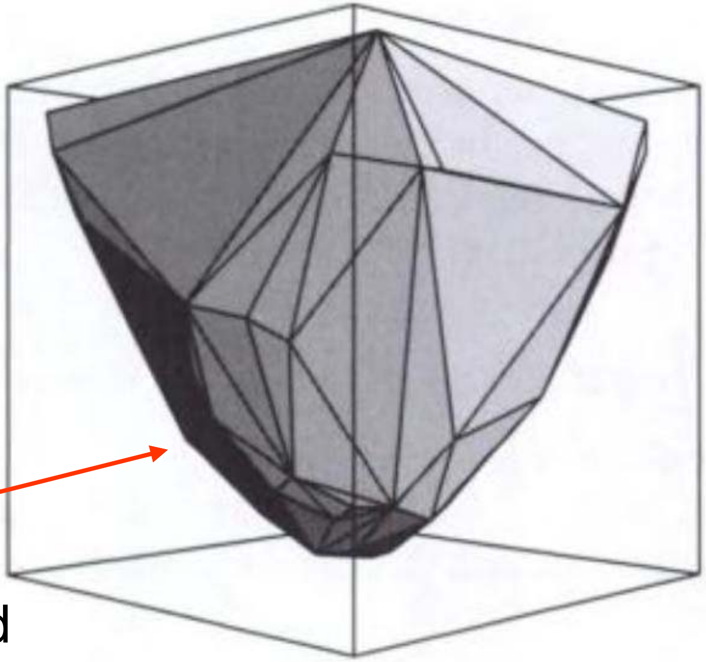
Vertical projection of points to paraboloid

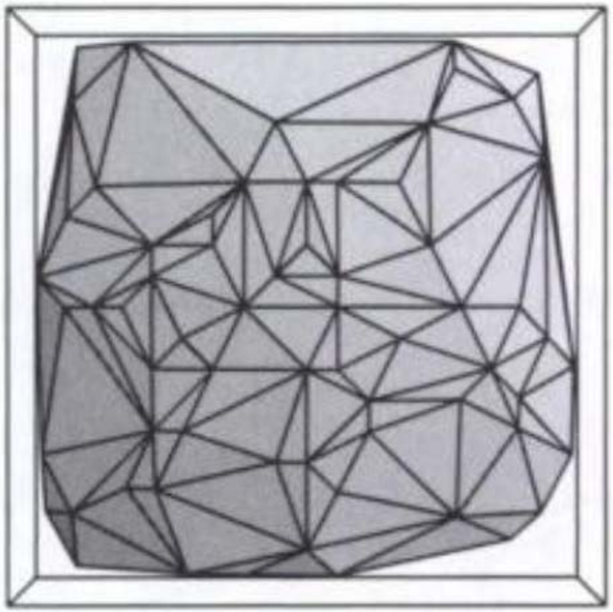
- Vertical projection of 2D point to paraboloid in 3D

$$(x, y) \rightarrow (x, y, x^2 + y^2)$$

- Lower convex hull
  = portion of CH visible from $z = -\infty$ (forms DT)

points on paraboloid (all on CH)





bottom view

[Rourke]

Relation between CH and DT

Delaunay condition (2D)

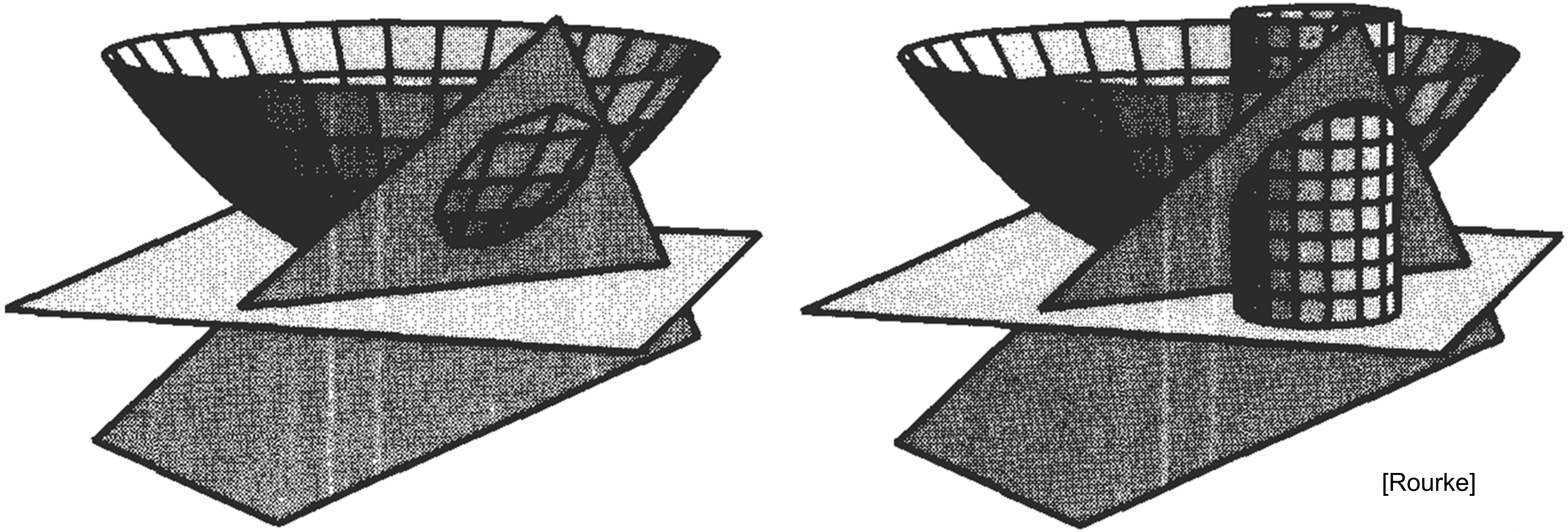
Points $p,q,r \in S$ form a Delaunay triangle **iff** the circumcircle of p,q,r is empty (contains no point)

Convex hull condition (3D)

Points $p',q',r' \in S'$ form a face of $CH(S')$ **iff** the plane passing through p',q',r' is supporting S'

- all other points lie to one side of the plane
- plane passing through p',q',r' is
  a supporting hyperplane of the convex hull CH(S')

Relation between CH and DT



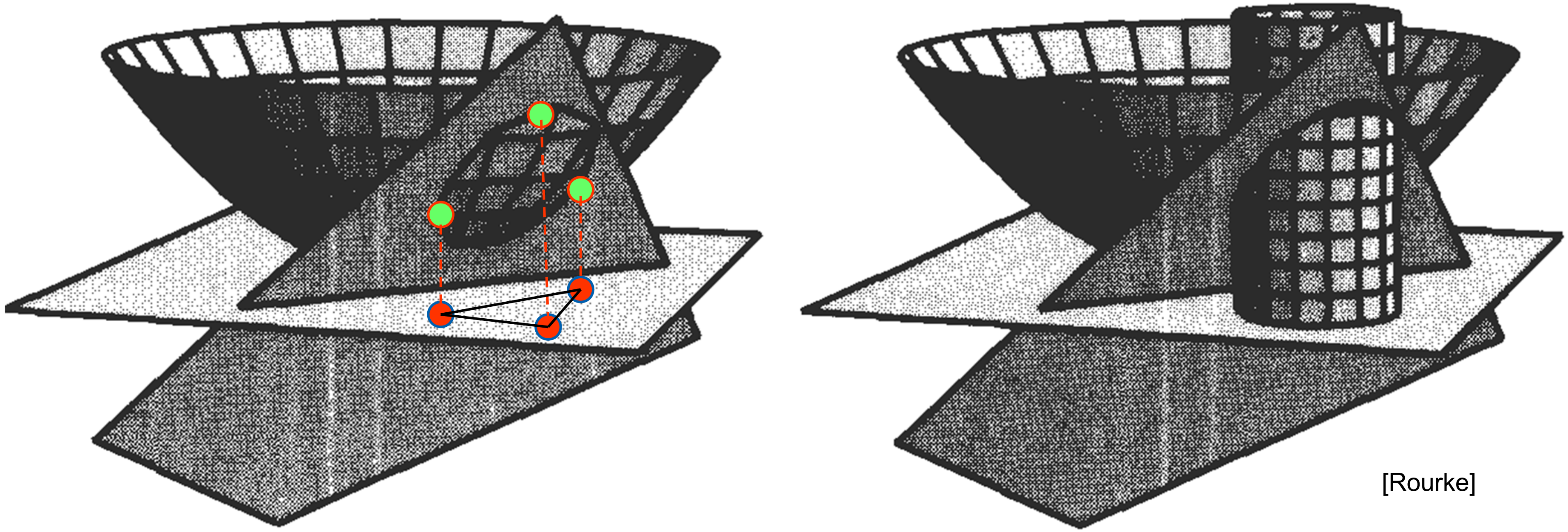
[Rourke]

4 distinct points p, q, r, s in the plane, and

p', q', r', s' be their projections onto the paraboloid $z = x^2 + y^2$

The point s lies within the circumcircle of pqr **iff** s' lies on the lower side of the secant plane passing through p', q', r'.

