VORONOI DIAGRAM
PART II

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Reiberg] and [Nandy]

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Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

Summary of the VD terms

- Site = input point, line segment, …
- Cell = area around the site, in VD_1 the nearest to site
- Edge, arc = part of Voronoi diagram
  (border between cells)
- Vertex = intersection of VD edges

Incremental construction – bounded cell

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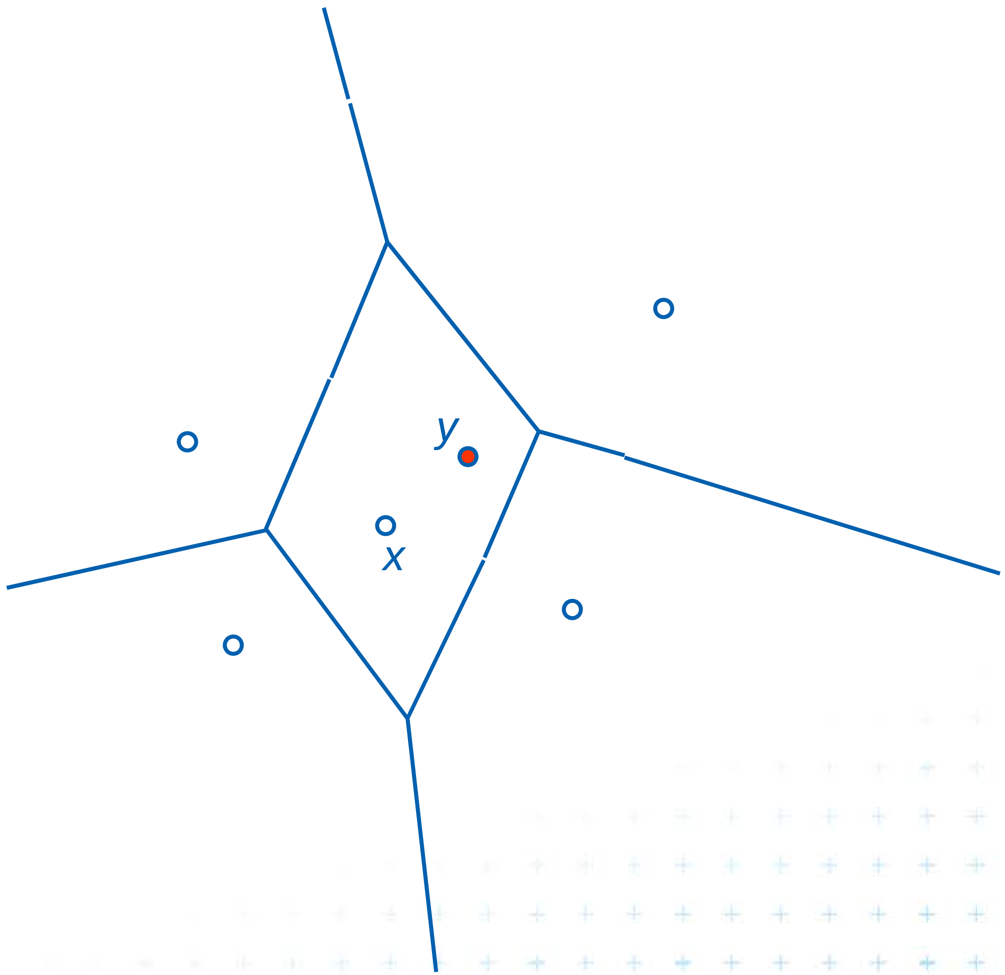
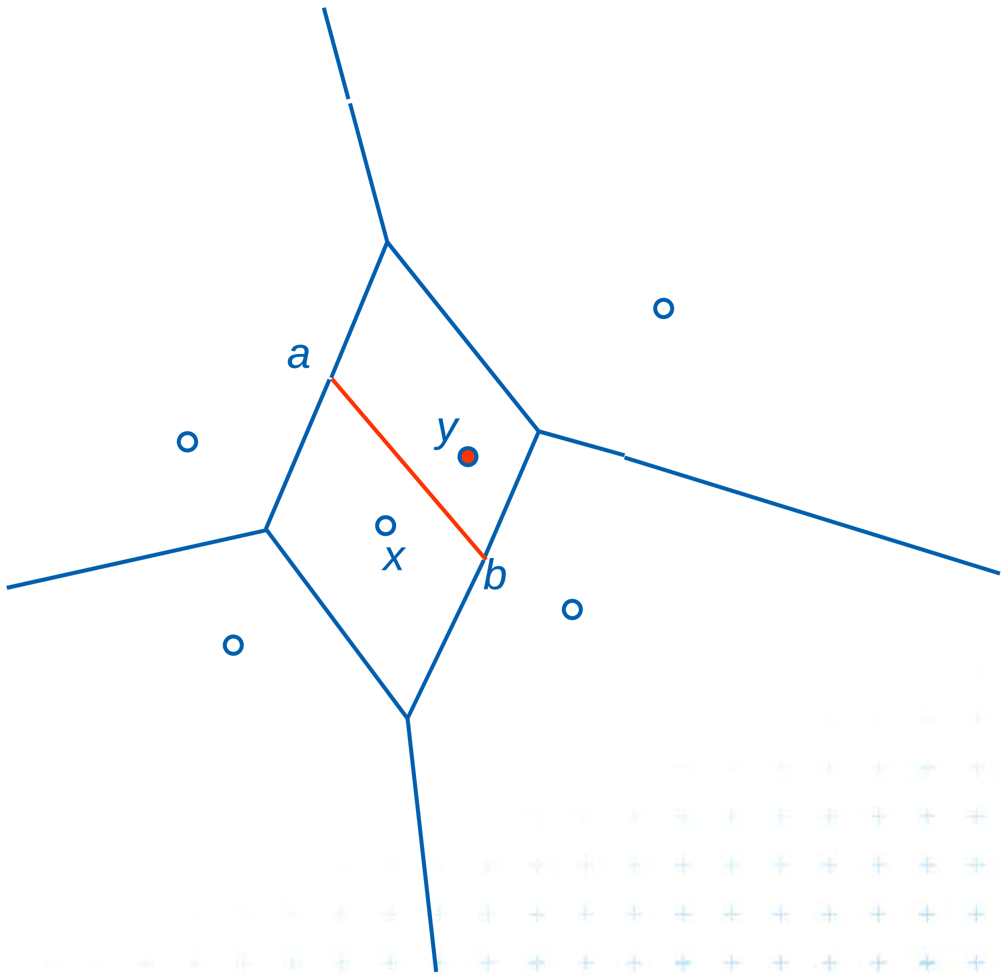


Incremental construction – bounded cell



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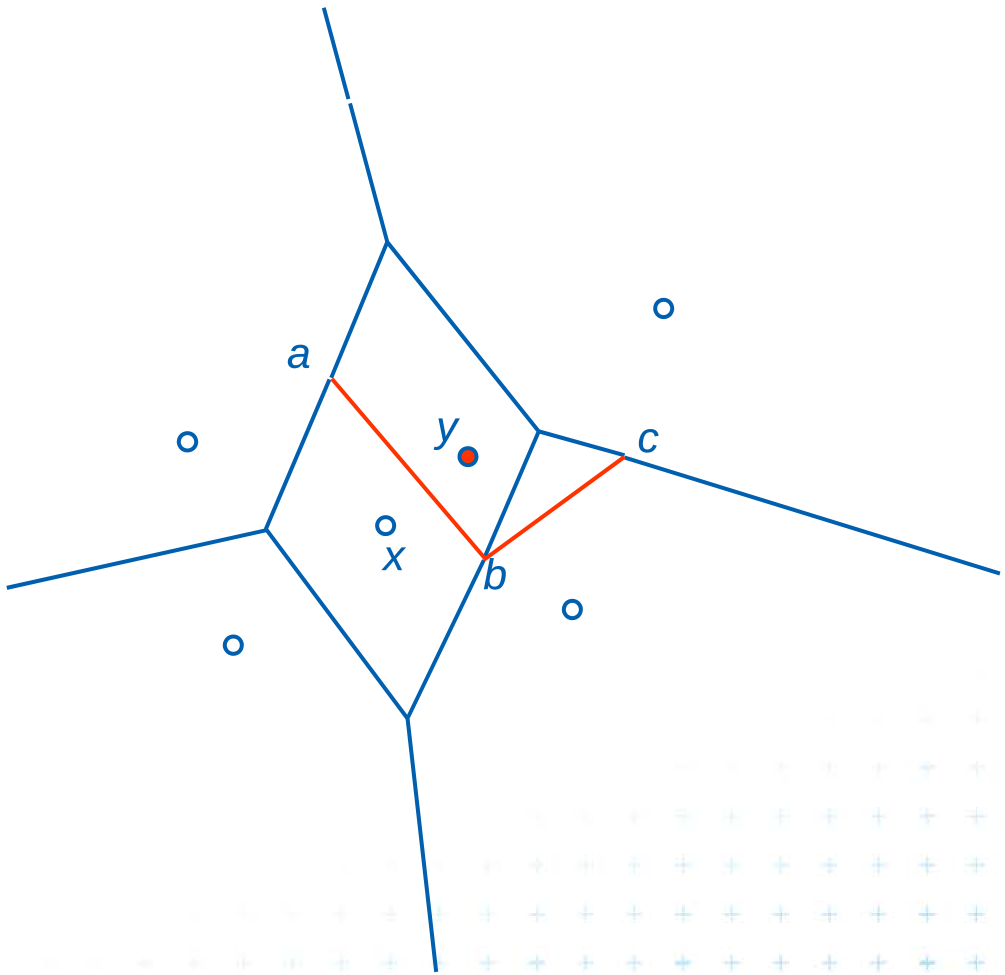

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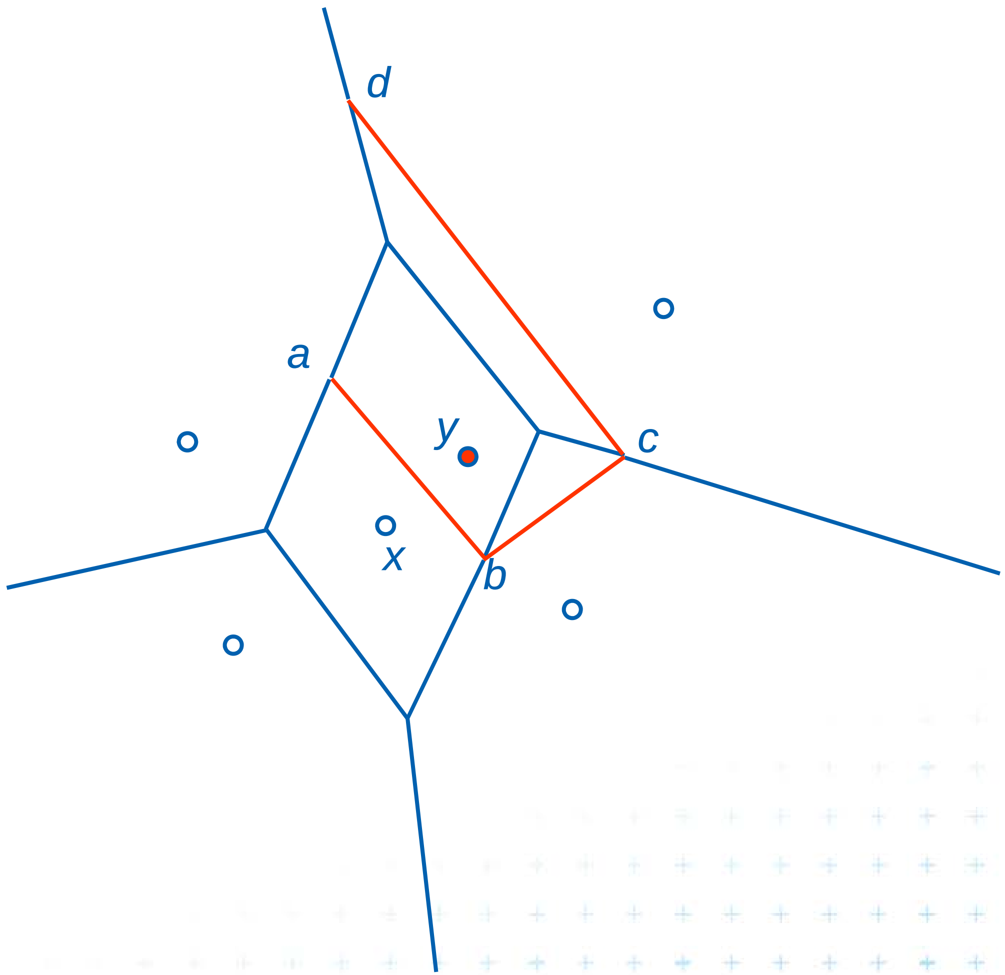

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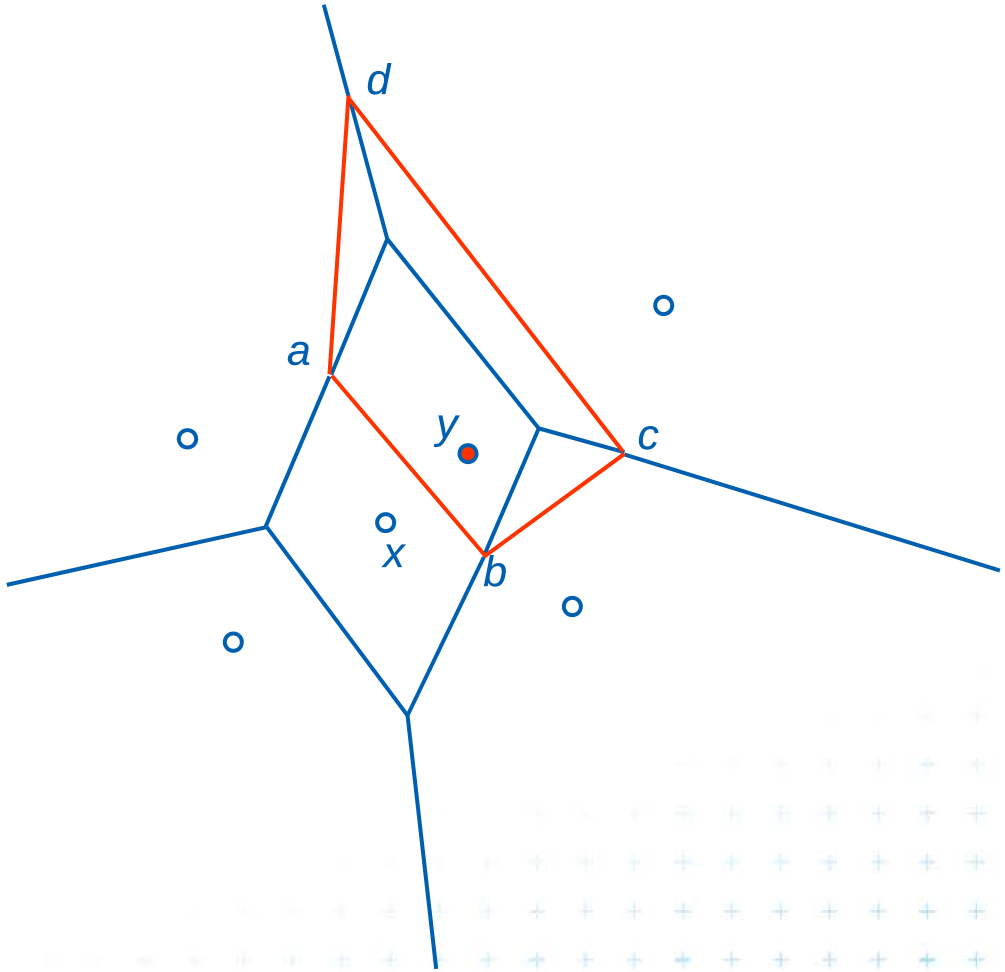

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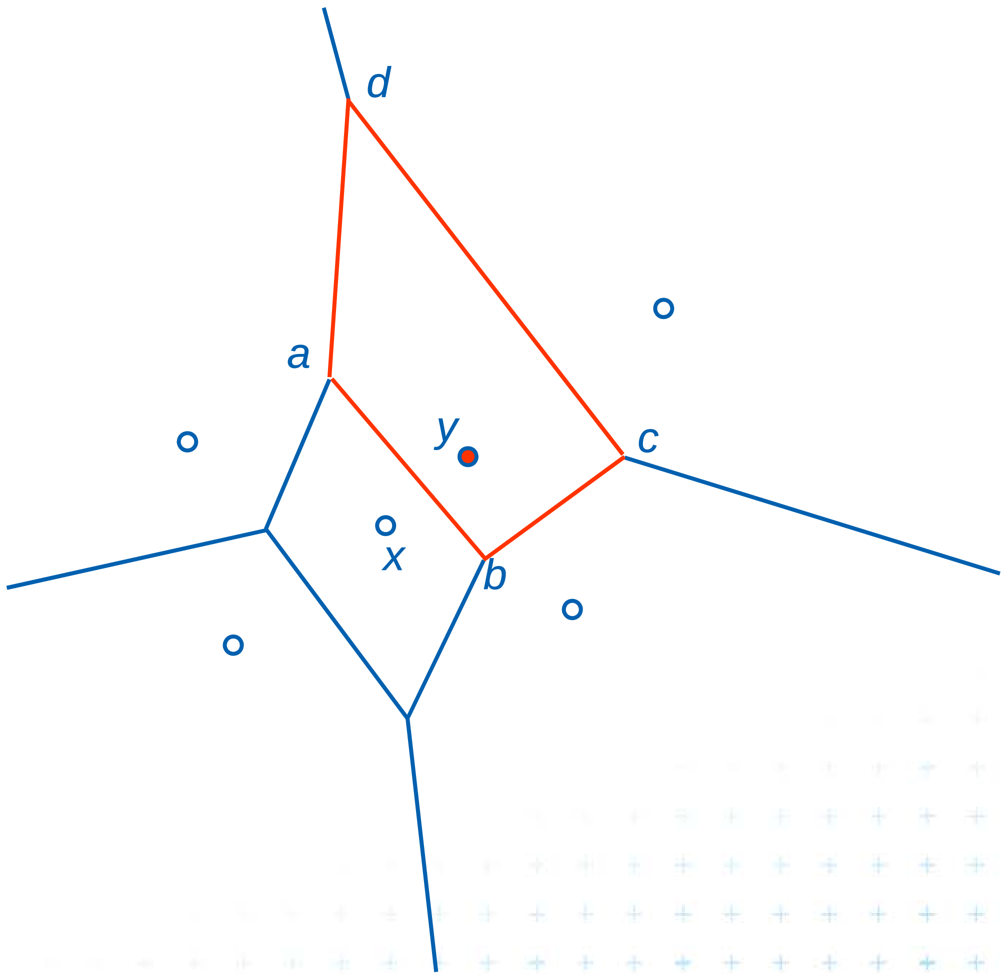

Incremental construction – unbounded cell

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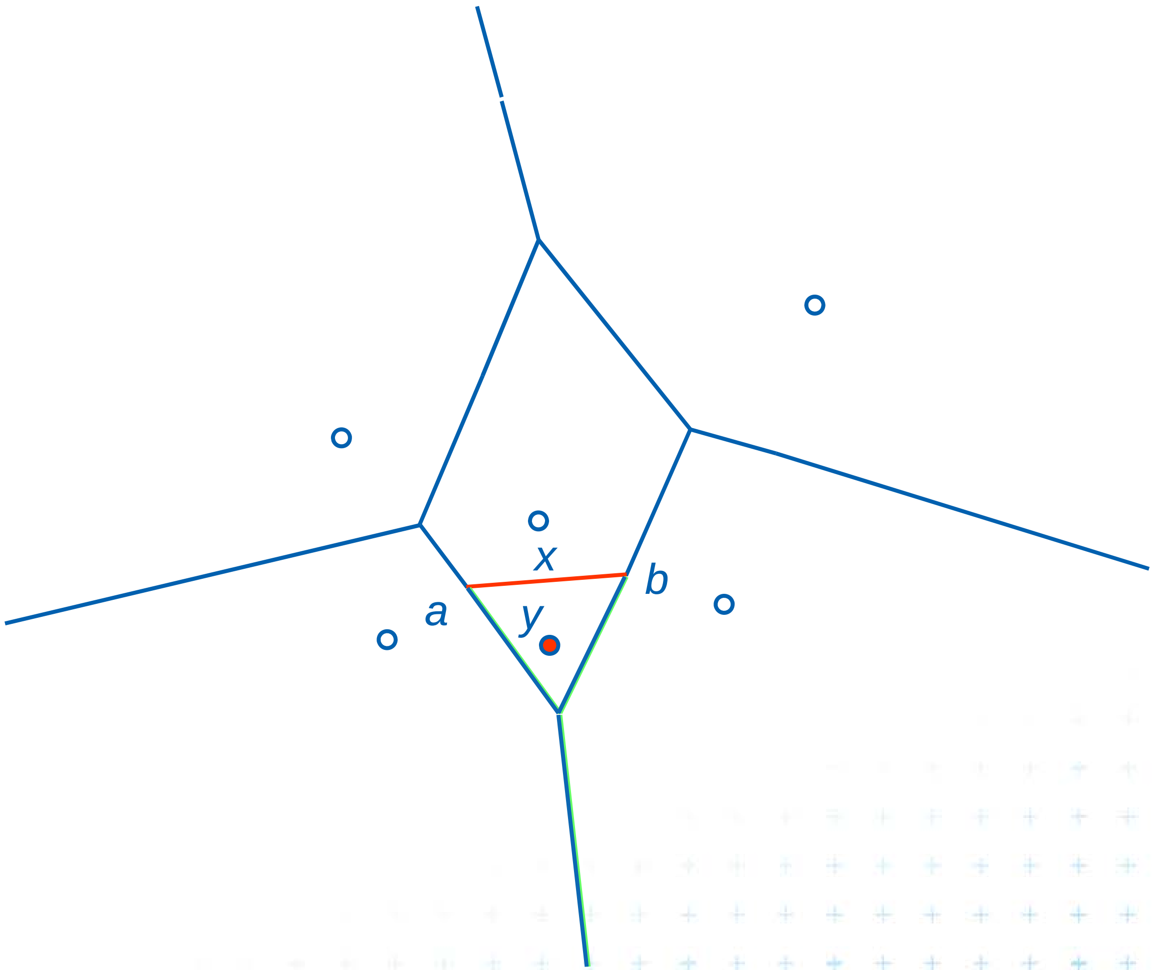

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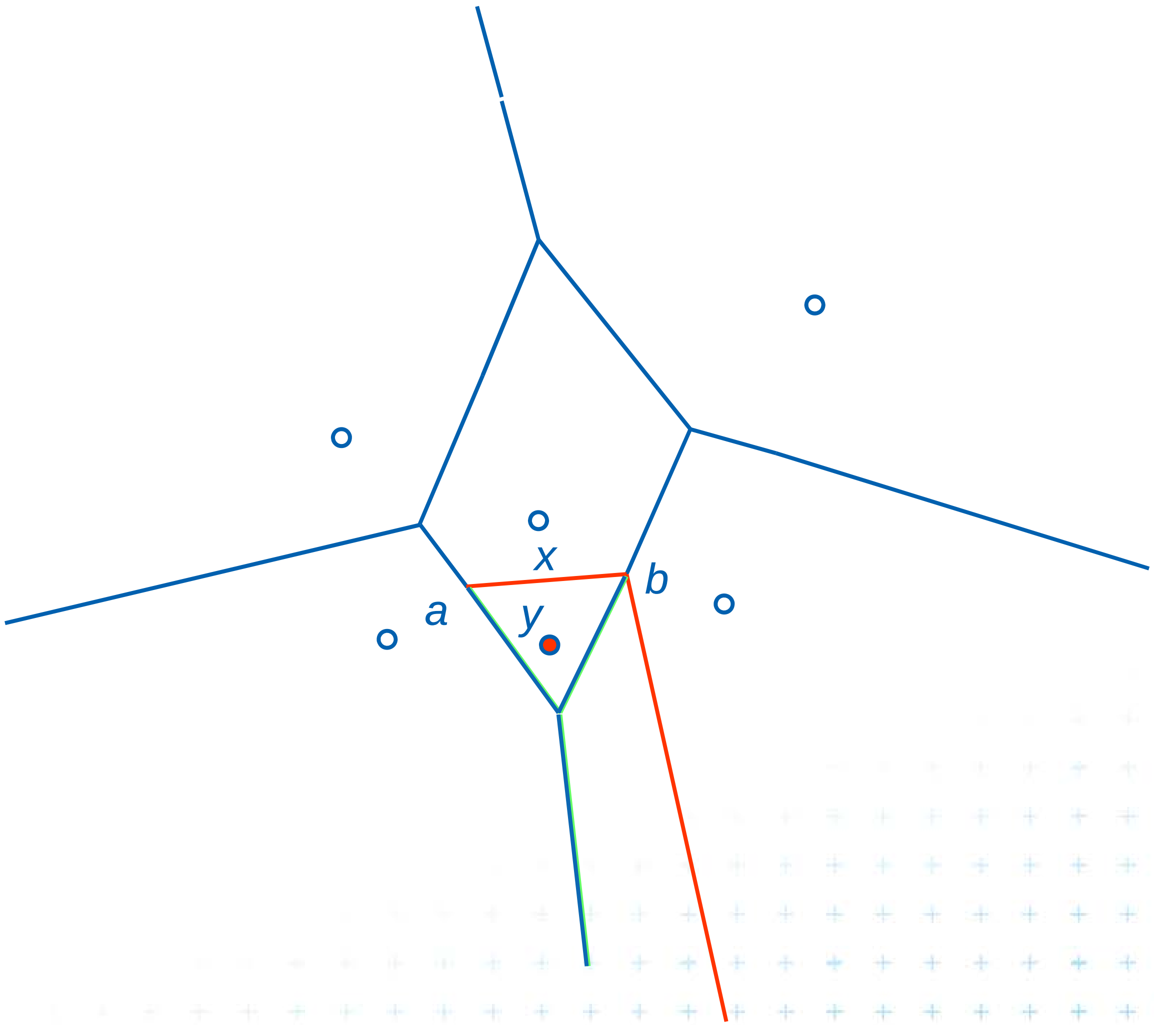

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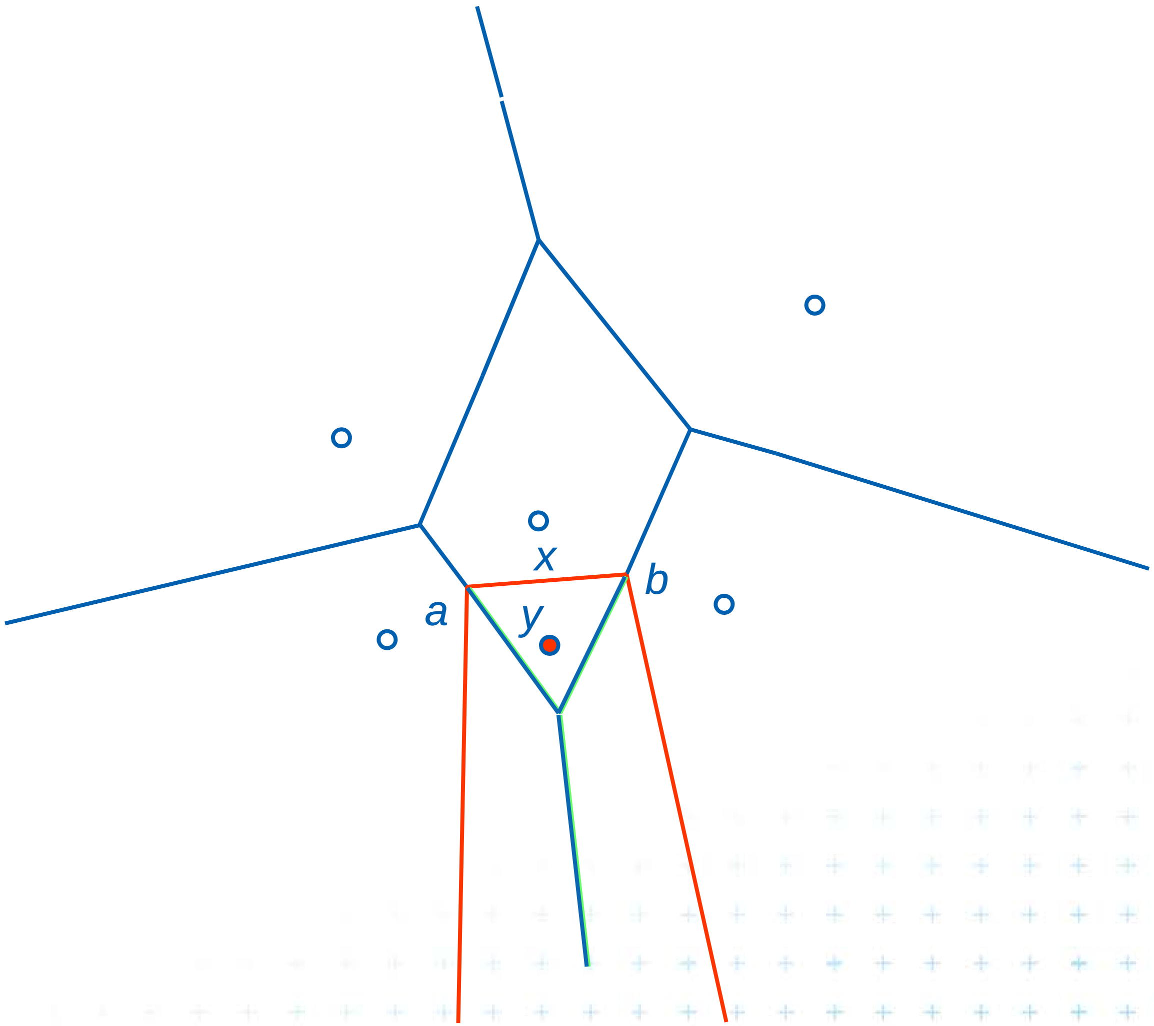

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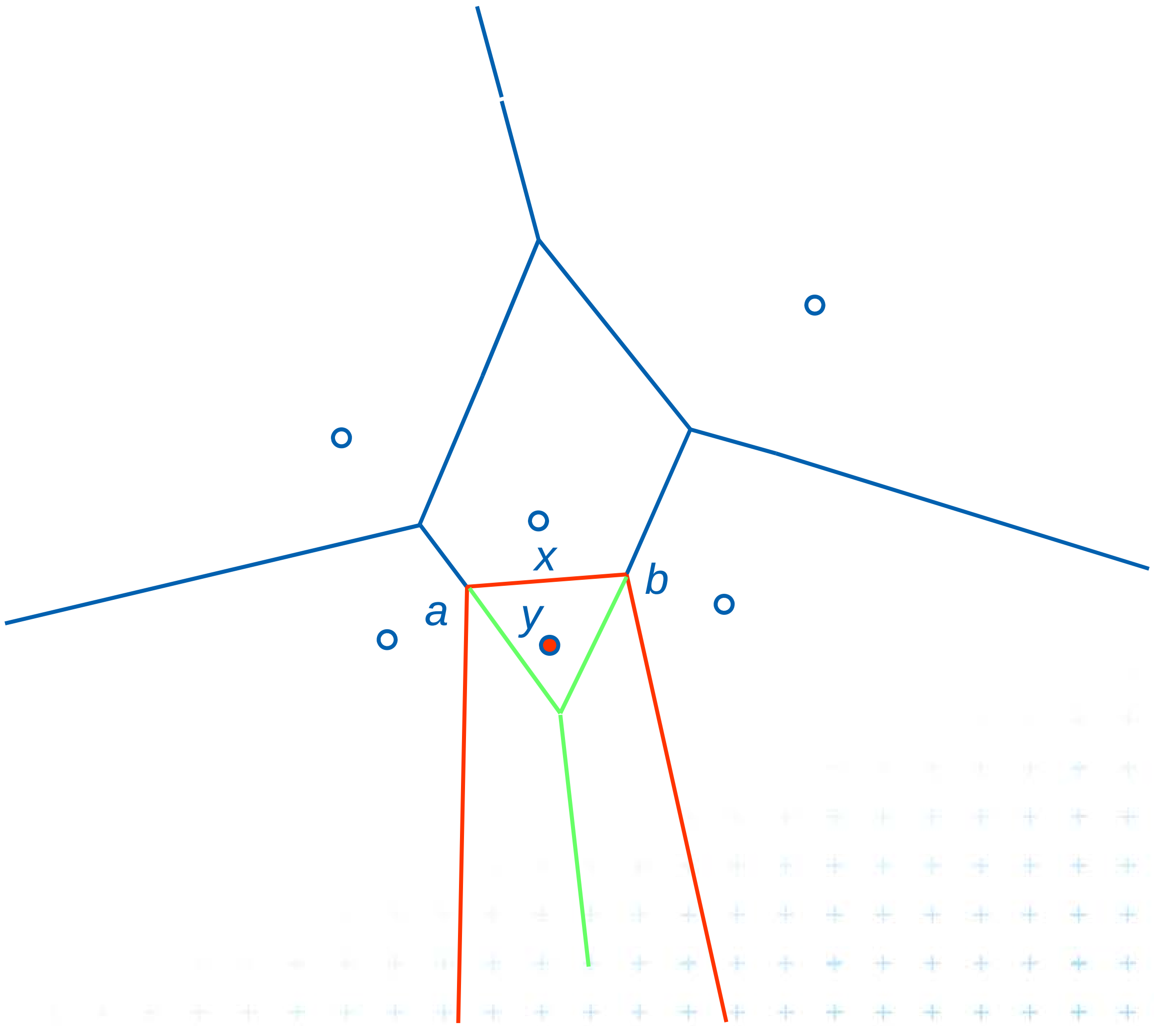

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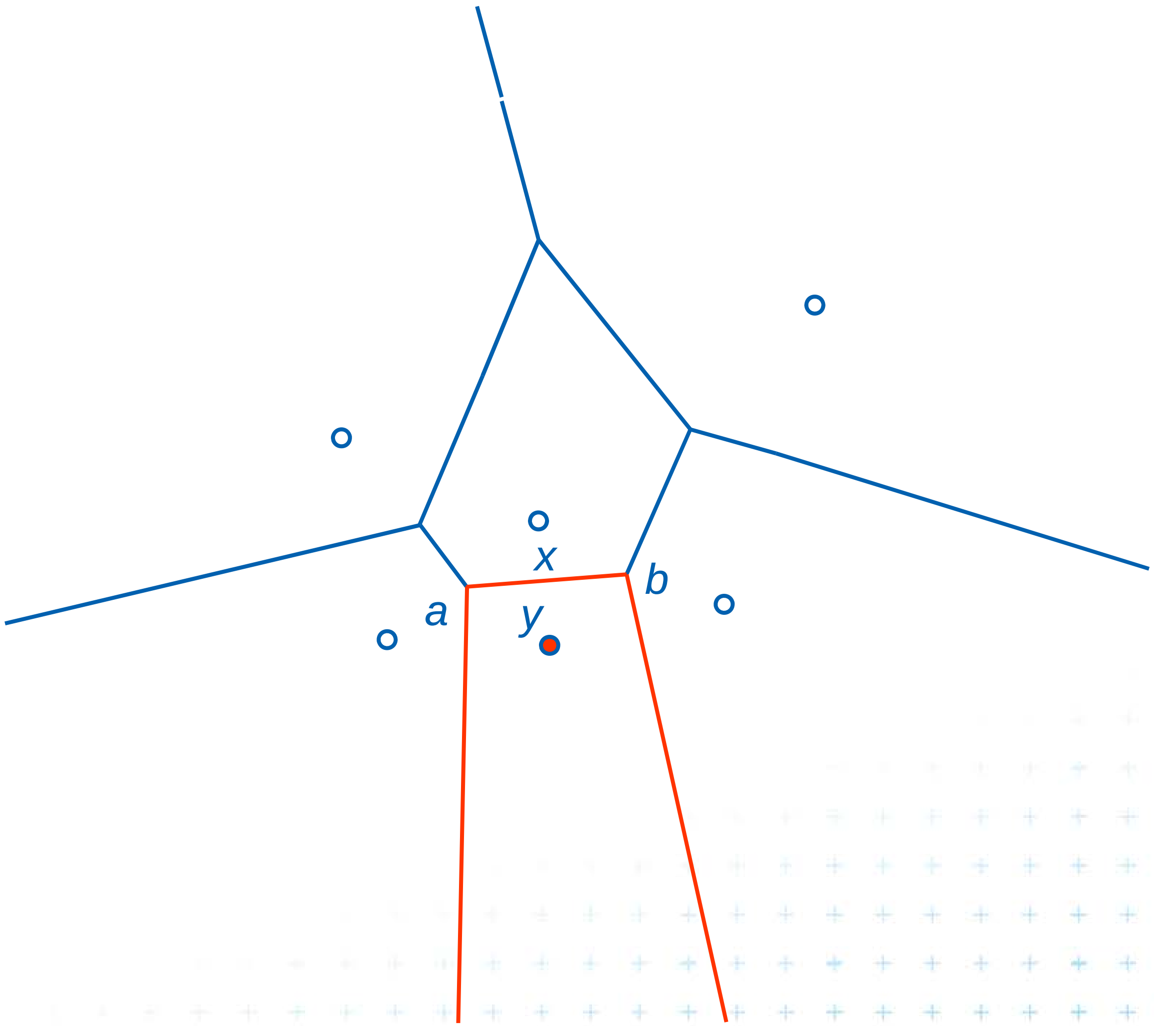

Incremental construction algorithm

InsertPoint(S, Vor(S), y) … **y = a new site**

Input: Point set S, its Voronoi diagram, and inserted point $y \notin S$

Output: VD after insertion of y

1. Find the site x in which cell point y falls, …O(log n)
2. Detect the intersections $\{a,b\}$ of bisector $L(x,y)$ with cell x boundary
   => create the first edge $e = ab$ on the border of site x …O(n)
3. Set start intersection point $p = b$, set new intersection c = undef
4. site z = neighbor site across the border with intersection b …O(1)
5. **while**(exists(p) *and* $c \neq a$) // trace the bisectors from b in one direction
   a. Detect intersection c of $L(y,z)$ with border of cell z
   b. Report Voronoi edge pc
   c. $p = c$, z=neighbor site across border with intersec. c

   …O(n^2)
5. **if**($c \neq a$) **then** // trace the bisectors from a in other direction
   a. p = a
   b. *Similarly as in steps 3,4,5 with a*

O(n^2) worst-case, O(n) expected time for some distributions

DCGI

Voronoi diagram of line segments

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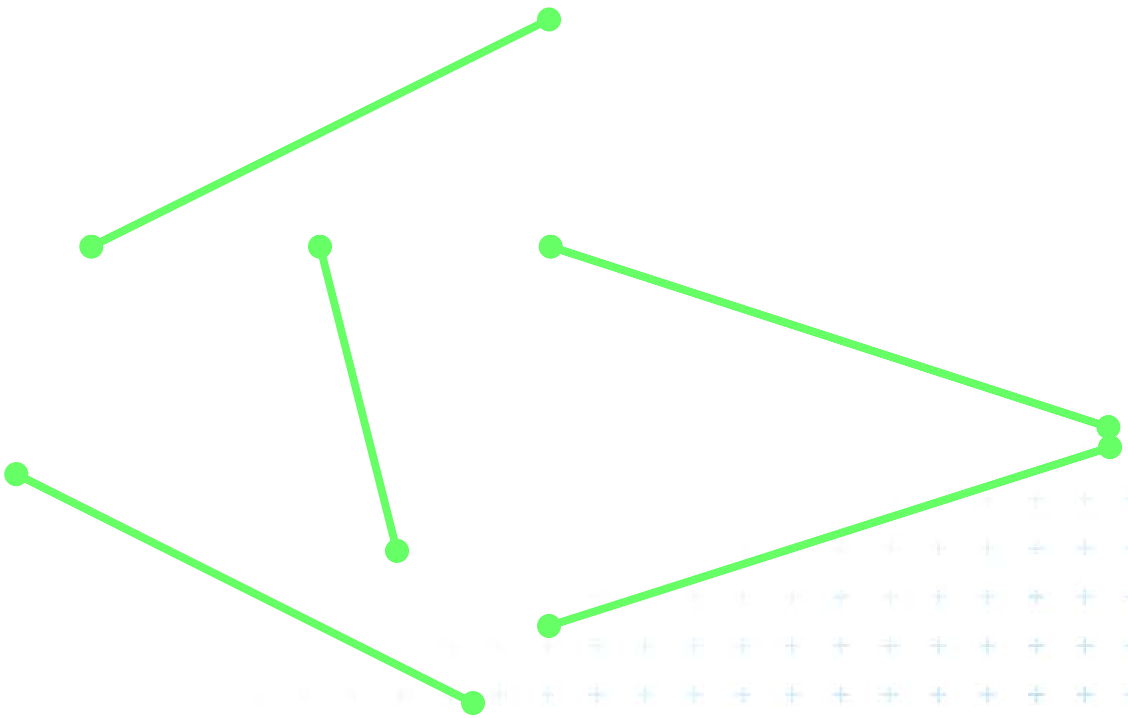
Input: $S = \{s_1, \ldots, s_n\}$ = set of n disjoint line segments (sites)



[Berg]

Voronoi diagram of line segments

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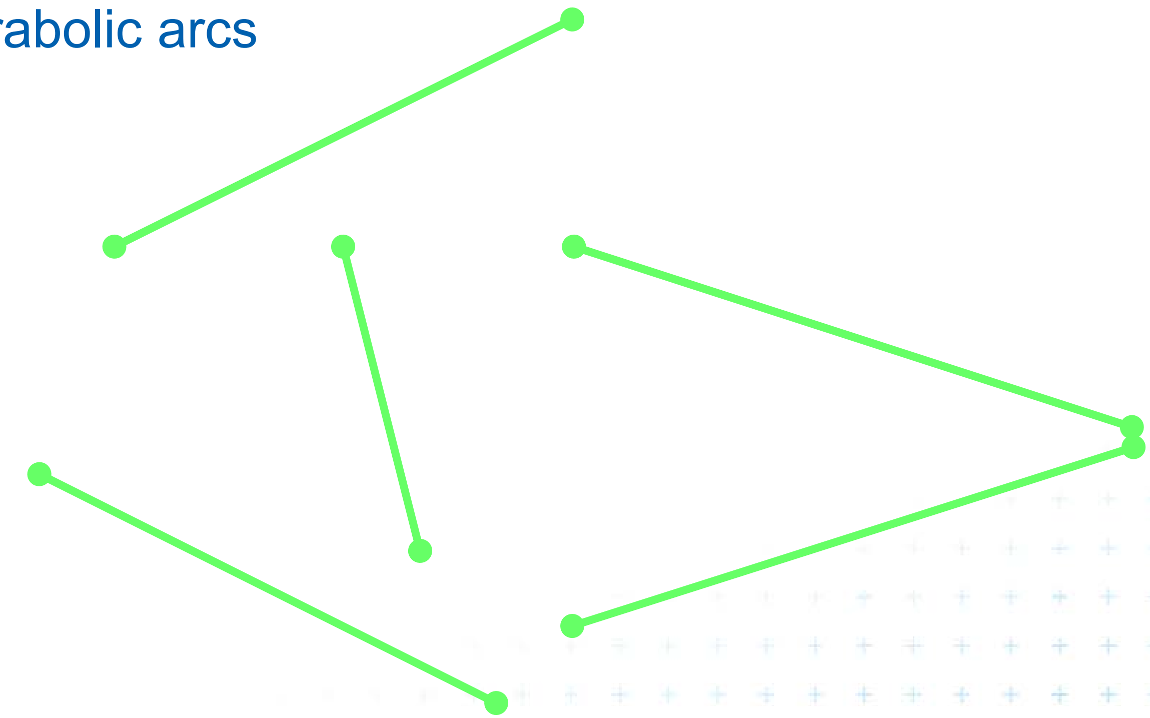
VD: line segments

parabolic arcs



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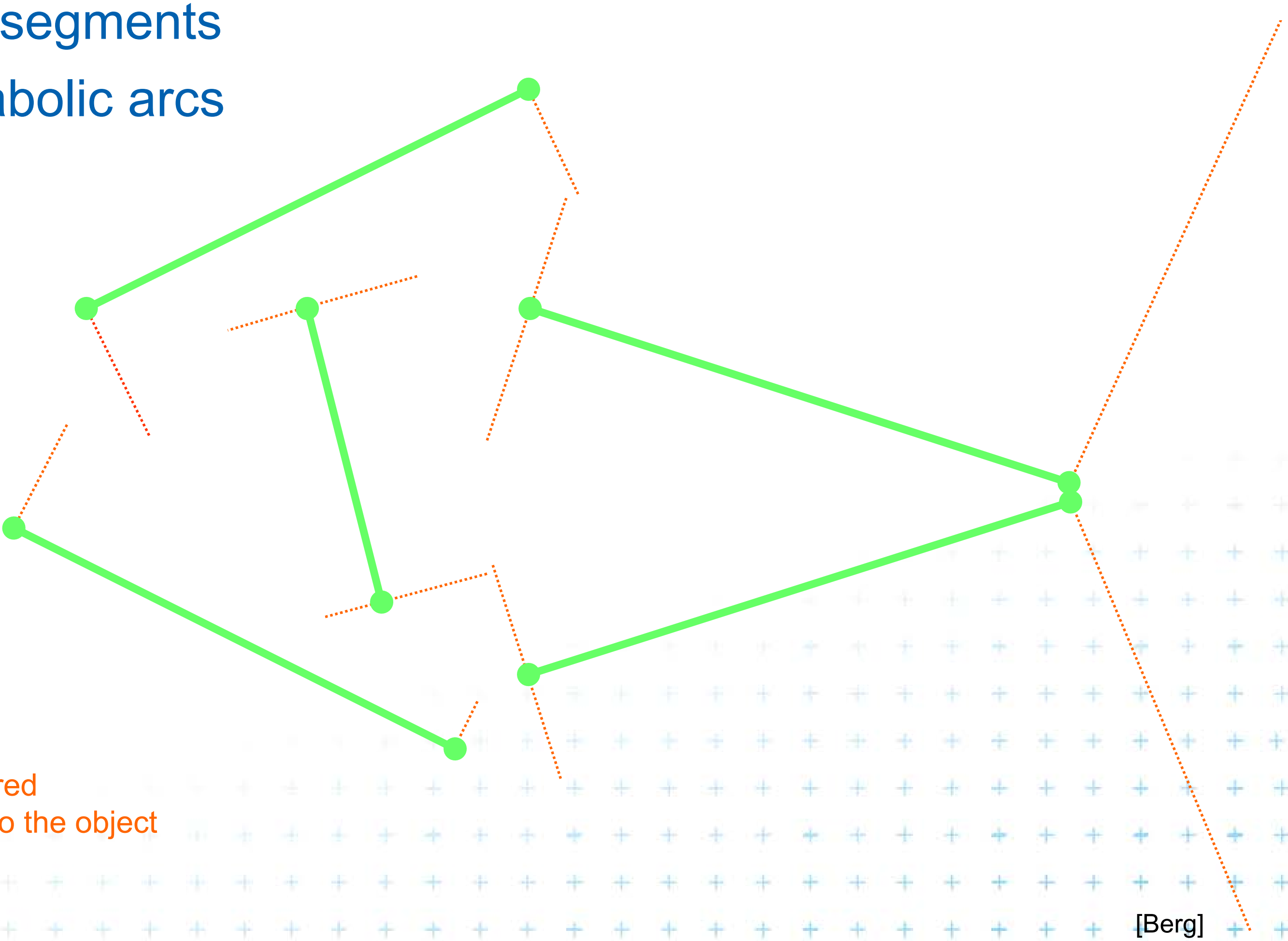

Distance measured perpendicularly to the object (line segment)

[Berg]

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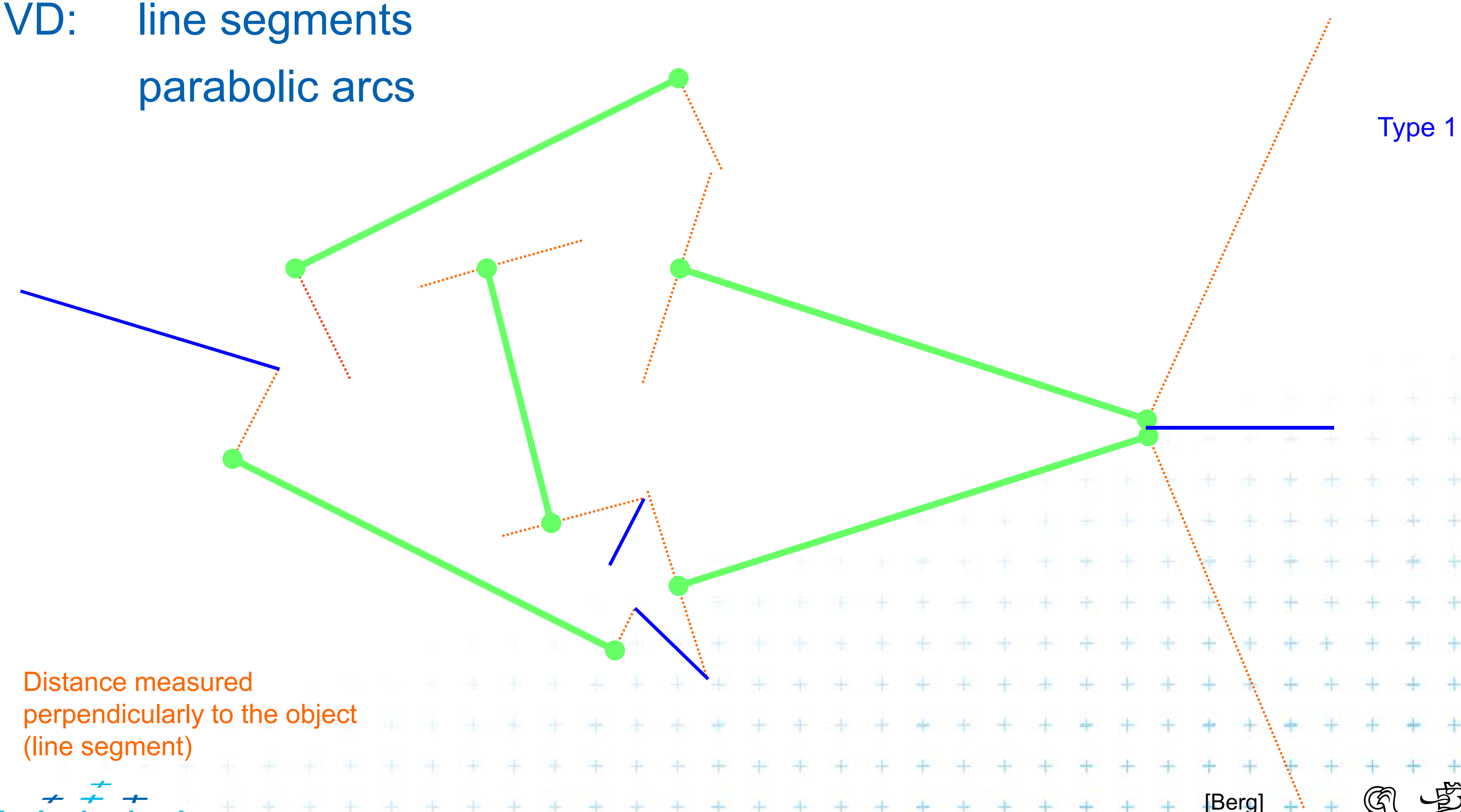

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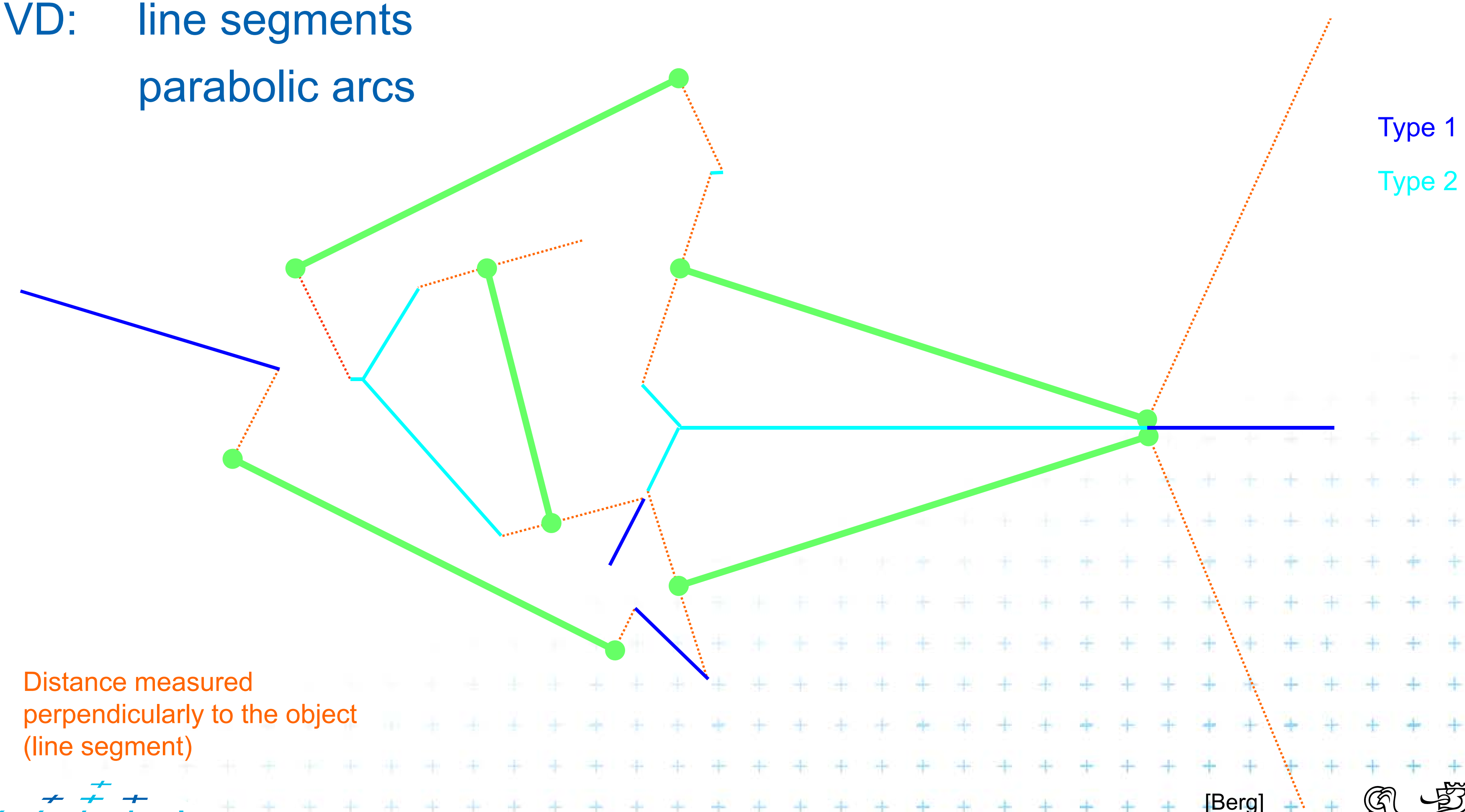

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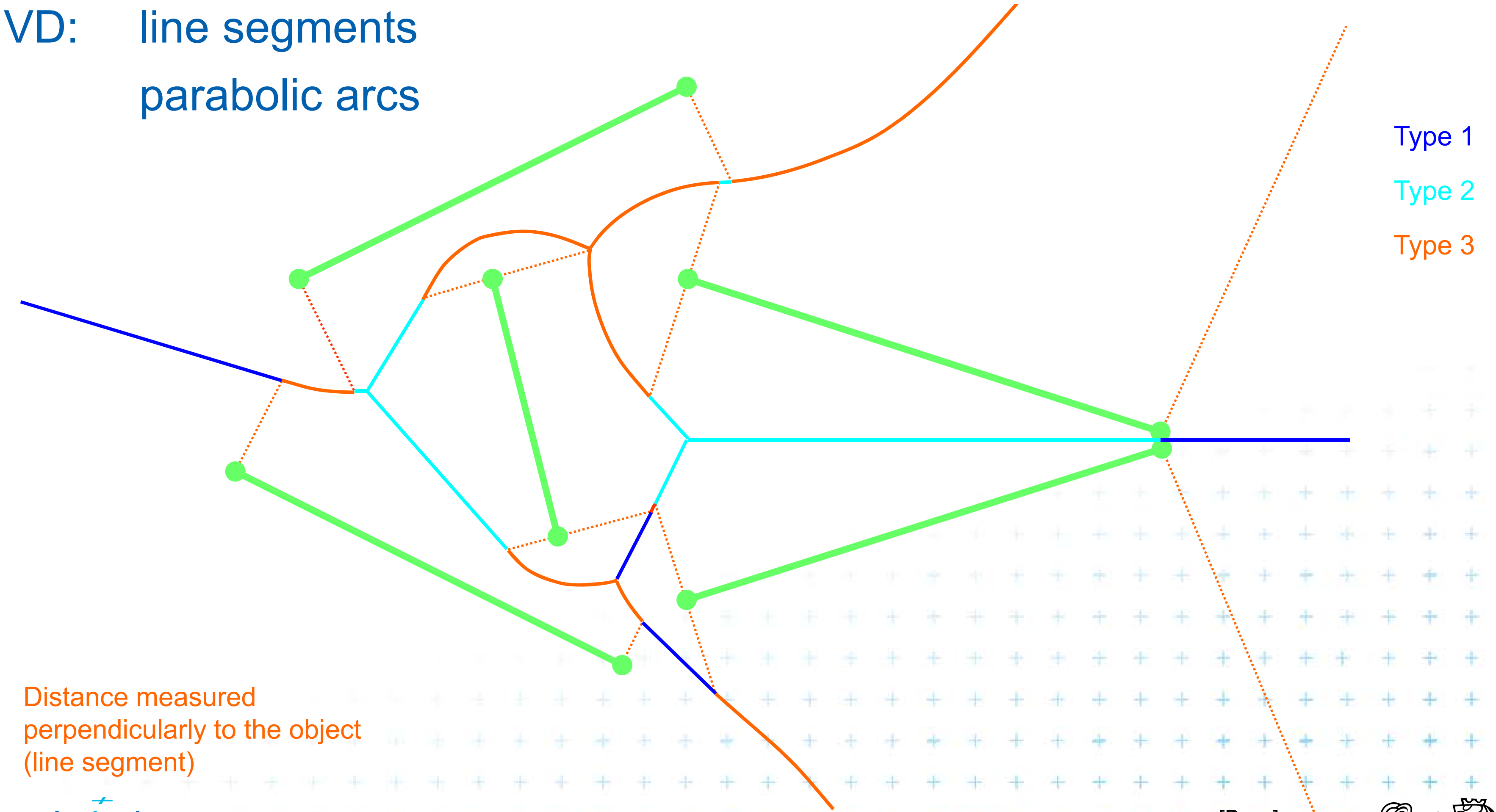

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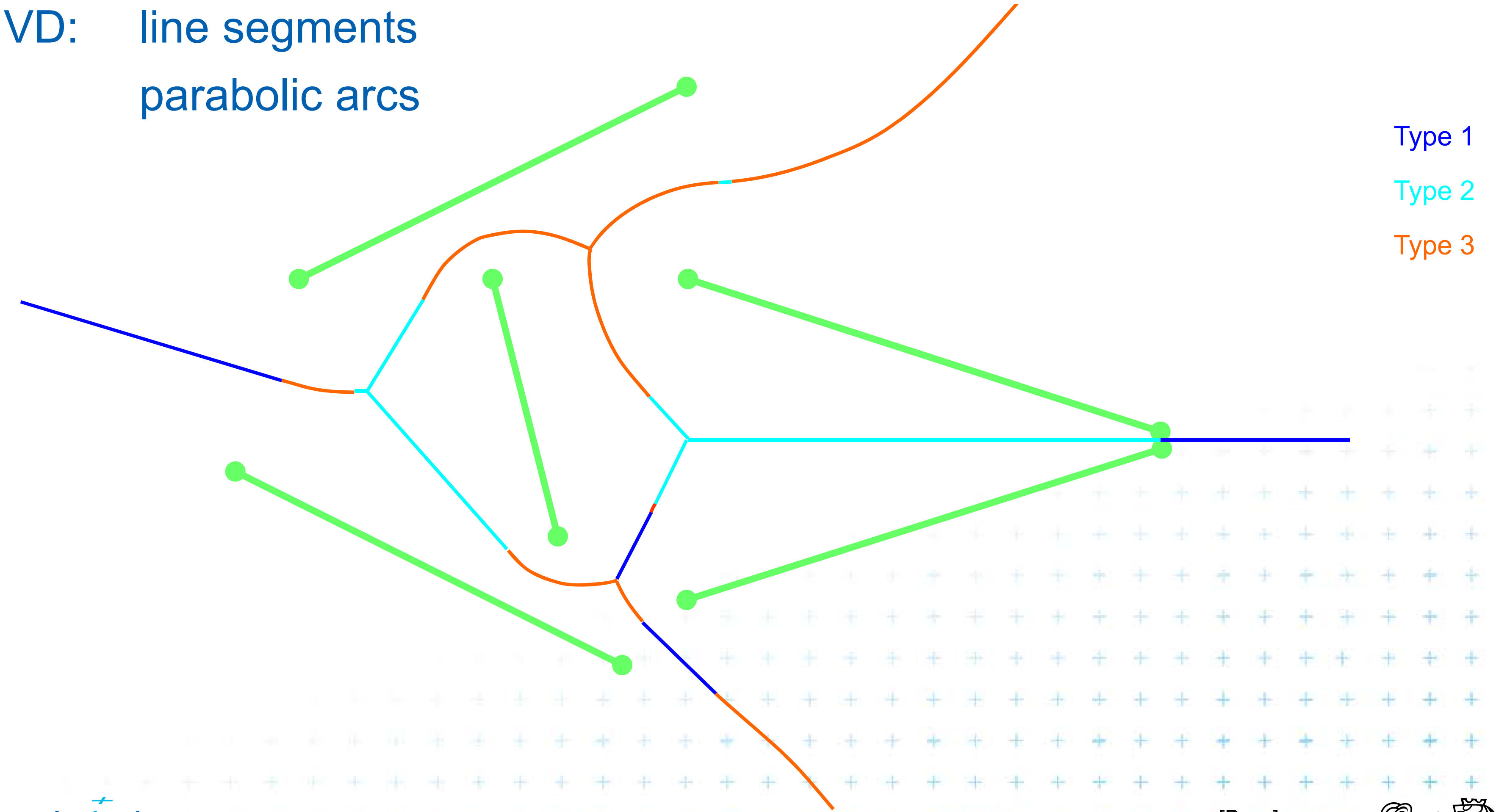
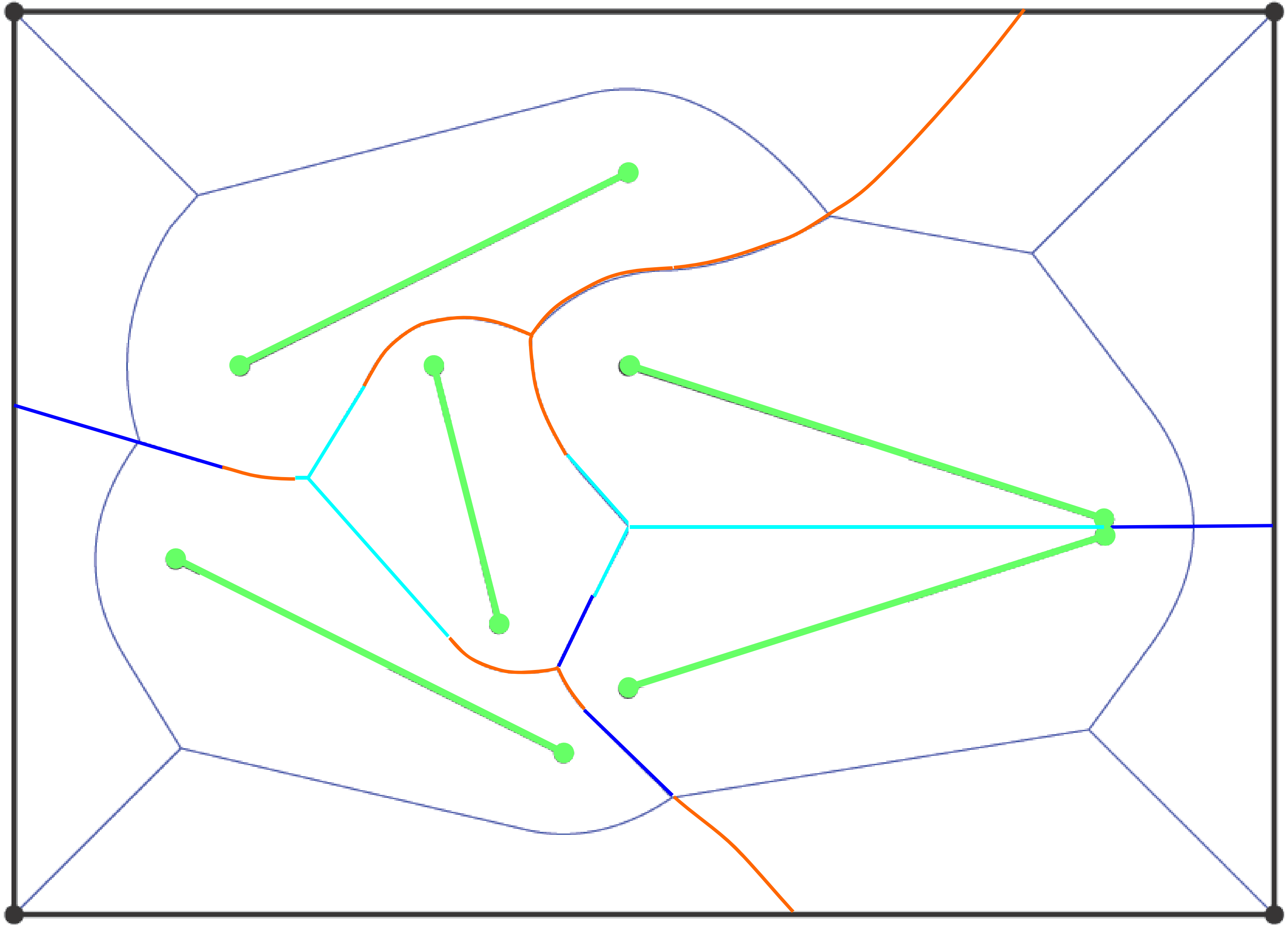


[Berg]

VD of line segments with bounding box



BBOX
=>
standard
DCEL

[Berg]

VD of 2 line-segments in detail

- Consists of line segments and parabolic arcs

  Distance from point-to-object is measured to the closest point on the object (perpendicularly to the object silhouette)

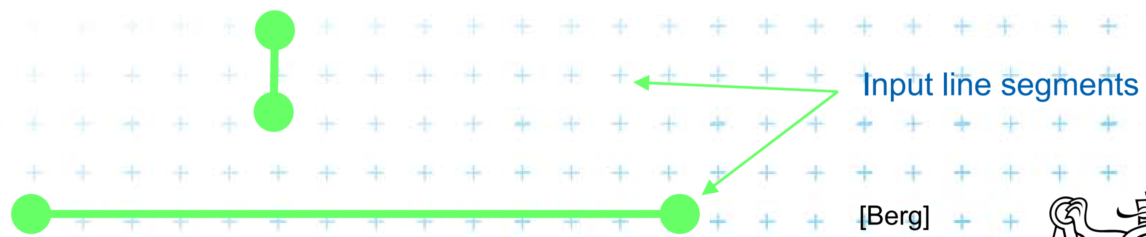
  – Line segment – bisector of end-points(1) or of interiors(2)

  – Parabolic arc – of point and interior(3) of a line segment

Type 1

Type 2

Type 3



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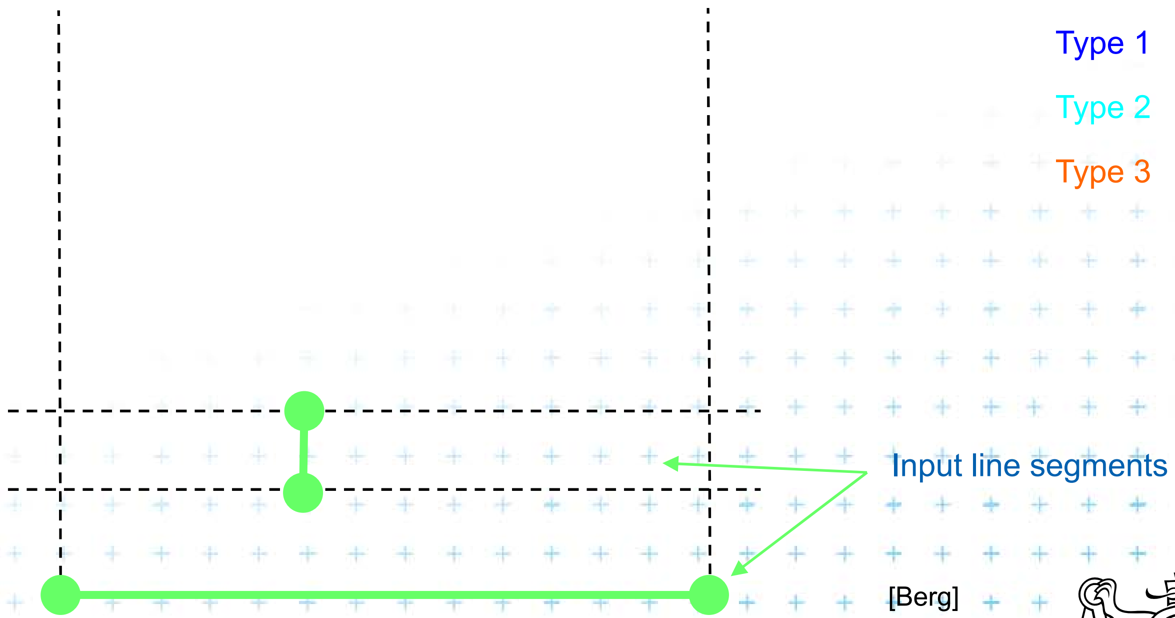

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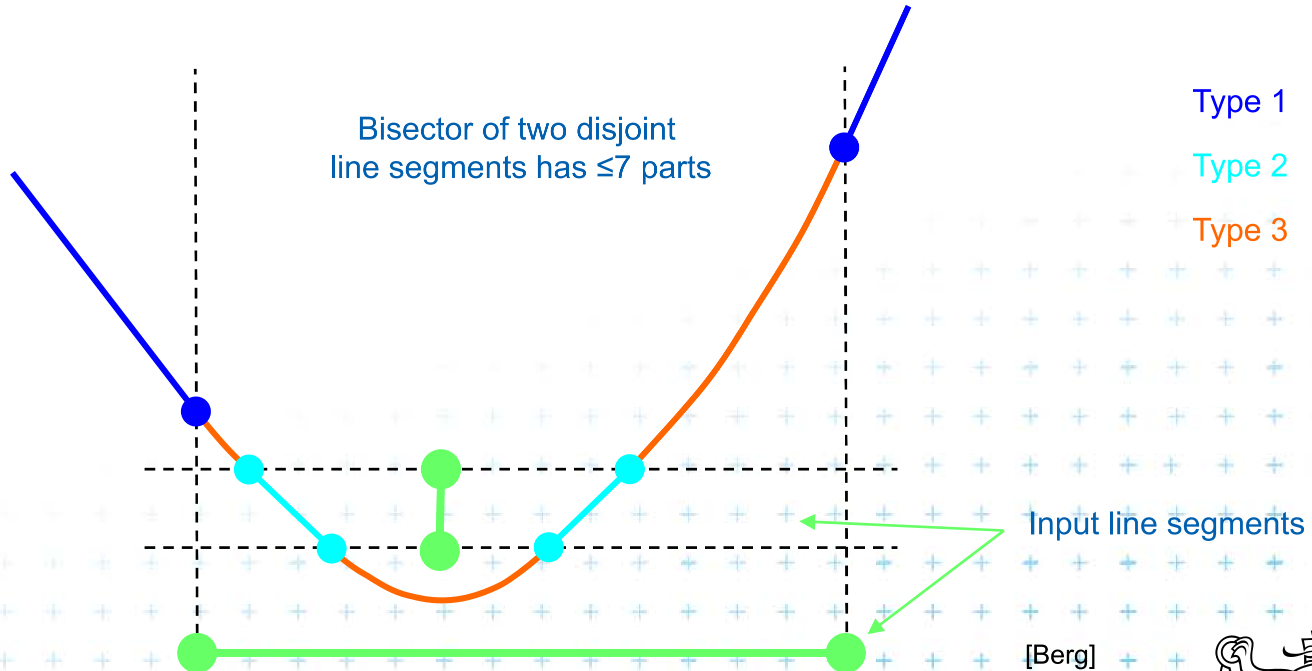

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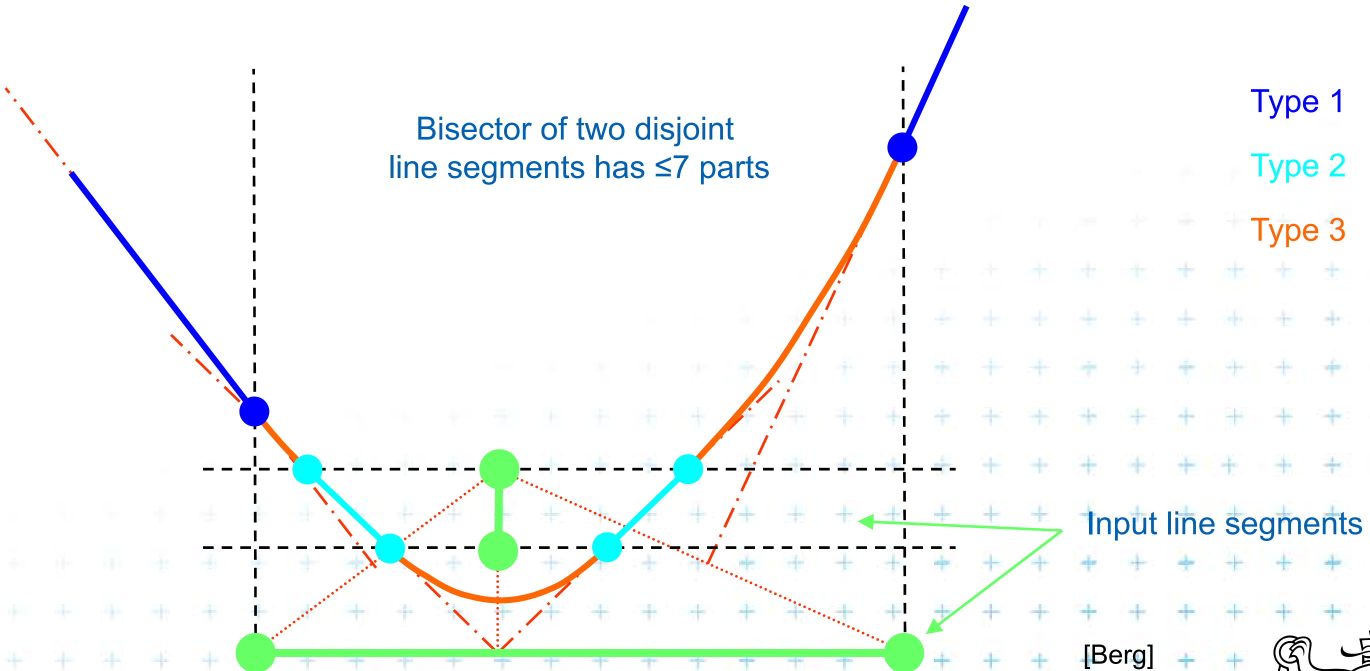

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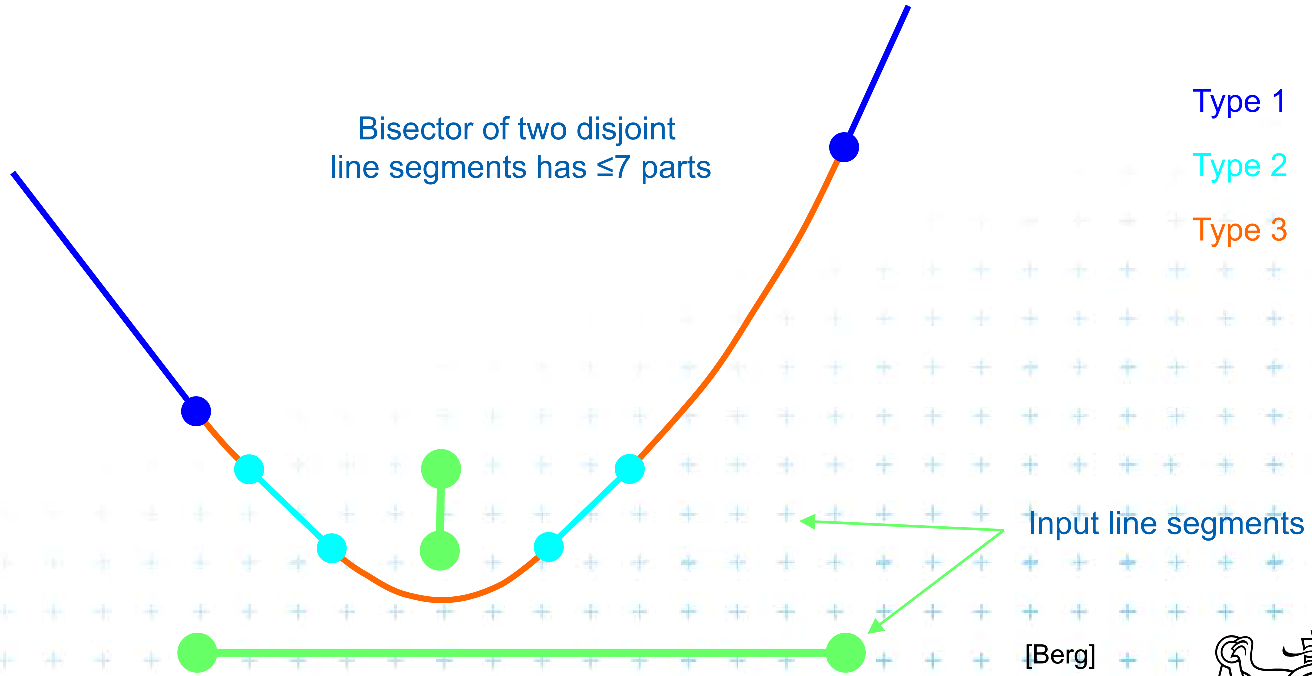
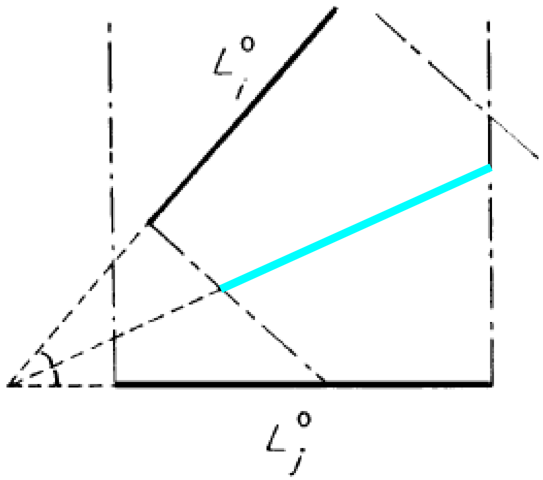


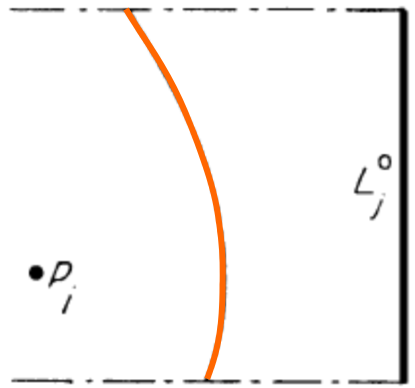
[Berg]

VD in greater details

Type 2



Type 3



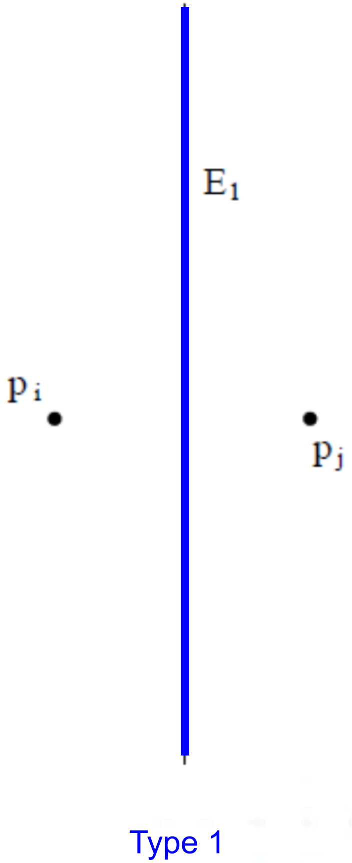
[Reiberg]

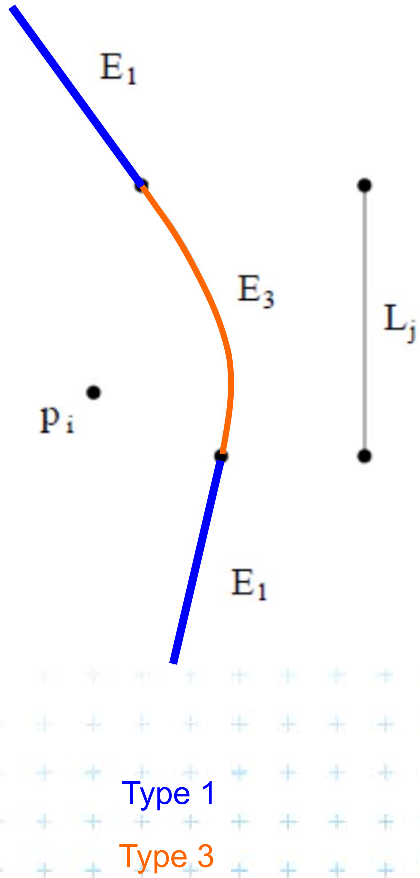
Bisector of two
line segment interiors
(in intersection of perpendicular slabs only)

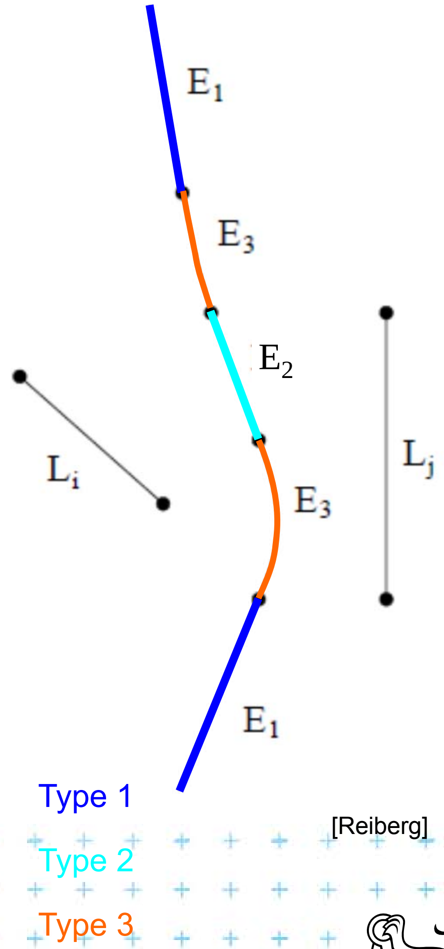
Bisector of (end-)point and
line segment interior

VD of points and line segments examples

2 points — Point & segment — 2 line segments



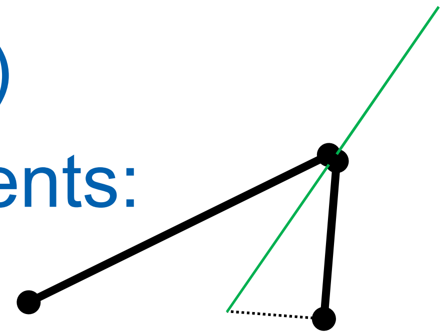


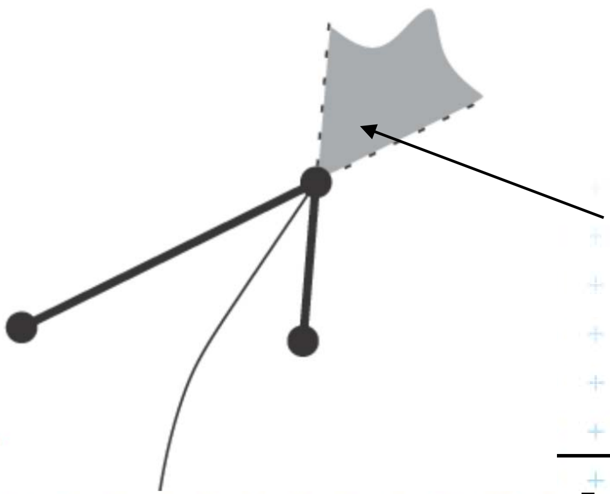


[Reiberg]

Voronoi diagram of line segments

- More complex bisectors of line segments
  - VD contains line segments and parabolic arcs
- Still combinatorial complexity of O(*n*)
- Assumptions on the input line segments:
  - non-crossing
  - strictly disjoint end-points (slightly shorten the segm.)