Relation between CH and DT



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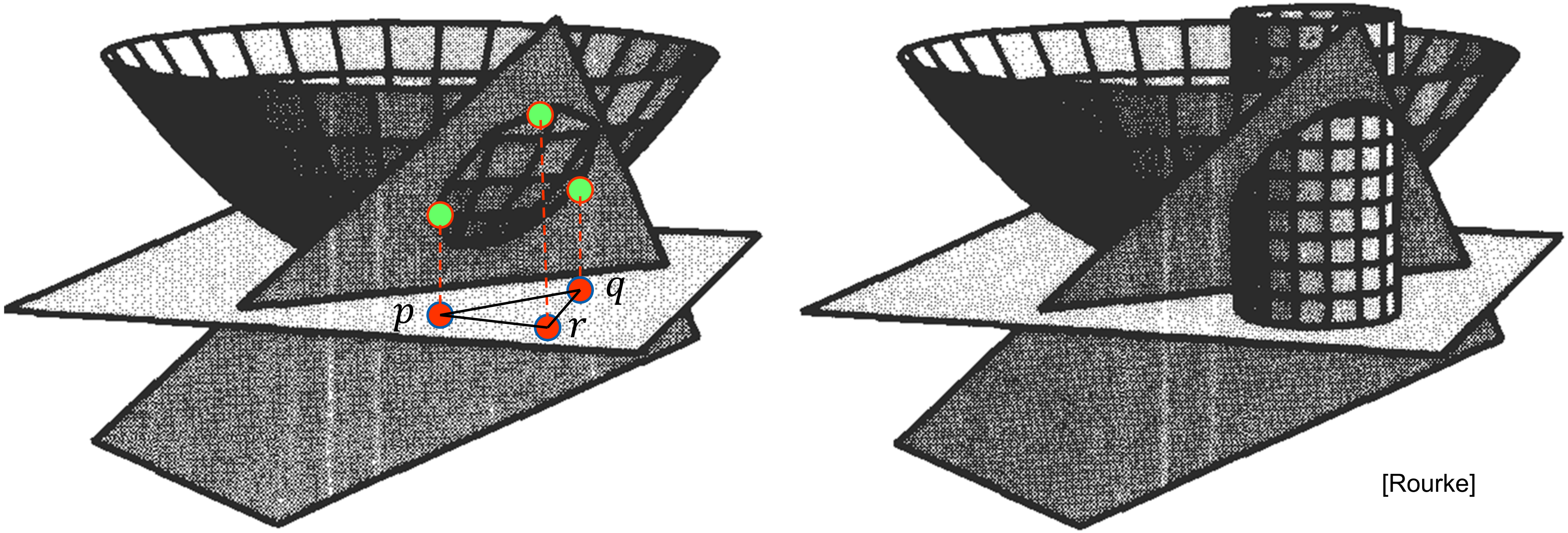

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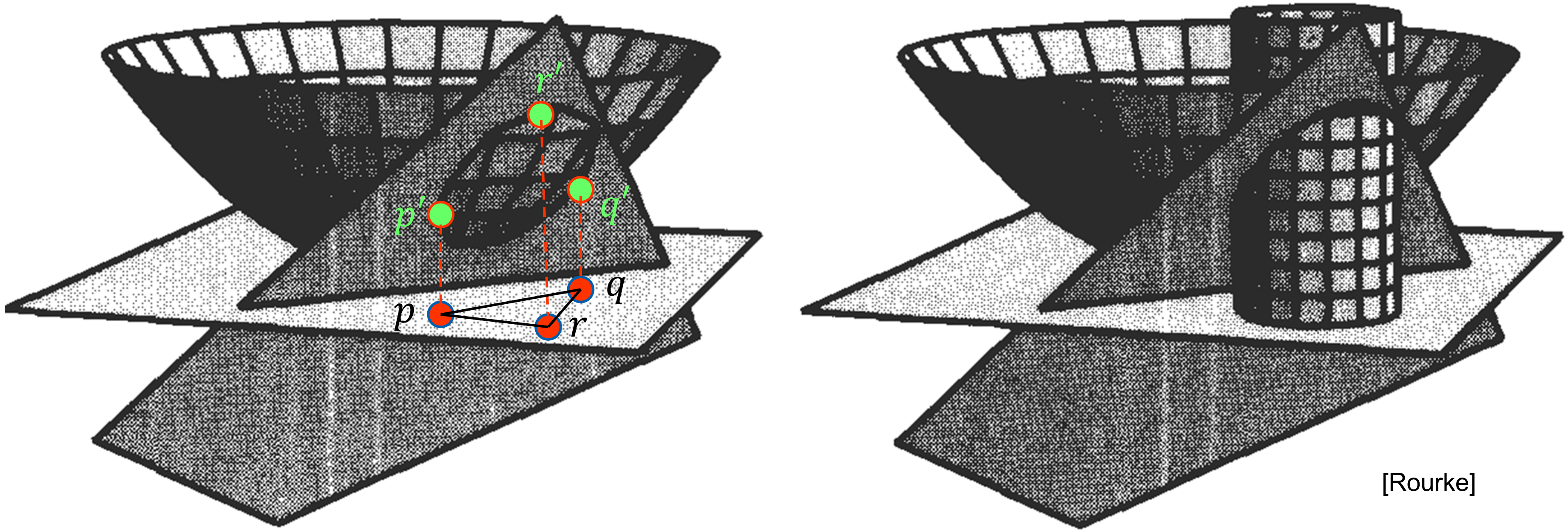

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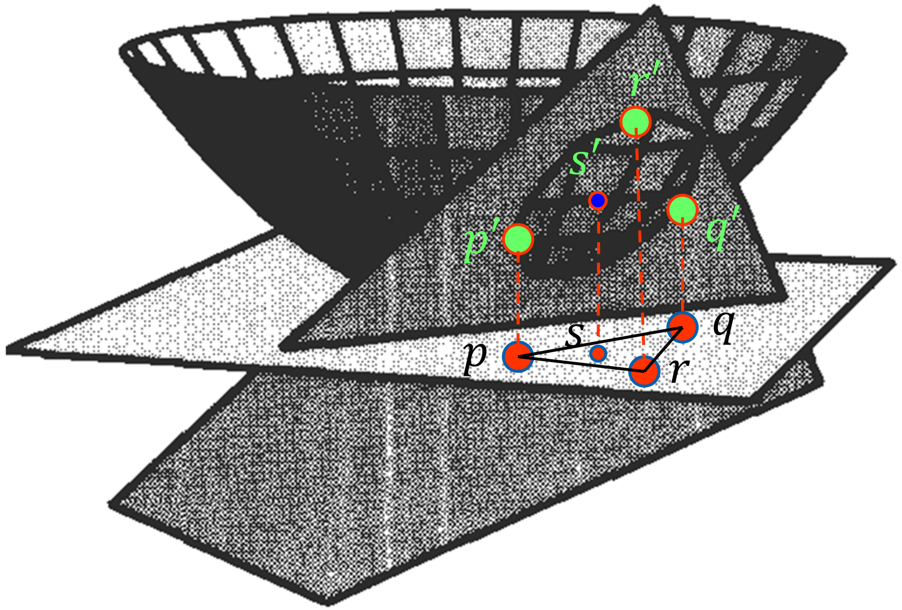