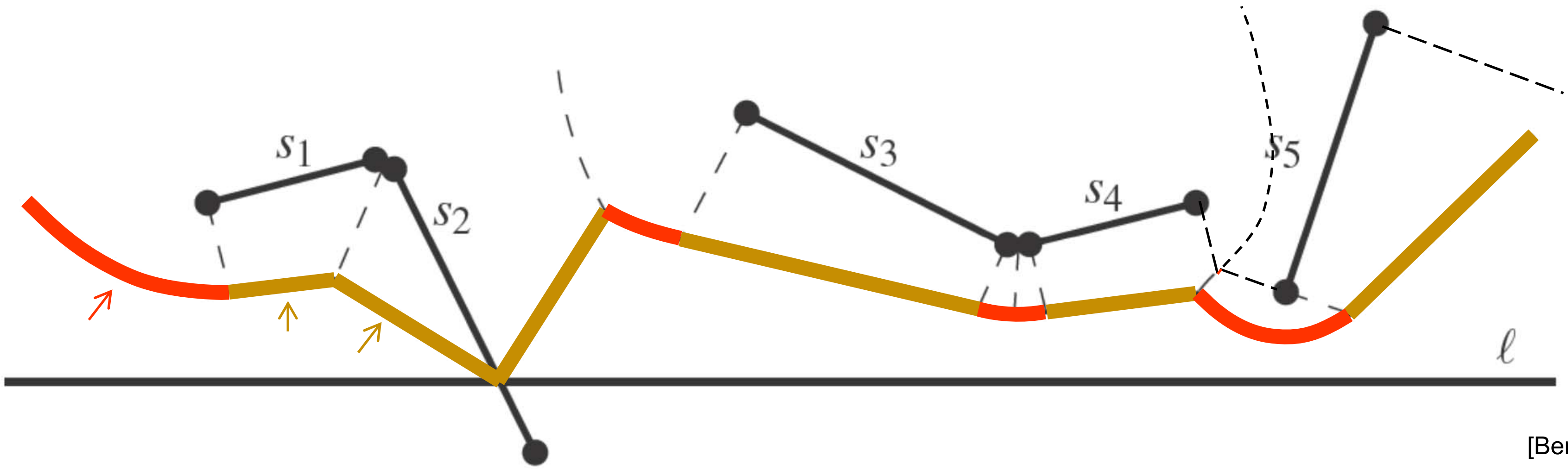


if(we allow touching segments)

Shared endpoints cause complication:
The whole region is equally close
to two line segments

Fortune's algorithm for line segments

Shape of beach line for line segments



[Berg]

Beach line = points with distance to the closest site above sweep line *l* equal to the distance to *l*

Beach line contains

- *parabolic arcs* when closest to a site end-point
- *straight line segments* when closest to a site interior (or just the part of the site interior above *l* if the site *s* intersects *l*)

(This is the shape of the beach line)

Beach line breakpoints types

Breakpoint p is equidistant from l and equidistant and closest to:

points

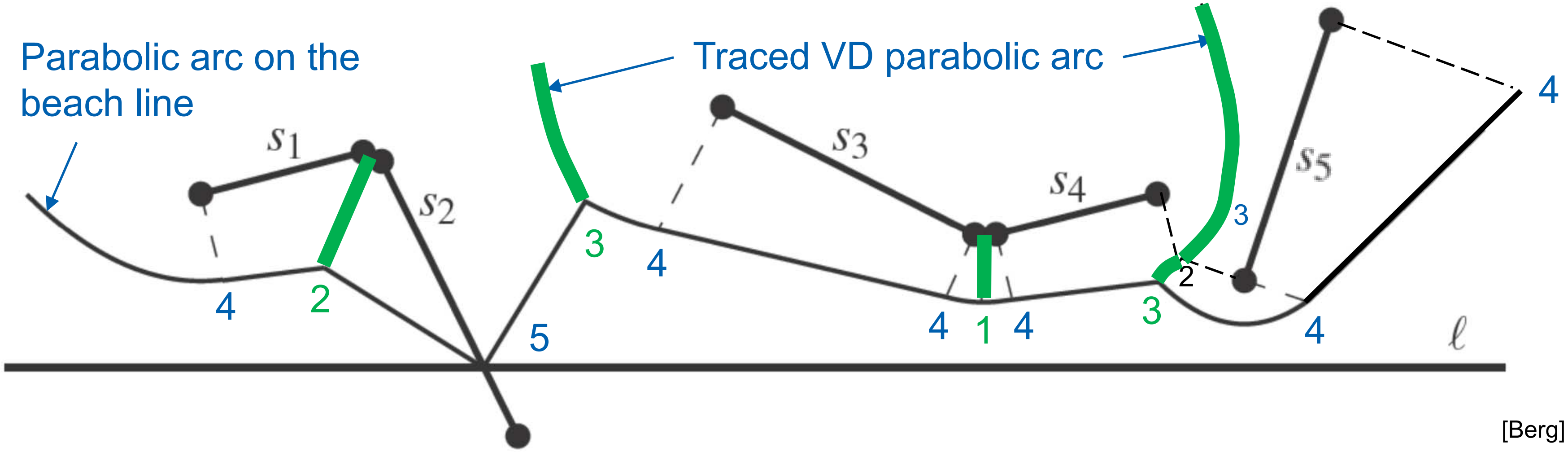
1. two site end-points => p traces a VD line segment

segments

2. two site interiors => p traces a VD line segment
3. end-point and interior => p traces a VD parabolic arc
4. one site end-point => p traces a line segment (border of the slab perpendicular to the site)
5. site interior intersects the scan line l => p = intersection, traces the input line segment

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram edge (are used by alg.only)

Breakpoints types - what they trace on VD



[Berg]

- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Site event – sweep line reaches an endpoint

I. At upper endpoint of

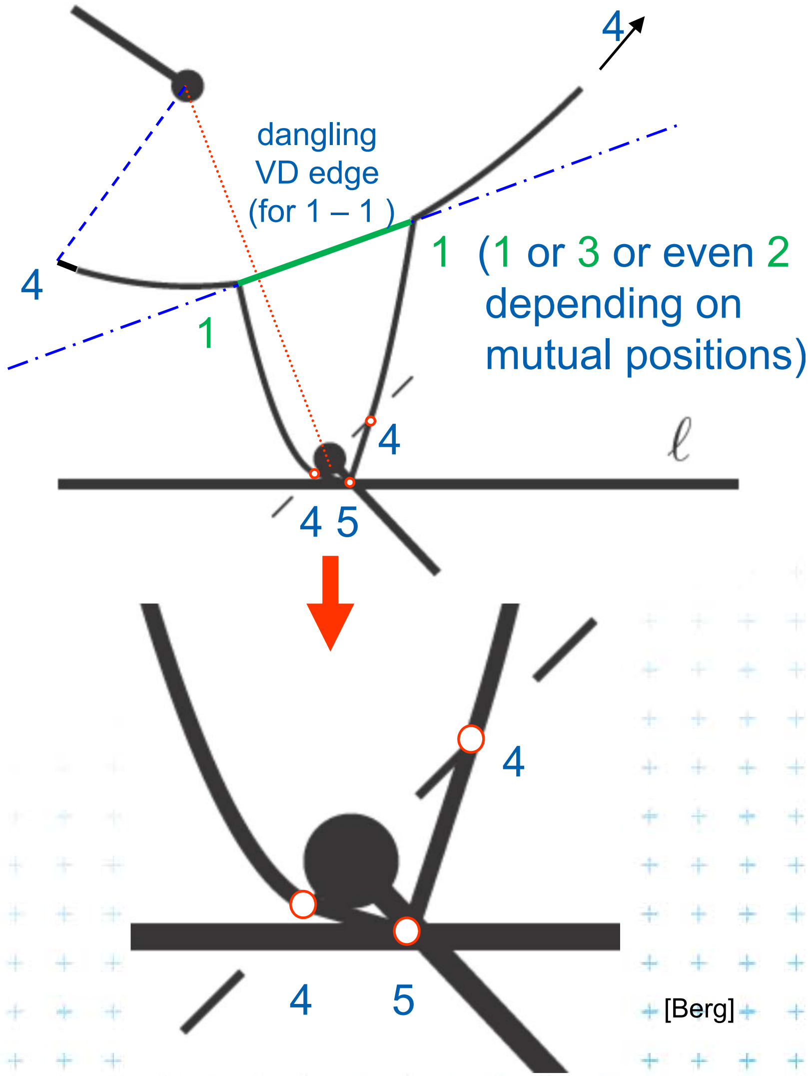
– Arc above is split into two

– four new arcs are created (2 segments + 2 parabolas)

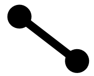
– Breakpoints for two segments are of type 4-5-4

– Breakpoints for parabolas depend on the surrounding sites

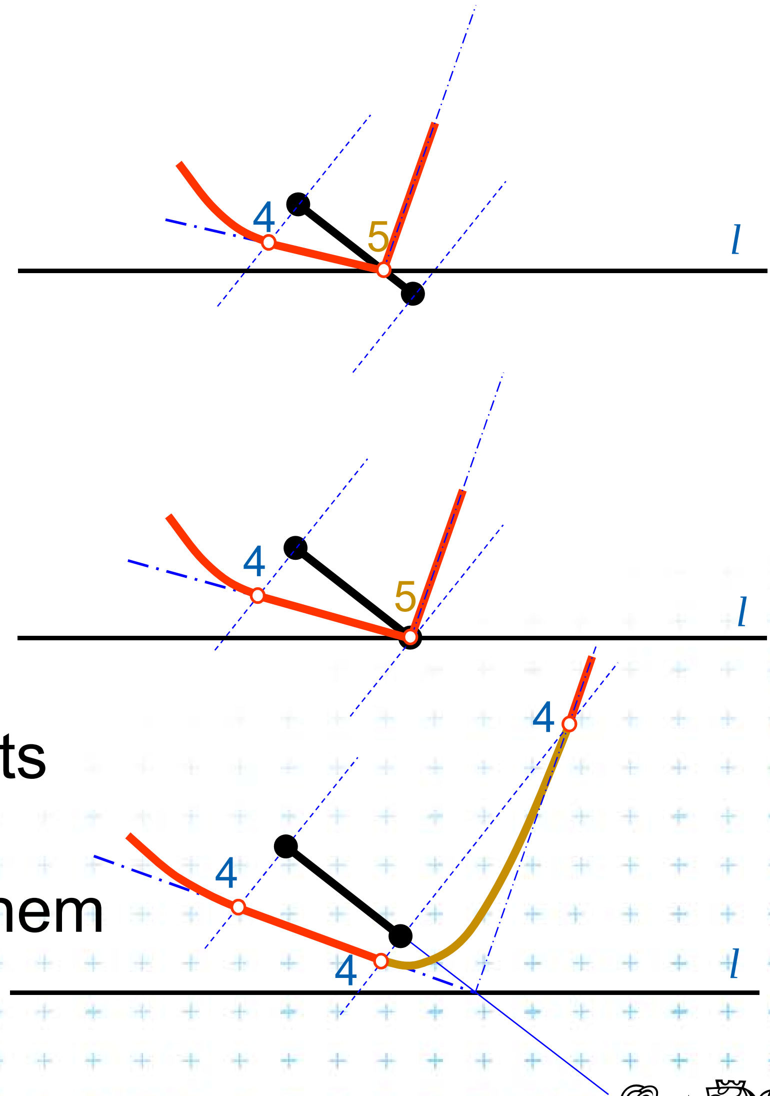
- Type 1 for two end-points
- Type 3 for endpoint and interior
- etc…



Site event – sweep line reaches an endpoint

II. At lower endpoint of 

- Intersection with interior (breakpoint of type 5)



- is replaced by two breakpoints (of type 4) with parabolic arc between them

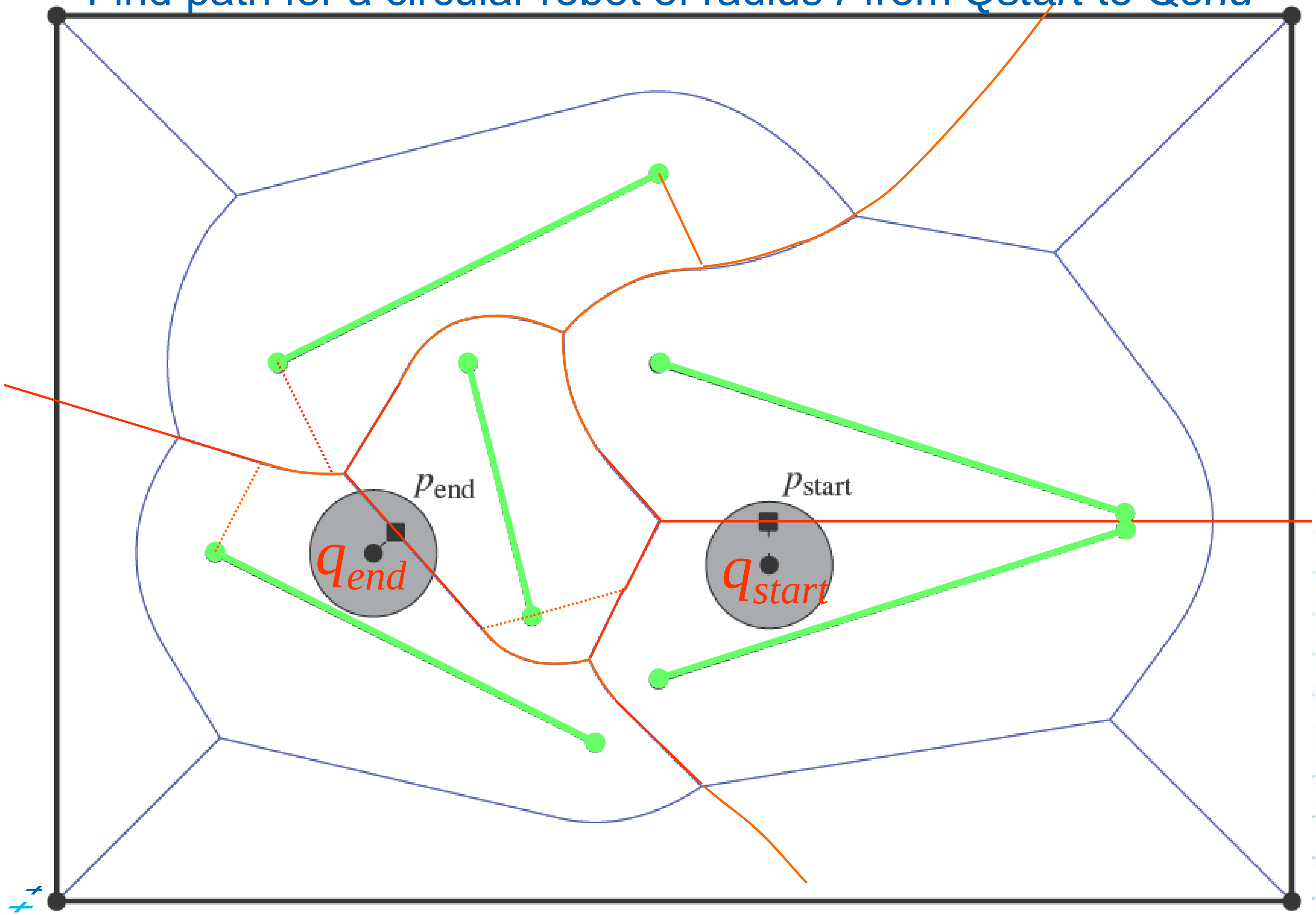
Circle event – lower point of circle of 3 sites

- Two breakpoints meet (on the beach-line)
- Solution depends on their type
  - Any of first three types (1,2,or 3) meet (circle event)
    - 3 sites involved – Voronoi vertex created
  - Type 4 (*segment interiors*) with something else
    - two sites involved – breakpoint changes its type
    - Voronoi vertex not created (Voronoi edge may change its shape)
  - Type 5 (*on segment*) with something else
    - never happens for disjoint segments (meet with type 4 happens before)

Motion planning example

Motion planning example - retraction Rušení hran

Find path for a circular robot of radius *r* from *Qstart* to *Qend*



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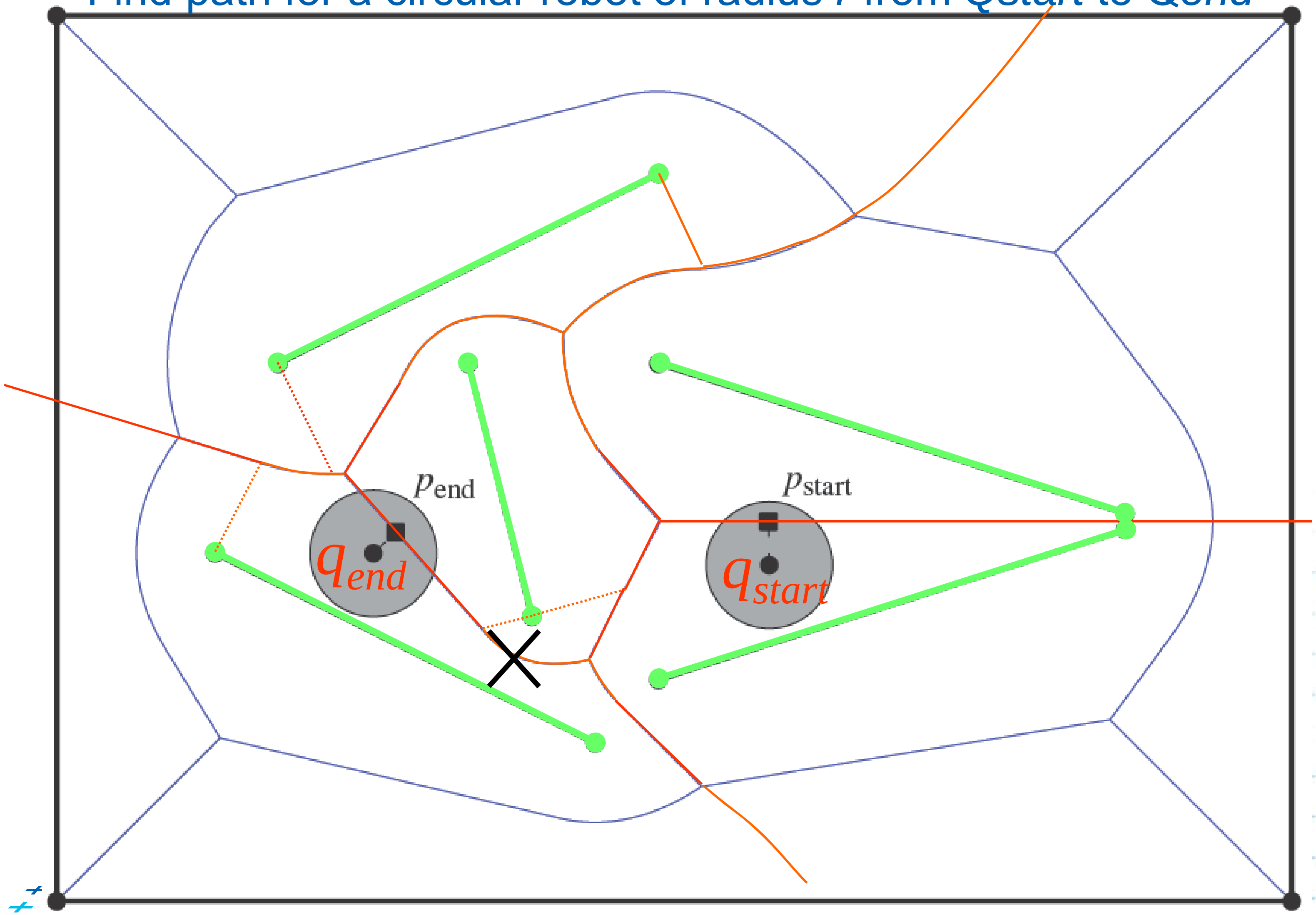

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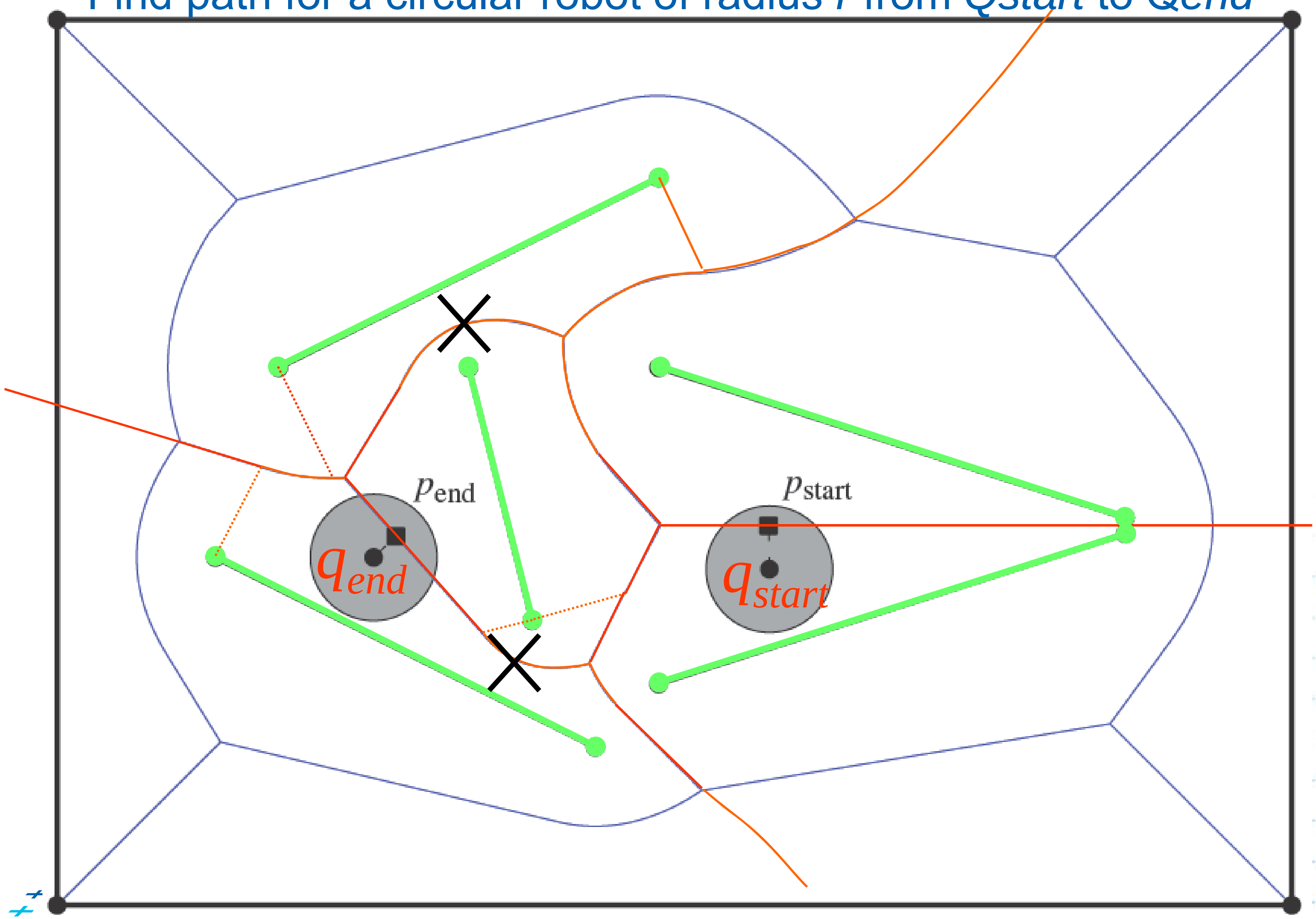

[Berg]

Motion planning example - retraction **Rušení hran**

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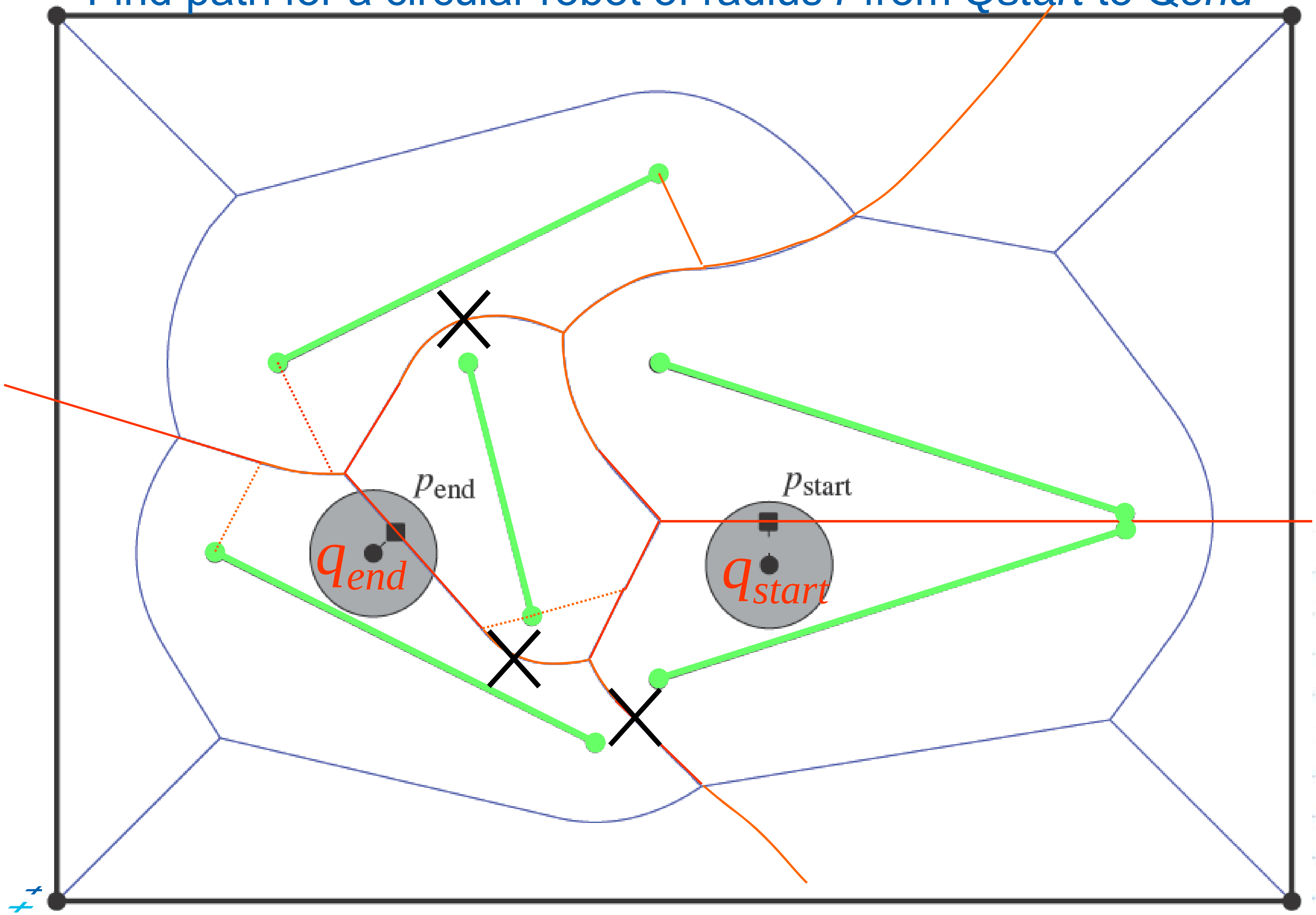

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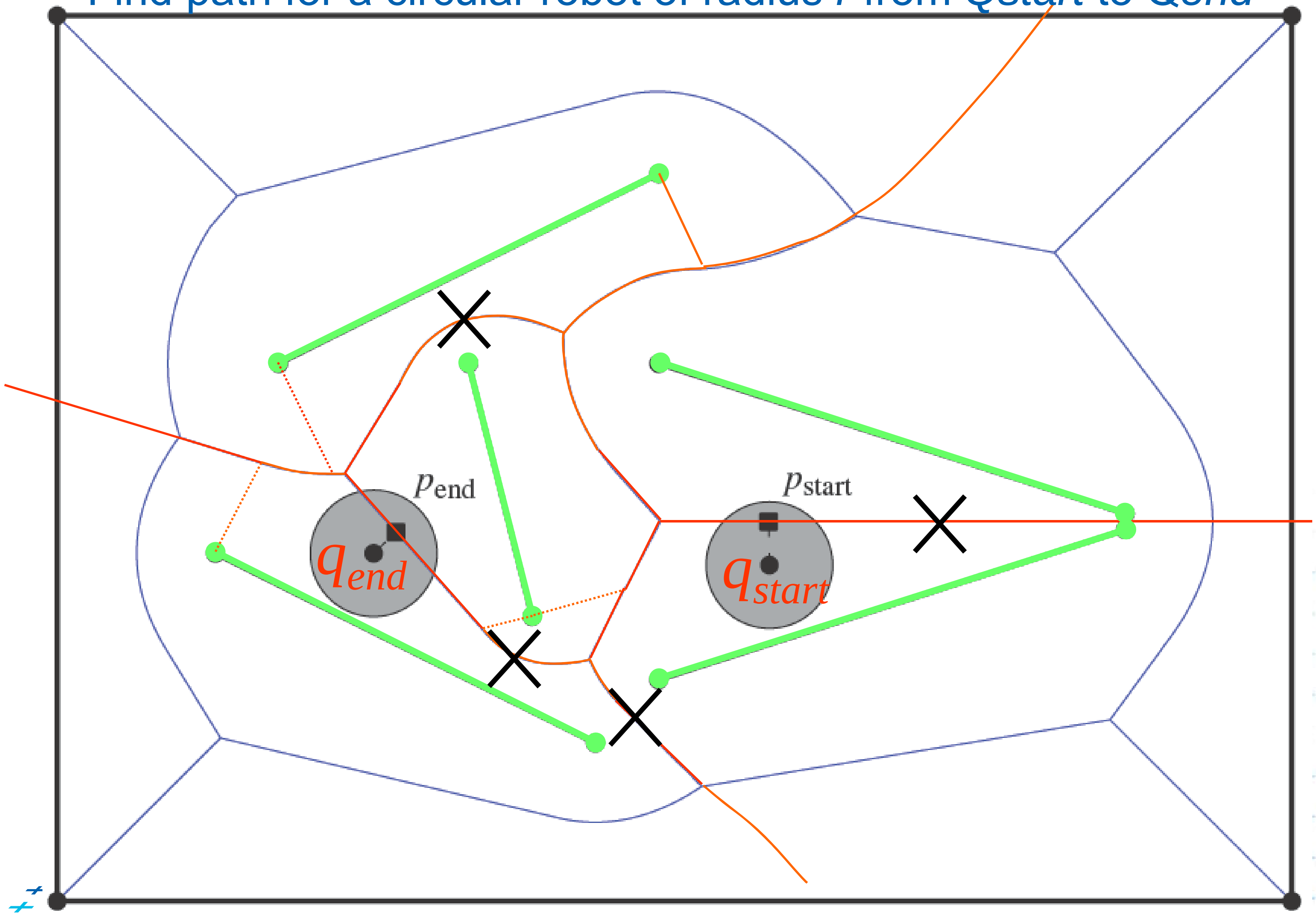

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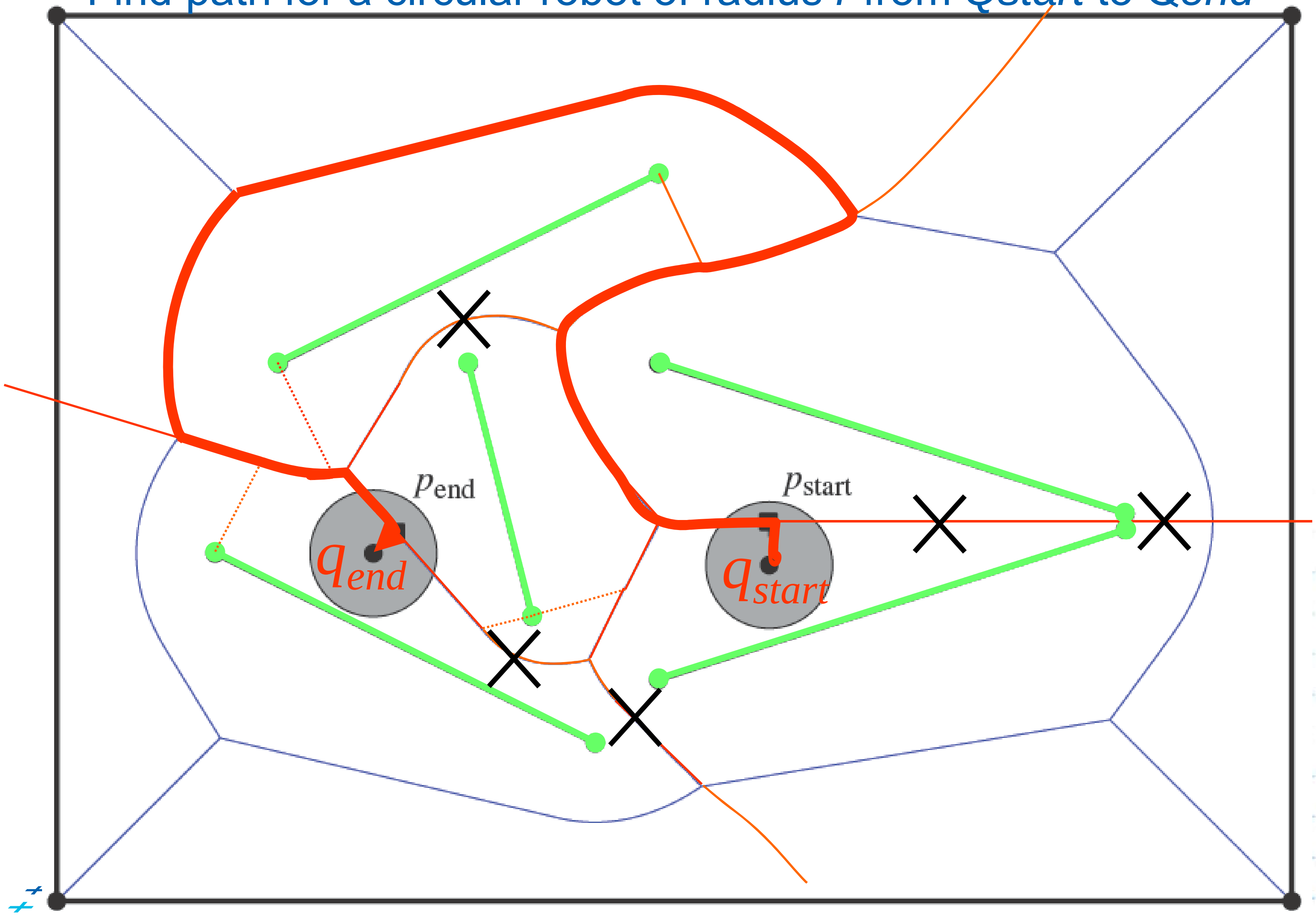


[Berg]

Motion planning example - retraction Rušení hran

Find path for a circular robot of radius r from Q_{start} to Q_{end}

- Create Voronoi diagram of line segments, take it as a graph
- Project Q_{start} to P_{start} *on VD and* Q_{end} to P_{end}
- Remove segments with distance to sites smaller than radius r of a robot
- Depth first search if path from P_{start} to P_{end} *exists*
- Report path $Q_{start} P_{start}$…*path*… P_{end} to Q_{end}
- $O(n \log n)$ time using $O(n)$ storage

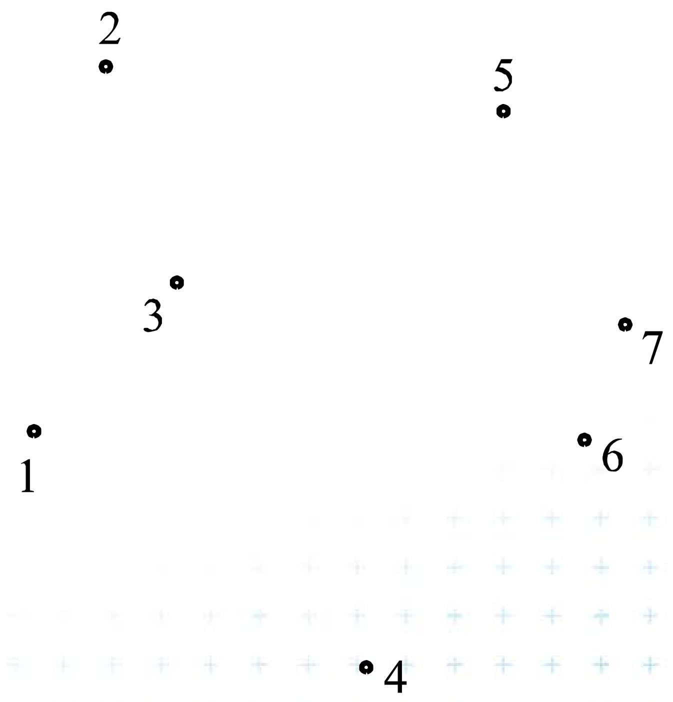
Higher order VD

Order-2 Voronoi diagram (nearest to two sites)

[Nandy]

Order-2 Voronoi diagram (nearest to two sites)

$V(p_i,p_j)$: the set of points of the plane closer to each of p_i and p_j than to any other site



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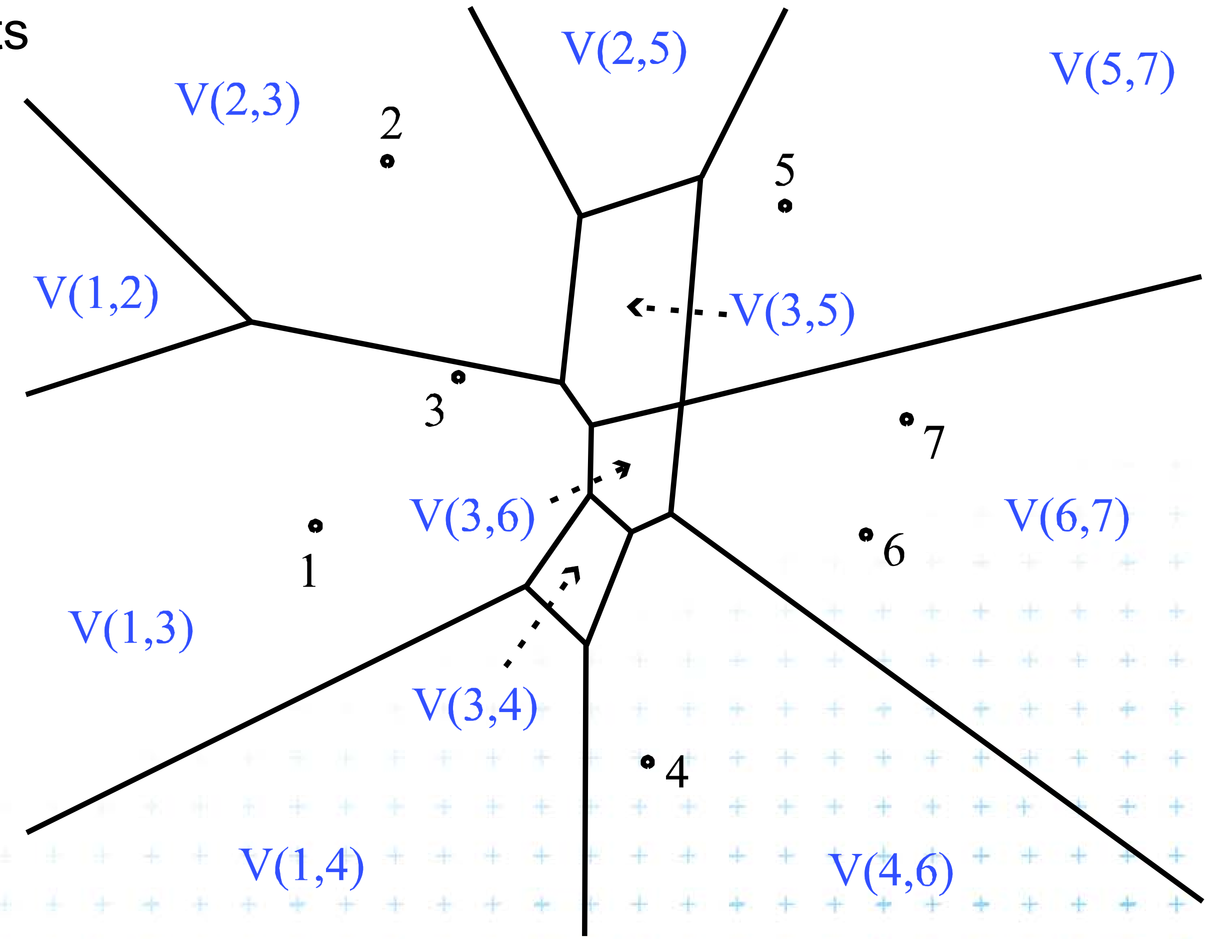

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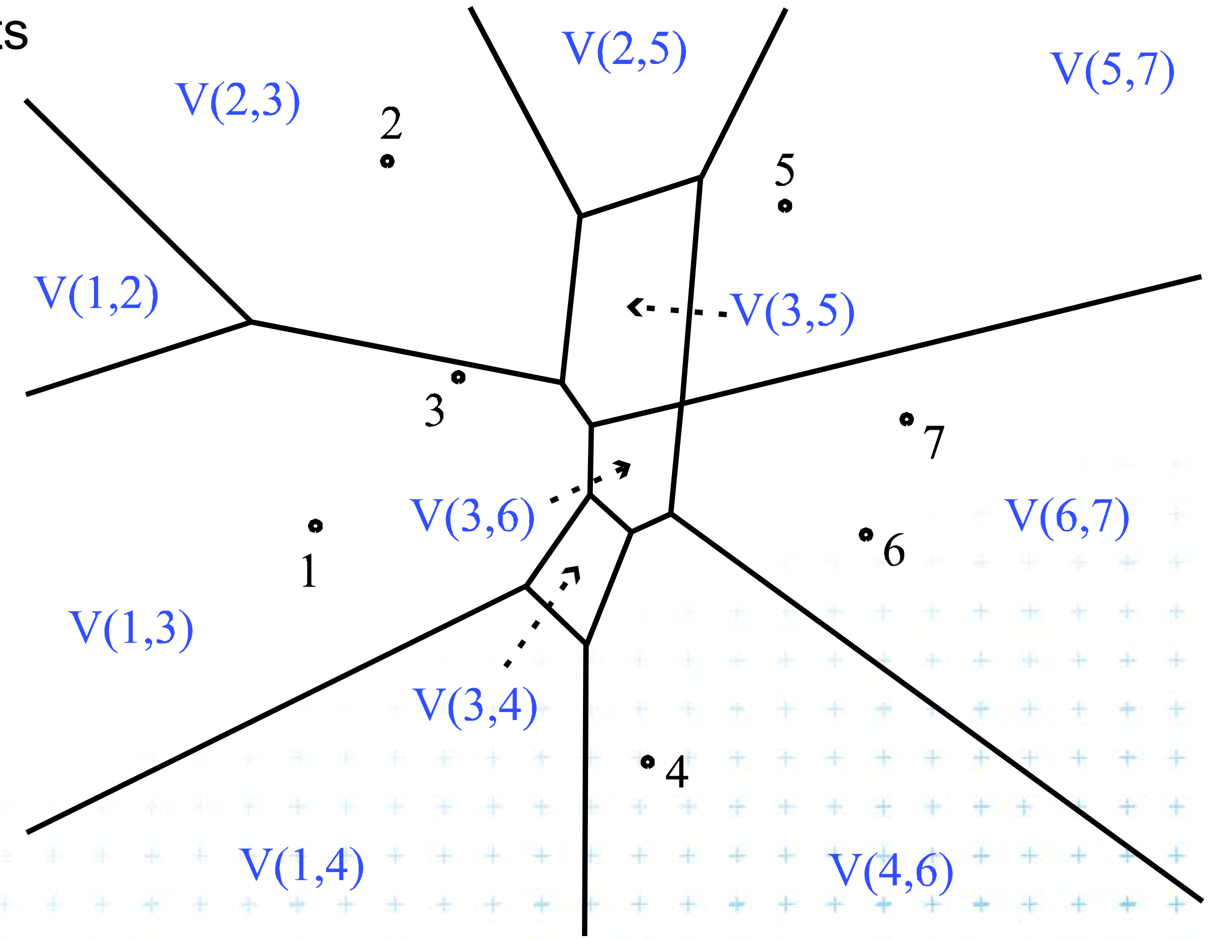
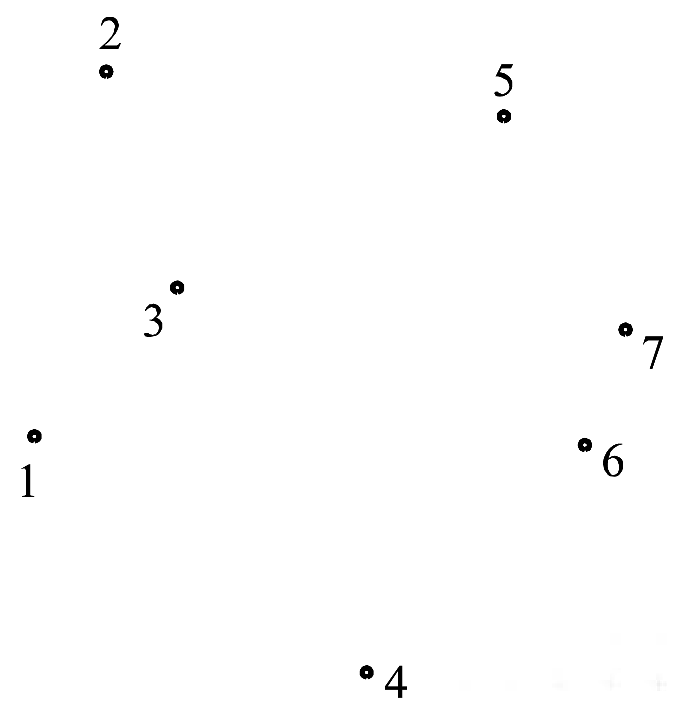


Property

The order-2 Voronoi regions are convex

[Nandy]

Construction of V(3,5) = V(5,3)

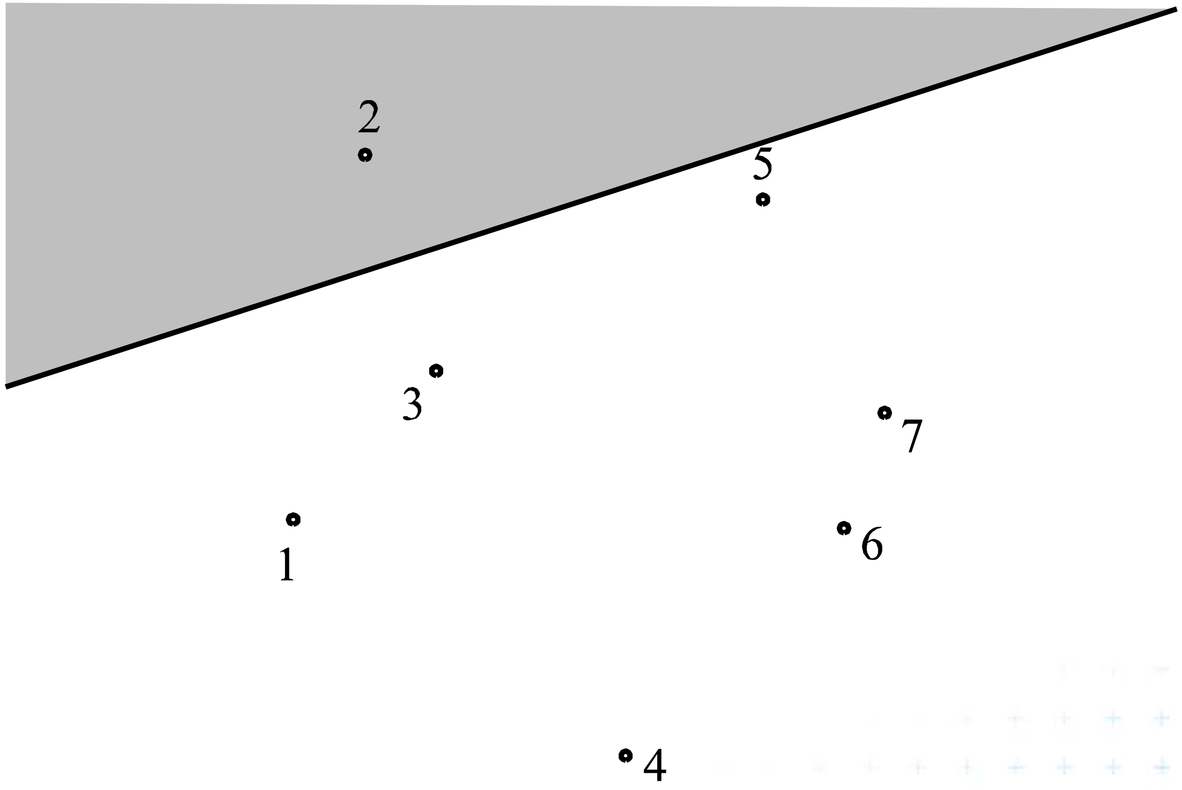


[Nandy]

Intersection of all halfplanes except $h(3,5)$ and $h(5,3)$

$$\underbrace{\bigcap_{x \neq 5} h(3,x)}\cap\underbrace{\bigcap_{x \neq 3} h(5,x)}$$

Construction of V(3,5) = V(5,3)

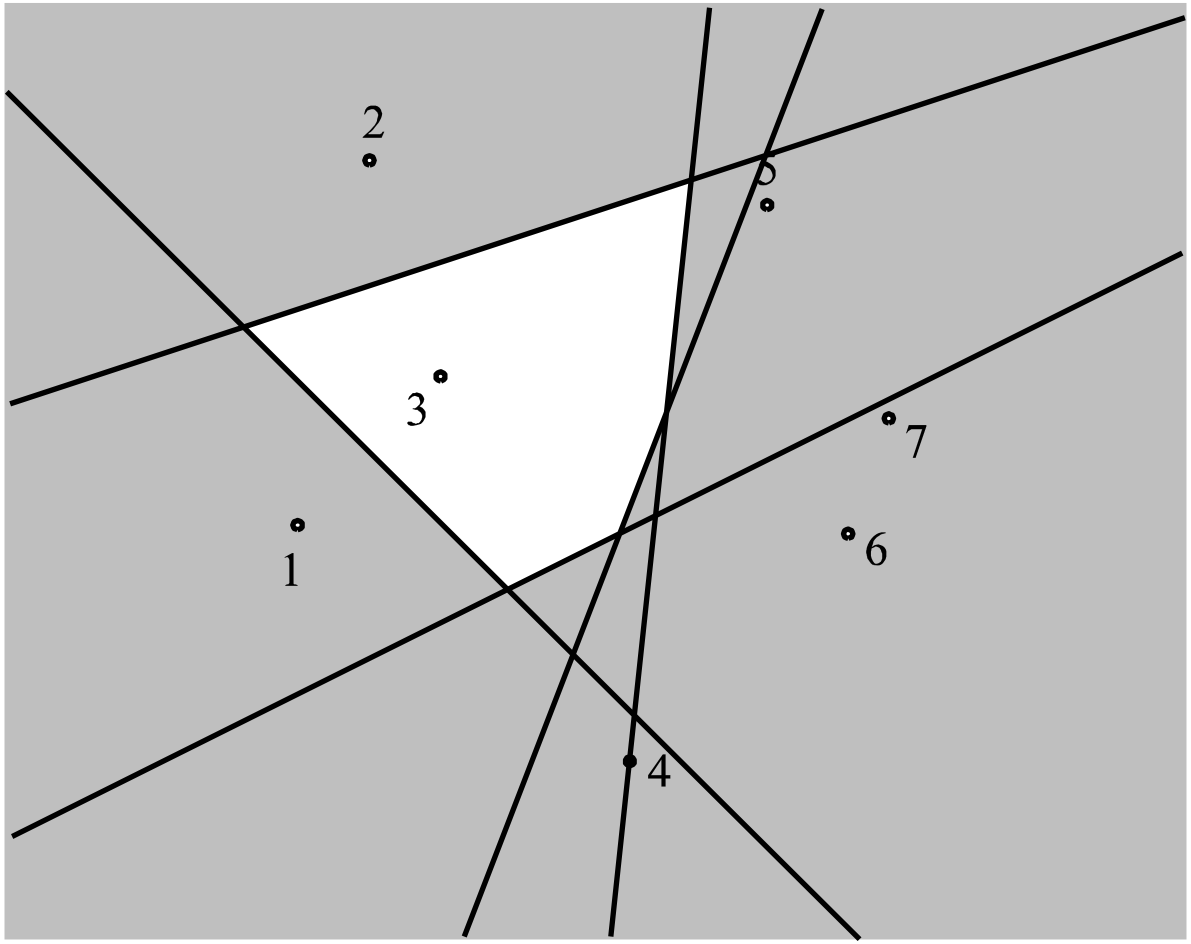


[Nandy]

Intersection of all halfplanes except *h*(3,5) and *h*(5,3)

$$\underbrace{\bigcap_{x\neq 5} h(3,x)}\cap\underbrace{\bigcap_{x\neq 3} h(5,x)}$$

Construction of V(3,5) = V(5,3)



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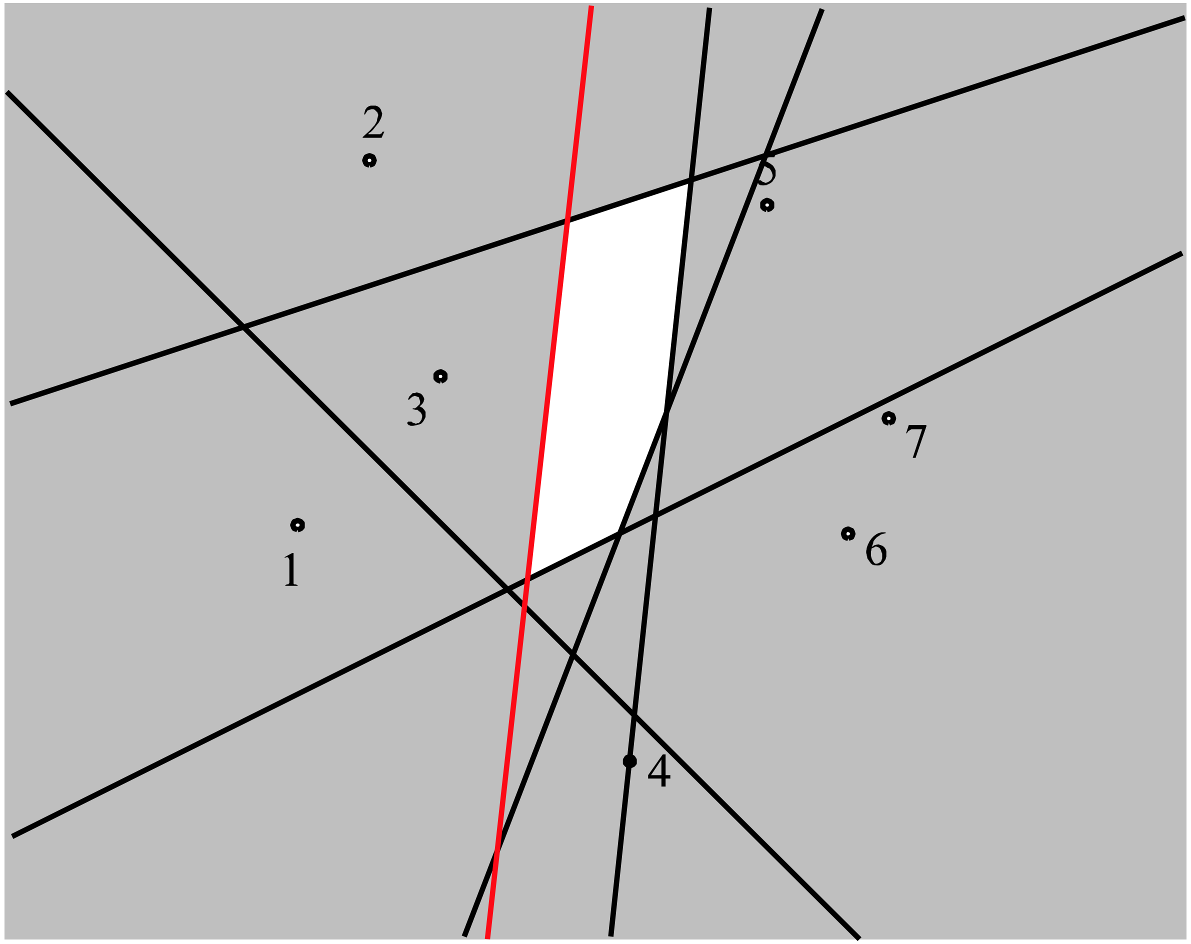

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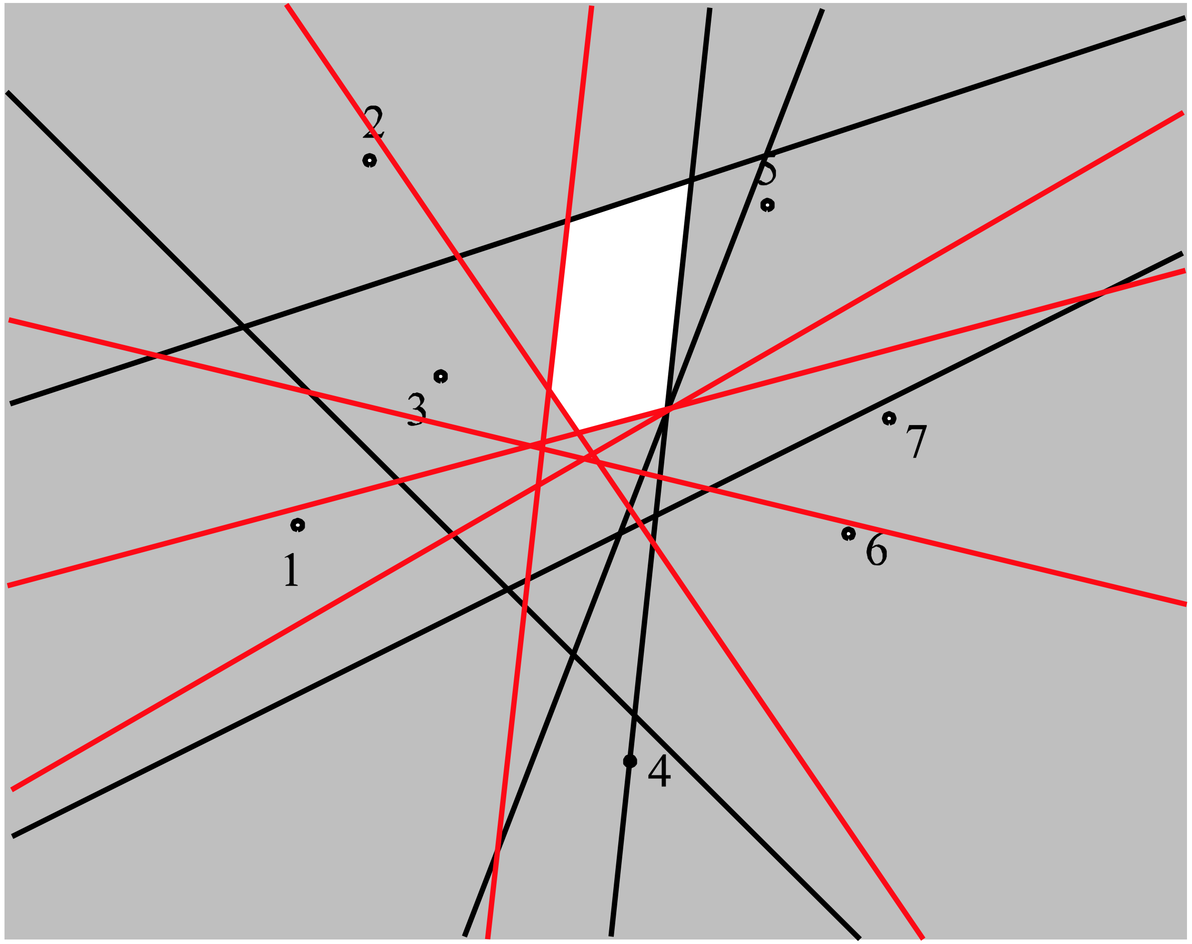

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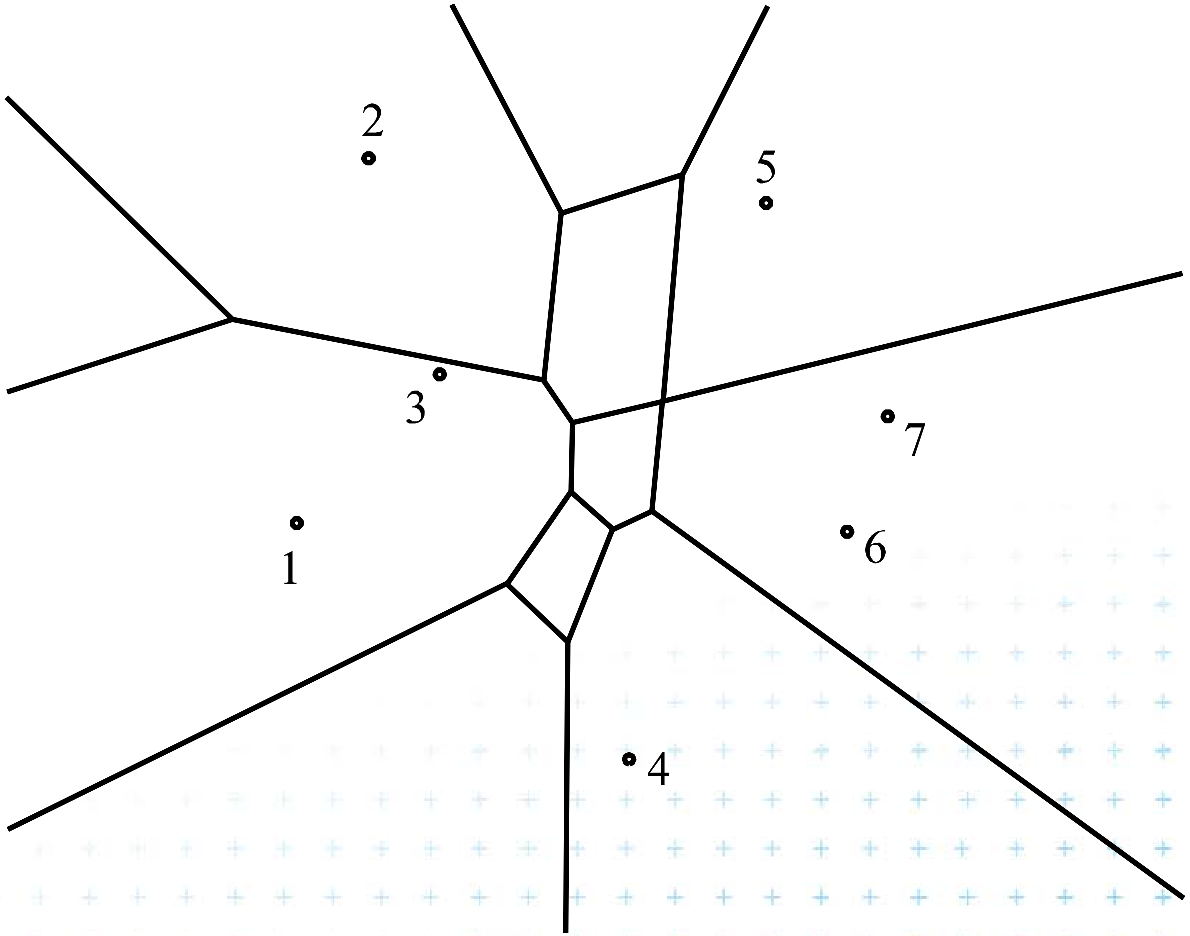
Order-2 Voronoi edges



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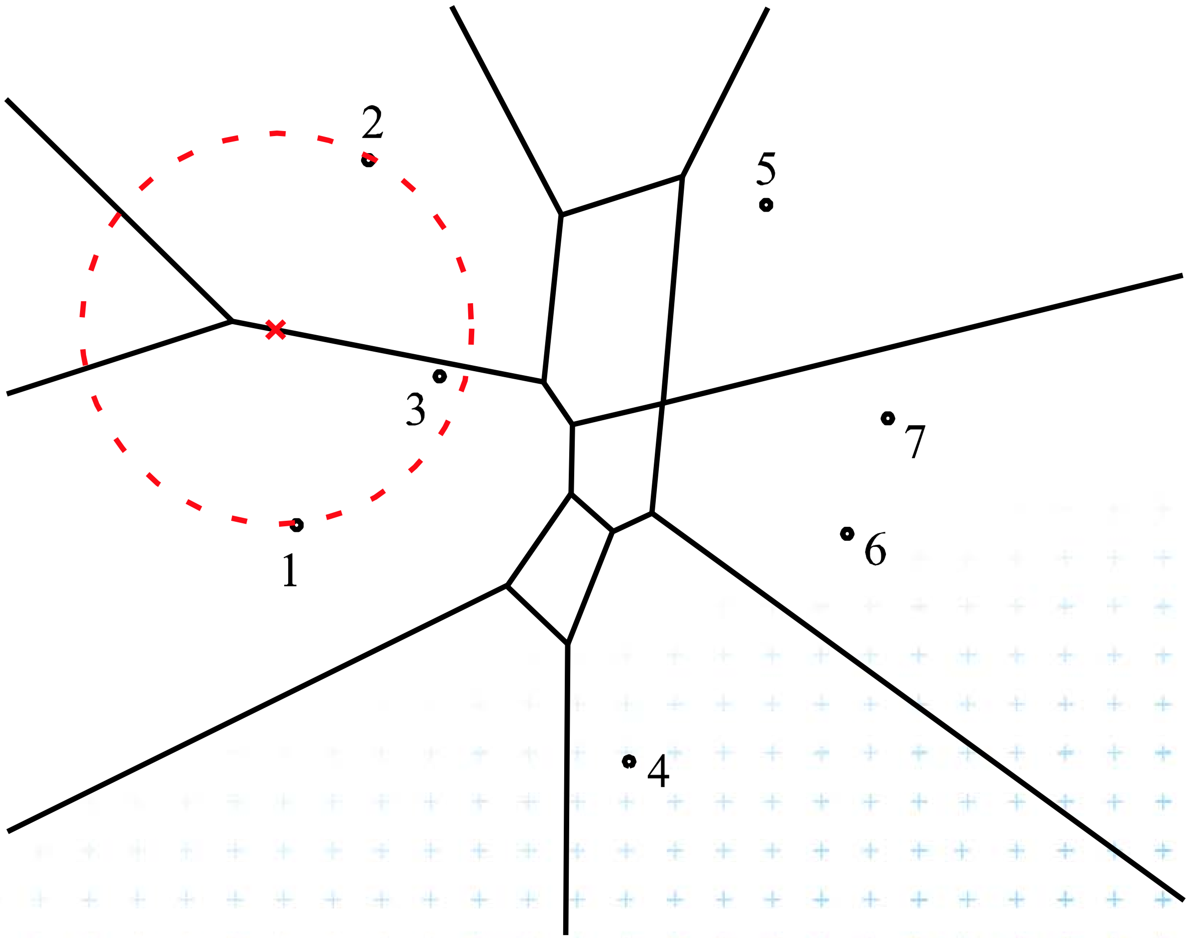
Order-2 Voronoi edges

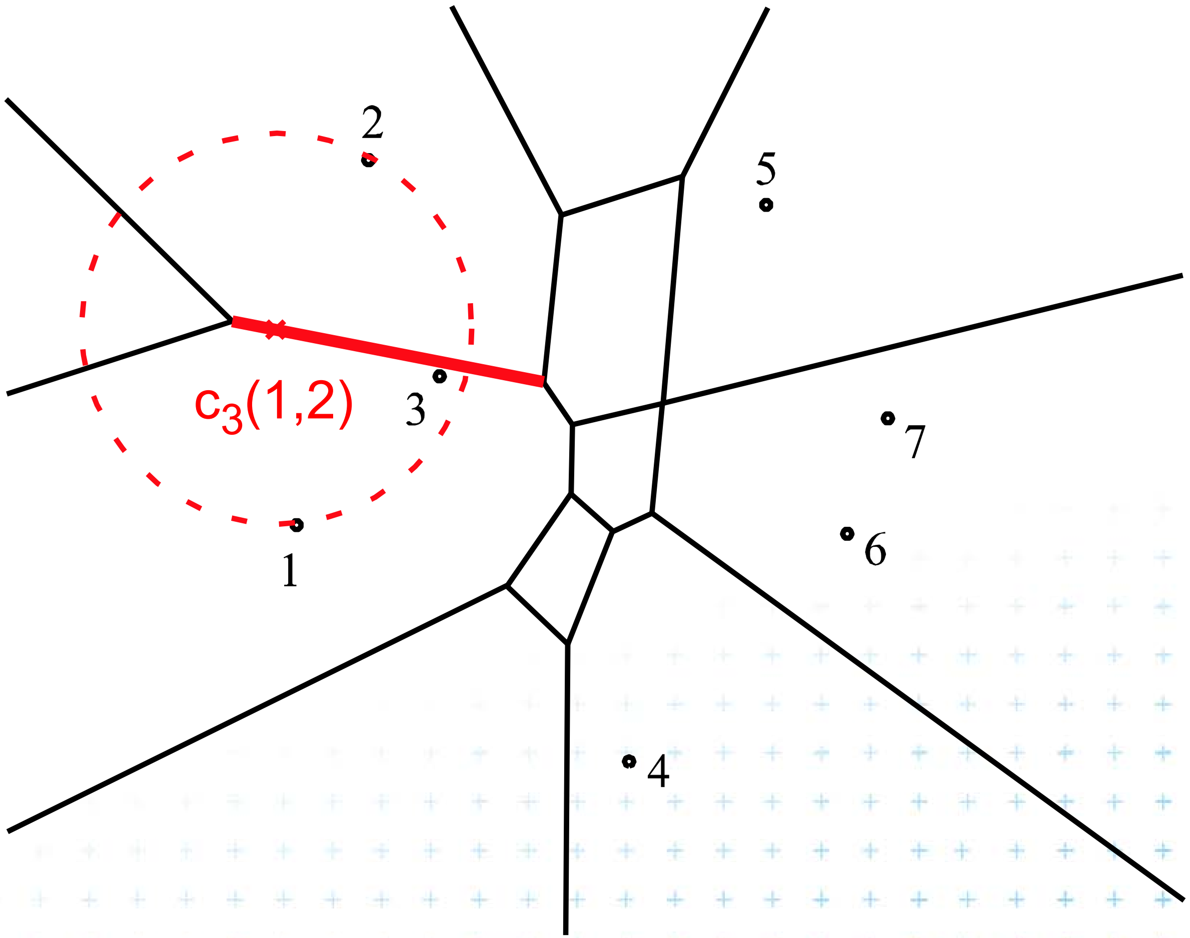


[Nandy]

Order-2 Voronoi edges

edge : set of centers of circles passing through 2 sites *s* and *t* and containing one site *p*

=> $c_p(s,t)$



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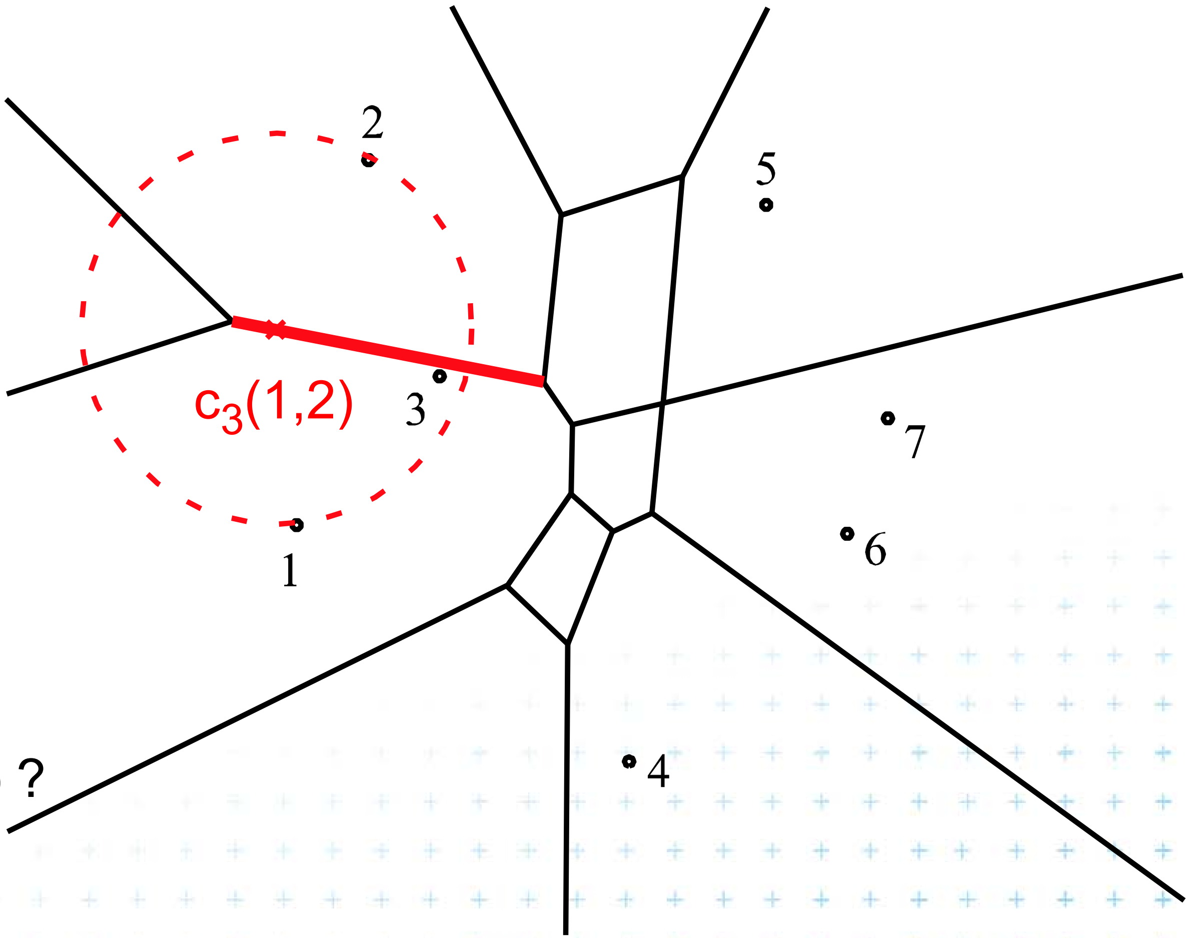

Question

Which are the regions on both sides of $c_p(s,t)$?

[Nandy]

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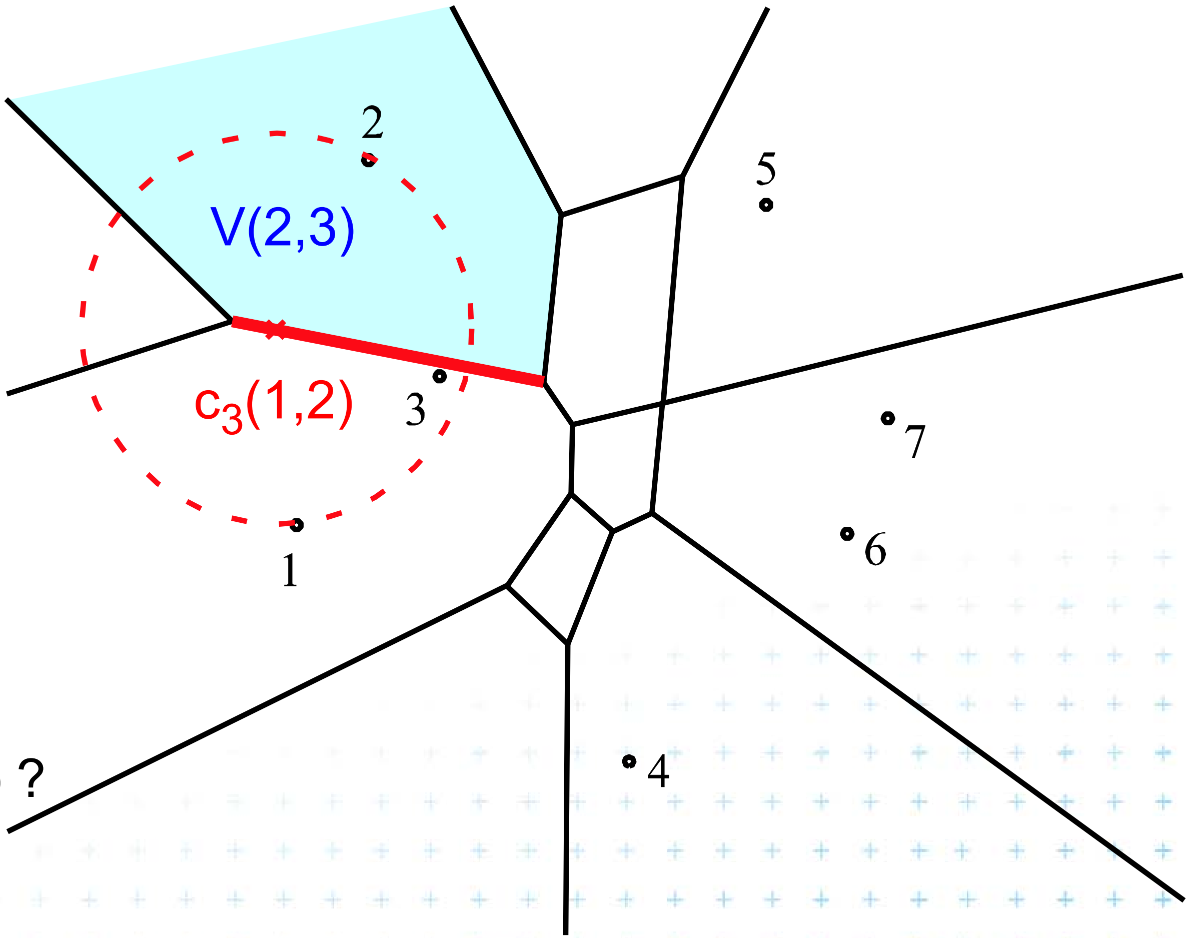

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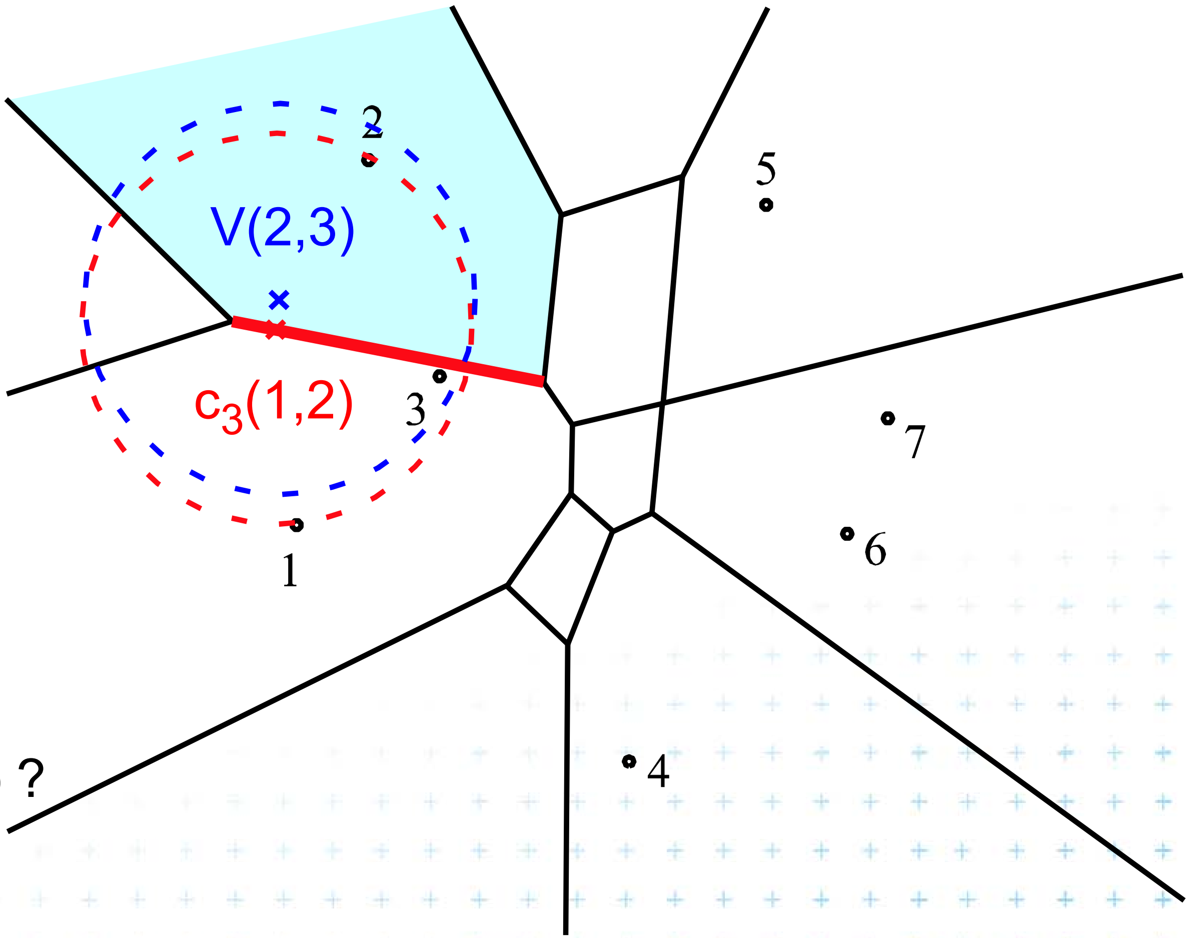

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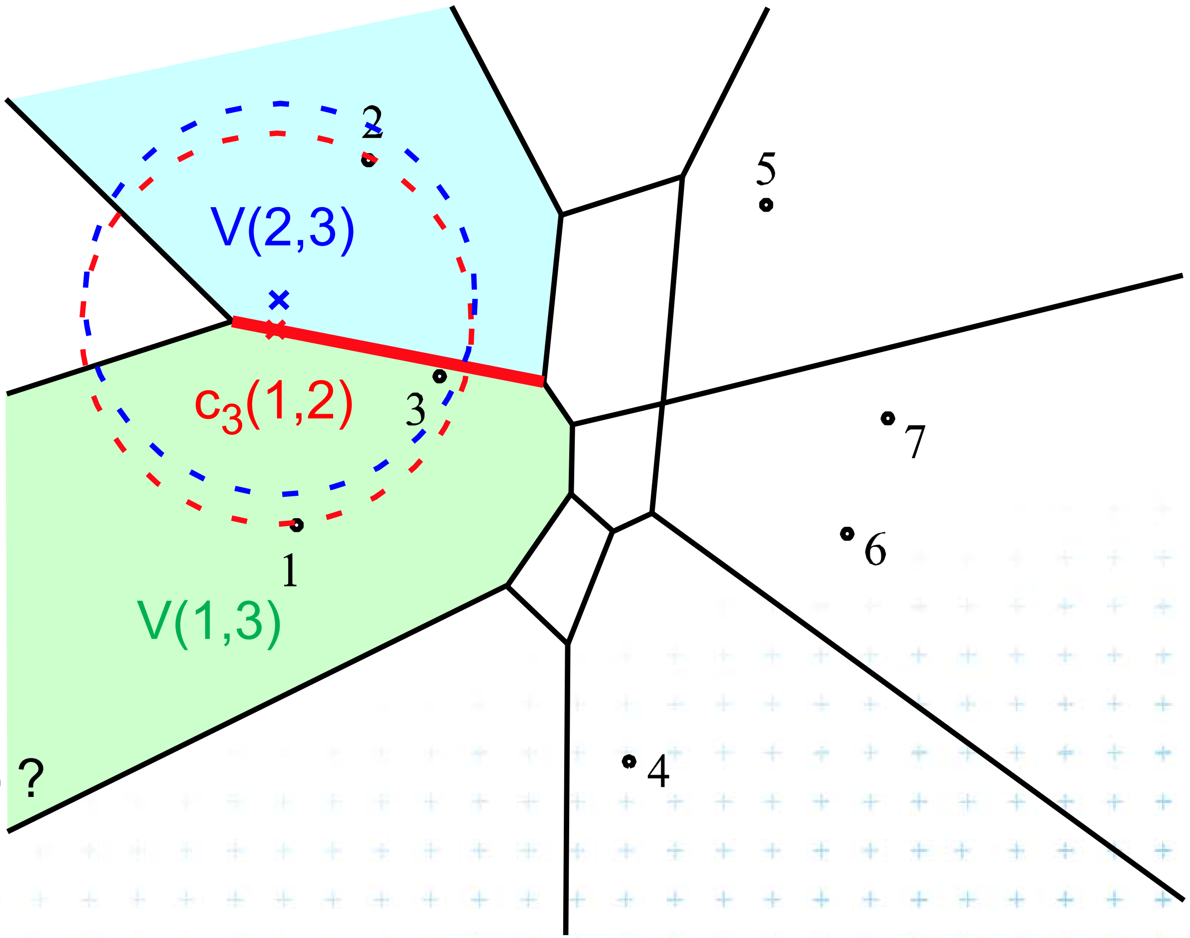

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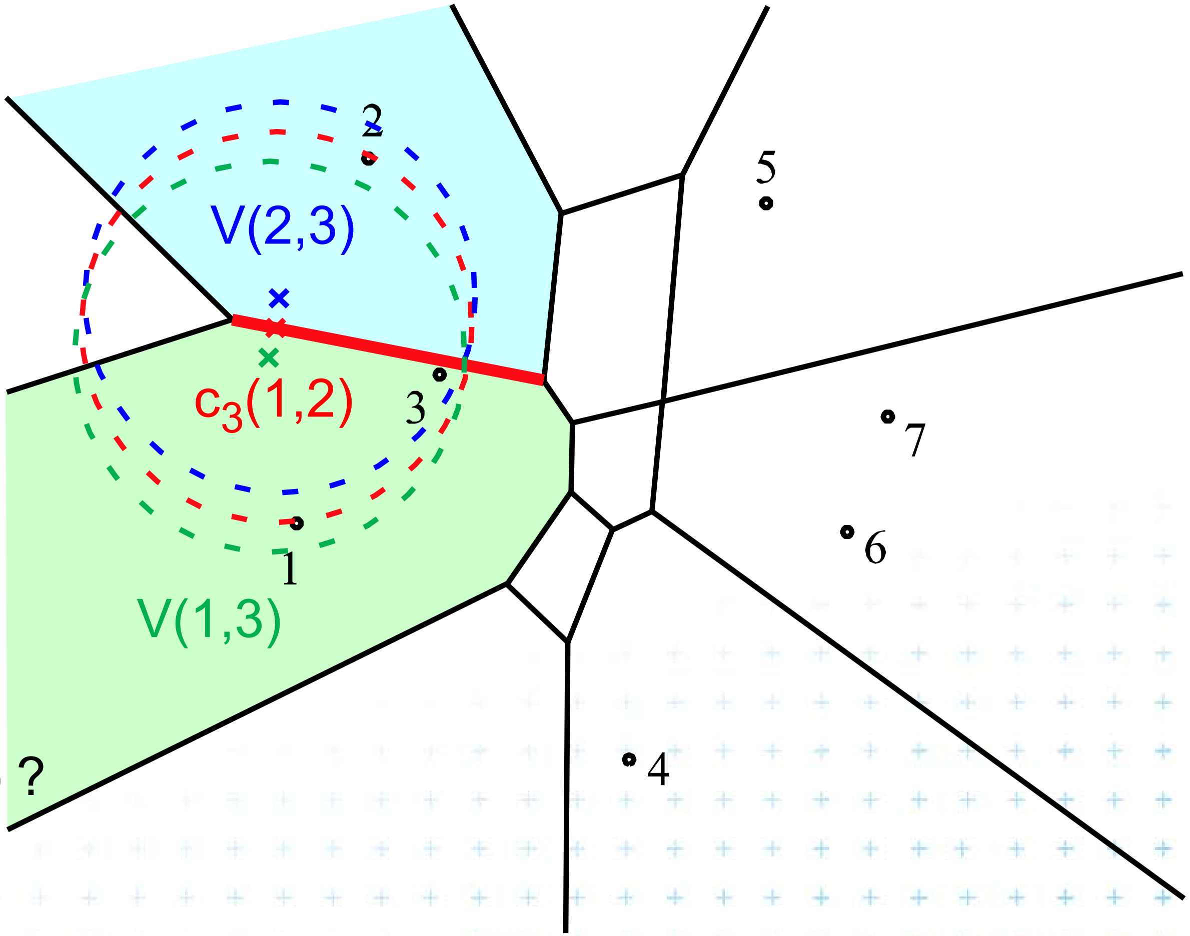

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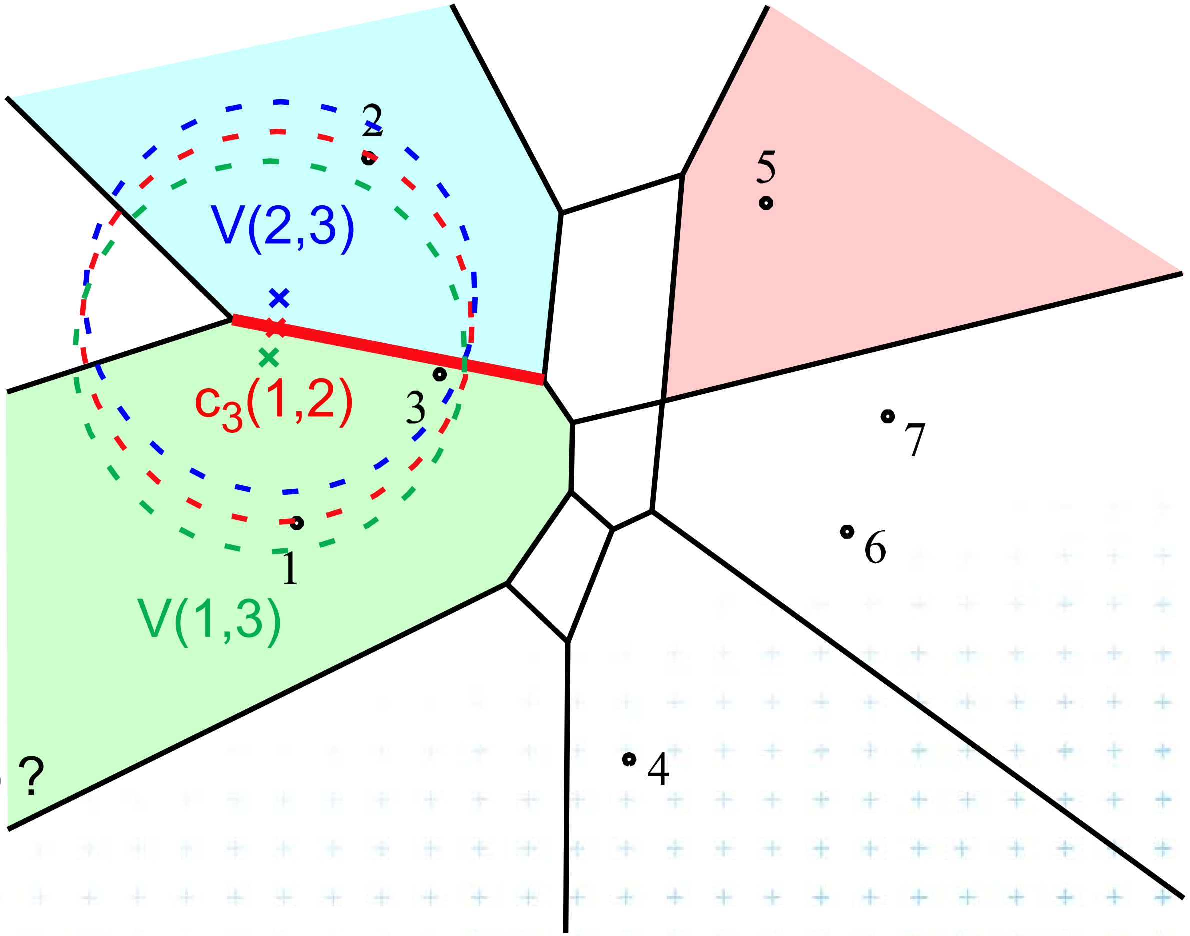

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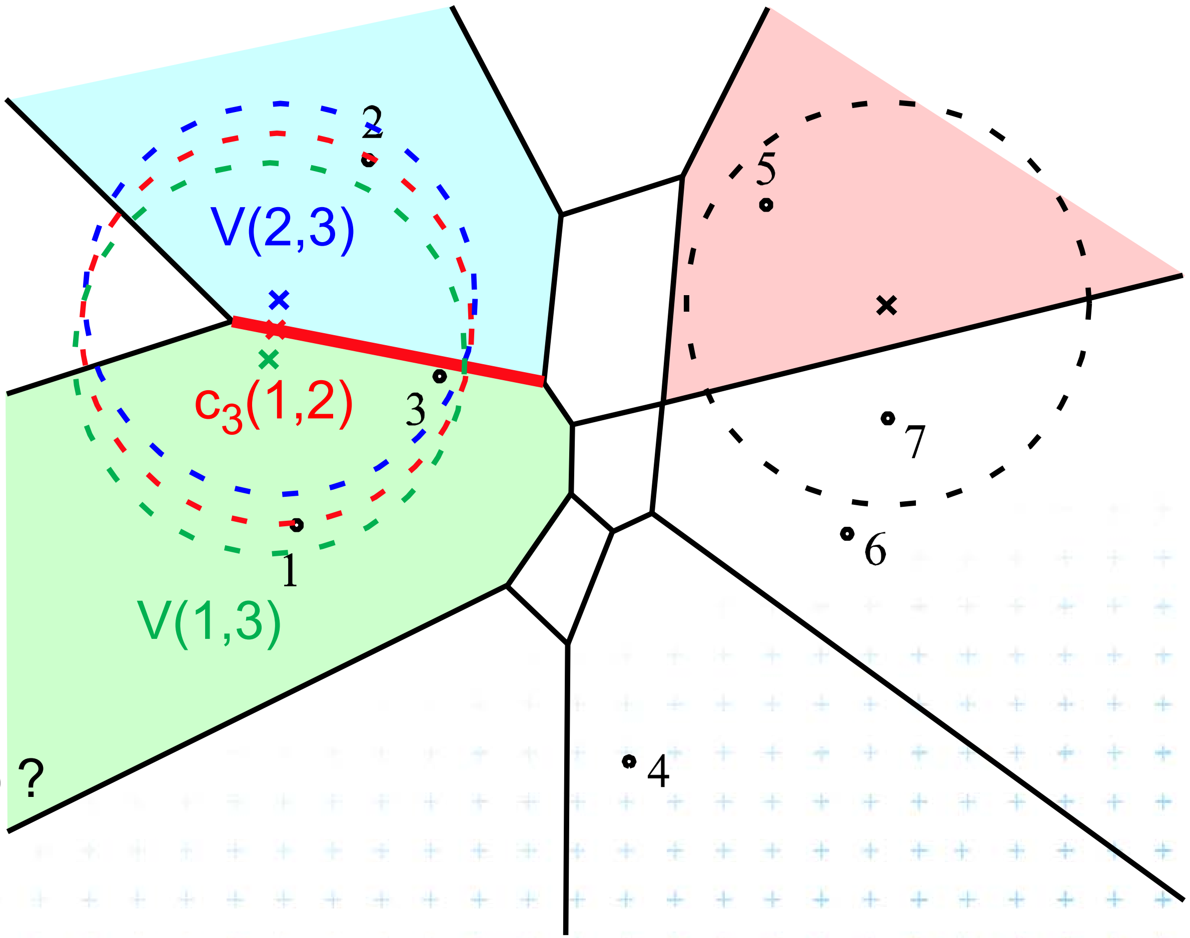