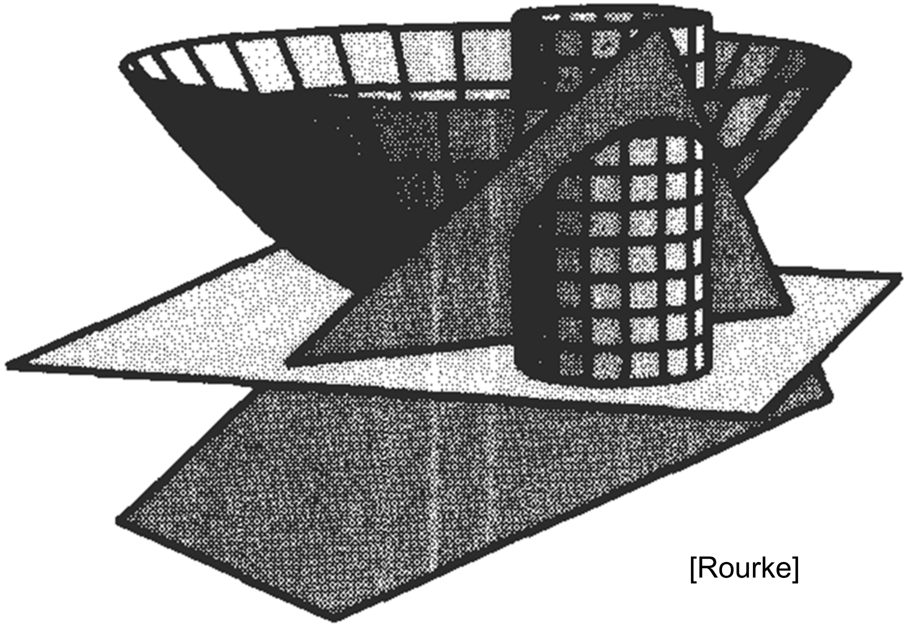

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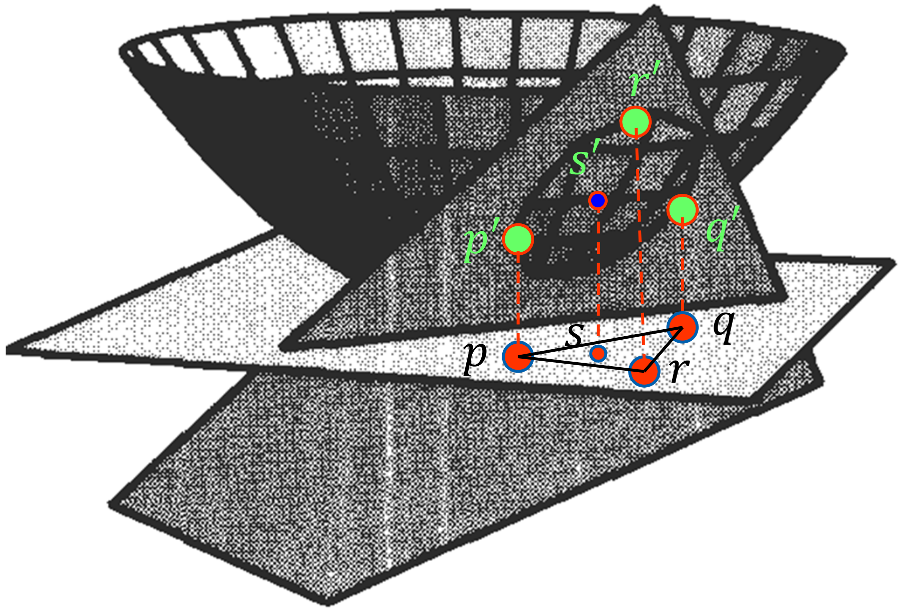

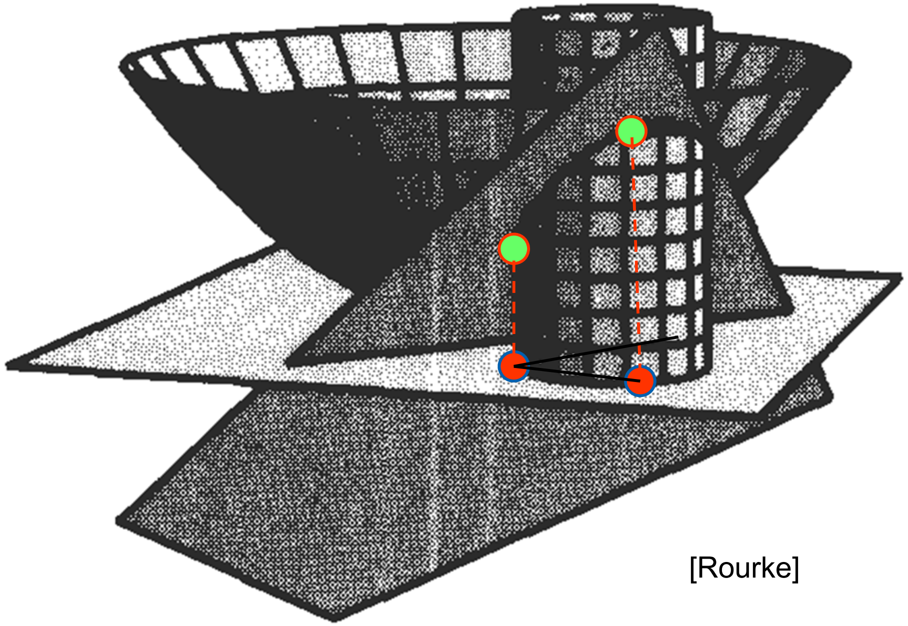

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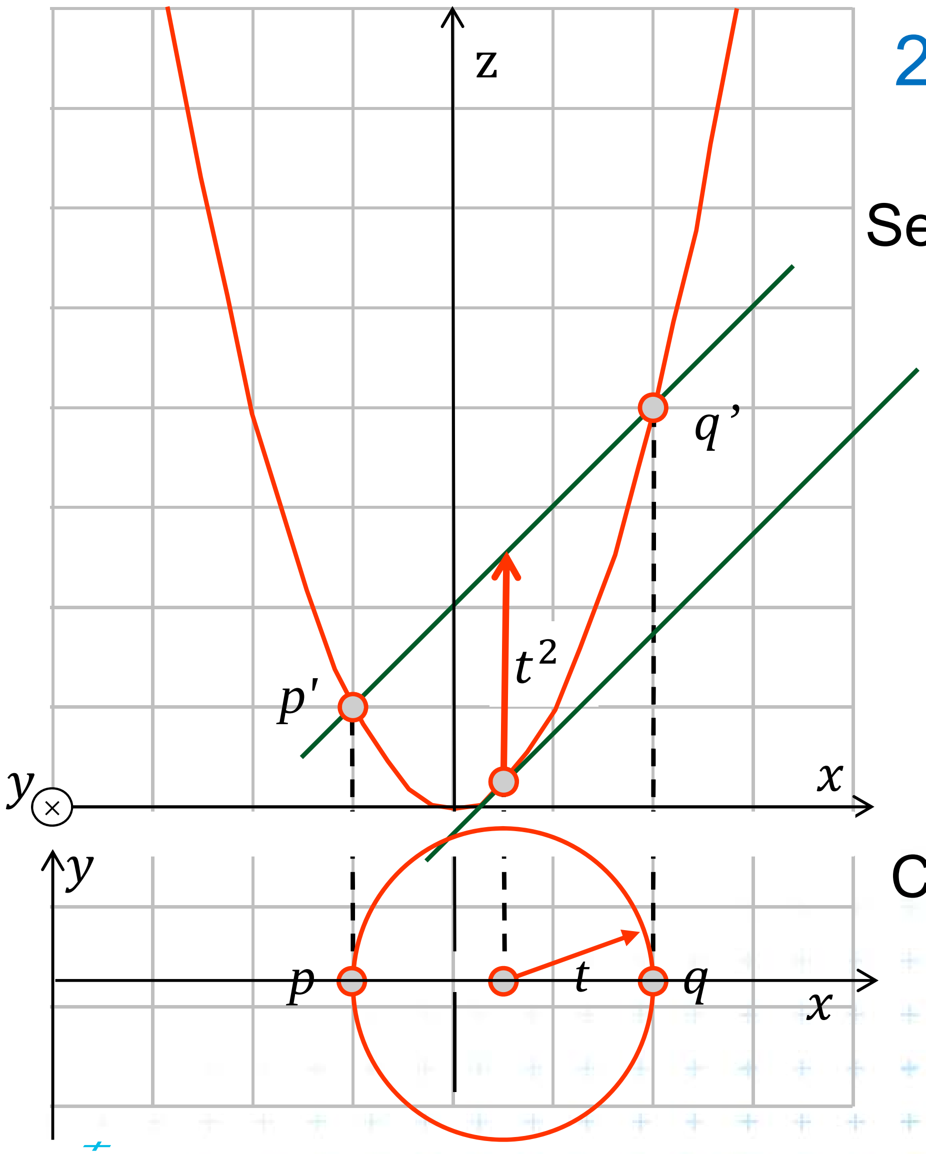
Tangent and secant planes



2D cross section of the paraboloid

Secant plane

Tangent plane

Circle in xy plane with radius t

Tangent plane to paraboloid

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x$$

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

[Mount]

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- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
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- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

- Evaluates to

[Mount]

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- Evaluates to

[Mount]

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- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

- Evaluates to $2a$

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

- Evaluates to $2a$ and

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point $\dfrac{\partial z}{\partial x} = 2x \qquad \dfrac{\partial z}{\partial y} = 2y$
- Evaluates to $2a$ and

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point $\frac{\partial z}{\partial x} = 2x \quad \frac{\partial z}{\partial y} = 2y$
- Evaluates to $2a$ and

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point $\quad \frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$
- Evaluates to $2a$ and $2b$

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point $\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$
- Evaluates to $2a$ and $2b$
- Plane:

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point $\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$
- Evaluates to $2a$ and $2b$
- Plane: $z = 2ax + 2by + \gamma$

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point $\frac{\partial z}{\partial x} = 2x$ $\frac{\partial z}{\partial y} = 2y$
- Evaluates to $2a$ and $2b$
- Plane: $z = 2ax + 2by + \gamma$

[Mount]

Tangent plane to paraboloid

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$
- Paraboloid $z = x^2 + y^2$
- Derivation at this point

$$\frac{\partial z}{\partial x} = 2x \qquad \frac{\partial z}{\partial y} = 2y$$

- Evaluates to $2a$ and $2b$
- Plane: $z = 2ax + 2by + \gamma$

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$$\gamma = -(a^2 + b^2)$$

- Tangent plane through point $(a, b, a^2 + b^2)$

[Mount]

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- Tangent plane through point $(a, b, a^2 + b^2)$

$$z = 2ax + 2by - (a^2 + b^2)$$

[Mount]

Plane intersecting the paraboloid (secant plane)

Non-vertical tangent plane through $(a, b, a^2 + b^2)$

$$z = 2ax + 2by - (a^2 + b^2)$$

- (project to 2D)

- This is a circle projected to 2D with center (a, b):

[Mount]

Plane intersecting the paraboloid (secant plane)

- Non-vertical tangent plane through $(a, b, a^2 + b^2)$

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- Shift this plane t^2 upwards

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- Shift this plane t^2 upwards –> secant plane intersects the paraboloid in an ellipse in 3D