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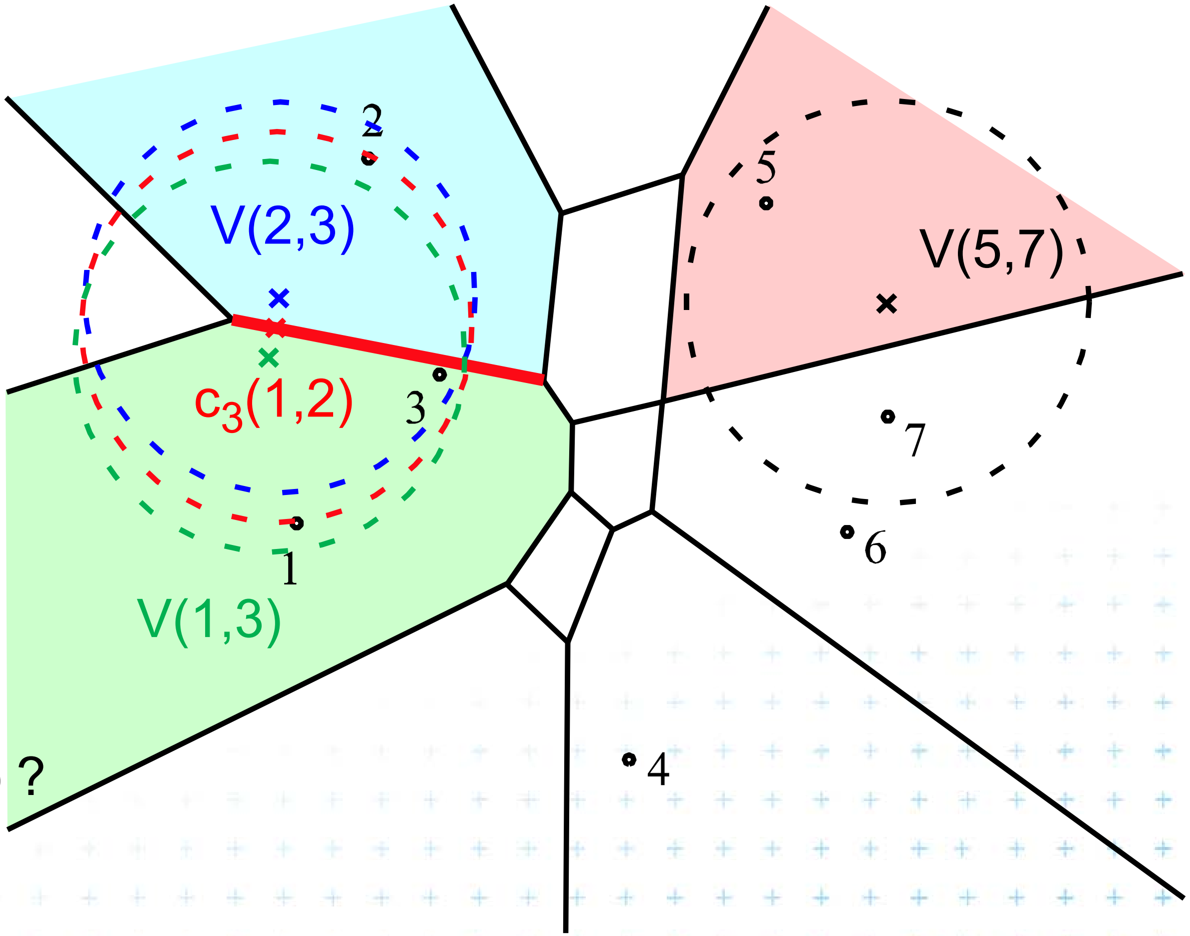

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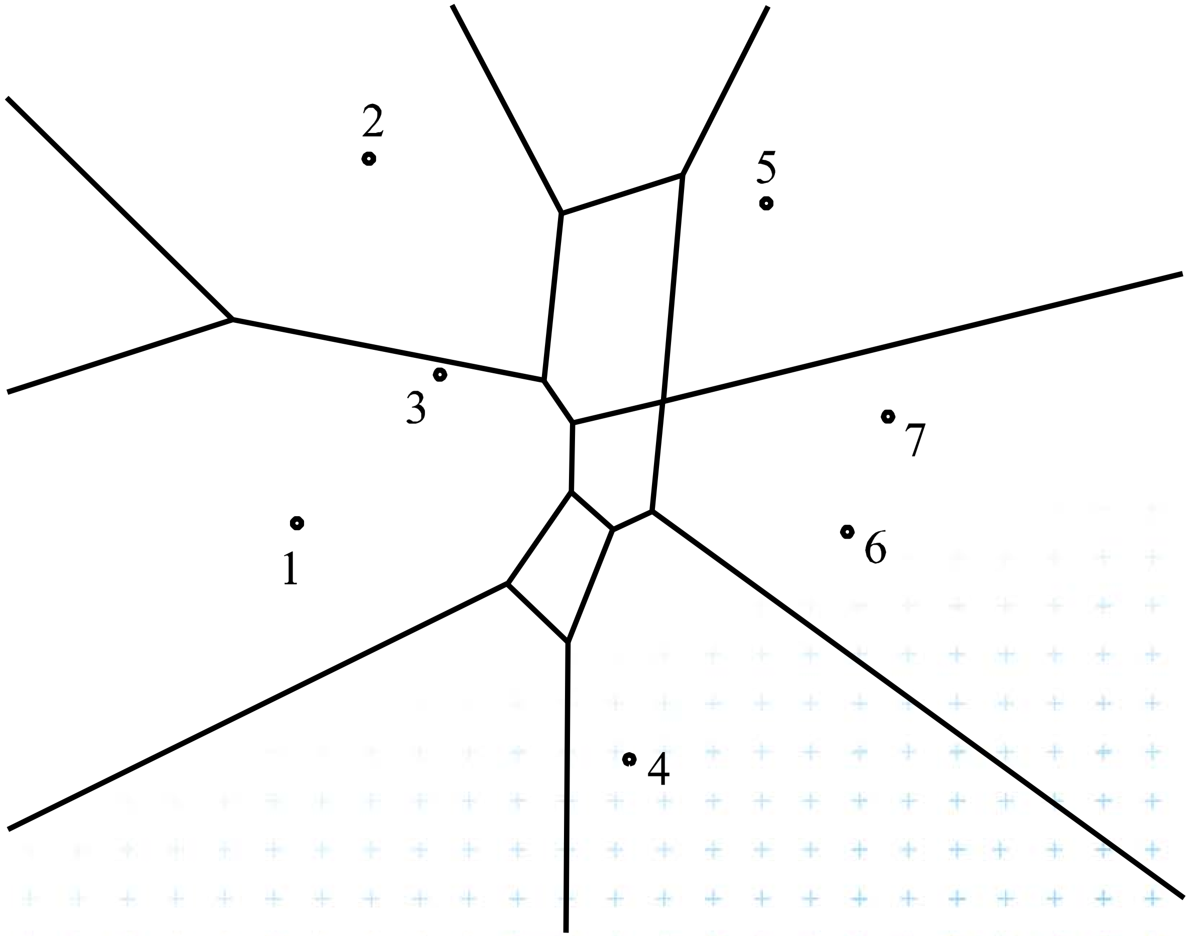
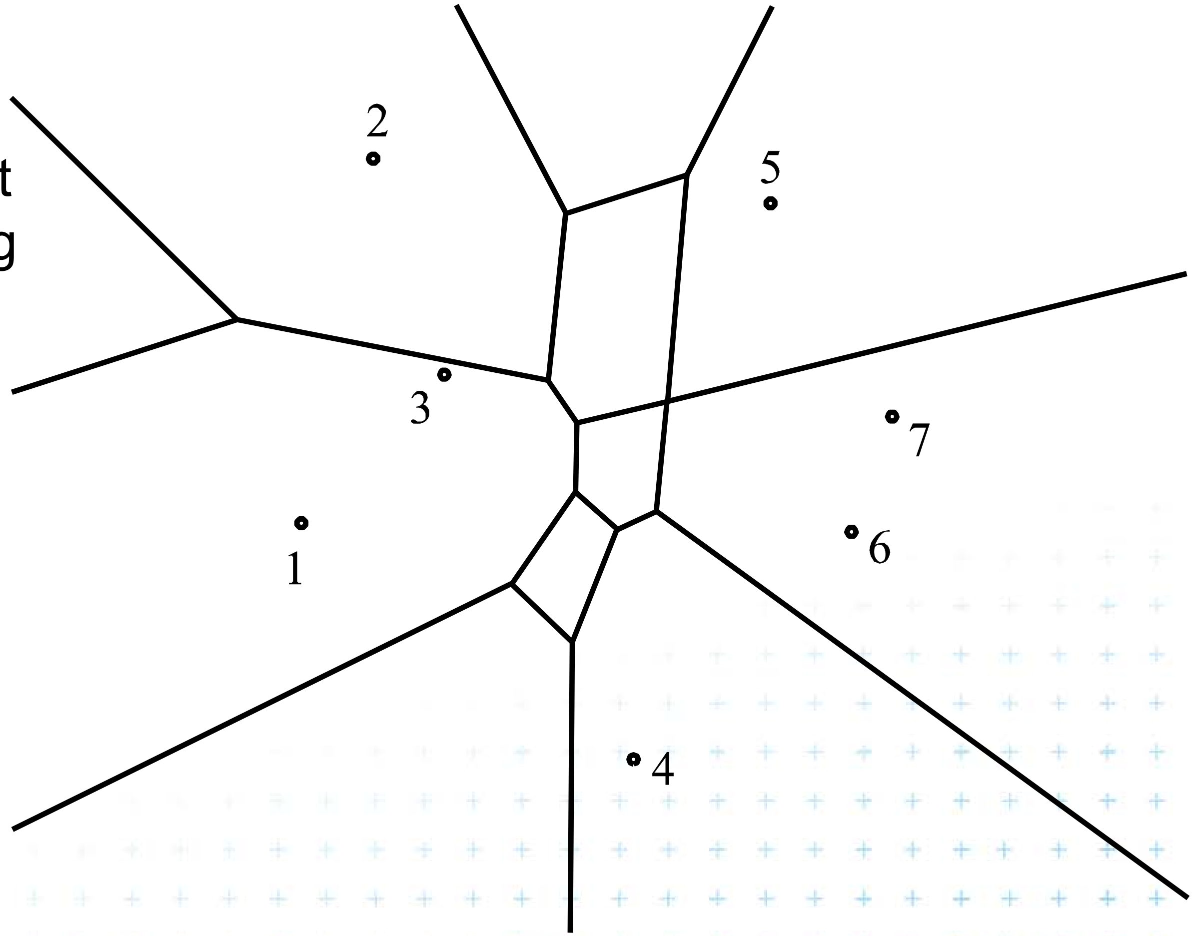
Order-2 Voronoi vertices



[Nandy]

Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing



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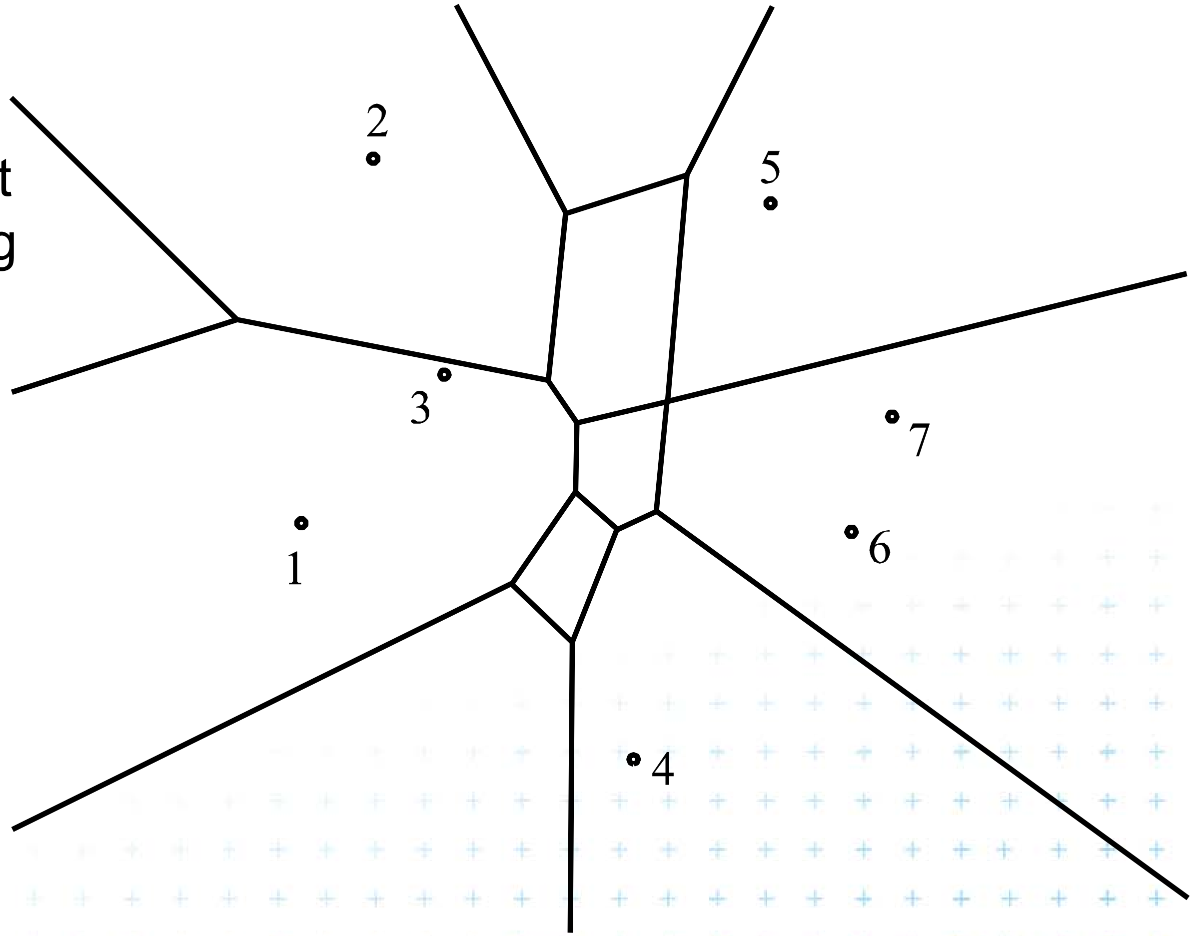
$u_5(2,3,7)$,



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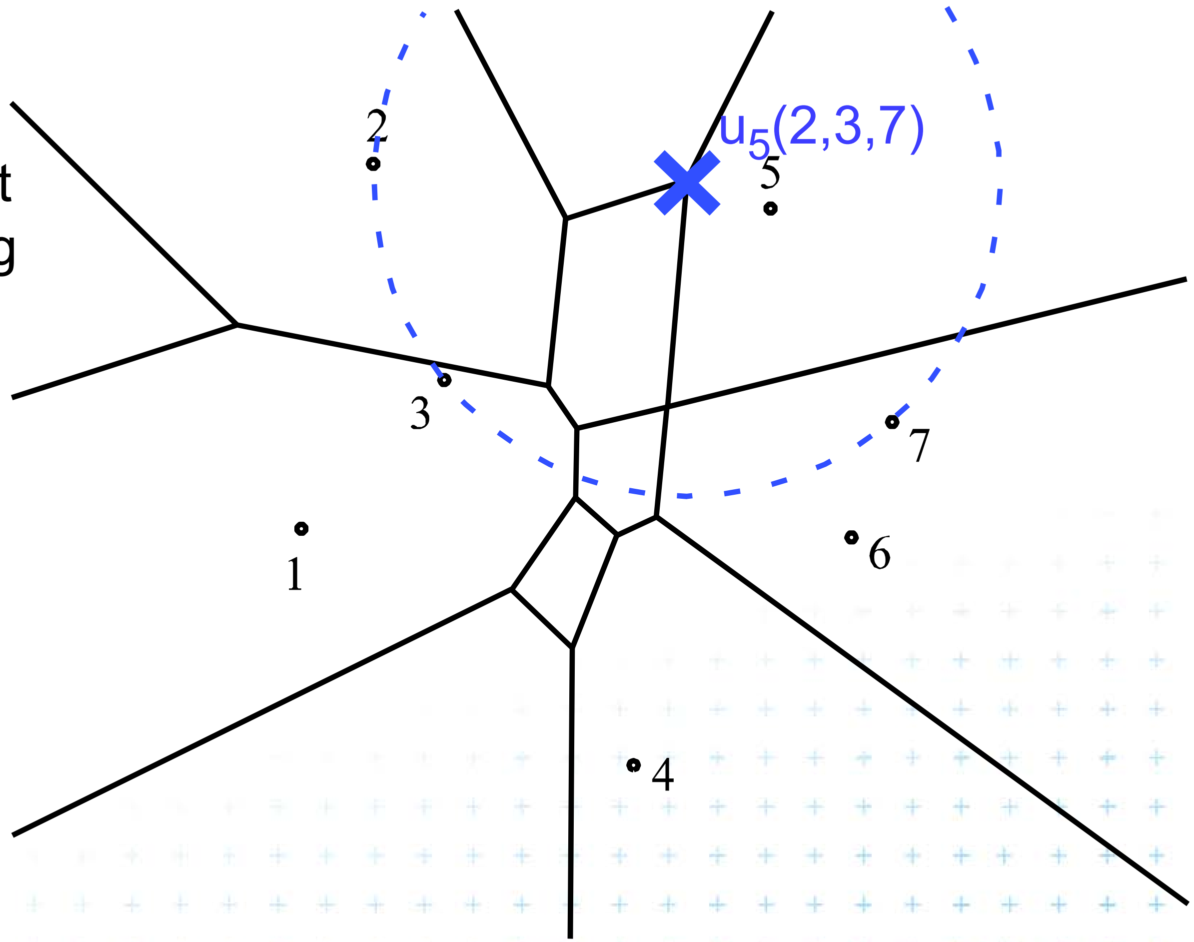

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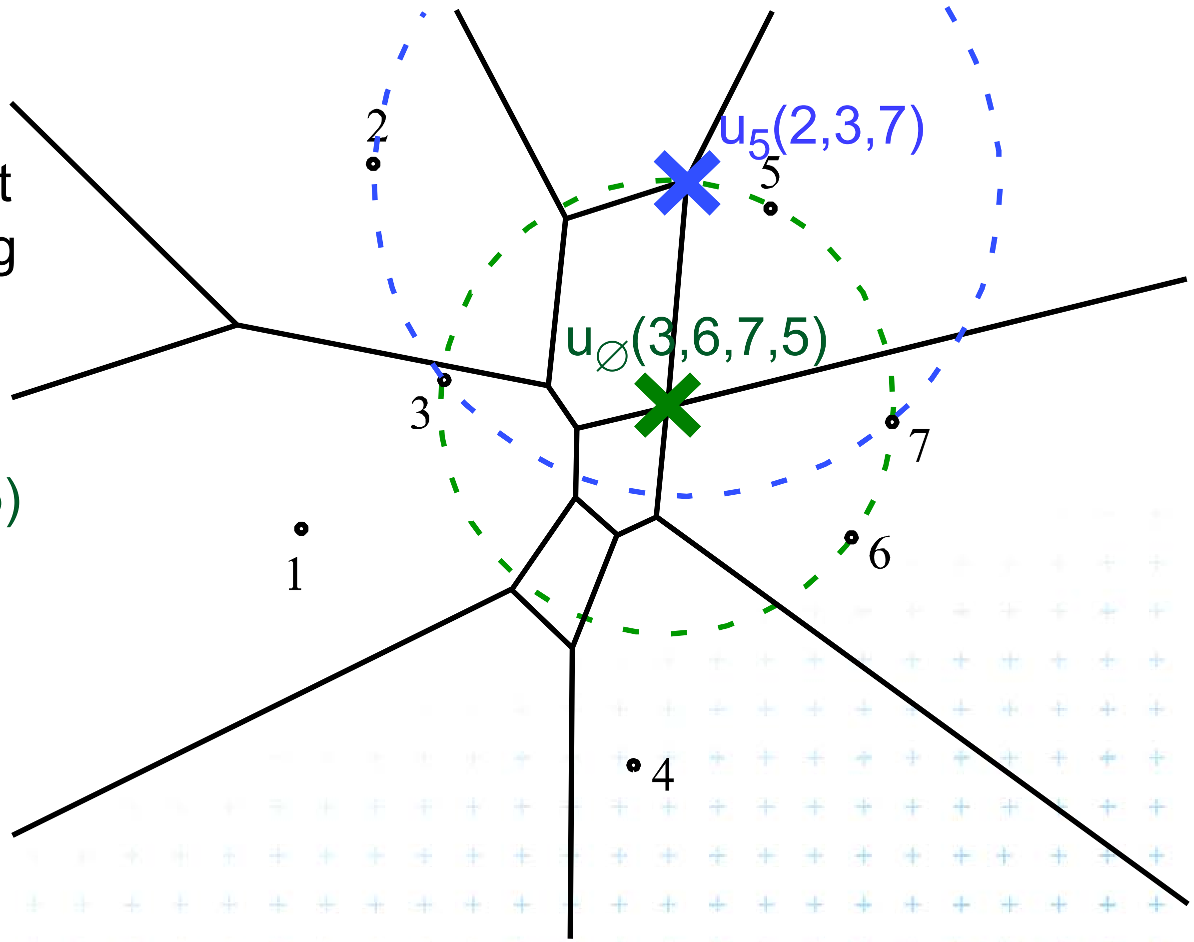
Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites Q and containing either site p or nothing

$\Rightarrow$ $u_p(Q)$ or $u_\varnothing(Q \cup p)$
$u_5(2,3,7)$, $u_\varnothing(3,6,7,5)$



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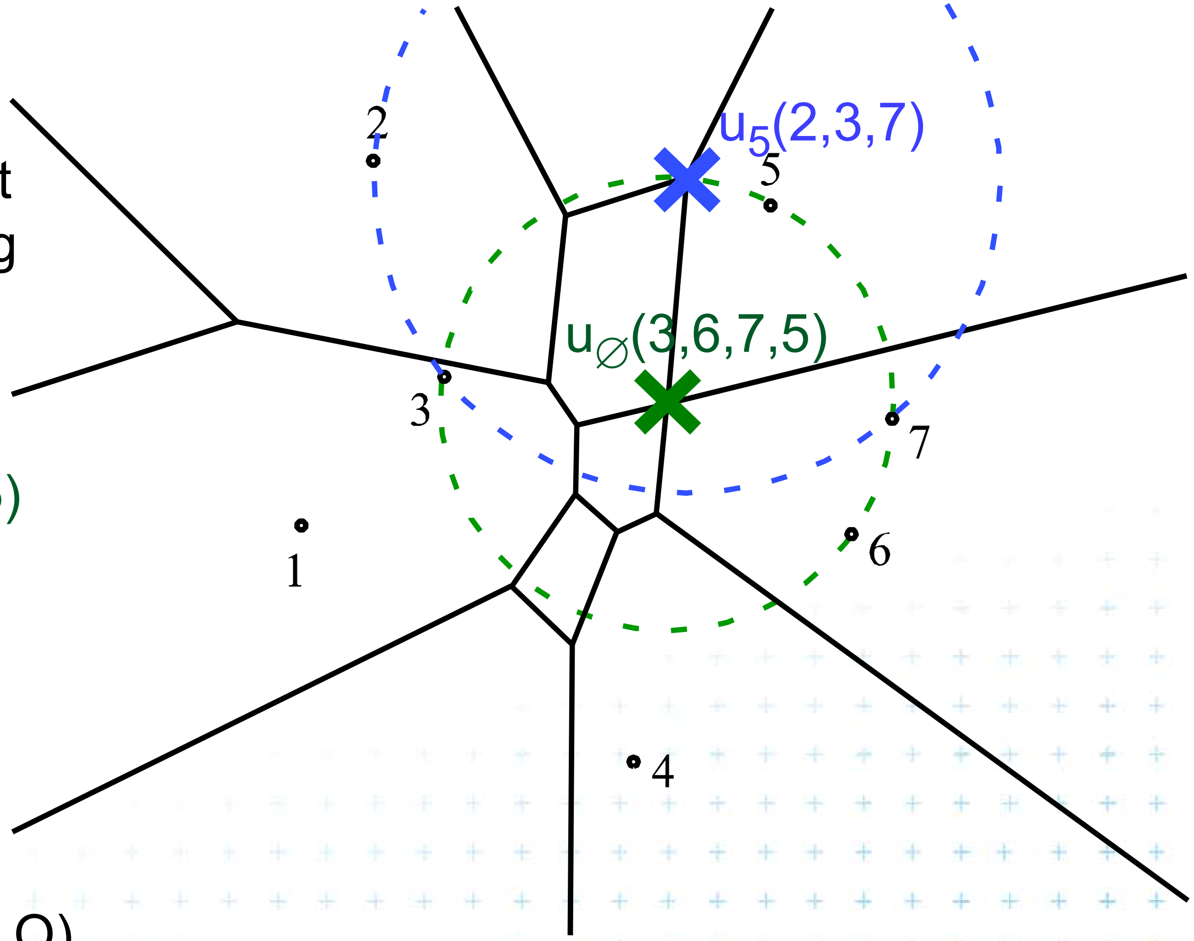


(circle circumscribed to Q)

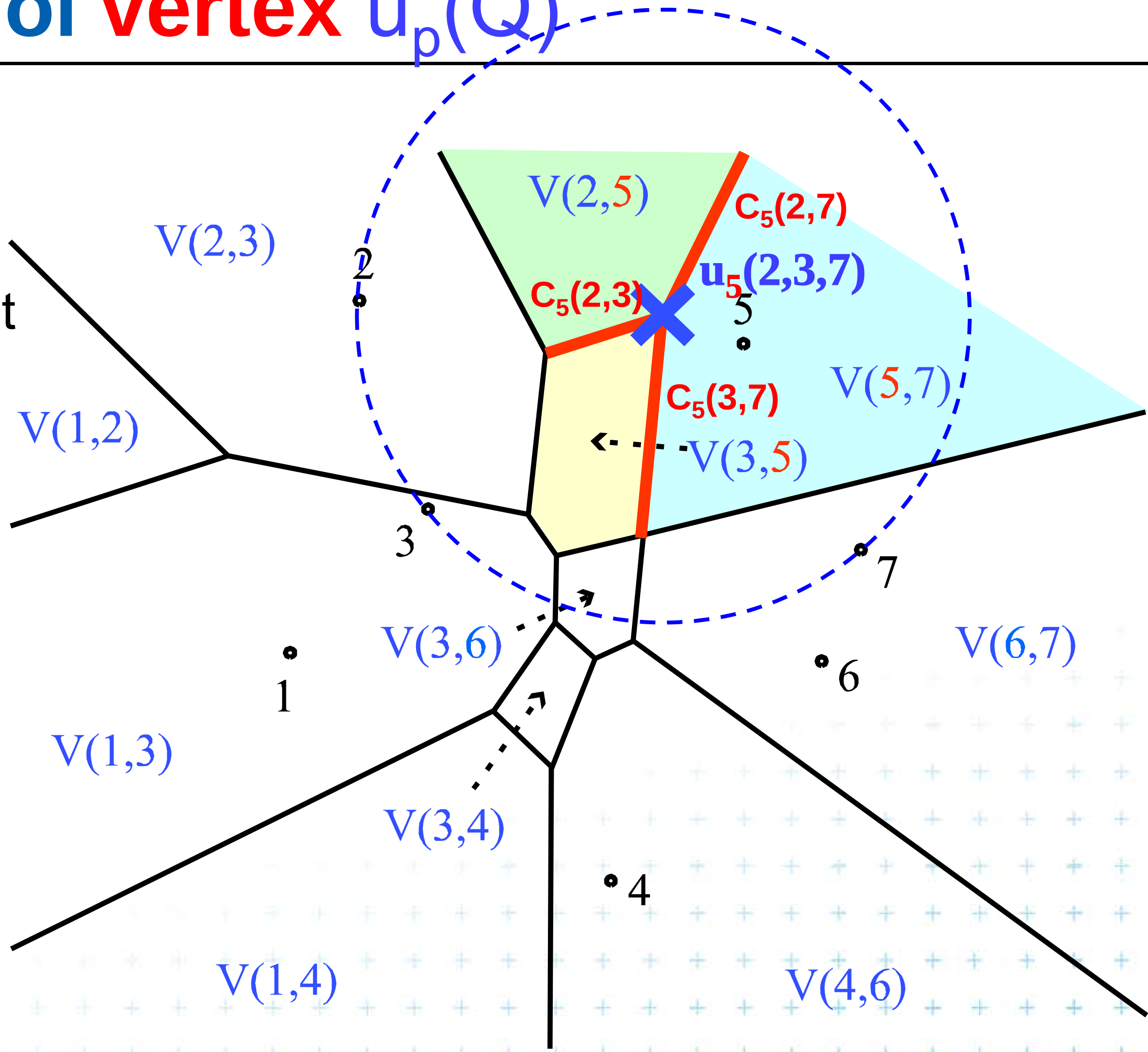
[Nandy]

Order-2 Voronoi vertex $u_p(Q)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_p(Q)$
$u_5(2,3,7)$

Cell 5 is inside for all incident edges:
$C_5(2,3)$
$C_5(2,7)$
$C_5(3,7)$
=> 5 is inside for circle with center in vertex



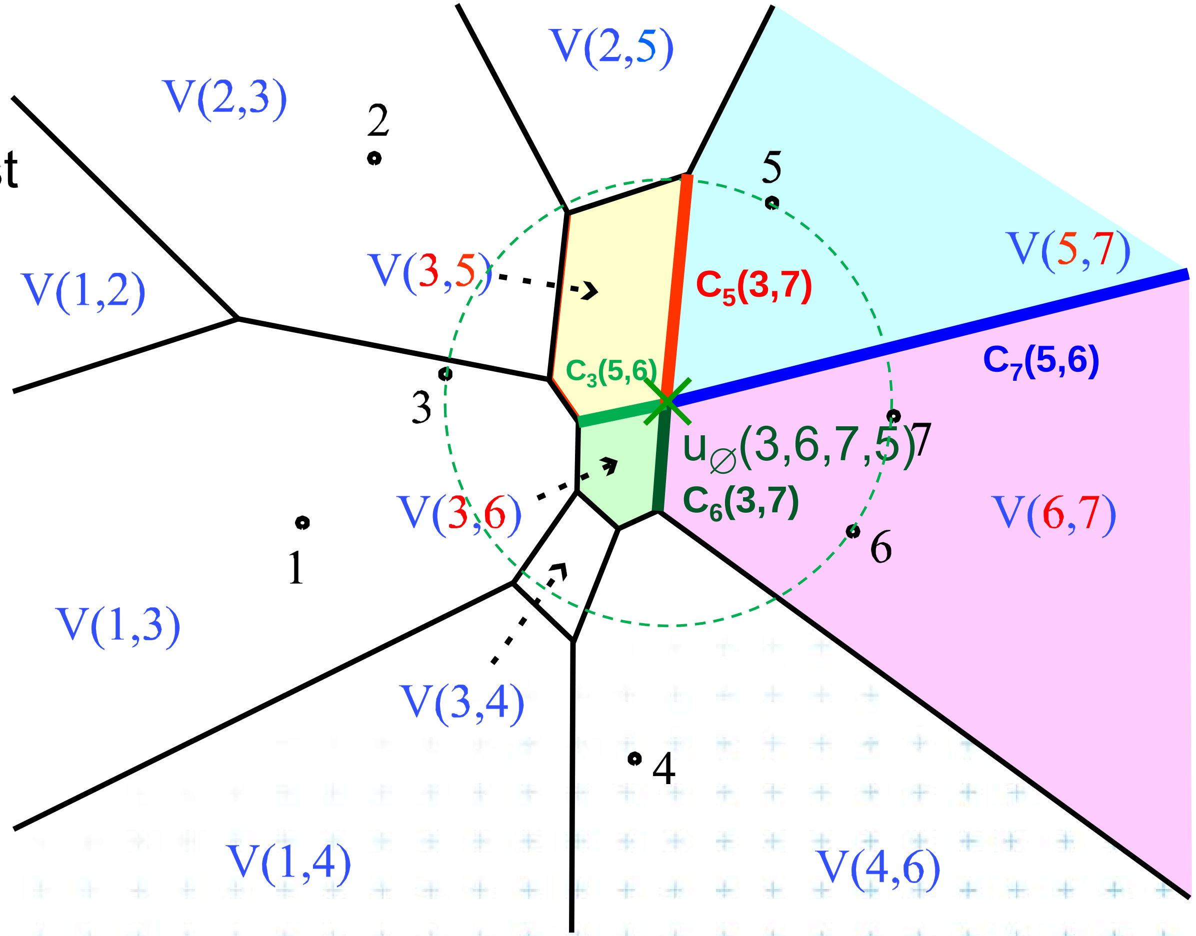
[Nandy]

Order-2 Voronoi **vertex** $u_{\varnothing}(Q \cup p)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_{\varnothing}(Q \cup p)$
$u_{\varnothing}(3,6,7,5)$

Cell 5 is not inside for all incident edges:
$C_5(3,7)$
$C_6(3,7)$
$C_3(5,6)$
$C_7(5,6)$
=> 5 on circle with center in vertex



[Nandy]

Order-k Voronoi Diagram

Theorem věta

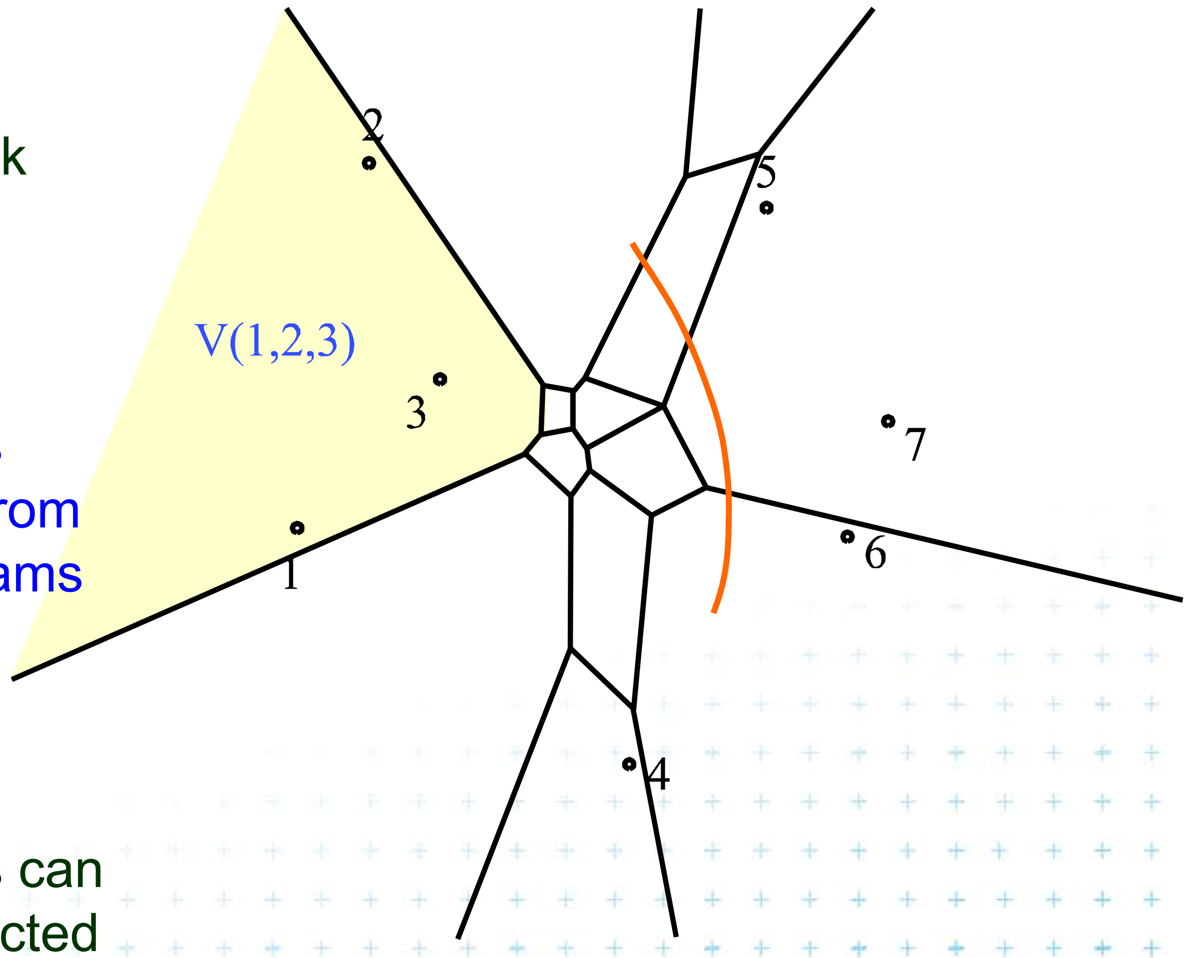
The size of the order-k diagrams is O(k(n-k))

Theorem věta

The order-k diagrams can be constructed from the order-(k-1) diagrams in O(k(n-k)) time

Corollary důsledek

The order-k diagrams can be iteratively constructed in $O(n \log n + k^2(n-k))$ time

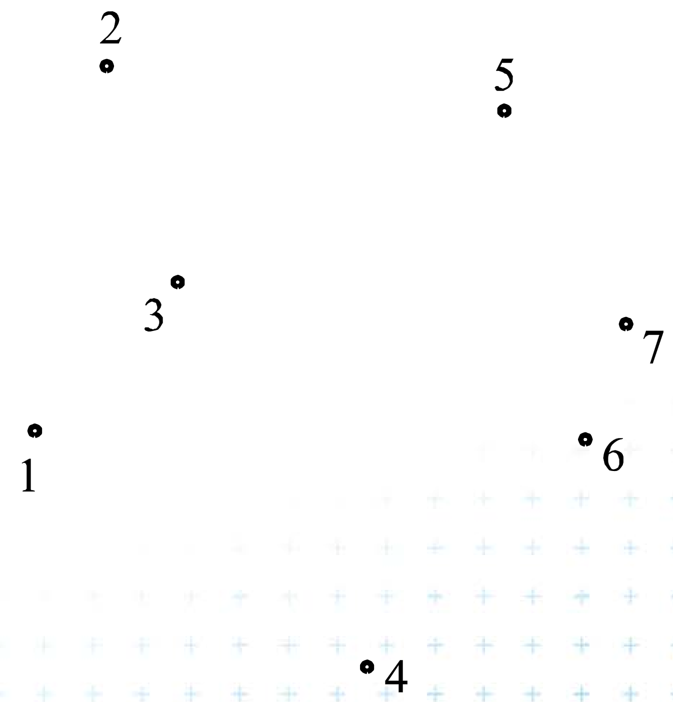


[Nandy]

Order n-1 VD
(Farthest-point Voronoi diagram)

Order n-1 = Farthest-point Voronoi diagram

cell $V_{-1}(7) = V_{n-1}(\{1,2,3,4,5,6\})$
= set of points in the plane farther from p_i=7 than from any other site



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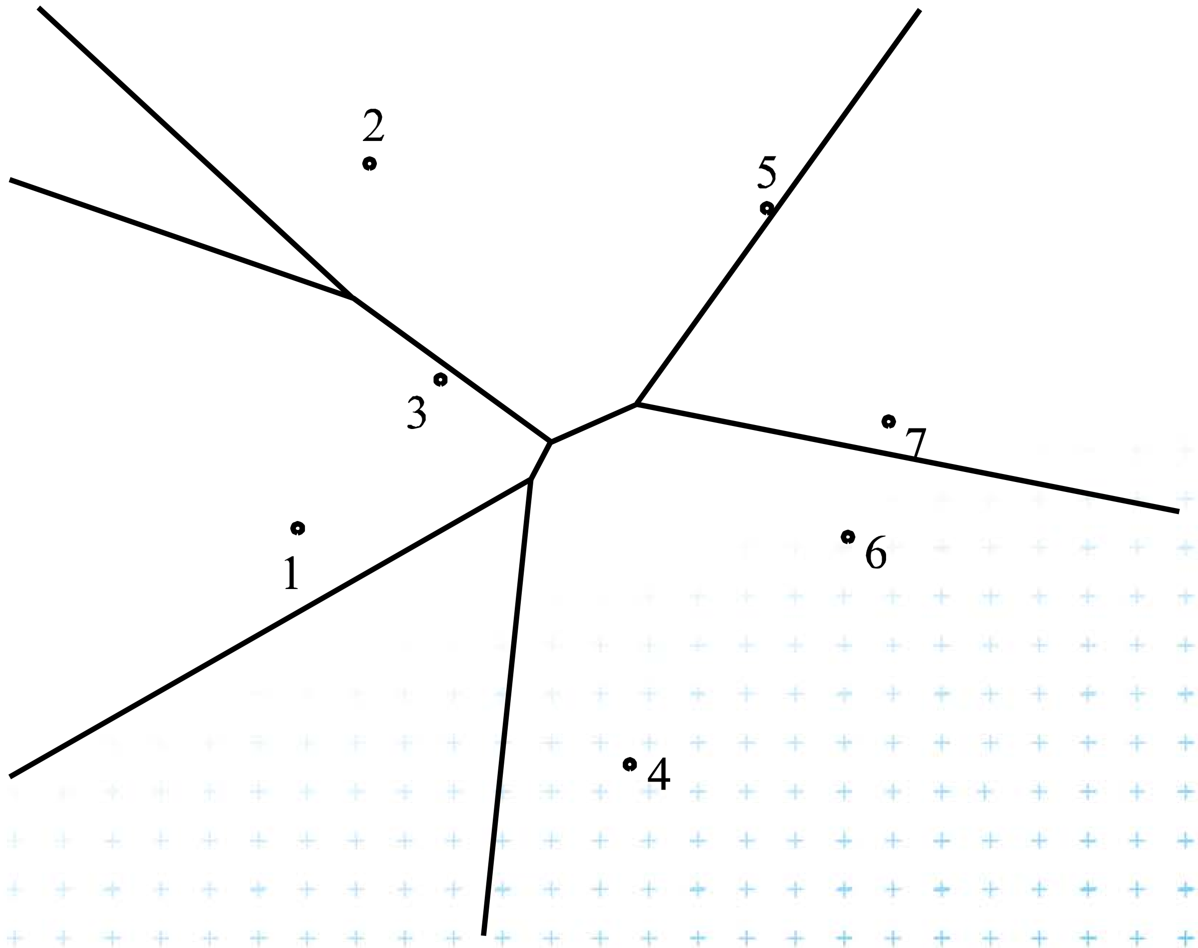

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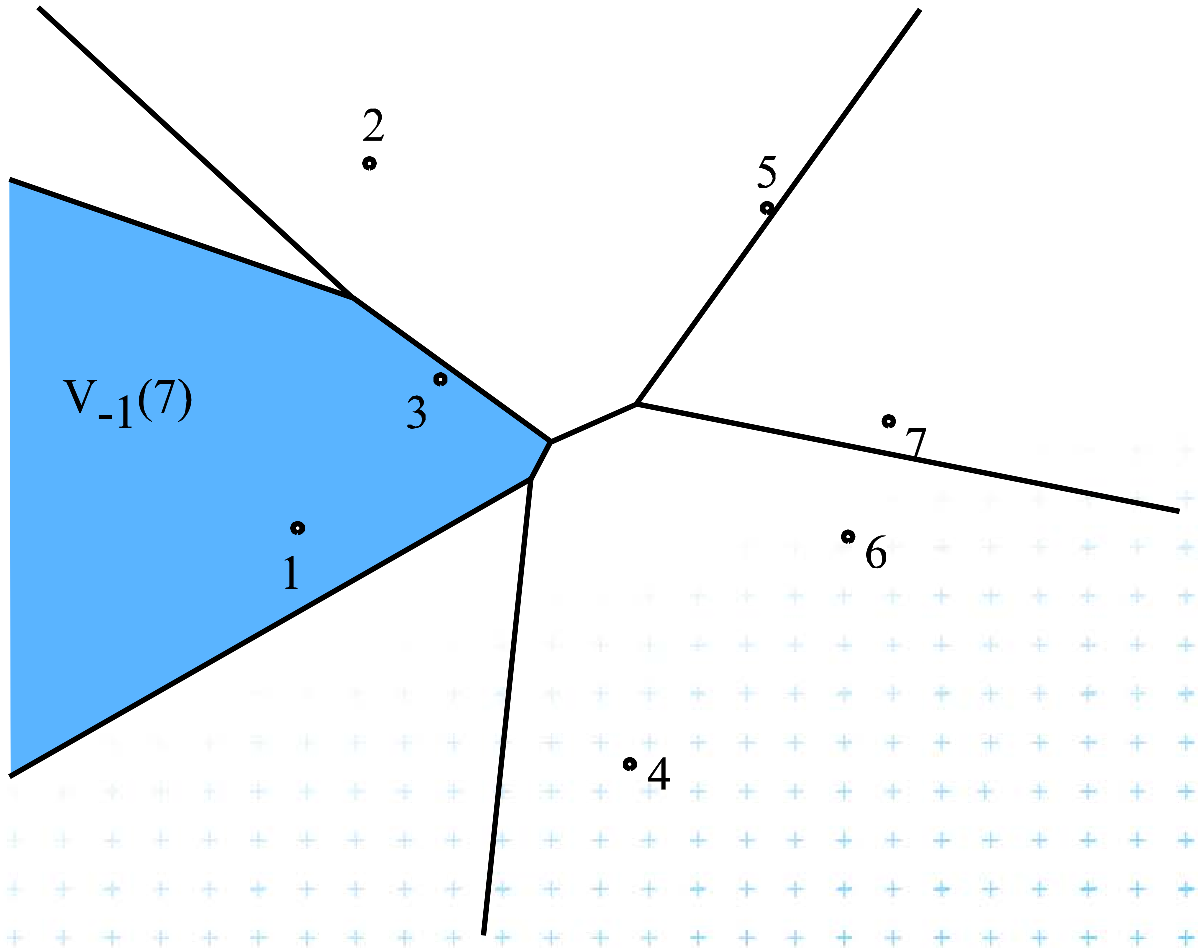