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- Eliminate z (project to 2D)

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[Mount]

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- This is a circle projected to 2D with center (a, b):

$$(x - a)^2 + (y - b)^2 = t^2$$

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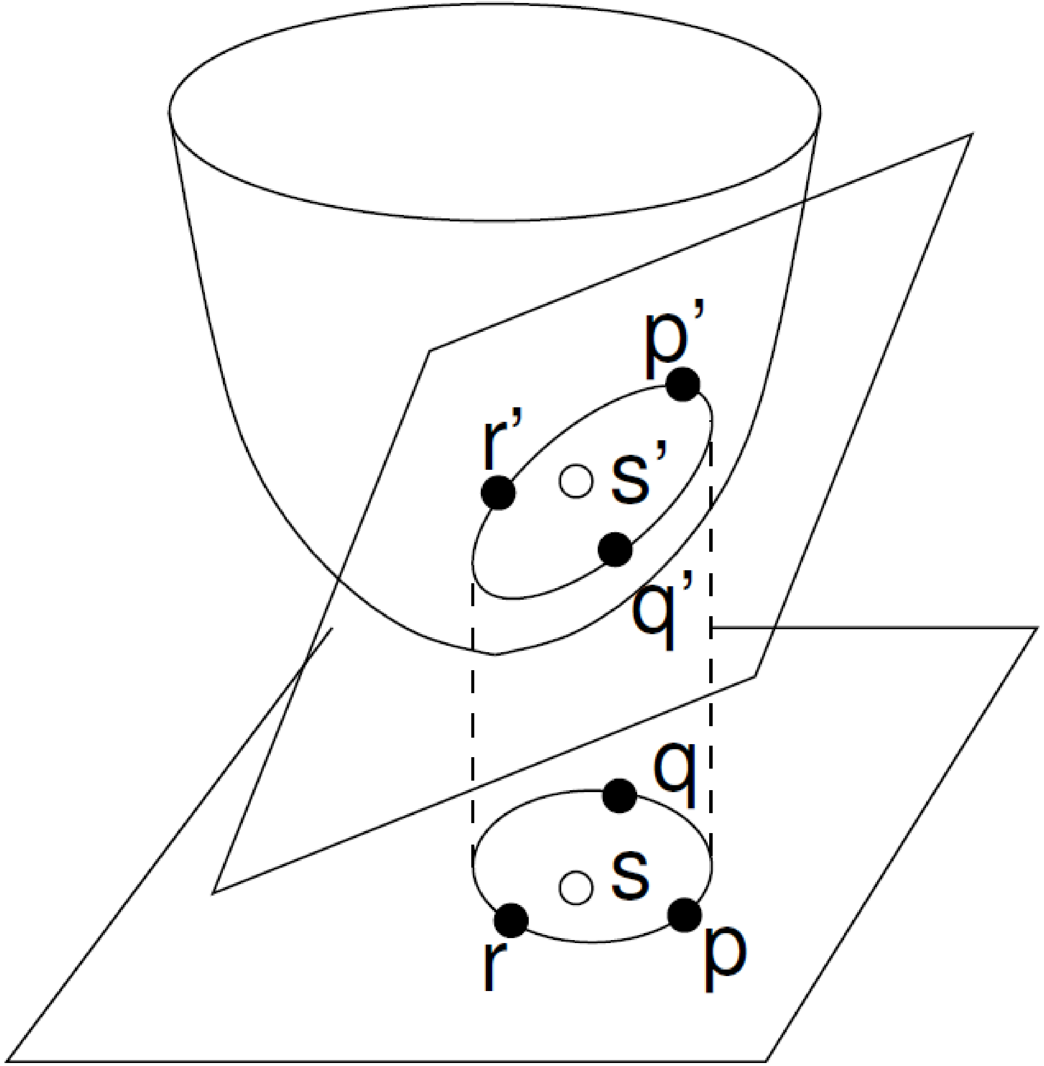
- This is a circle projected to 2D with center (a, b):

$(x - a)^2 + (y - b)^2 = t^2$ and radius t

[Mount]

Secant plane defined by three points



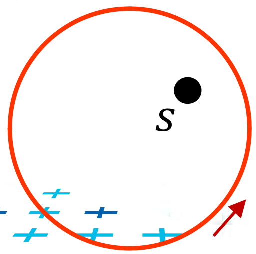
DT in 2D → CH in 3D

[Mount]

Test inCircle – meaning in 3D

- Points *p,q,r* are counterclockwise in the plane
- Test, if *s* lies in the circumcircle of $\triangle pqr$ is equal to
  - = test, weather *s'* lies within a lower half space of the plane passing through *p',q',r'* (3D)
  - = test, if quadruple *p',q',r',s'* is positively oriented (3D)
  - = test, if *s lies* to the left of the oriented circle through *pqr* (2D)

$$\mathrm{in}(p,q,r,s) = \det \begin{pmatrix} p_x & p_y & p_x^2+p_y^2 & 1 \\ q_x & q_y & q_x^2+q_y^2 & 1 \\ r_x & r_y & r_x^2+r_y^2 & 1 \\ s_x & s_y & s_x^2+s_y^2 & 1 \end{pmatrix} > 0.$$



[Mount]

Delaunay triangulation and inCircle test

- DT splits each quadrangle by one of its two diagonals
- For a valid diagonal, the fourth point is not inCircle
  - => the fourth point is right from the oriented circumcircle (outside)
  - => inCircle(….) < 0 for CCW orientation
- inCircle(P,Q,R,S) = inCircle(P,R,S,Q) = – inCircle(P,Q,S,R) = – inCircle(S,Q,R,P)

inCircle(…) > 0

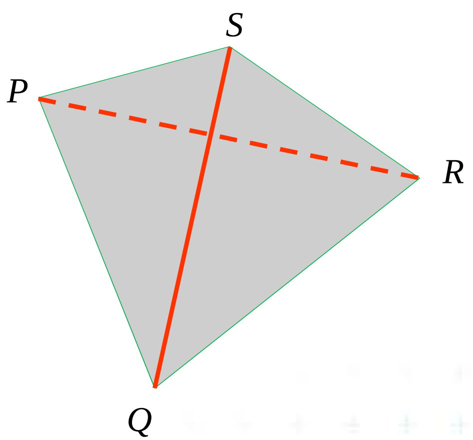
P - - - - - - *R*

Invalid diagonal

inCircle(…) < 0

S —— *Q*

Valid diagonal



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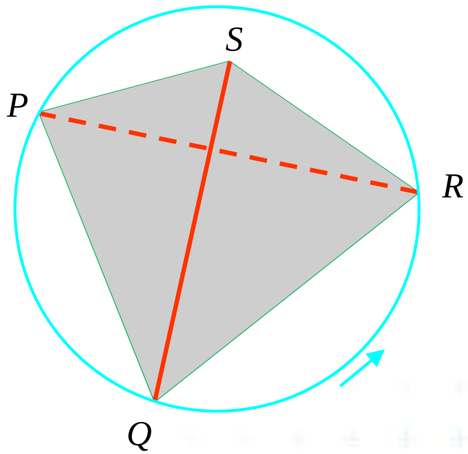
inCircle(…) > 0	inCircle(…) < 0
P - - - - - *R*	*S* ——— *Q*
Invalid diagonal	Valid diagonal



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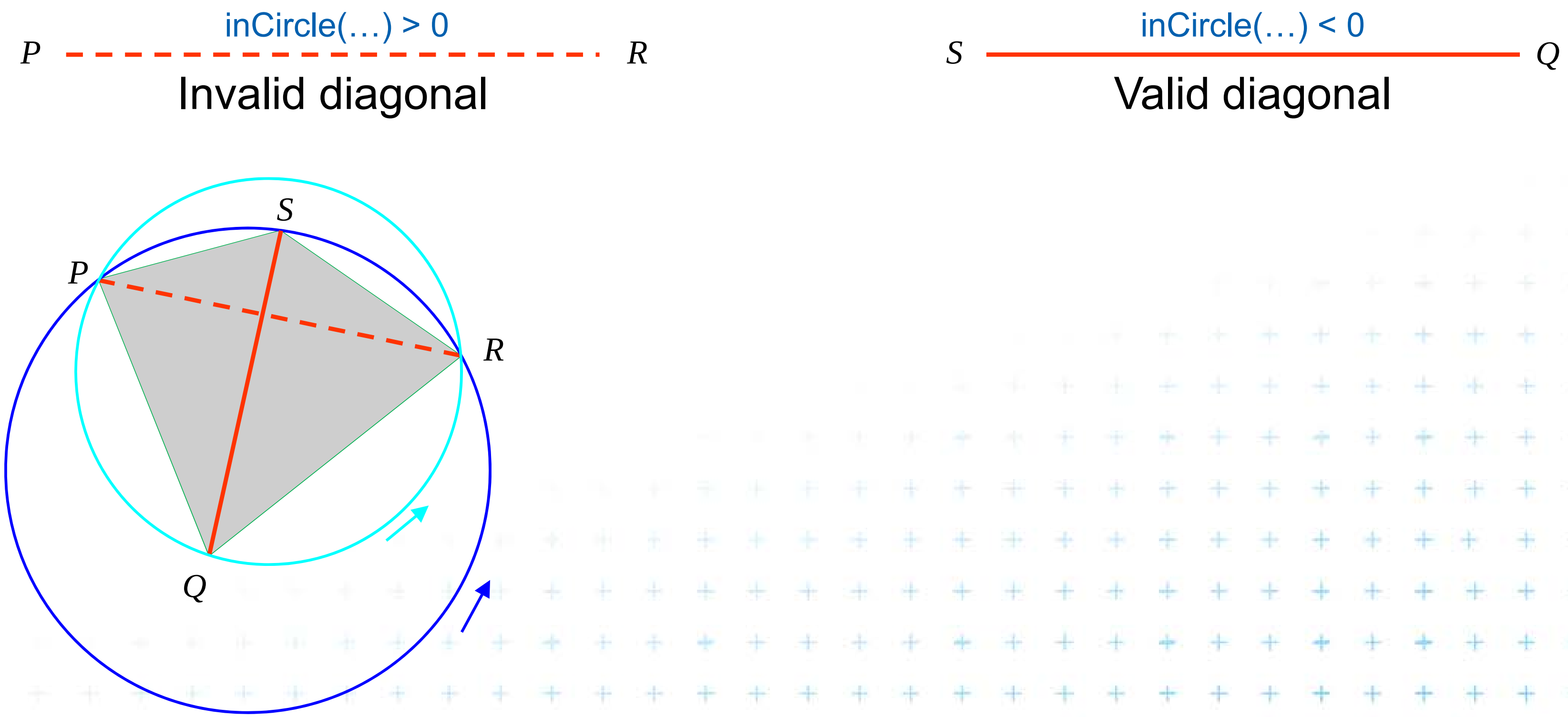