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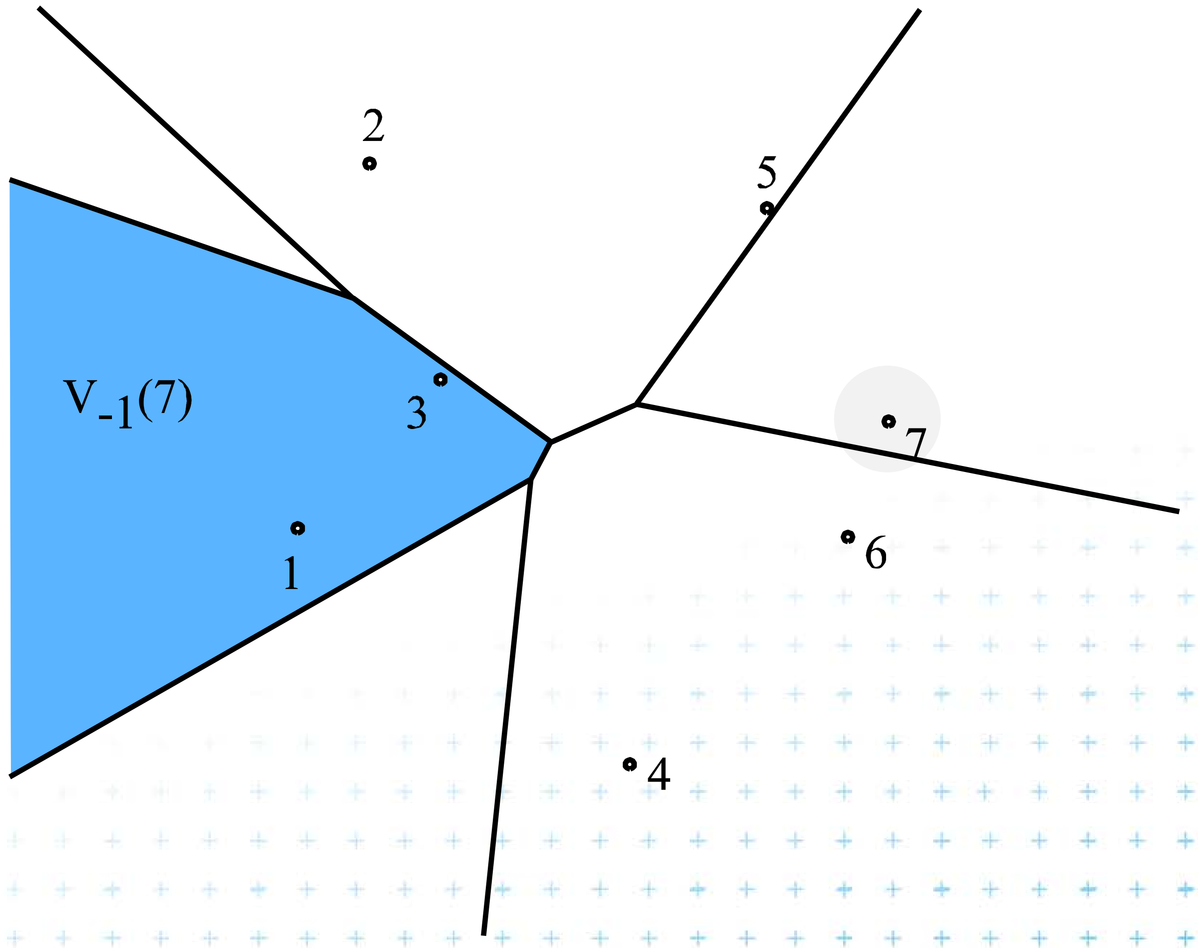

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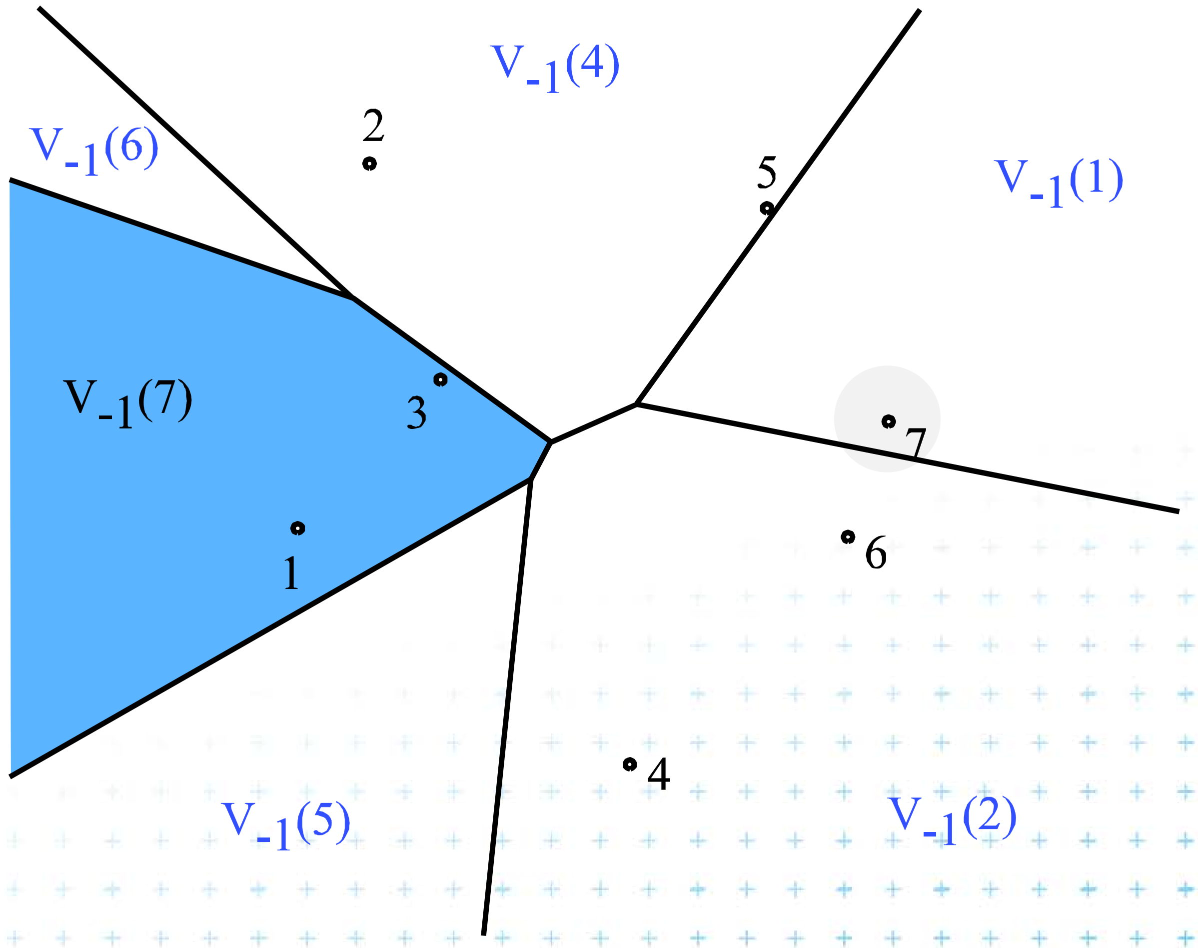
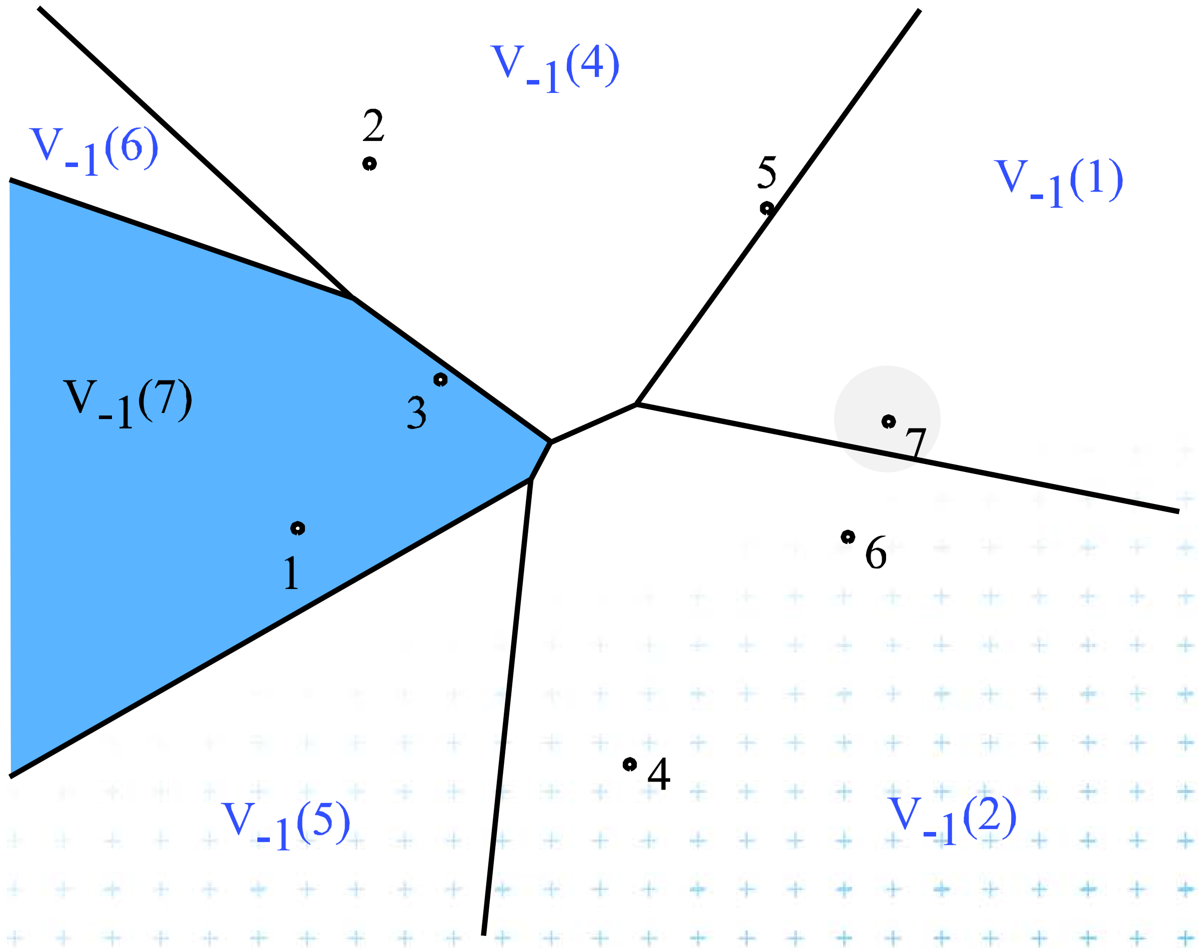


[Nandy]

Order n-1 = Farthest-point Voronoi diagram

cell $V_{-1}(7) = V_{n-1}(\{1,2,3,4,5,6\})$
= set of points in the plane farther from $p_i=7$ than from any other site

$Vor_{-1}(P) = Vor_{n-1}(P)$
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



[Nandy]

Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell
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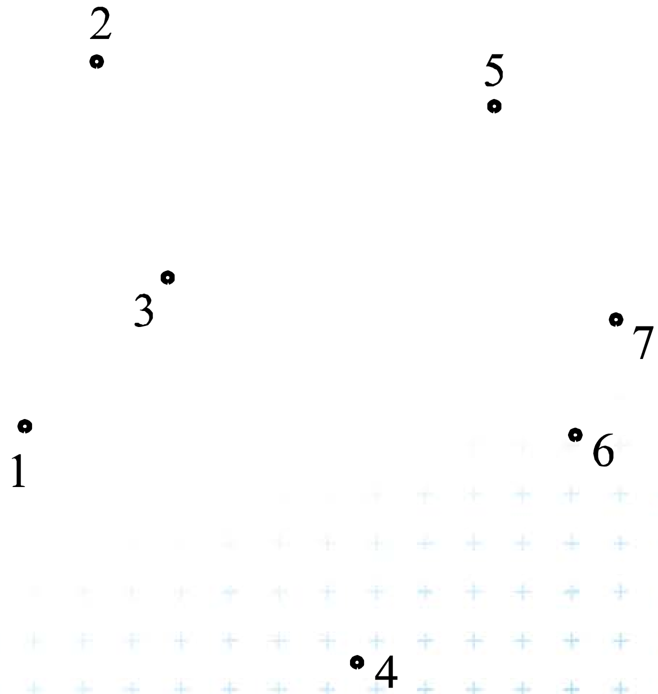

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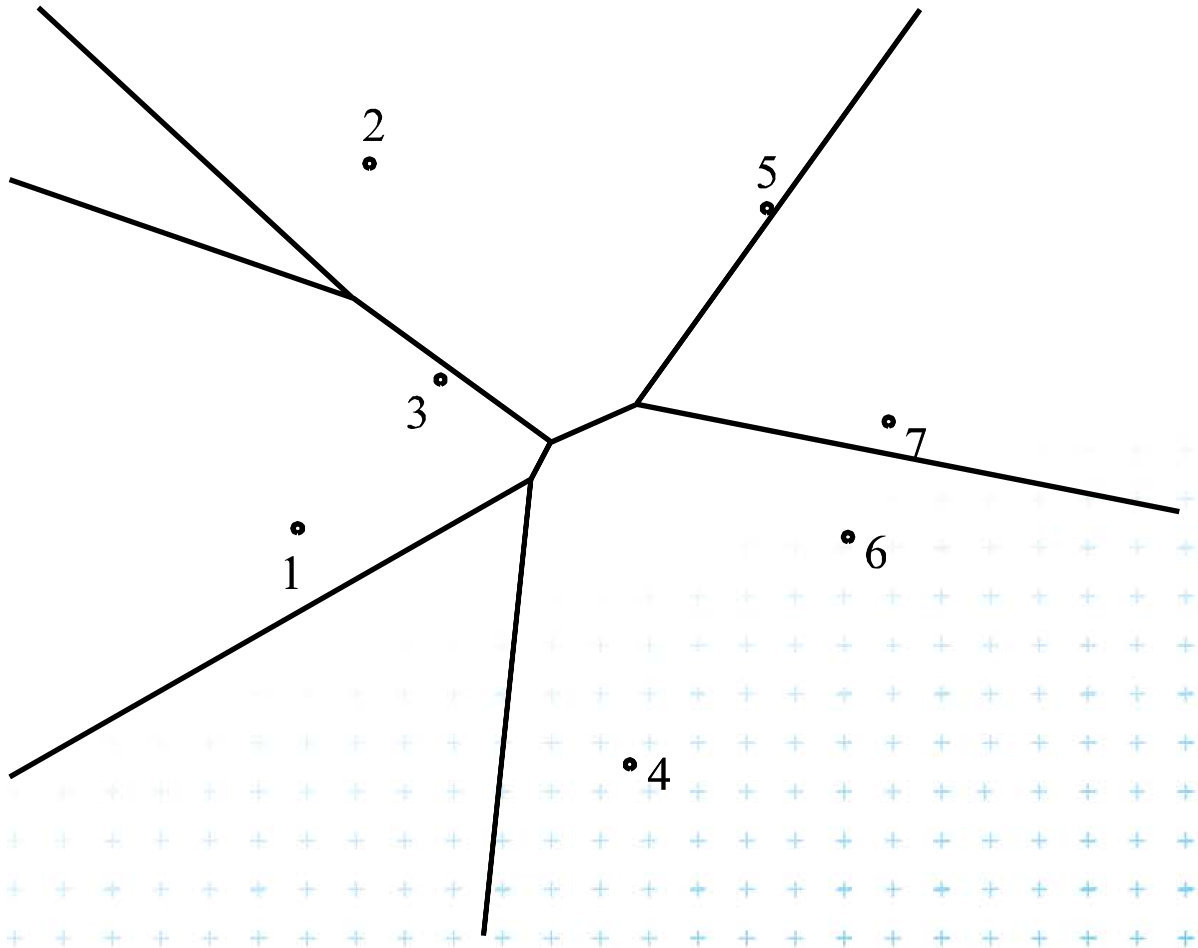

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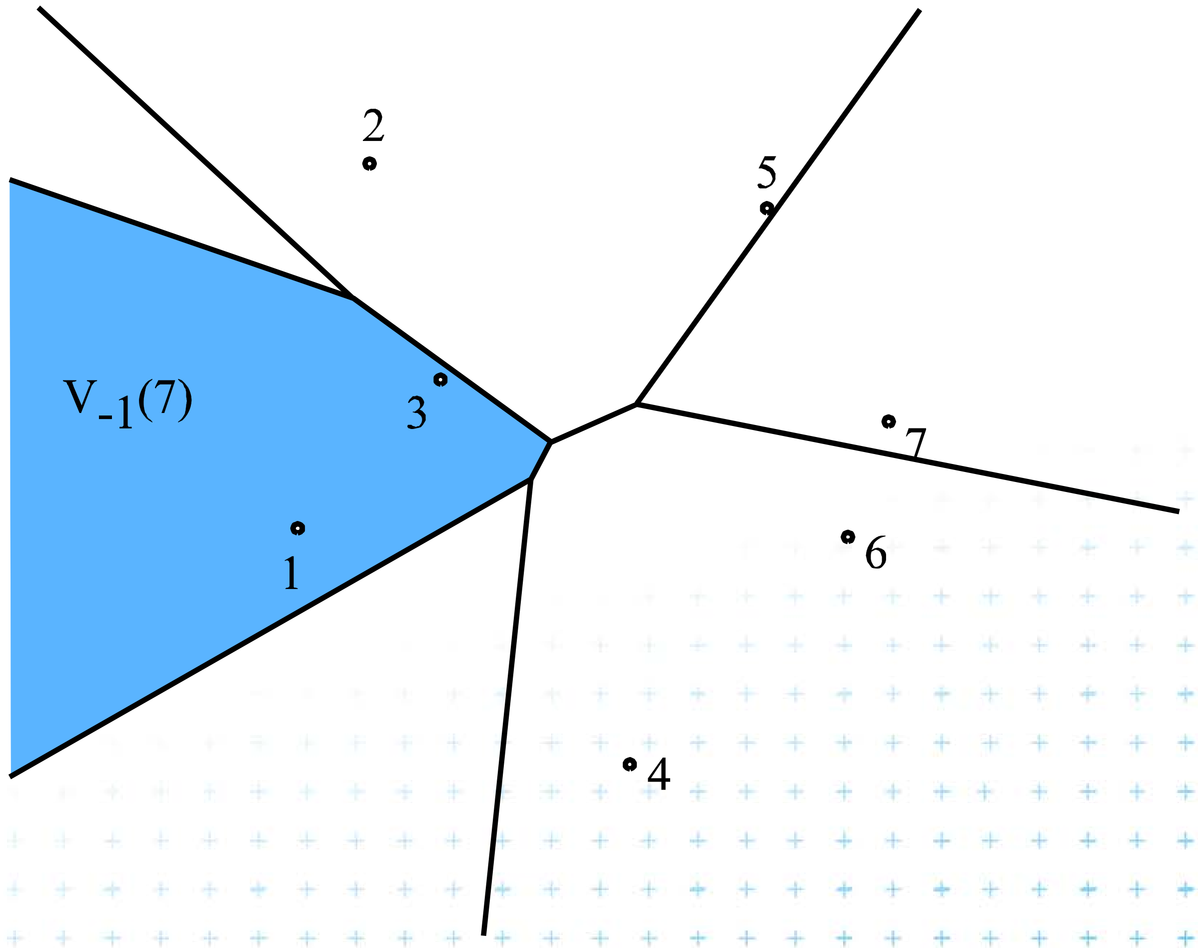

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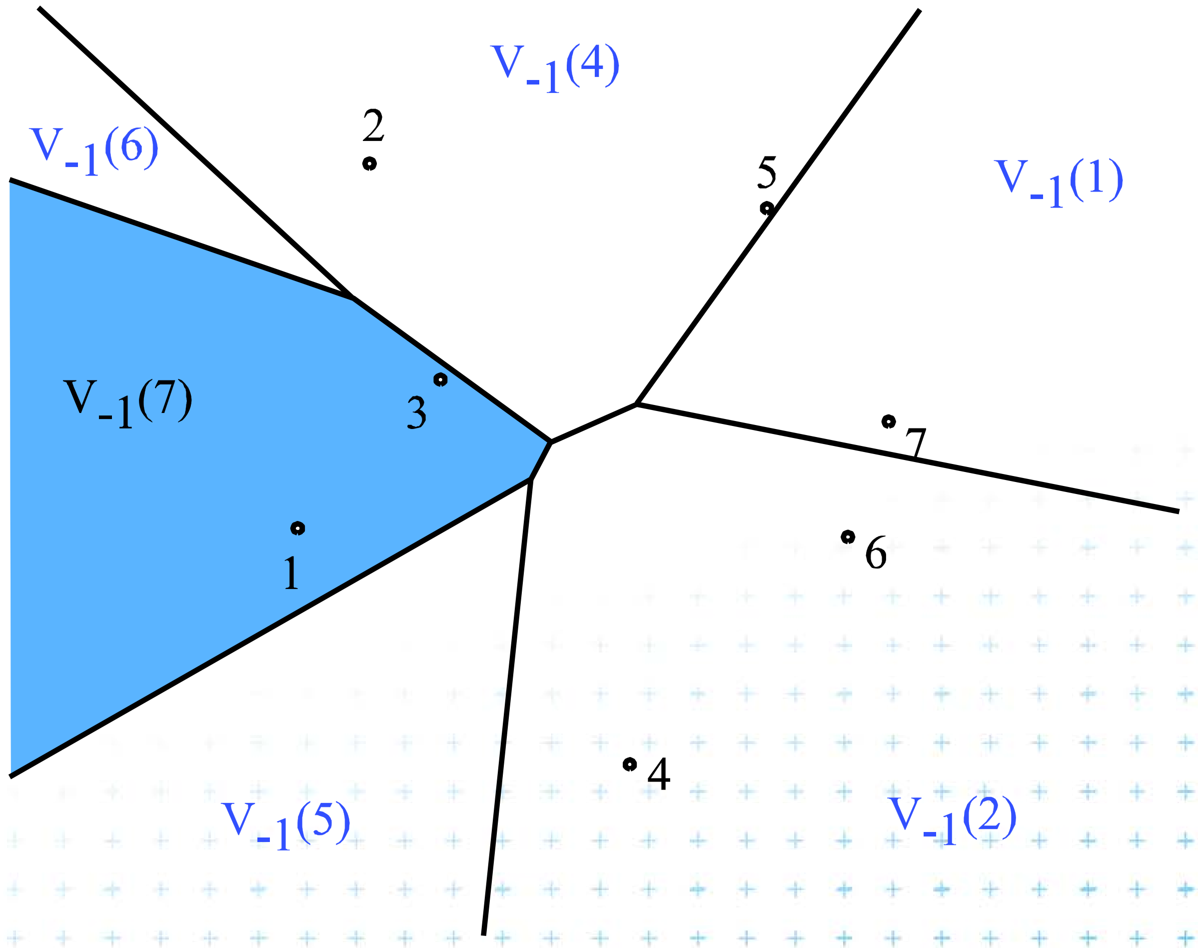

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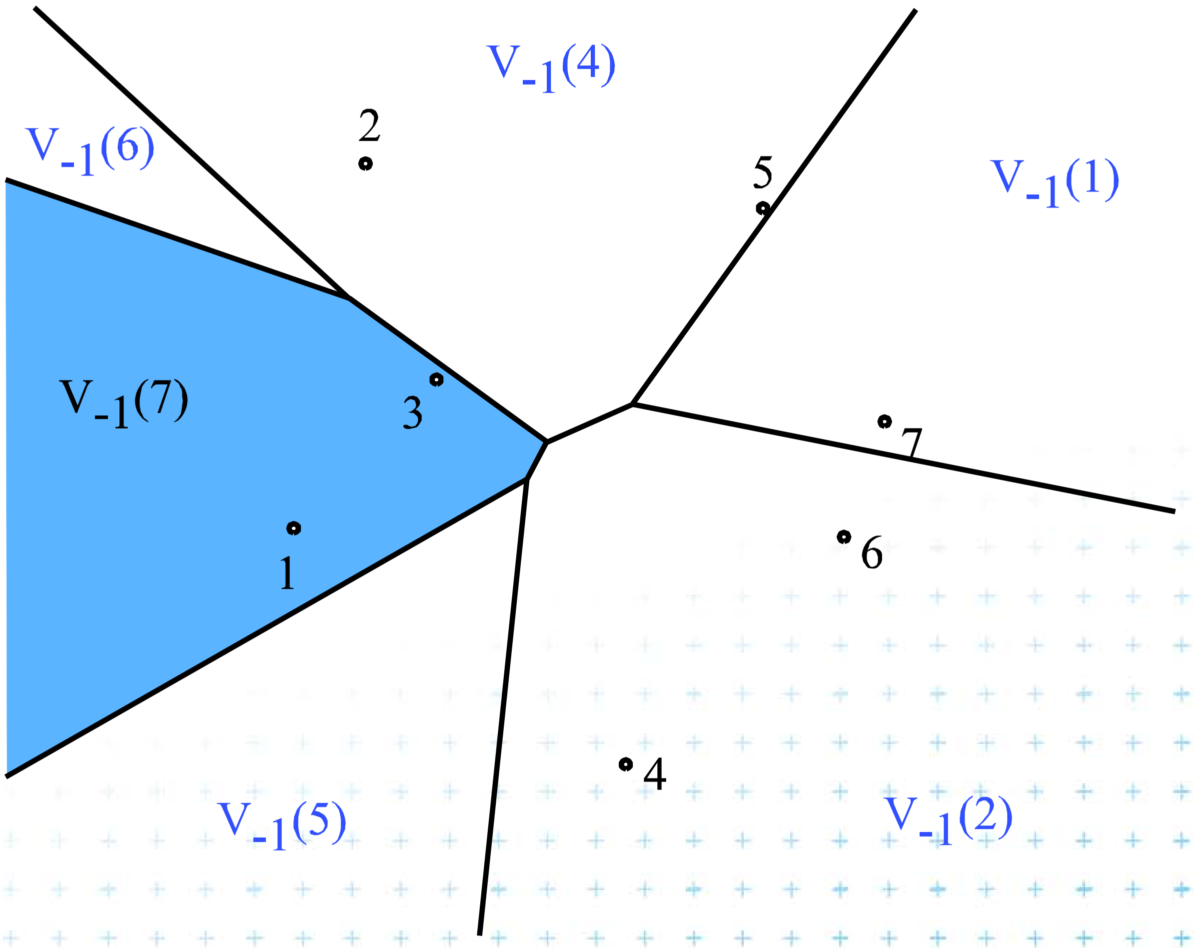


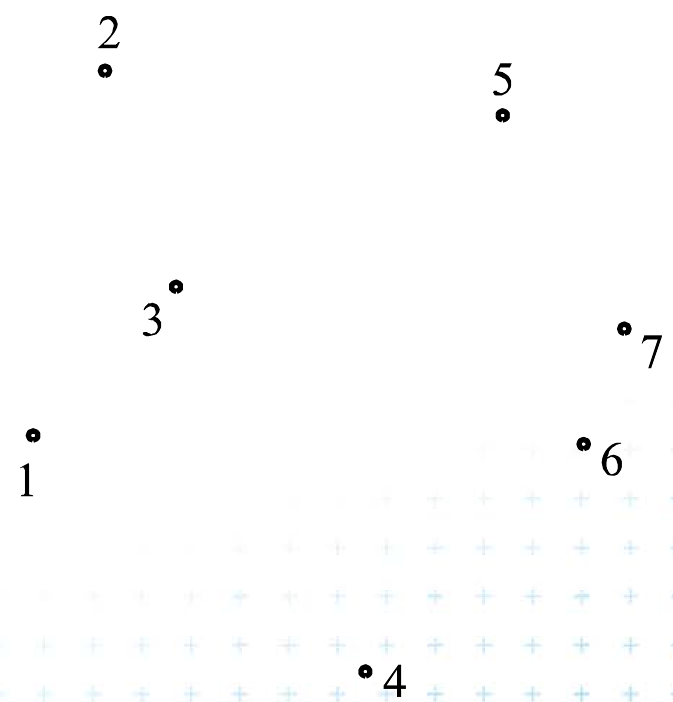
[Nandy]

Farthest-point Voronoi region (cell)

Computed as intersection of halfplanes, but we take "other sides" of bisectors

Construction of $V_{-1}(7)$

$$V_{-1}(y) = \bigcap_{x=1}^{n} h(y,x),\ y \neq x$$



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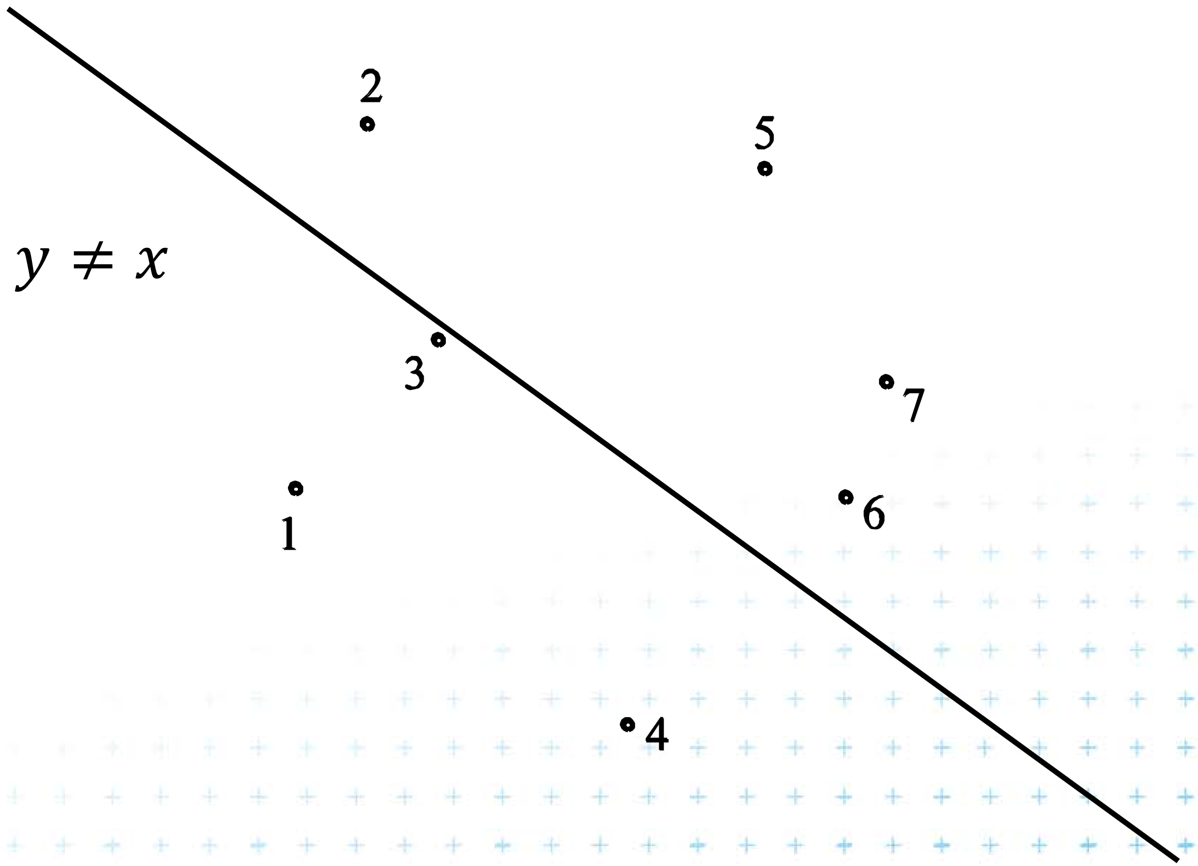

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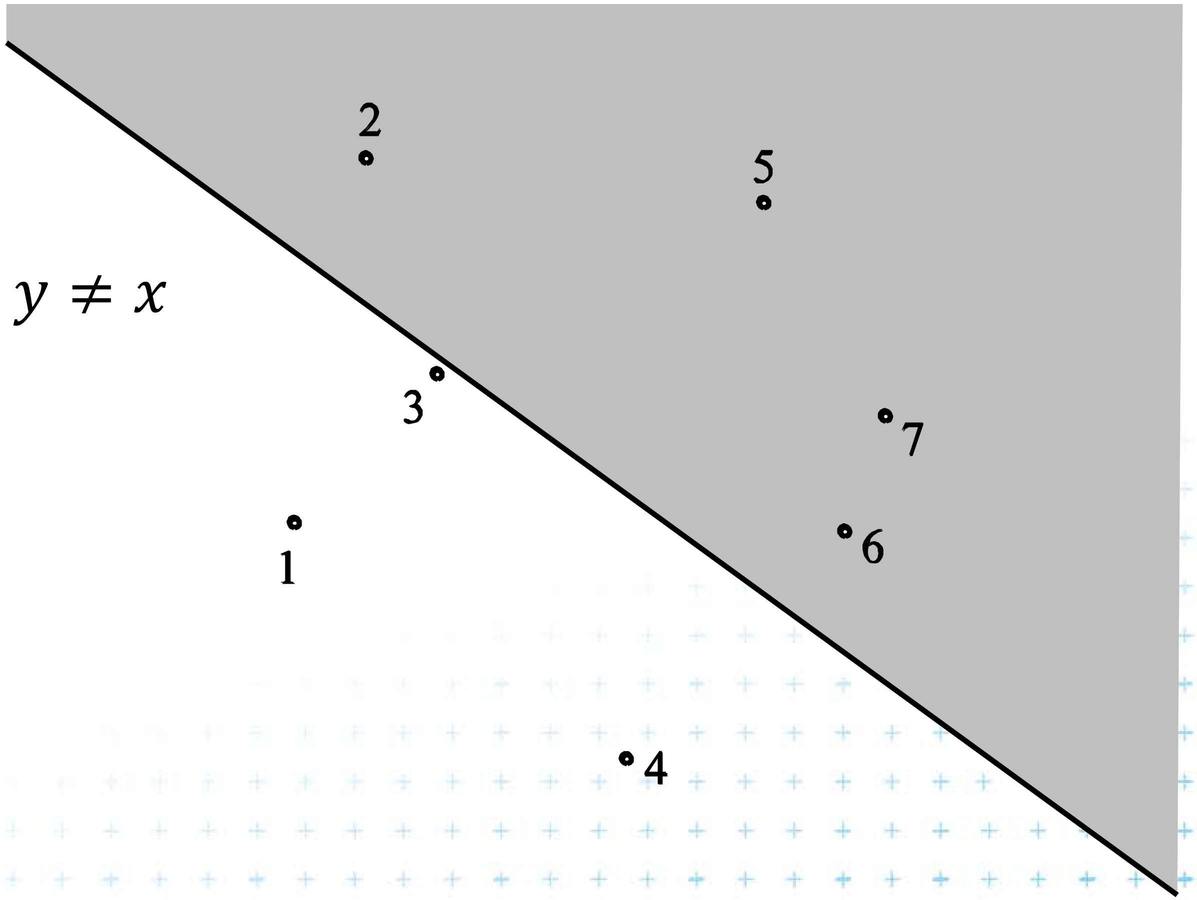

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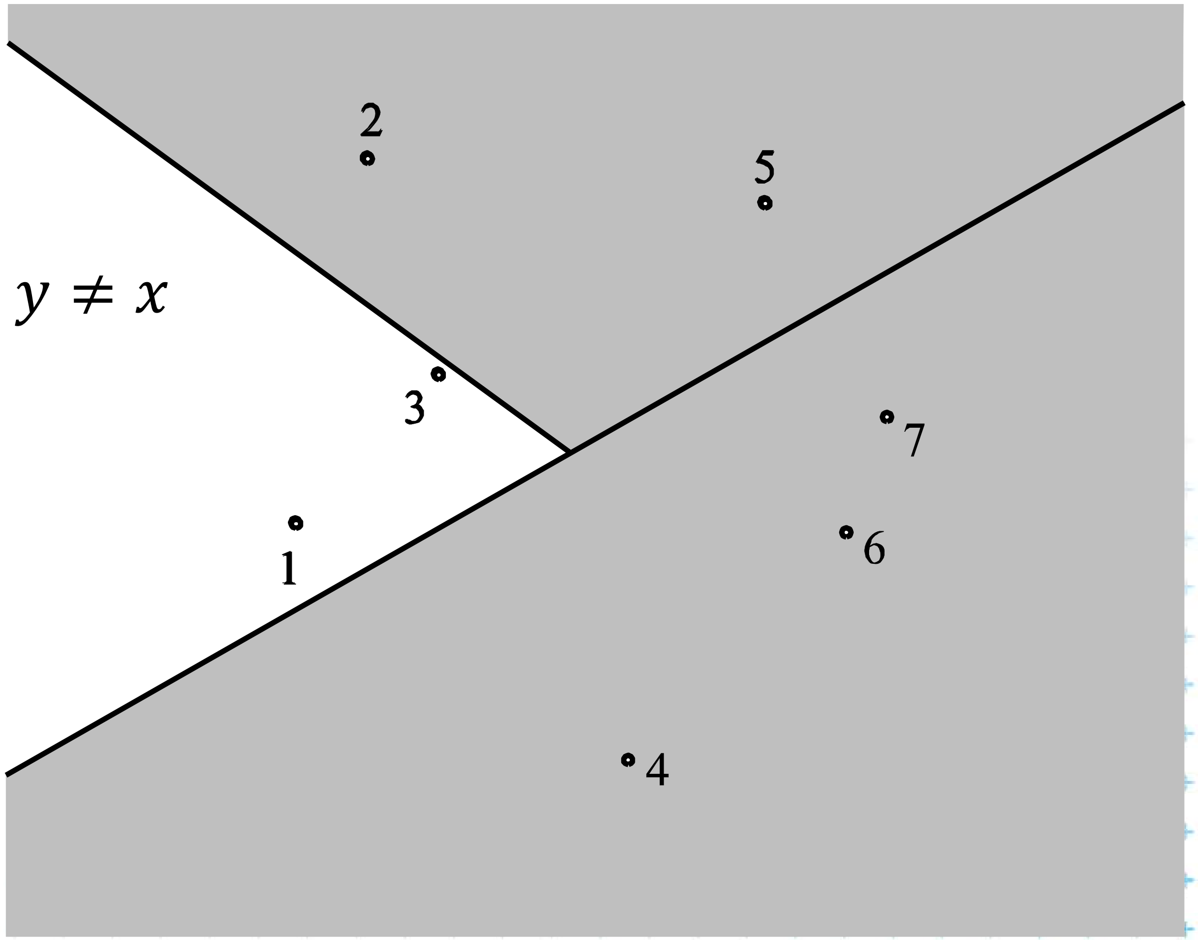

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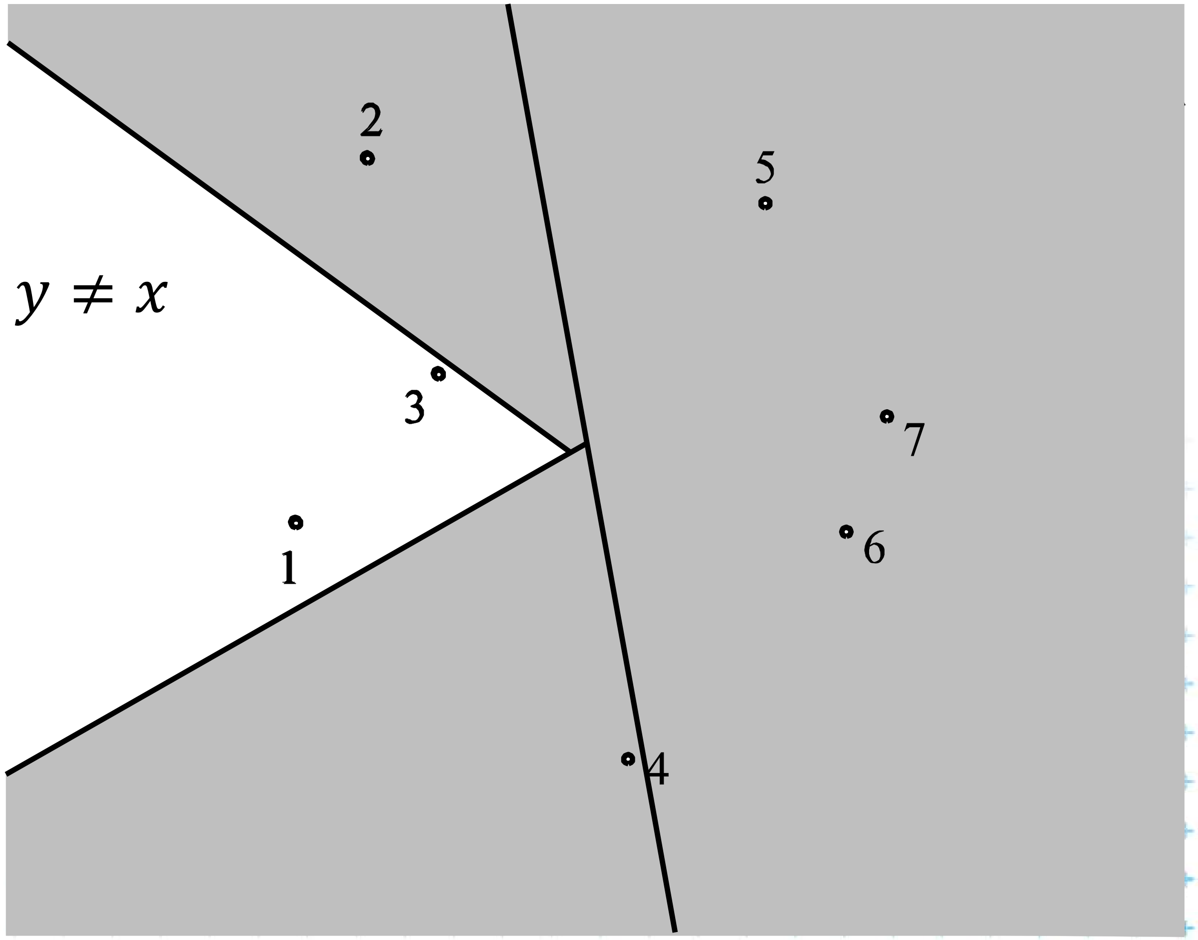

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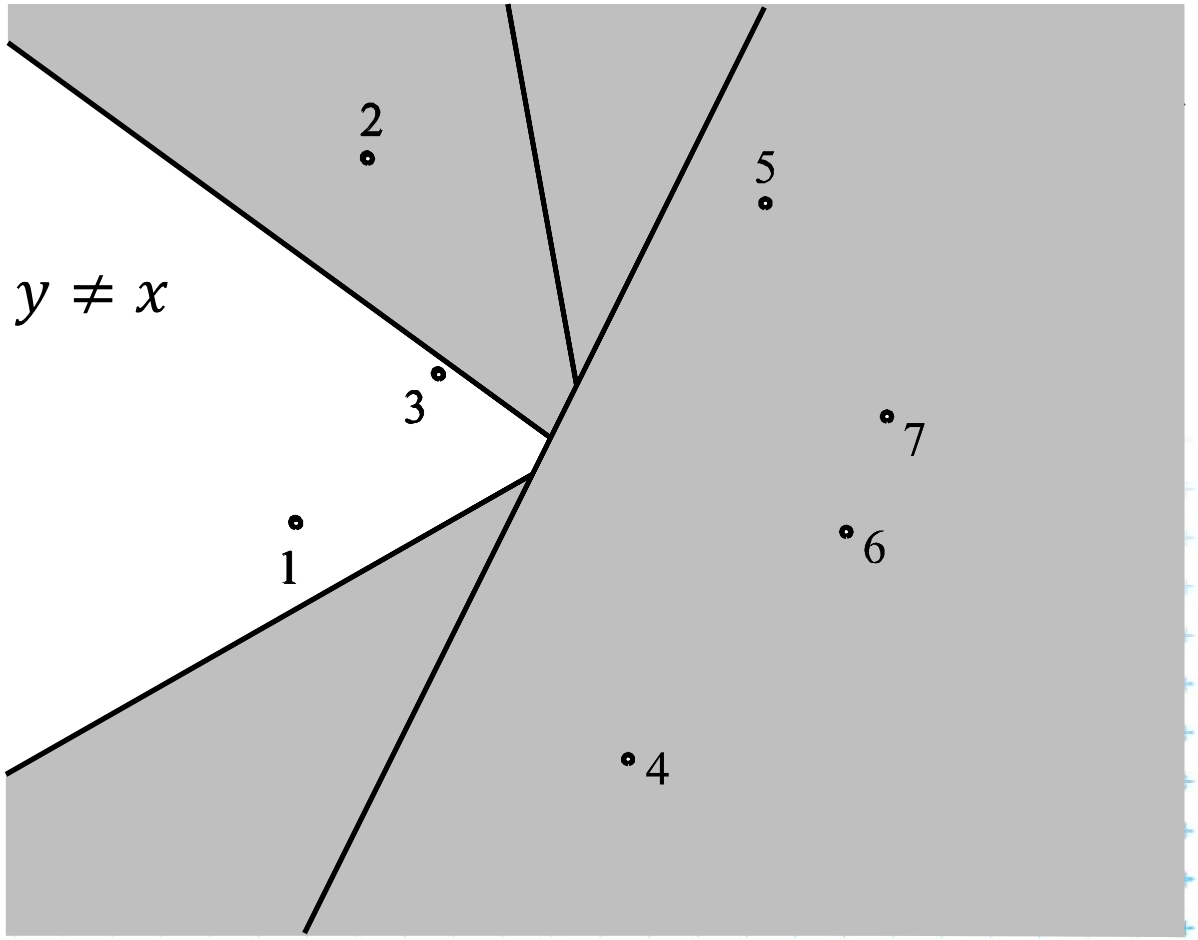