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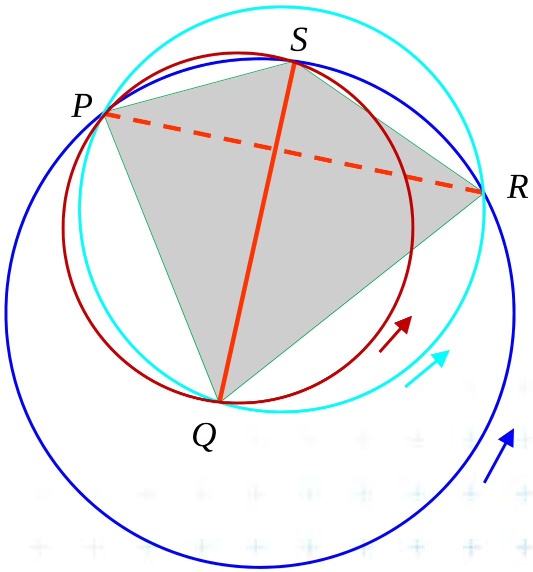

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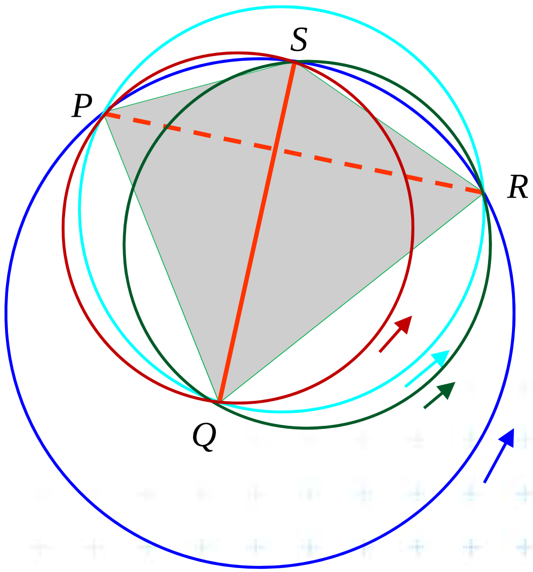
P - - - - - - - - - - - *R*

Invalid diagonal

inCircle(…) < 0

S ——————— *Q*

Valid diagonal



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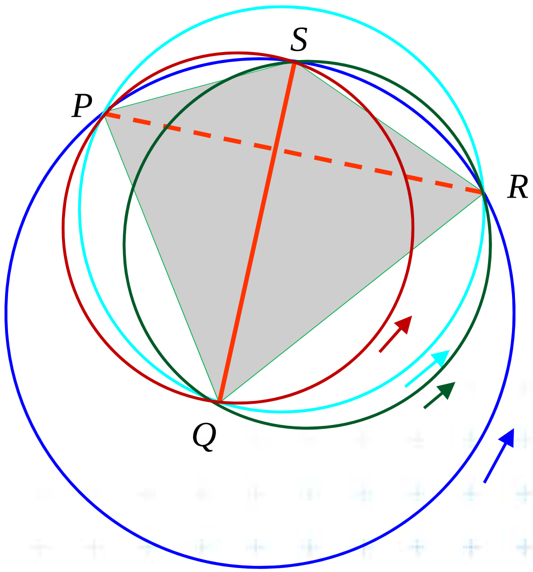
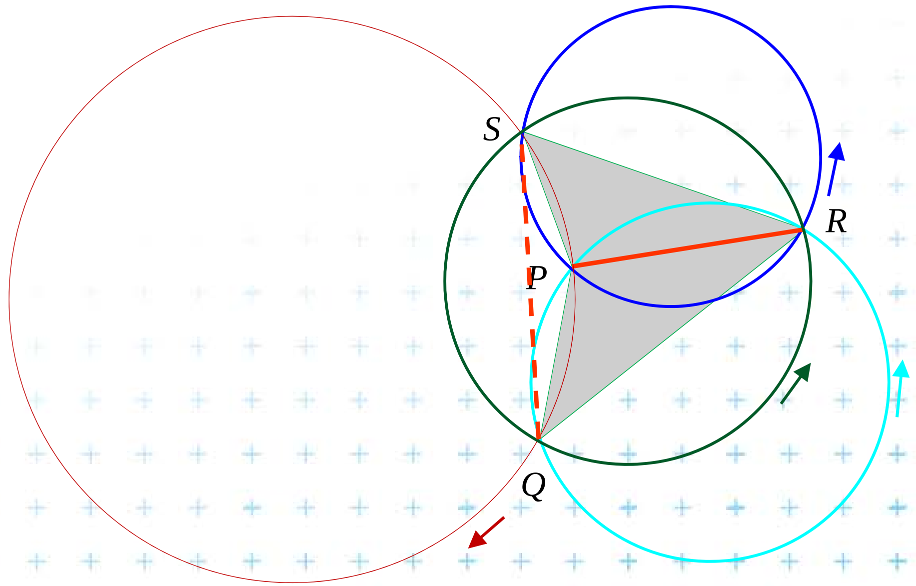
inCircle(…) > 0	inCircle(…) < 0
P - - - - - - R	S ——— Q
Invalid diagonal	Valid diagonal

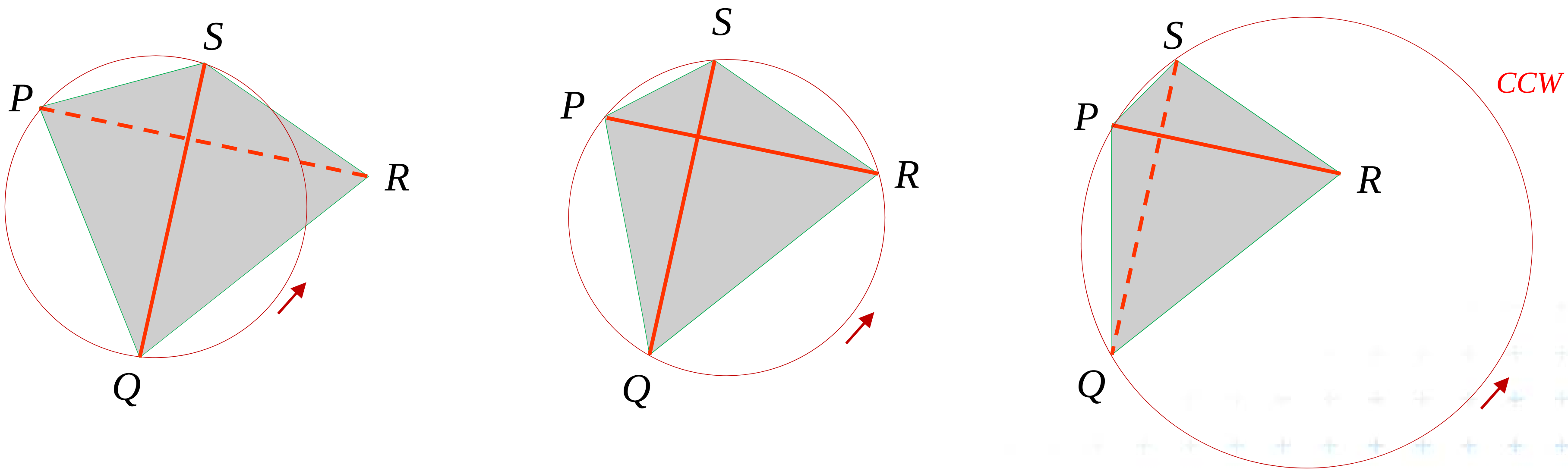




inCircle test detail

Point *P* moves right toward point *R*

We test position of *R* in relation to oriented circle (*P*,*Q*,*S*)



inCircle(P,Q,S,R) < 0	inCircle(P,Q,S,R) = 0	inCircle(P,Q,S,R) > 0
R is right (out)	*R* is on the circle	*R* is left (in)
diagonal QS is valid	both QS and PR are valid	QS is invalid