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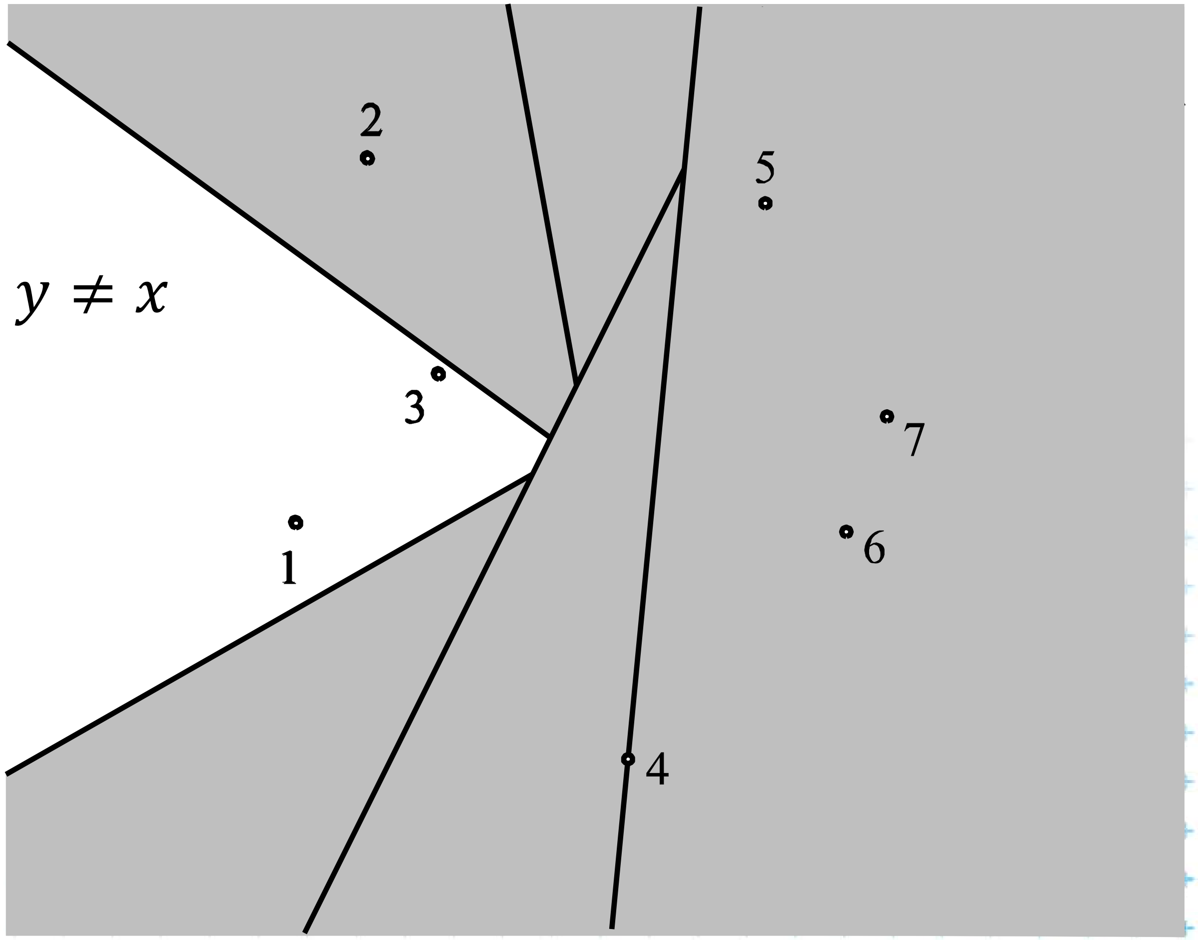

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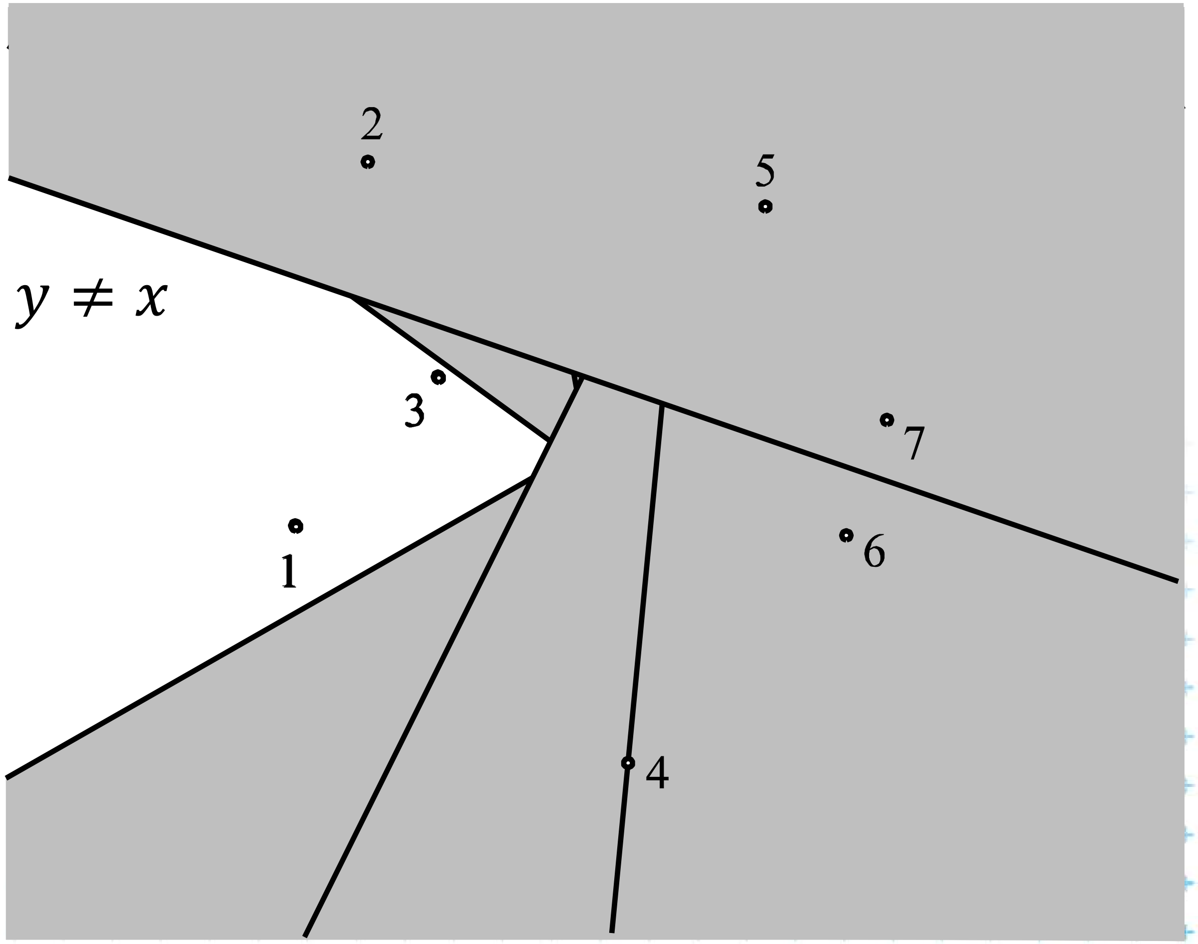

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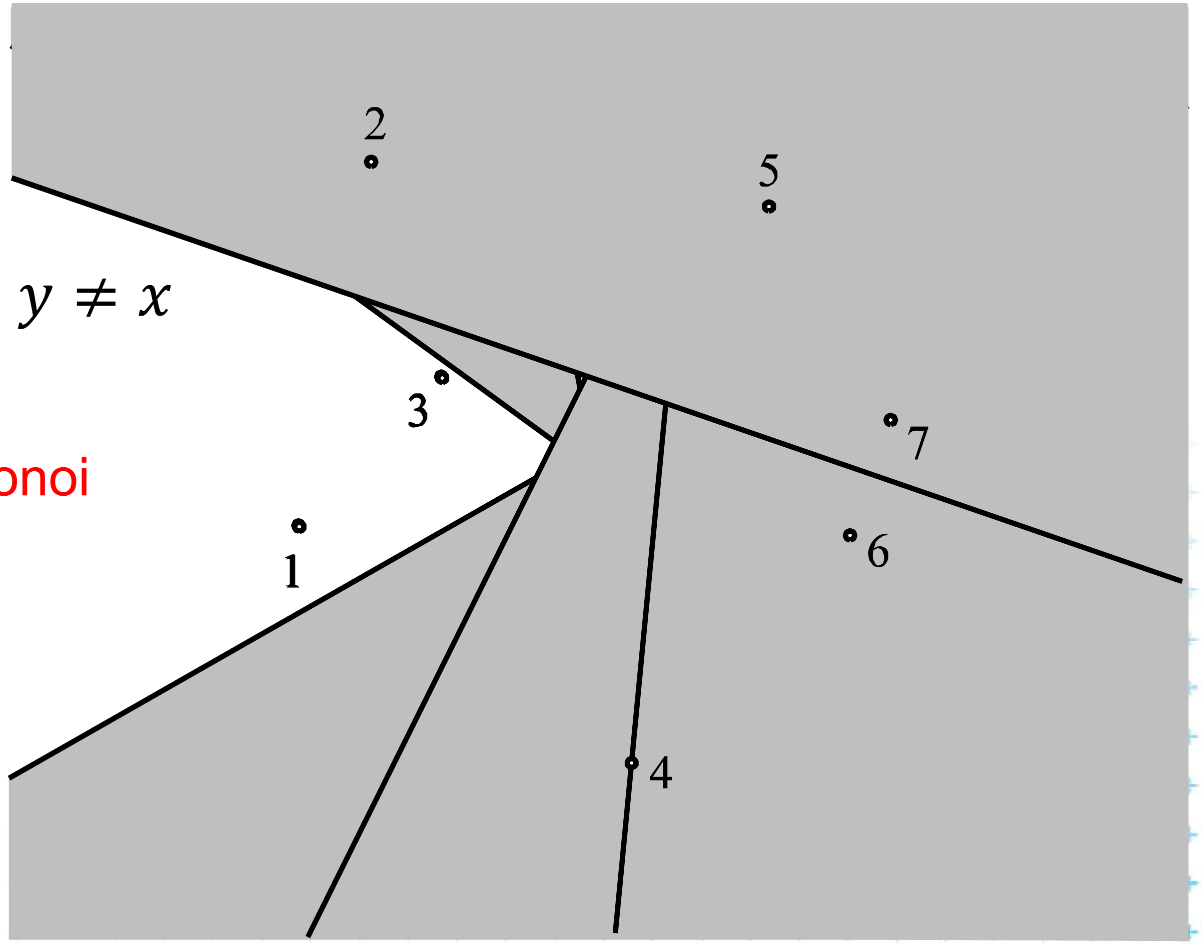


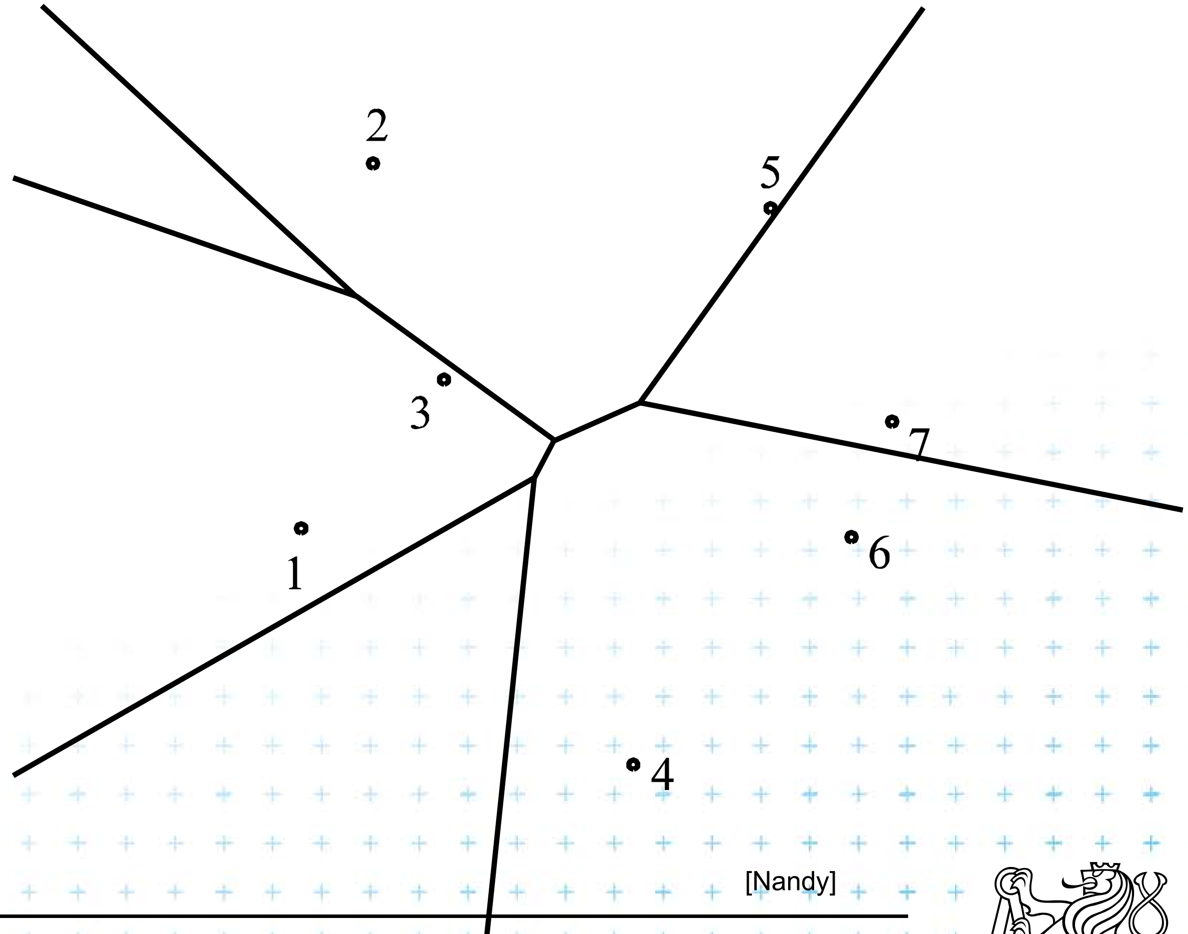
Property

The farthest point Voronoi regions are convex and unbounded

[Nandy]

Farthest-point Voronoi region

Properties:

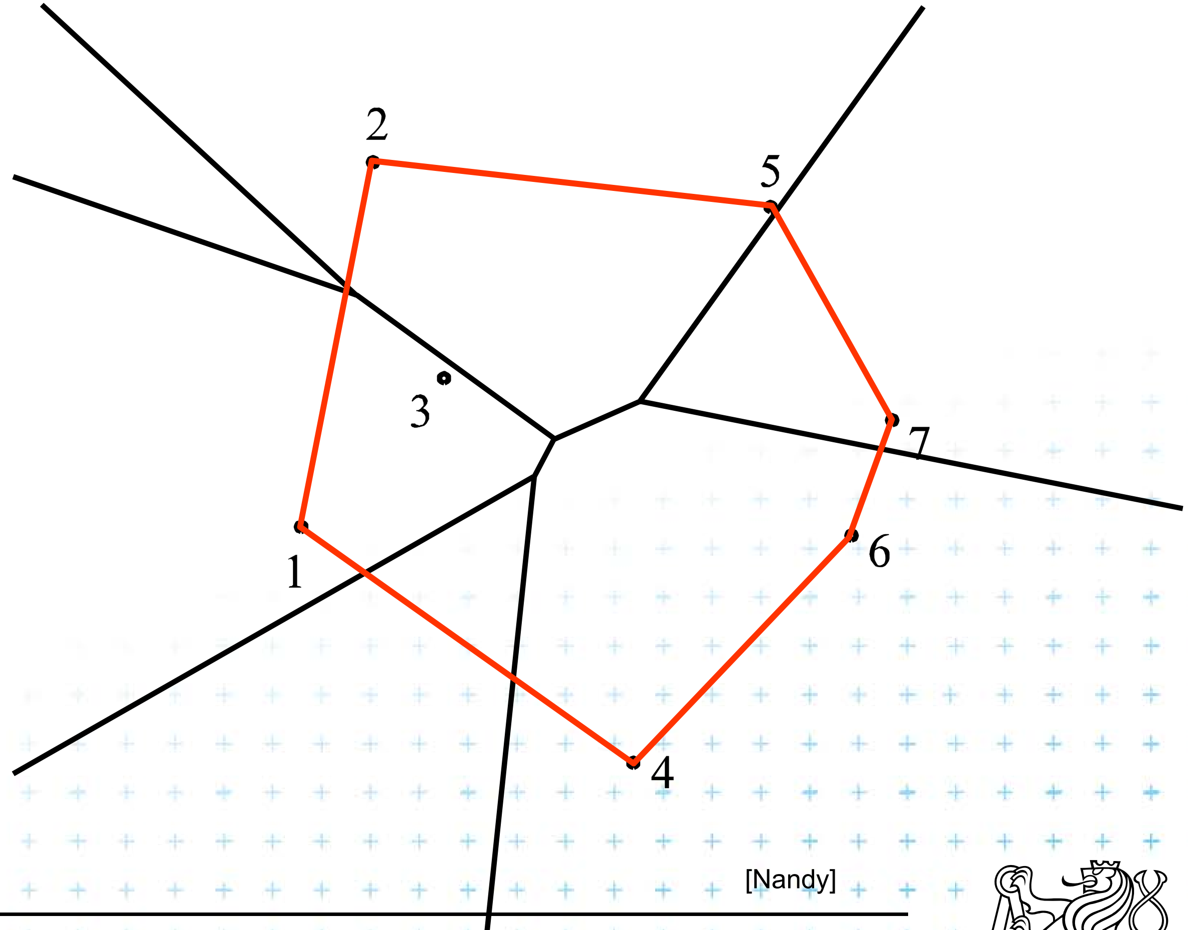


[Nandy]

Farthest-point Voronoi region

Properties:

- Only vertices of the convex hull have their cells in farthest Voronoi diagram

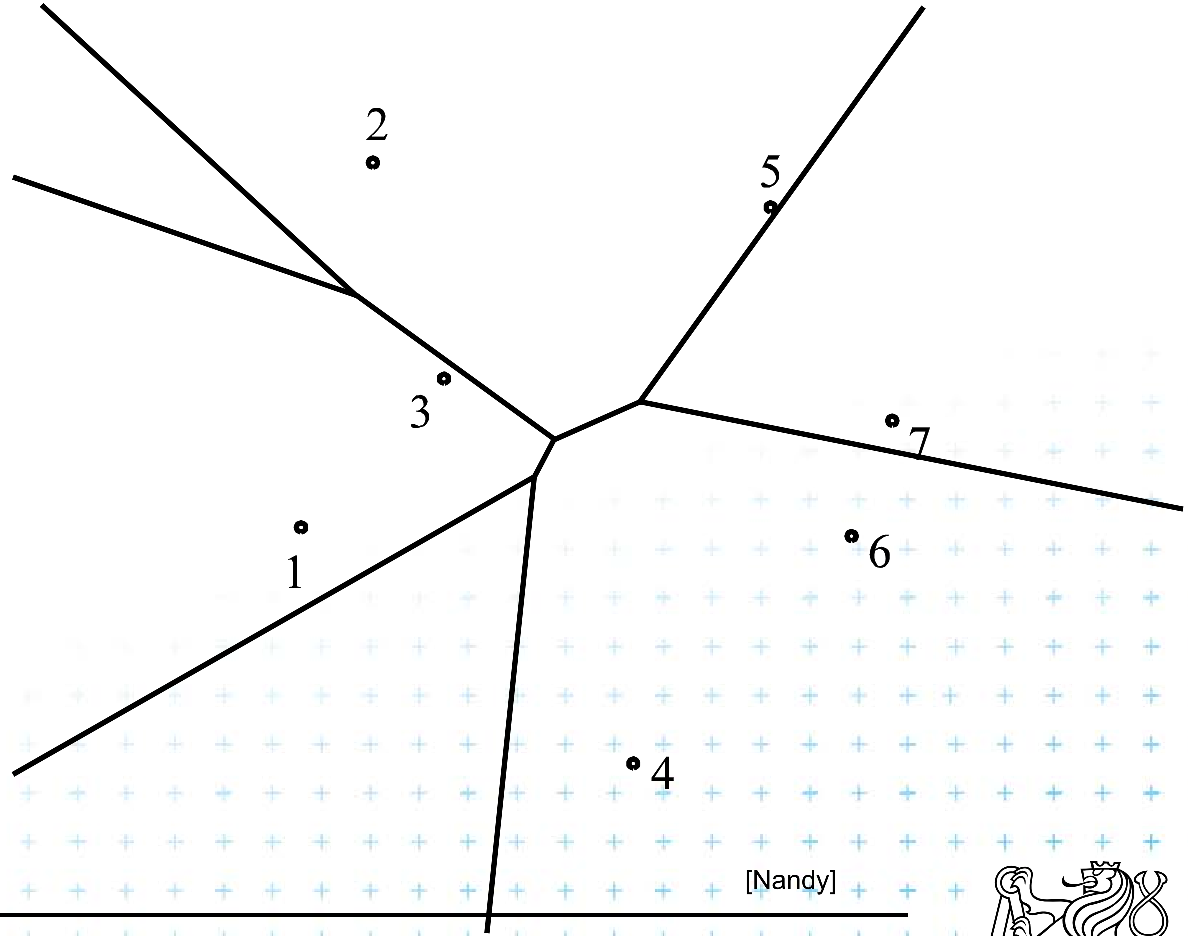


[Nandy]

Farthest-point Voronoi region

Properties:

- Only vertices of the convex hull have their cells in farthest Voronoi diagram

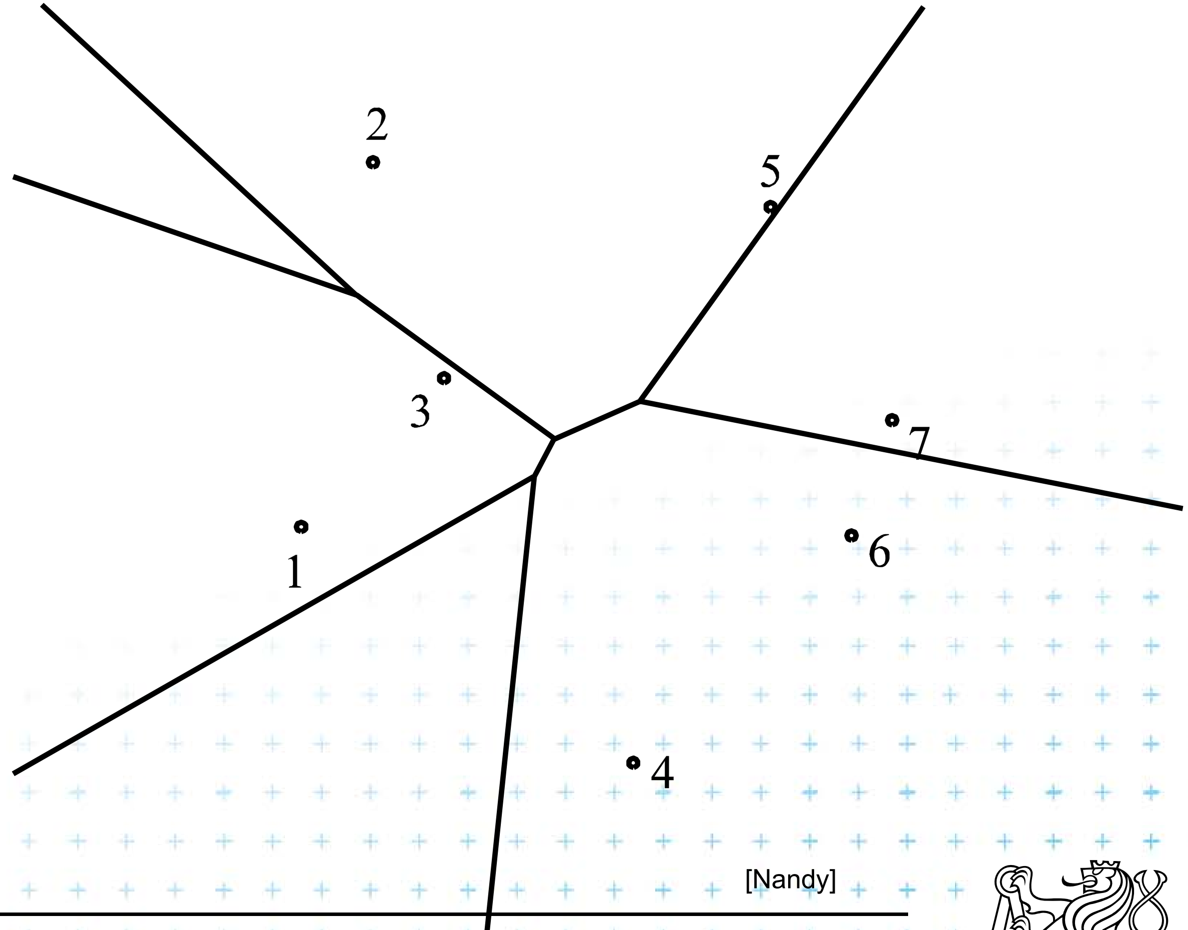


[Nandy]

Farthest-point Voronoi region

Properties:

- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded



[Nandy]

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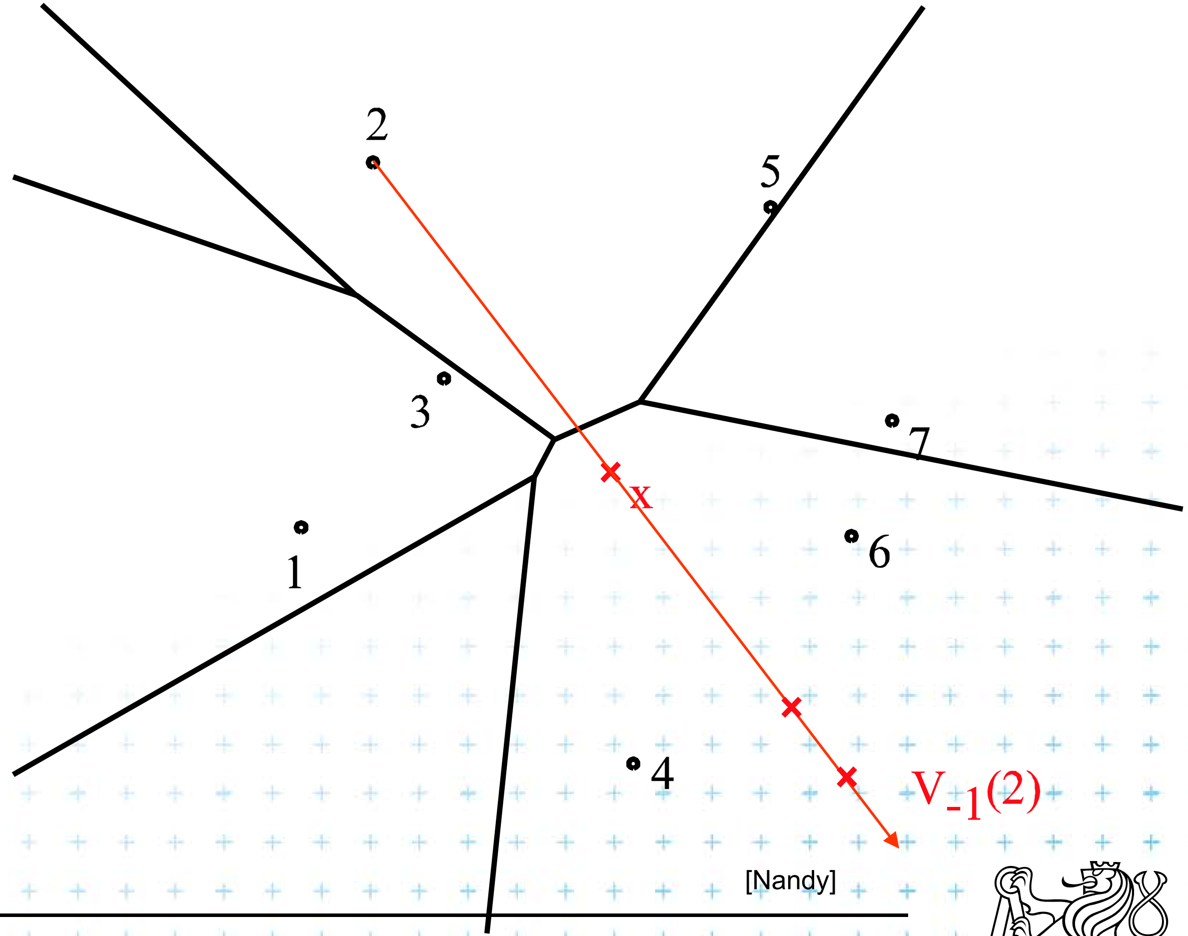

[Nandy]

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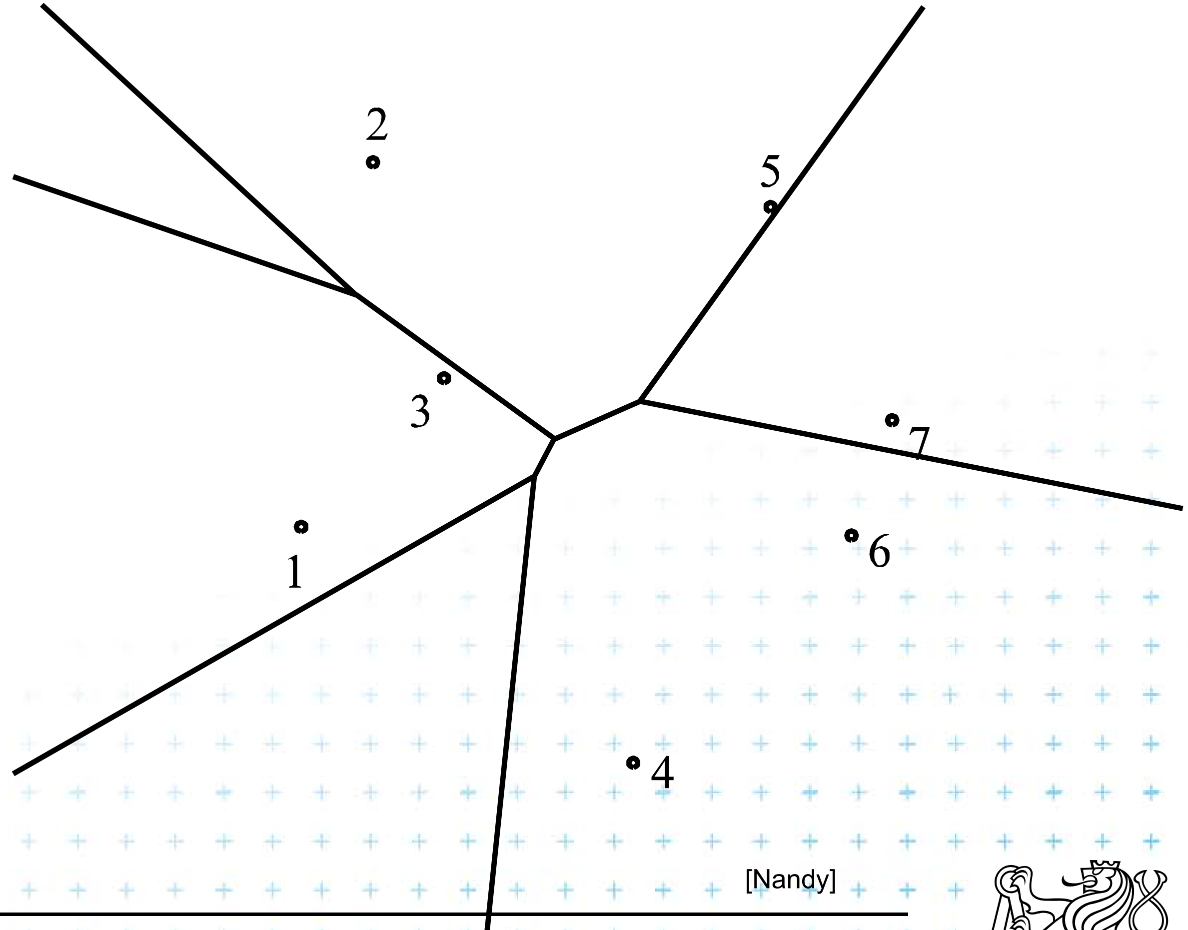
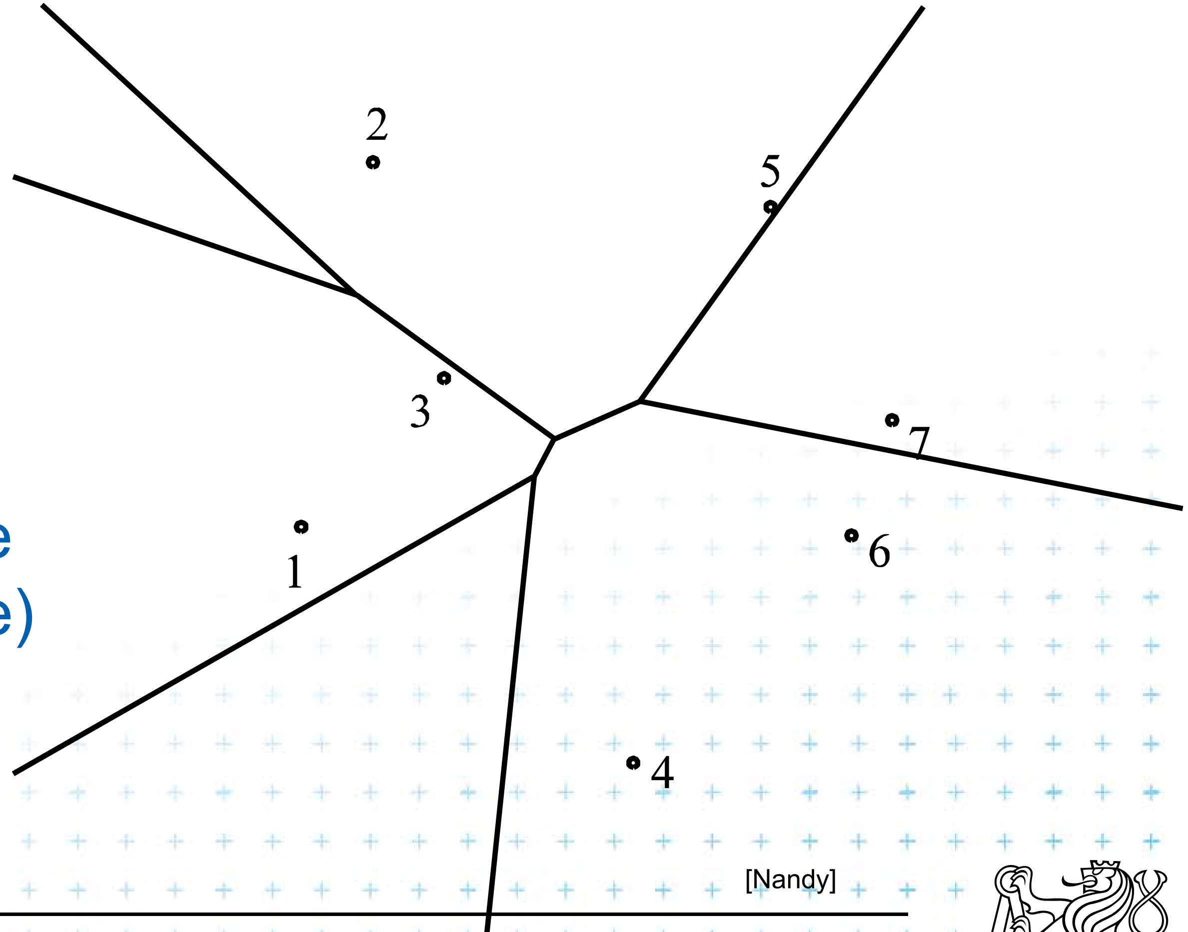


[Nandy]

Farthest-point Voronoi region

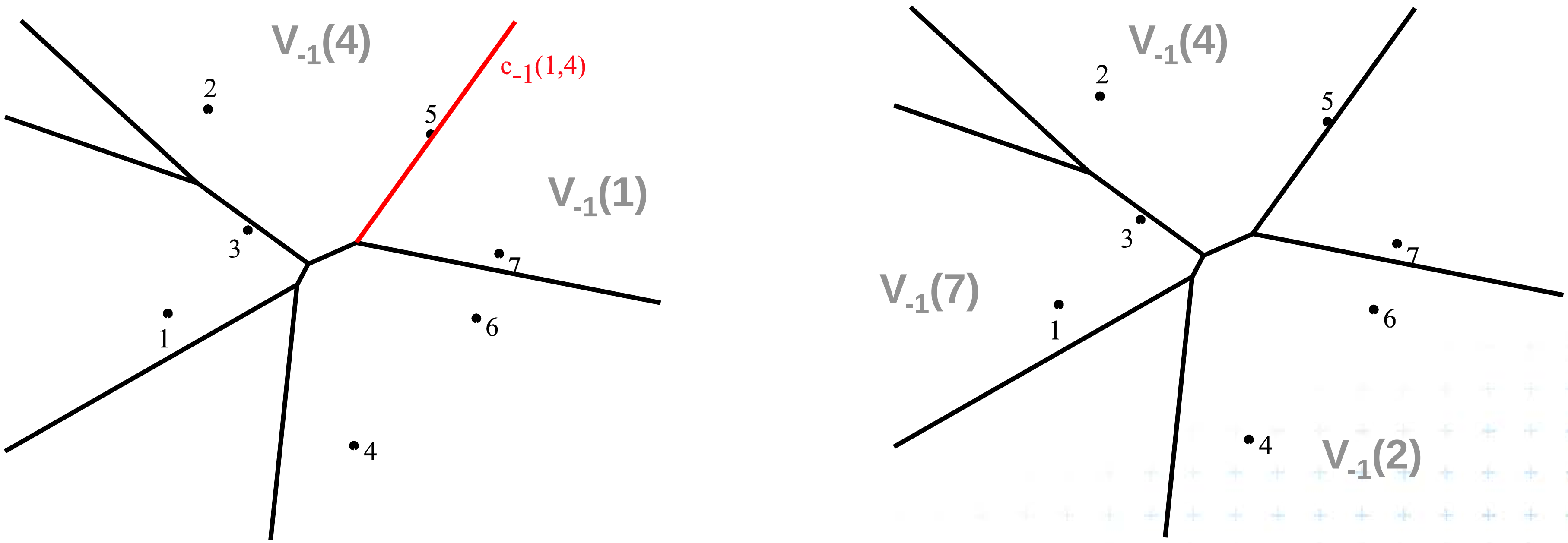
Properties:

- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded
- The farthest point Voronoi edges and vertices form a tree (in the graph sense)



[Nandy]

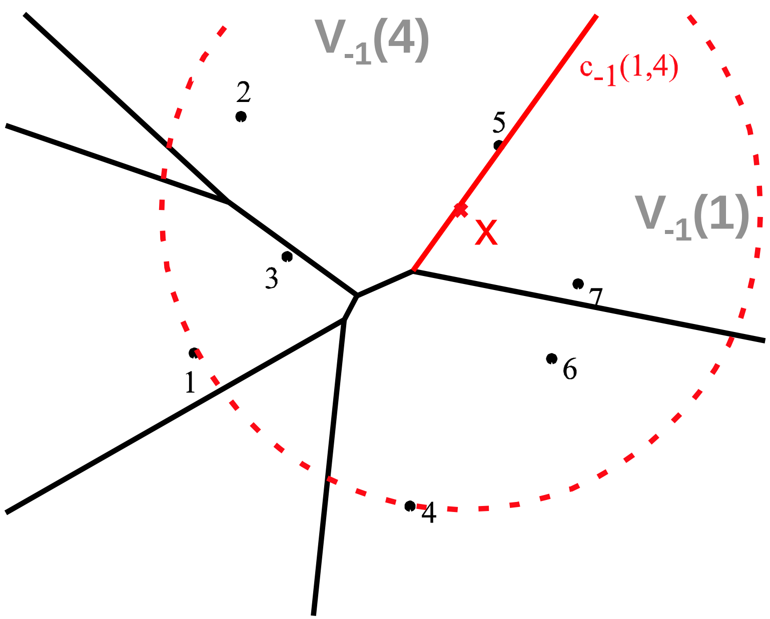
Farthest point Voronoi edges and vertices

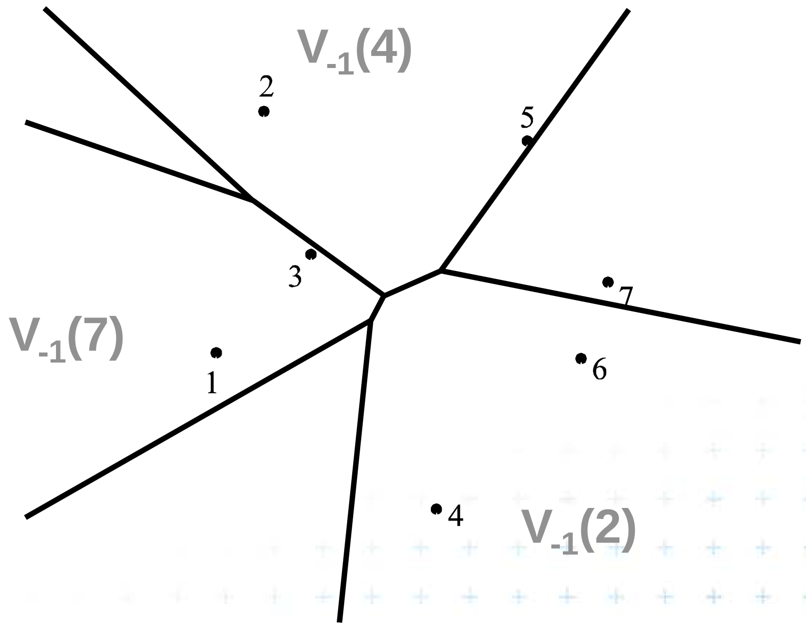


edge : set of points equidistant from 2 sites and closer to all the other sites

[Nandy]

Farthest point Voronoi edges and vertices





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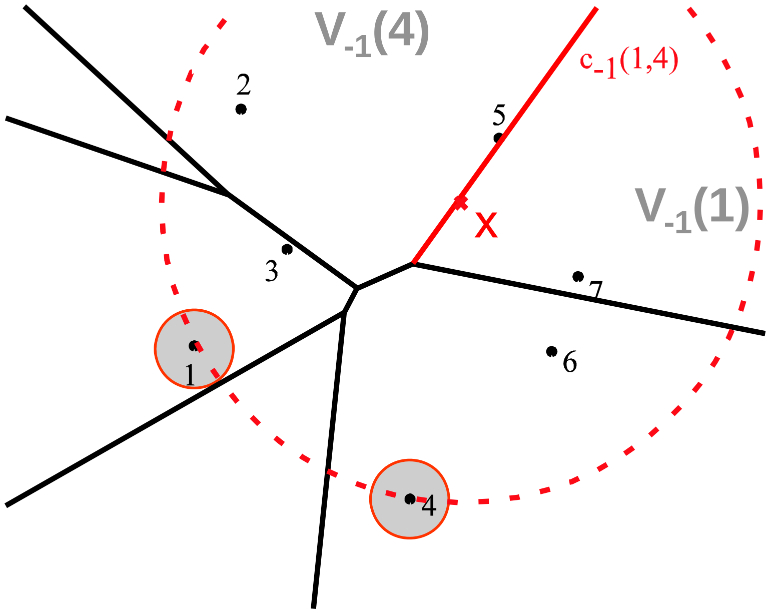
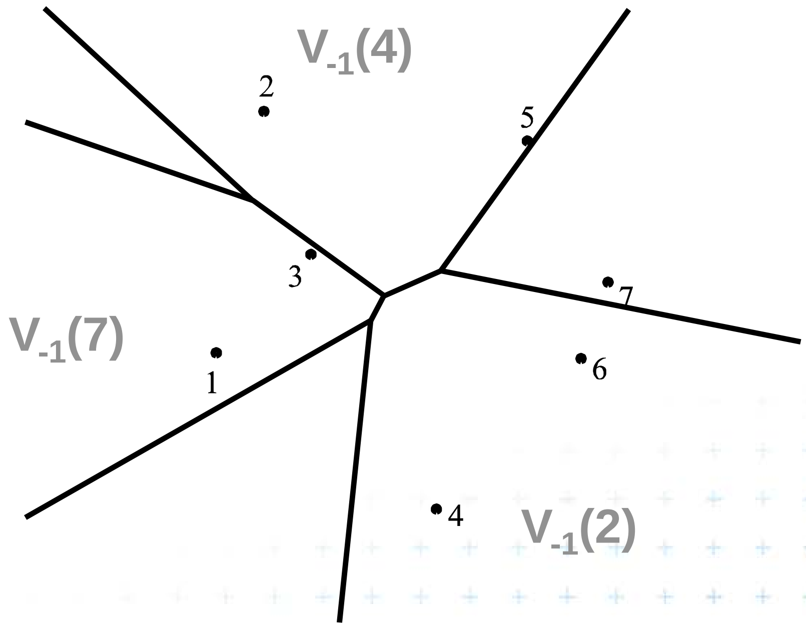
[Nandy]