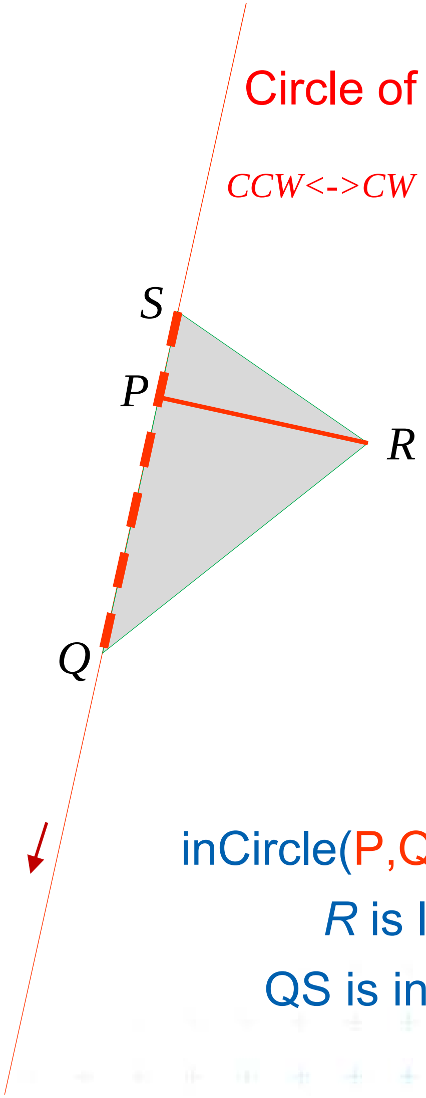
Invalid diagonal

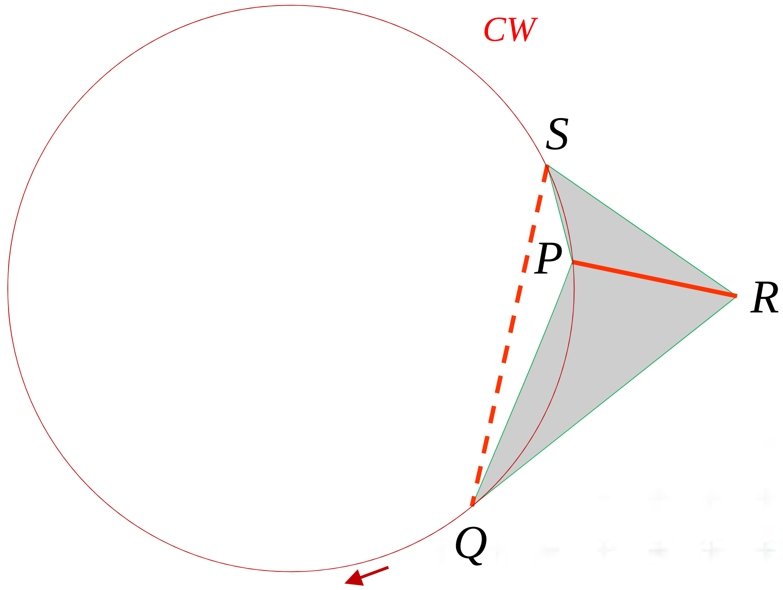
Valid diagonal

inCircle test detail

Circle of infinite diameter

The circle flipped its orientation





inCircle(P,Q,S,R) > 0

R is left

QS is invalid

inCircle(P,Q,S,R) > 0

R is left

QS is invalid

Invalid diagonal

Valid diagonal

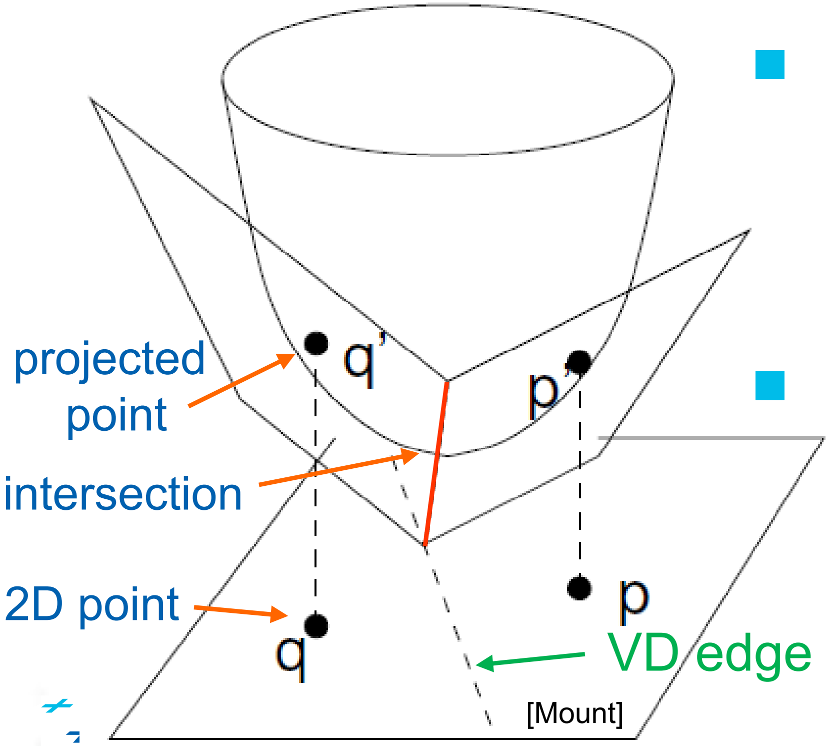
An the Voronoi diagram?

- VD and DT are dual structures
- Points and lines in the plane
  are dual to
  points and planes in 3D space
- VD of points in the plane
  can be transformed to
  intersection of halfspaces in 3D space

Voronoi diagram as upper envelope in R^{d+1}

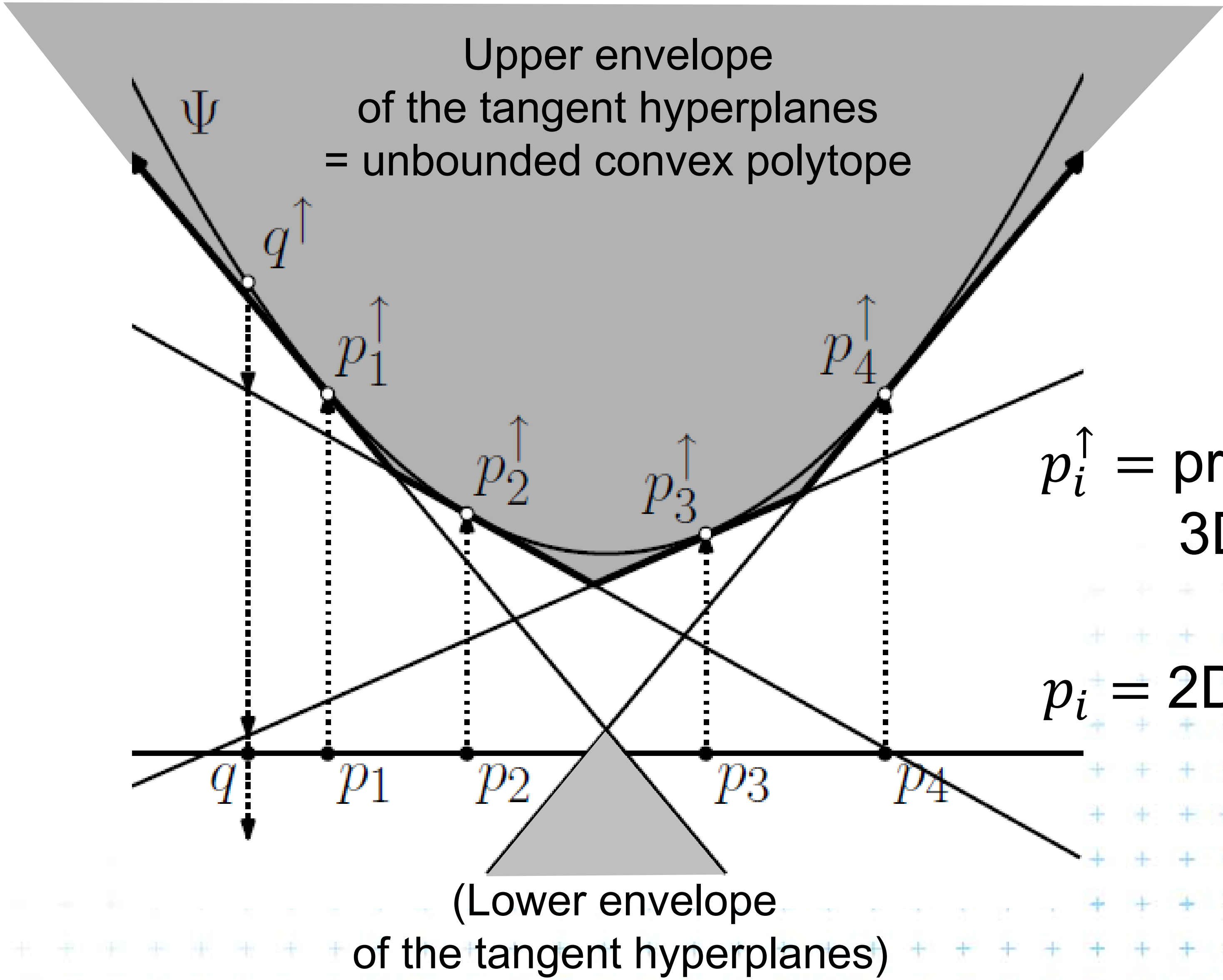
- For each point $p = (a, b)$ a tangent plane $H(p)$ to the paraboloid is $z = 2ax + 2by - (a^2 + b^2)$
- $H^+(p)$ is the set of points above this tangent plane

$$H^+(p) = \{(x, y, z) \mid z \geq 2ax + 2by - (a^2 + b^2)\}$$



- VD of points in the plane can be computed as intersection of halfspaces $H^+(p_i)$ in 3D
- This intersection of halfspaces = unbounded convex polyhedron = upper envelope of halfspaces $H^+(p_i)$

Upper envelope of planes (a 2D cross section)



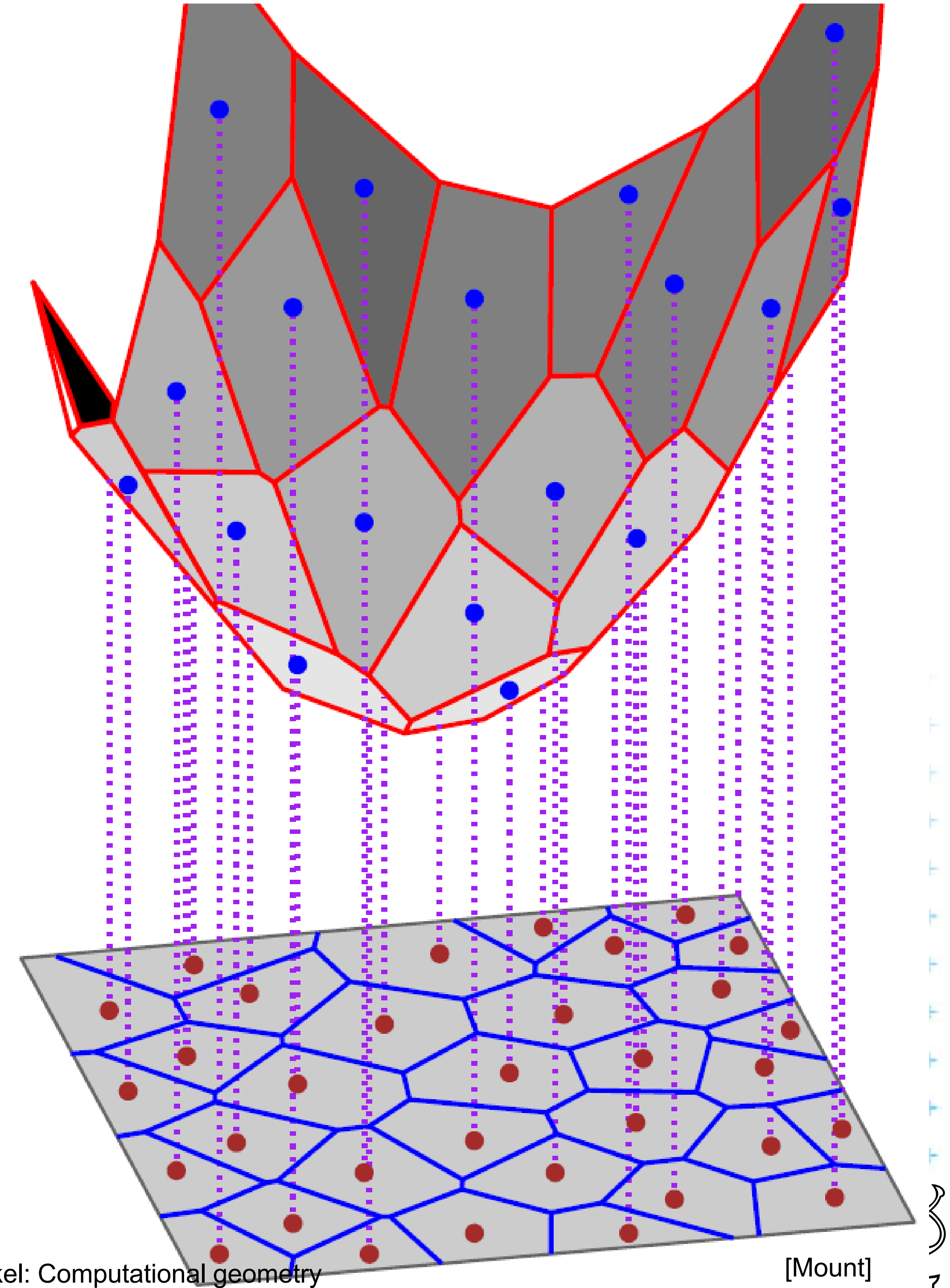
$p_i^{\uparrow}$ = projected 3D points

p_i = 2D points

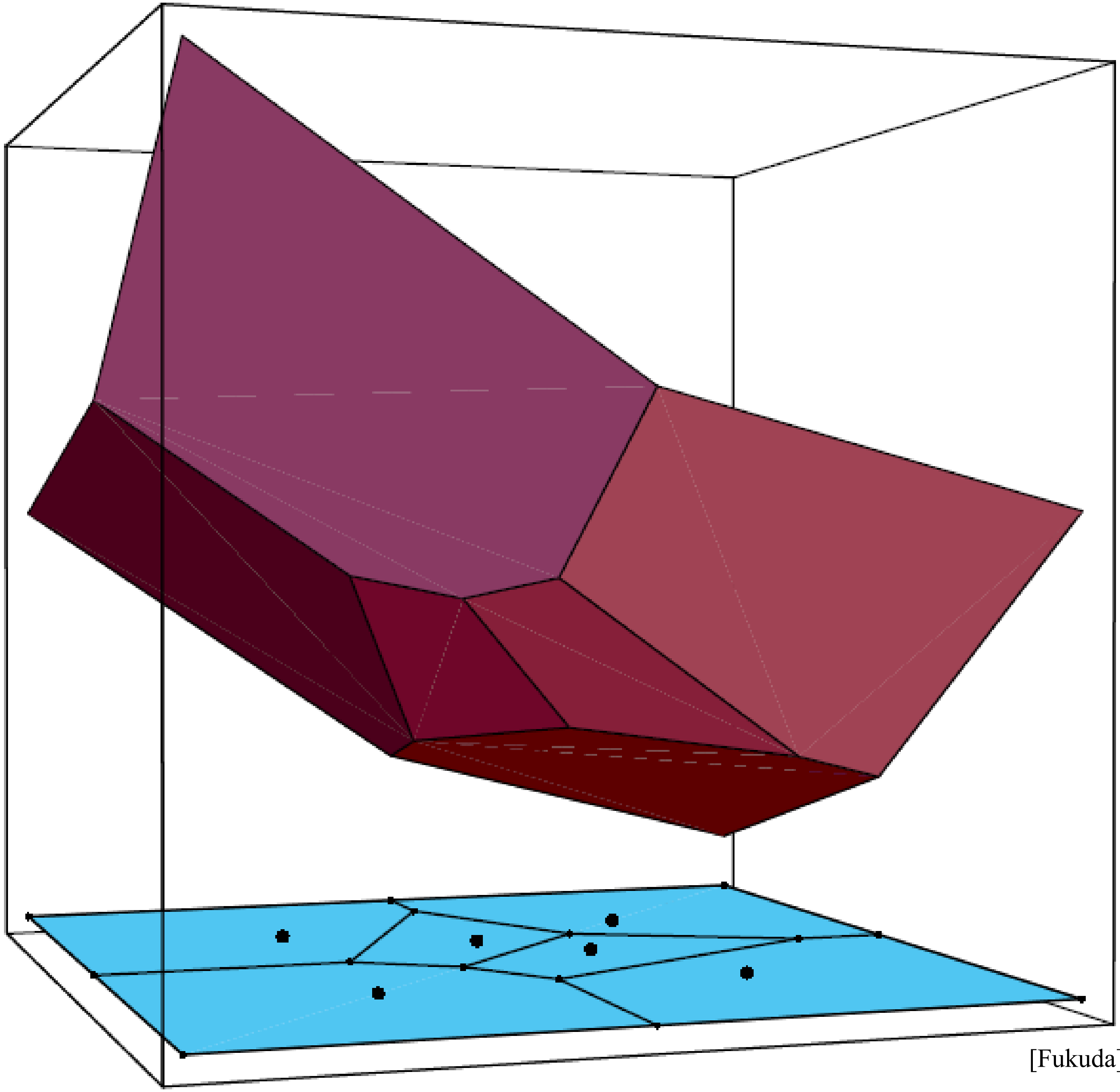
[Mount]

Projection to 2D

- Upper envelope of tangent hyperplanes (through sites projected upwards to the cone)
- Projected to 2D gives Voronoi diagram

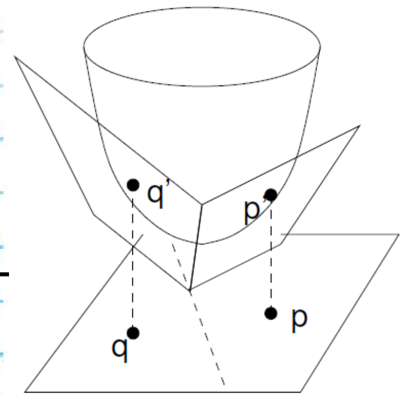

[Mount]

Voronoi diagram as upper envelope in 3D



[Fukuda]

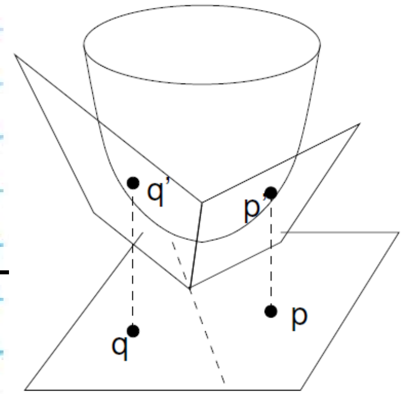
Derivation of projected Voronoi edge



[Mount]

Derivation of projected Voronoi edge

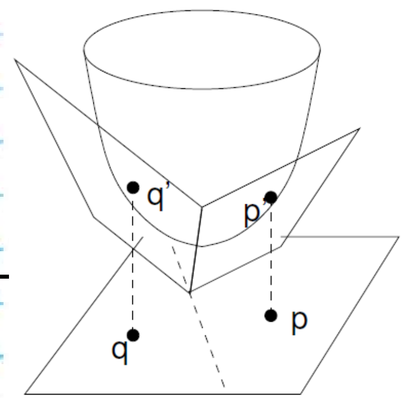
- 2 points: $p = (a, b)$ and in the plane



[Mount]

Derivation of projected Voronoi edge

- 2 points: $p = (a, b)$ and $q = (c, d)$ in the plane

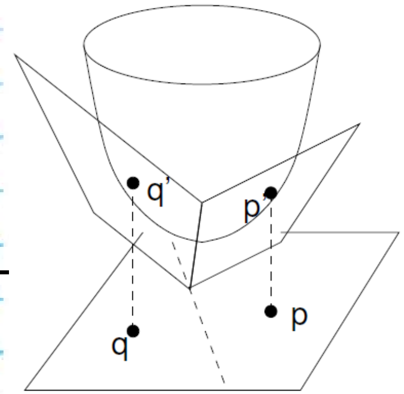


[Mount]

Derivation of projected Voronoi edge

- 2 points: $p = (a, b)$ and $q = (c, d)$ in the plane

  2 tangent planes
  to paraboloid



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Derivation of projected Voronoi edge

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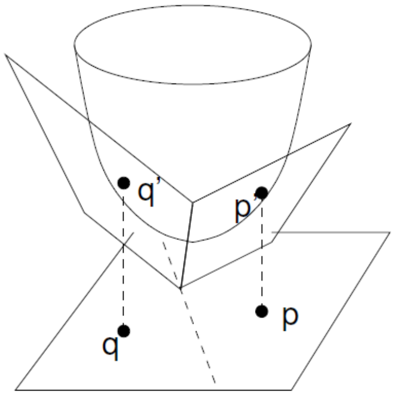
  to paraboloid

[Mount]



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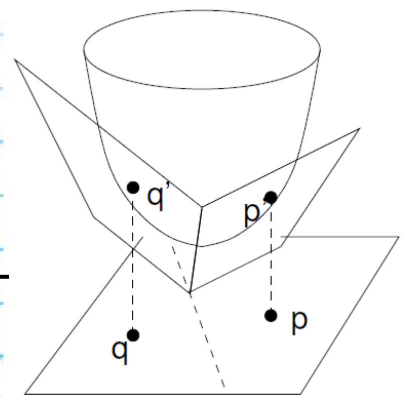
  $$z = 2ax + 2by - (a^2 + b^2)$$
  $$z = 2cx + 2dy - (c^2 + d^2)$$



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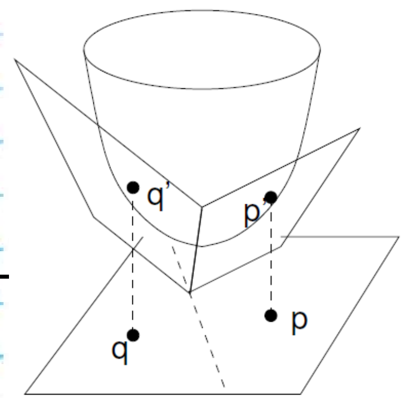
- Intersect the planes, project onto xy (eliminate *z*)



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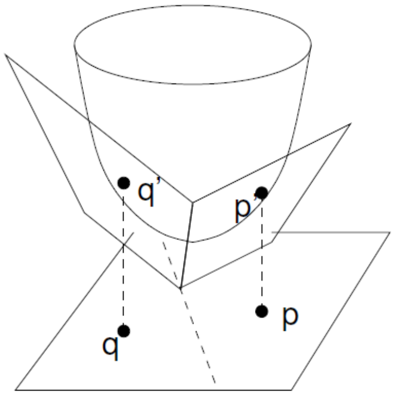

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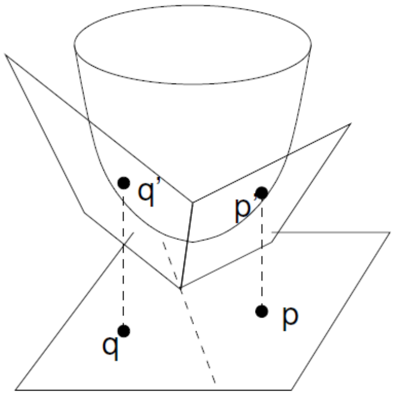
  $$x(2a - 2c) + y(2b - 2d) = (a^2 - c^2) + (b^2 - d^2)$$



[Mount]

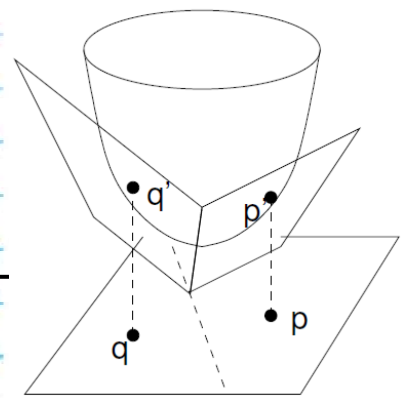
Derivation of projected Voronoi edge

- 2 points: $p = (a, b)$ and $q = (c, d)$ in the plane

  2 tangent planes to paraboloid
  $$z = 2ax + 2by - (a^2 + b^2)$$
  $$z = 2cx + 2dy - (c^2 + d^2) \qquad /(-)$$

- Intersect the planes, project onto xy (eliminate *z*)
  $$x(2a - 2c) + y(2b - 2d) = (a^2 - c^2) + (b^2 - d^2)$$
- This line passes through midpoint between p and q

[Mount]



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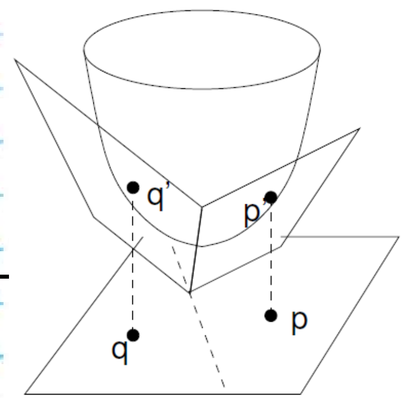

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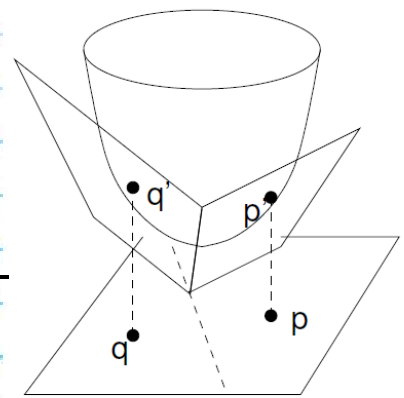

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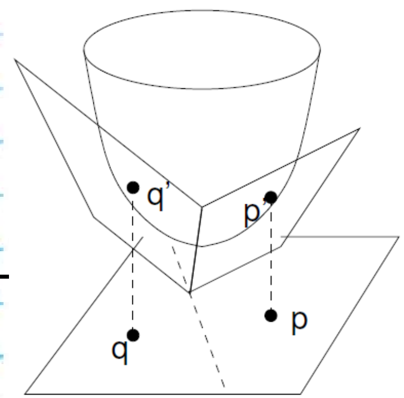

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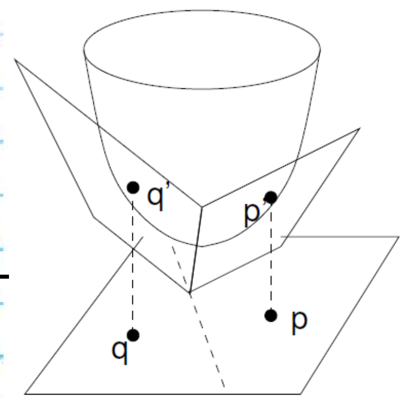
[Mount]



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