Farthest point Voronoi edges and vertices

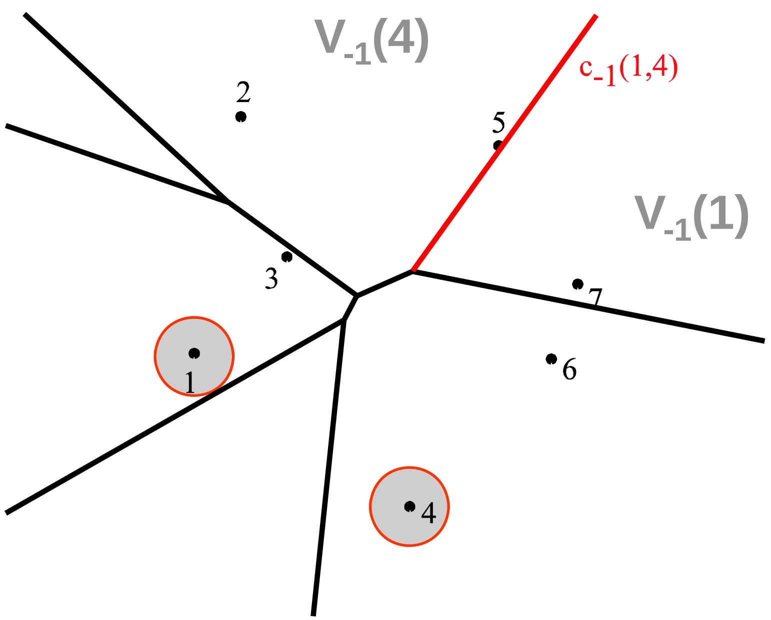


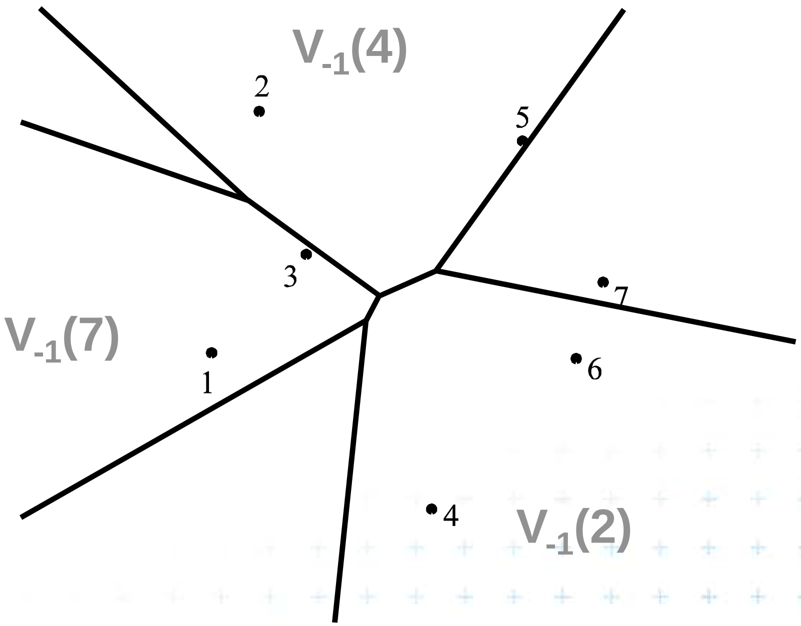


edge : set of points equidistant
from 2 sites and closer to
all the other sites

[Nandy]

Farthest point Voronoi edges and vertices





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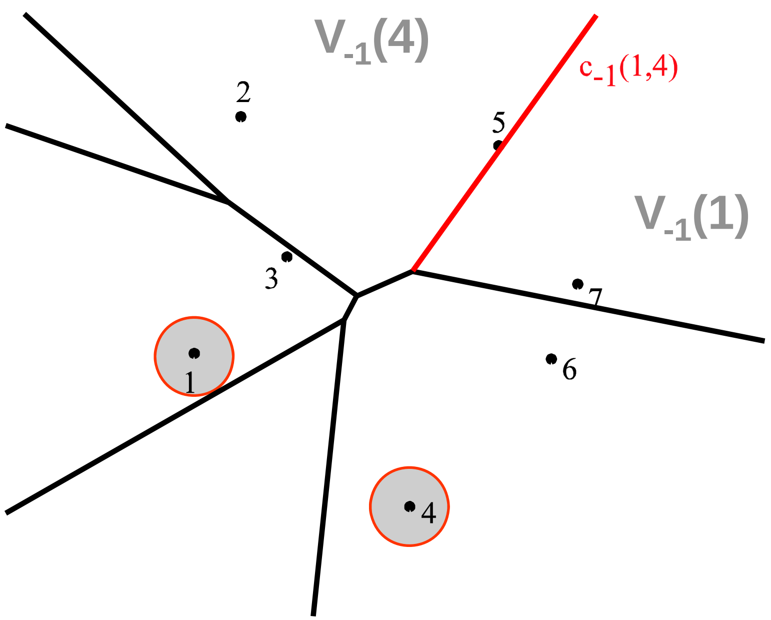
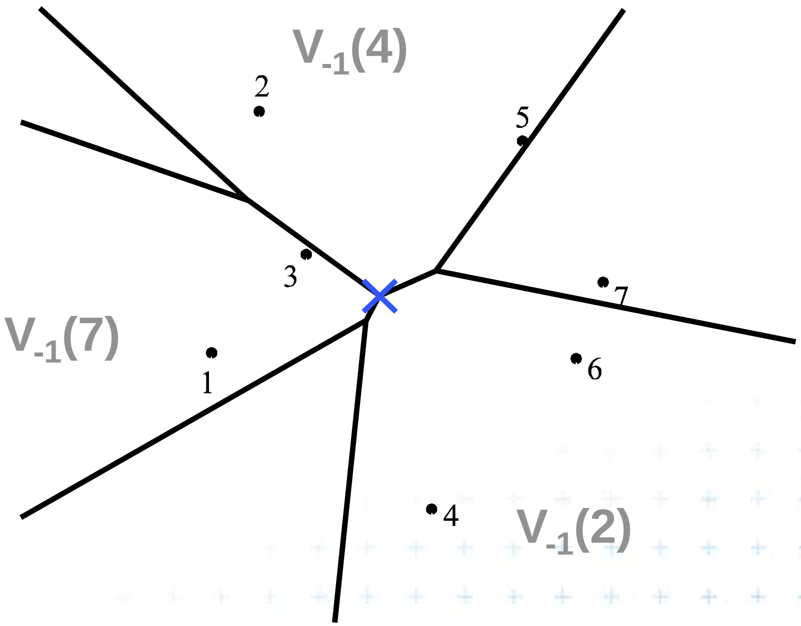
[Nandy]

Farthest point Voronoi edges and vertices





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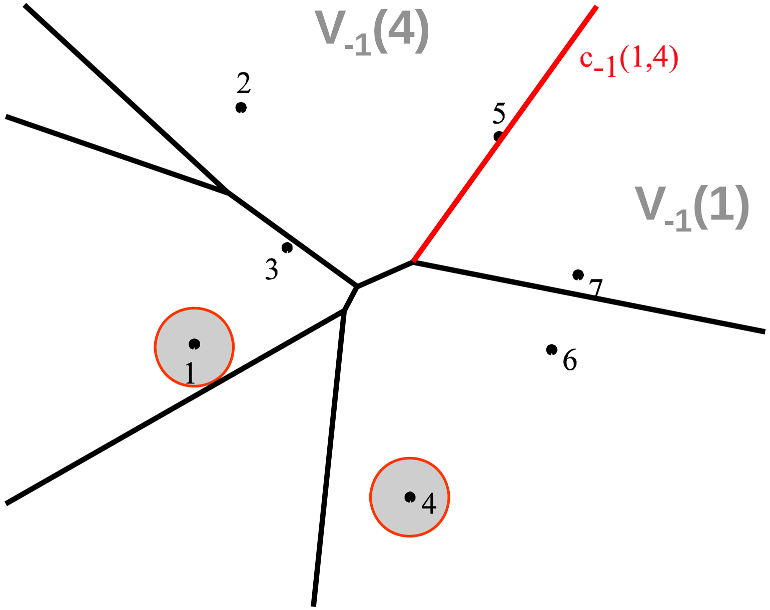
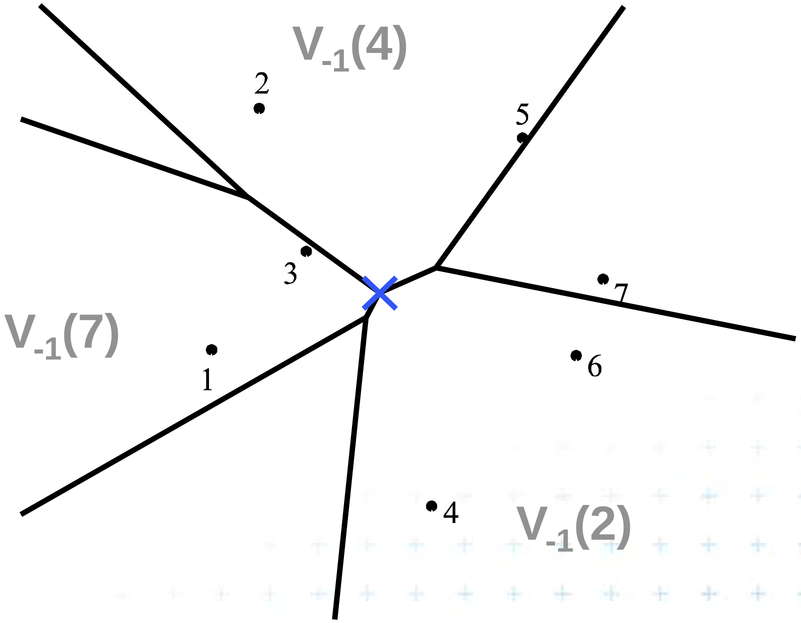
Farthest point Voronoi edges and vertices



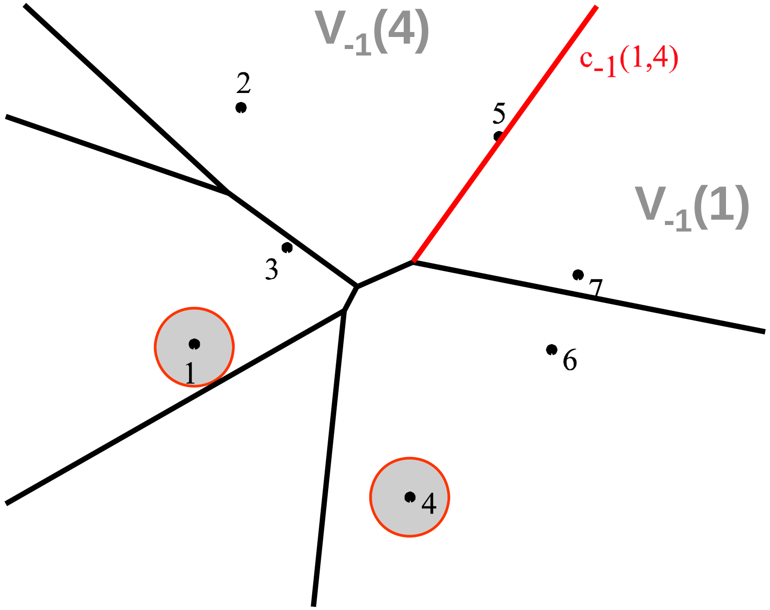


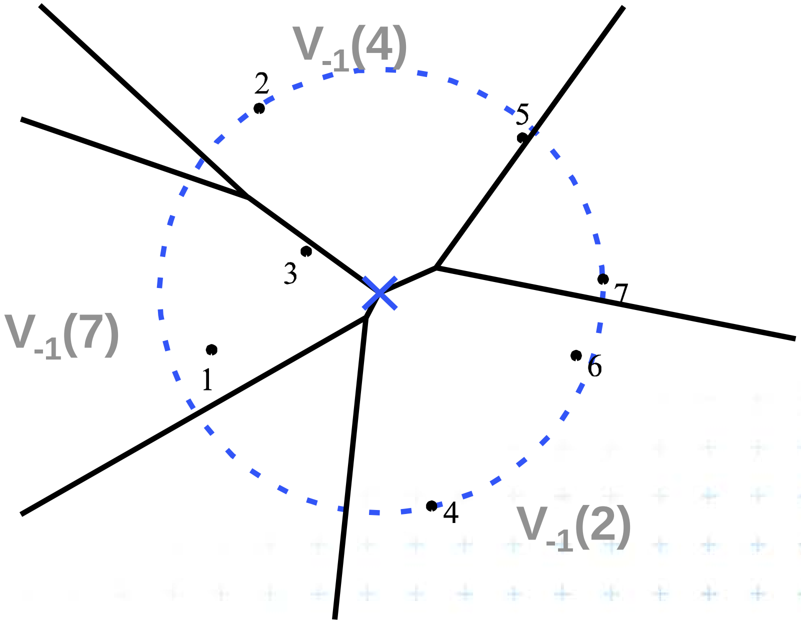
edge : set of points equidistant from 2 sites and closer to all the other sites

vertex : point equidistant from at least 3 sites and closer to all the other sites

[Nandy]

Farthest point Voronoi edges and vertices





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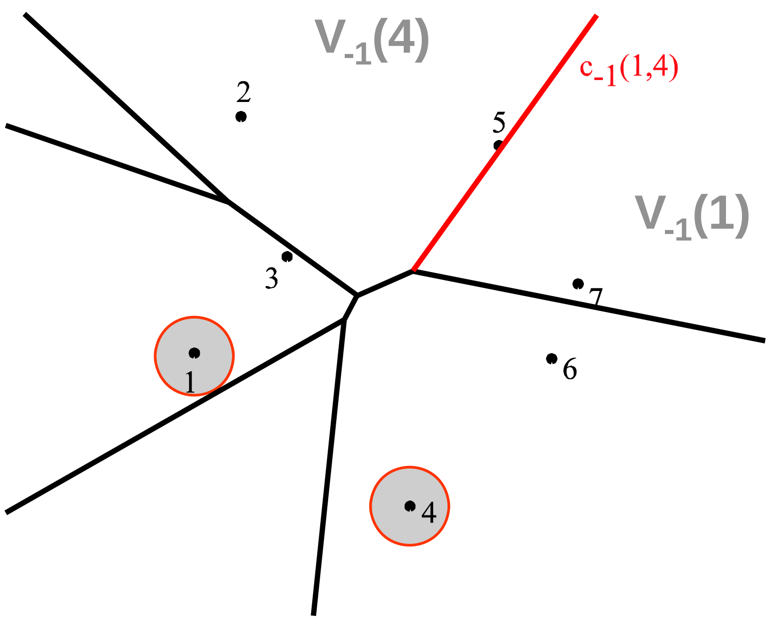

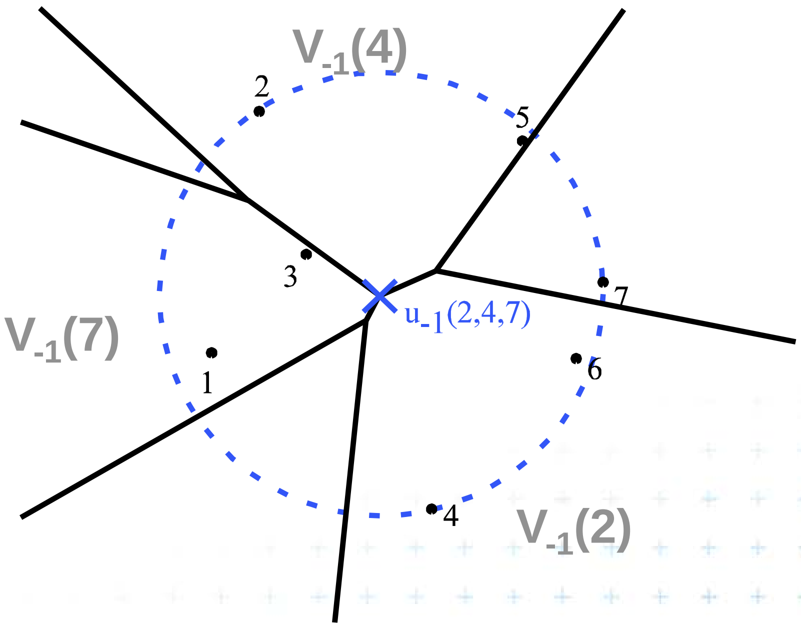

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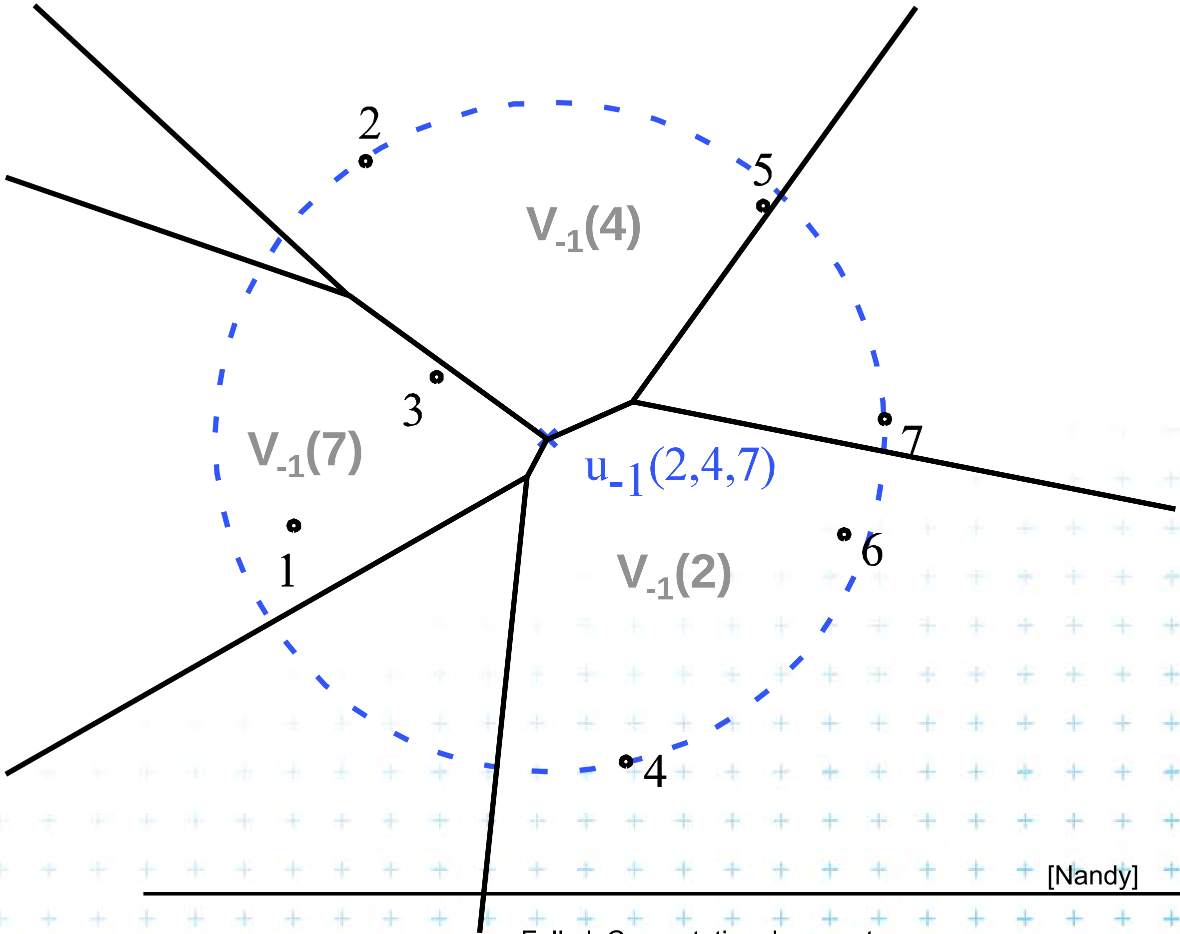
[Nandy]

Application of $Vor_{-1}(P)$: Smallest enclosing circle

- Construct $Vor_{-1}(P)$ and find minimal circle with center in $Vor_{-1}(P)$ vertices or on edges



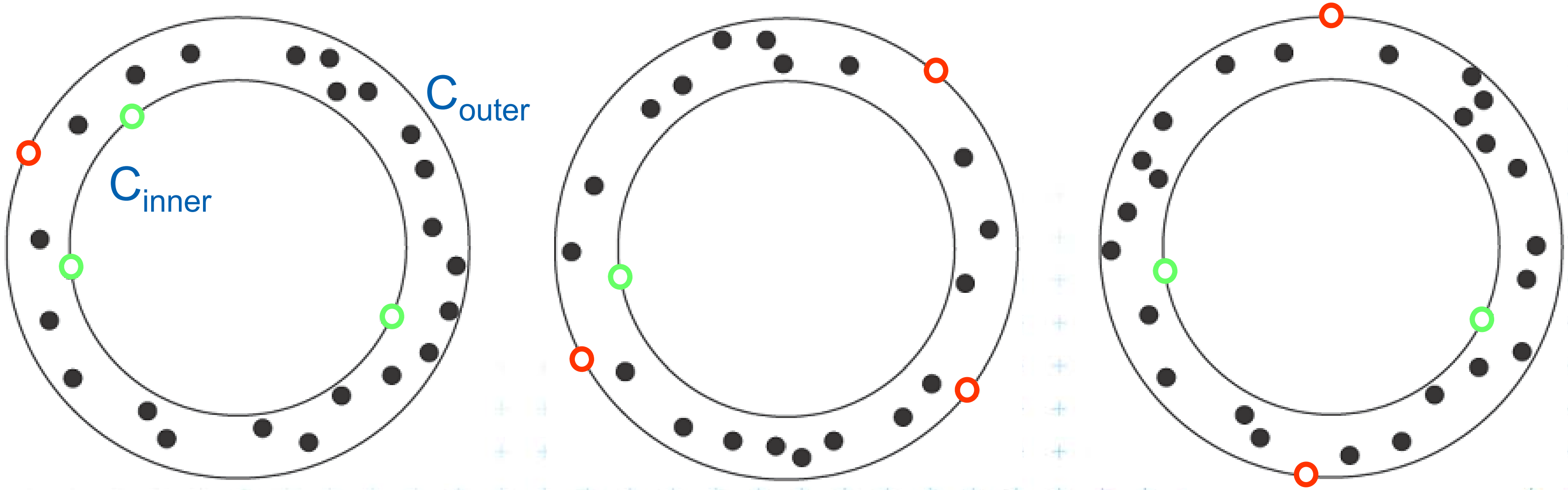
[Nandy]

Farthest-point Voronoi diagrams example

Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus mezikruží s nejmenší šířkou (region between two concentric circles C_{inner} and C_{outer})

Three cases to test – one will win:



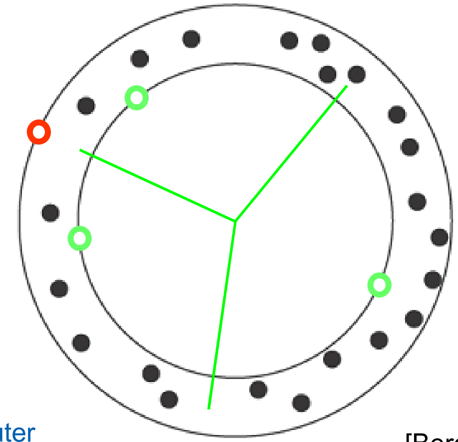
a) 3 in – 1 out b) 1 point in – 3 out c) 2 in – 2 out

[Berg]

Smallest width annulus – cases with 3 pts

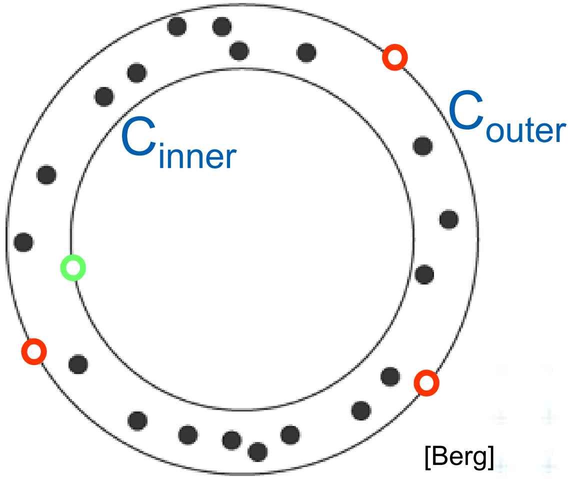
a) C_{inner} contains at least 3 points

- Center is the *vertex of normal Voronoi diagram* (1st order VD)
- The remaining point on C_{outer} in O(n) for each vertex
  - ⇒ not the largest (inscribed) empty circle - as discussed on seminar as we must test all VD vertices in combination with point on C outer
  - ⇒ $O(n^2)$



[Berg]

3 in – 1 out



[Berg]

1 point in – 3 out

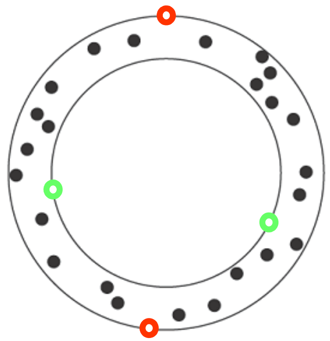
b) C_{outer} contains at least 3 points

- Center is the *vertex of the farthest Voronoi diagram*
- The remaining point on C_{inner} in O(n)
  - ⇒ not the smallest enclosing circle - as discussed on seminar as we must test all vertices **in combination** with point on C inner
  - ⇒ $O(n^2)$

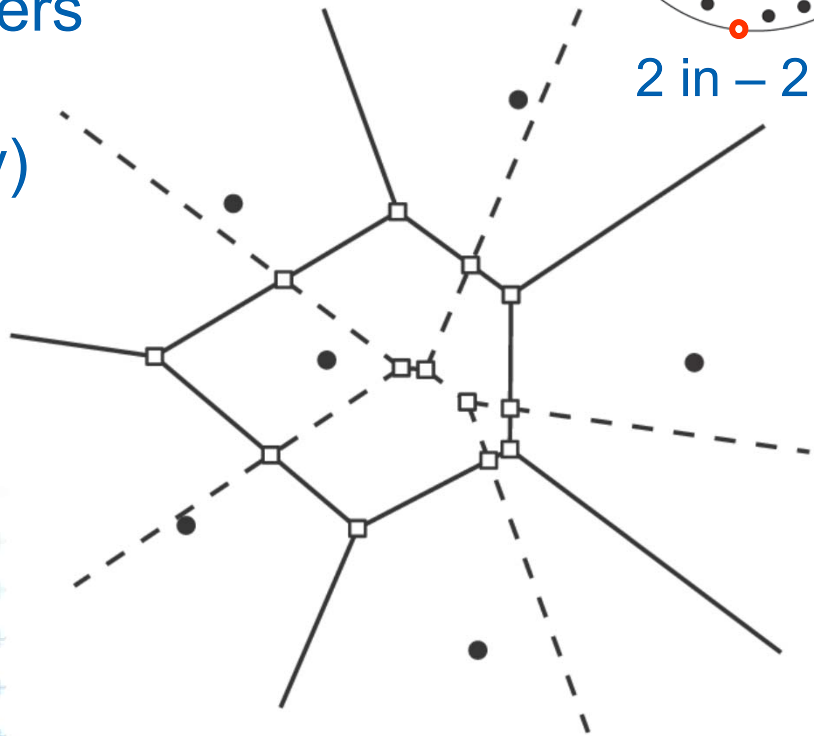
Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi (__) and farthest-point Voronoi (- - -) diagrams
  => $O(n^2)$ candidates for centers
  (we need only vertices, not the complete overlay)
- annulus computed in O(1) from center and 4 points (same for all 3 cases)
- $O(n^2)$



2 in – 2 out



[Berg]

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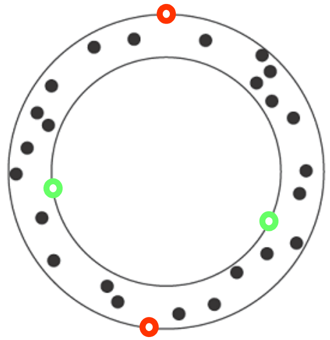

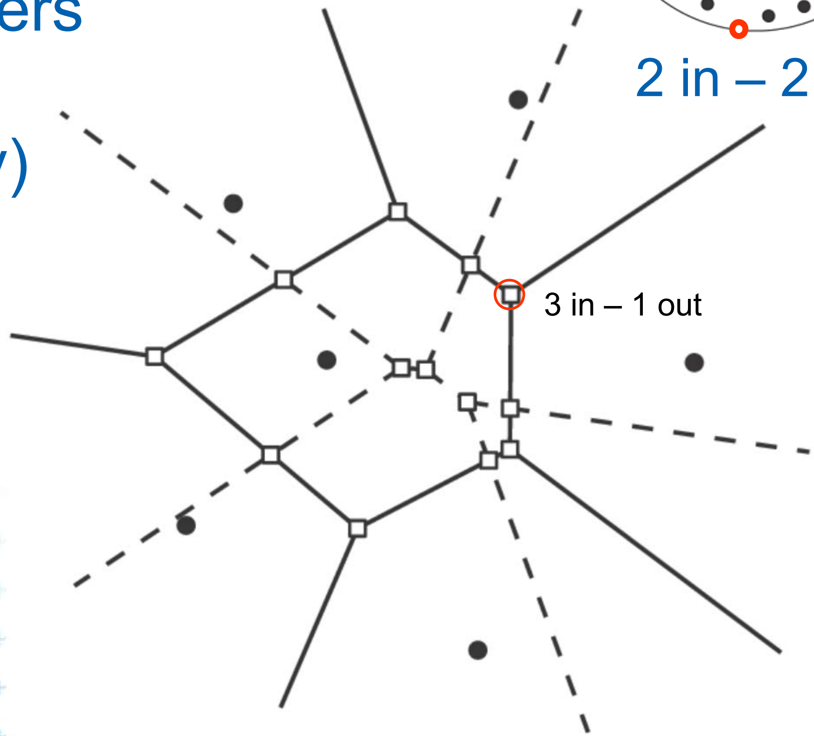

[Berg]

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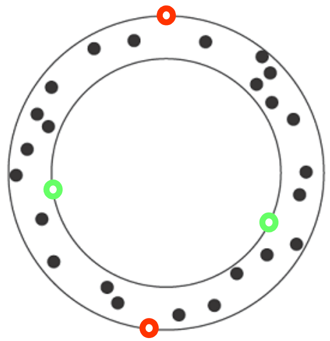

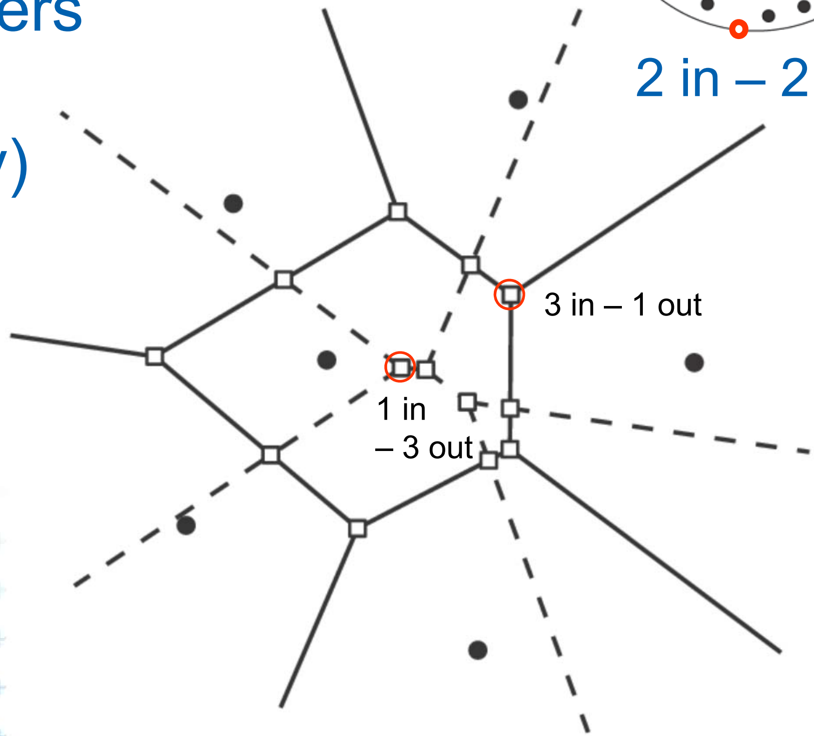

[Berg]

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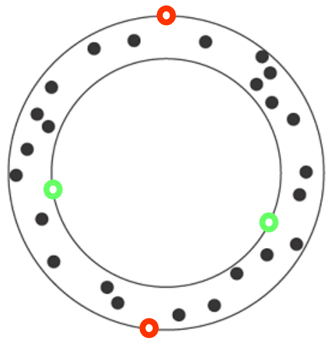
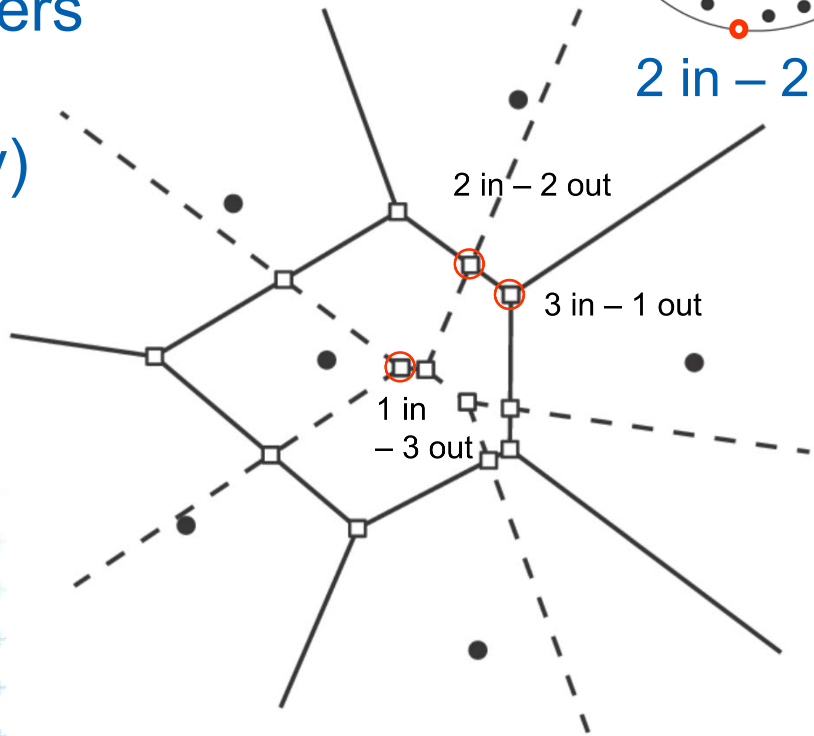


2 in – 2 out



[Berg]

Smallest width annulus

Smallest-Width-Annulus

Input: Set P of n points in the plane

Output: Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram Vor(P) and farthest-point Voronoi diagram $\mathrm{Vor}_{-1}(P)$ of P
2. For each vertex of Vor(P) (r) determine the *farthest point* (R) from P => $O(n)$ sets of four points defining candidate annuli – case a)
3. For each vertex of $\mathrm{Vor}_{-1}(P)$ (R) determine the *closest point* (r) from P => $O(n)$ sets of four points defining candidate annuli – case b)
4. For every pair of edges Vor(P) and $\mathrm{Vor}_{-1}(P)$ test if they intersect => another set of four points defining candidate annulus – c)
5. For all candidates of all three types chose the smallest-width annulus

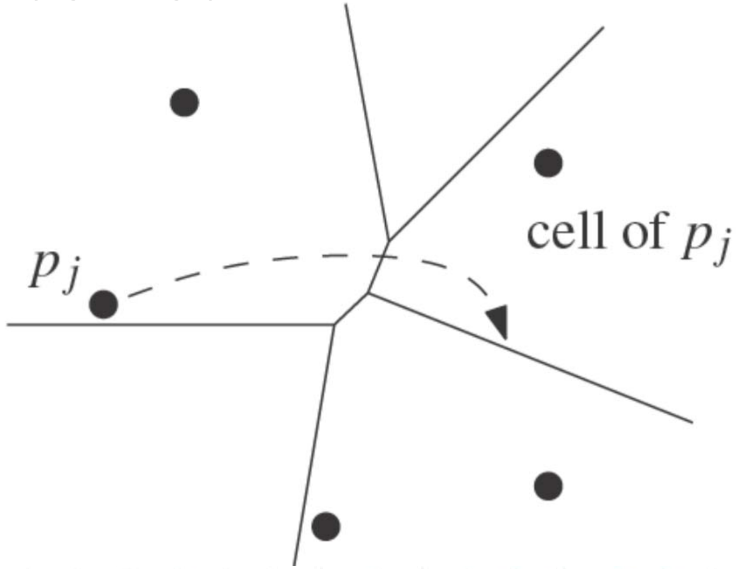
1. $O(n \log n)$
2. $O(n^2)$
3. $O(n^2)$
4. $O(n^2)$
5. $O(n^2)$

$O(n^2)$ time using $O(n)$ storage

Order n-1 VD construction

Modified DCEL for farthest-point Voronoi d

- Half-infinite edges -> we adapt DCEL
- Half-edges with origin in infinity
  - Special vertex-like record for origin in infinity
  - Store direction instead of coordinates
  - Next(e) or Prev(e) pointers undefined
- For each inserted site p_j
  - store a pointer to the most CCW half-infinite half-edge of its cell in DCEL



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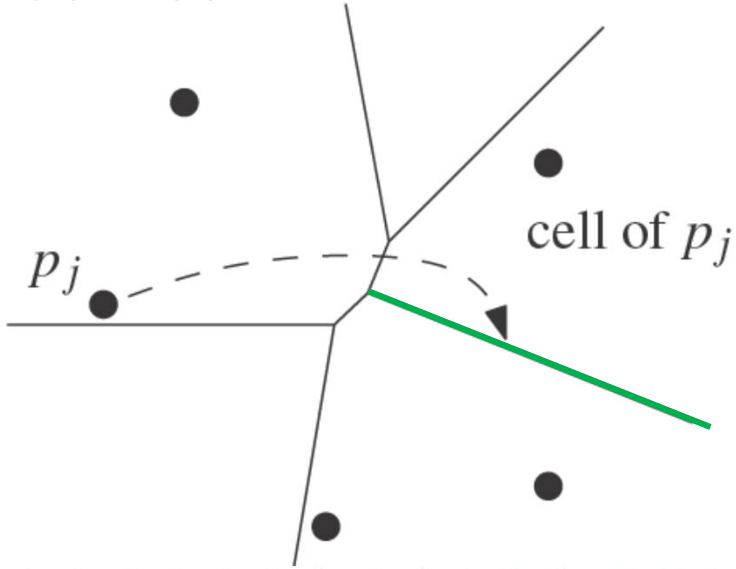

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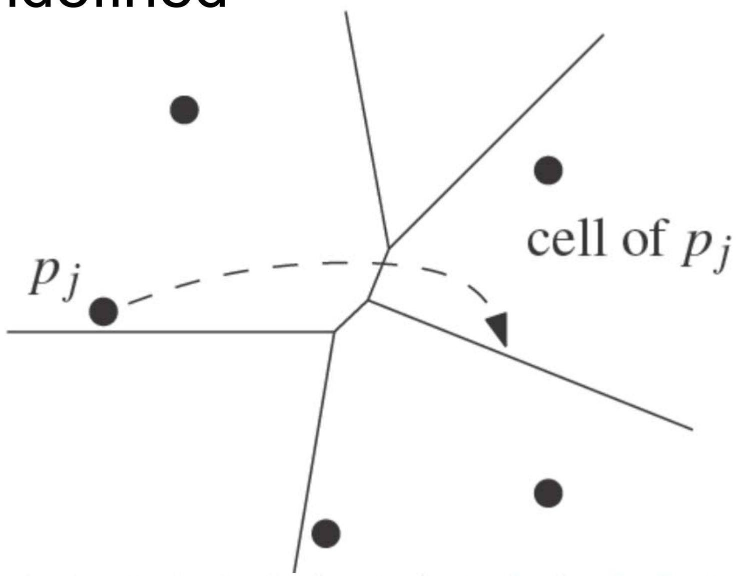
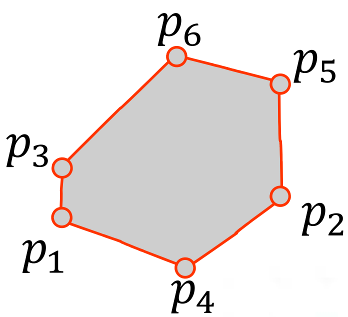
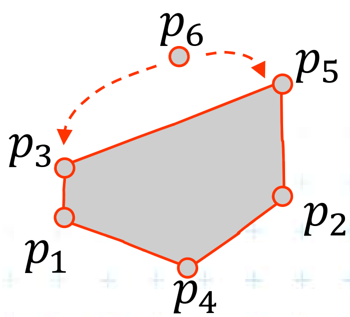


Idea of the algorithm

1. Create the convex hull
   and number the CH points randomly
2. Remove the points starting in the last of this random order and store $cw(p_i)$ and $ccw(p_i)$ points at the time of removal.
3. Include the points back and compute V_{-1}





p_i	$ccw(p_i)$	$cw(p_i)$
p_6	p_3	p_5
p_5	p_3	p_2
…		

Farthest-point Voronoi d. construction

Farthest-pointVoronoi O(nlog n) time in O(n) storage

Input: Set of points *P* in plane

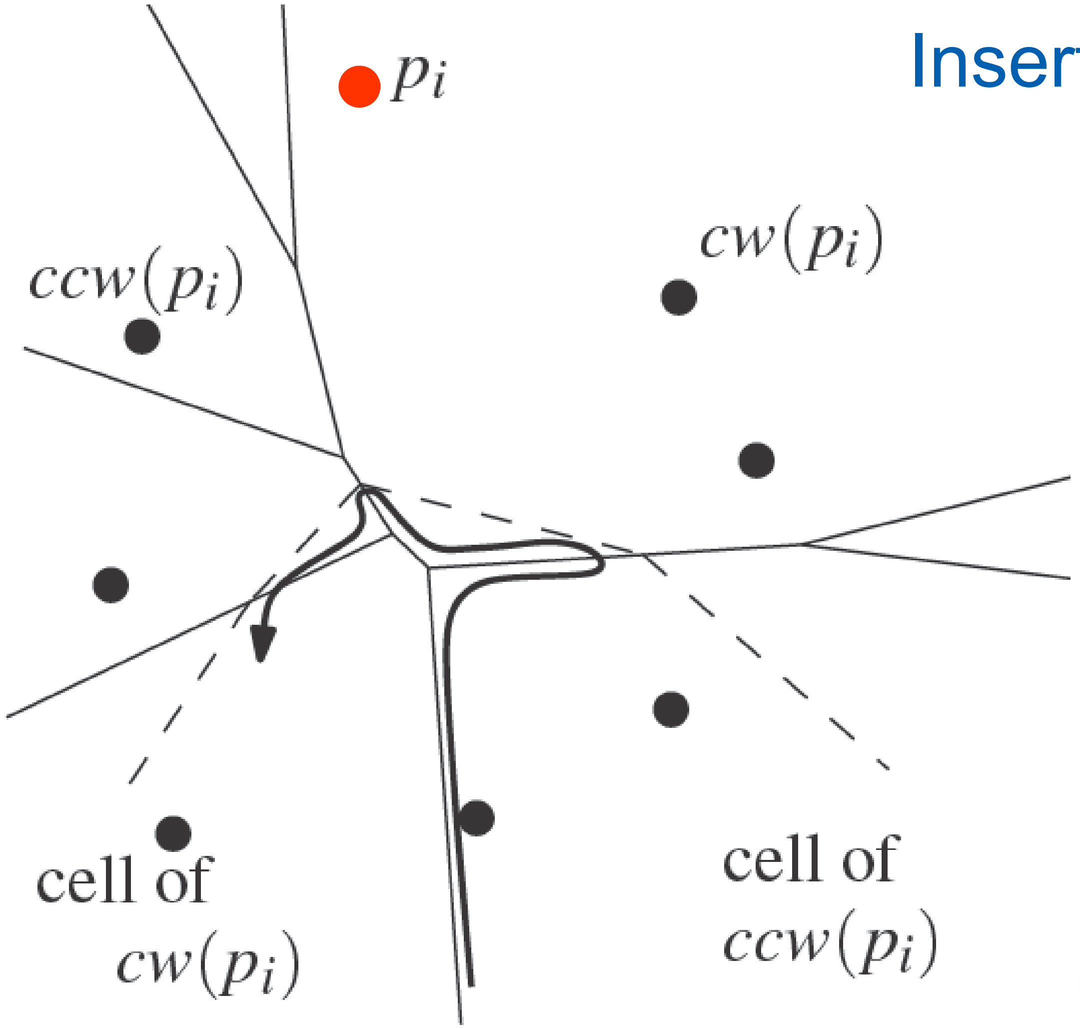
Output: Farthest-point VD $Vor_{-1}(P)$

1. Compute convex hull of *P*
2. Put points in CH(*P*) of *P* in random order $p_1,\ldots,p_h$
3. Remove $p_h, \ldots, p_4$ from the cyclic order (around the CH). When removing p_i, store the neighbors: $cw(p_i)$ and $ccw(p_i)$ at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $Vor_{-1}(\{ p_1, p_2, p_3 \})$ as init
5. **for** i = 4 **to** *h* **do**
6. Add site p_i to $Vor_{-1}(\{ p_1, p_2, \ldots, p_{i-1} \})$ between site $cw(p_i)$ and $ccw(p_i)$
7. - start at most CCW edge of the cell $ccw(p_i)$
8. - continue CW to find intersection with bisector($ccw(p_i)$, p_i)
9. - trace borders of Voronoi cell p_i in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell p_i

DCGI

Farthest-point Voronoi d. construction

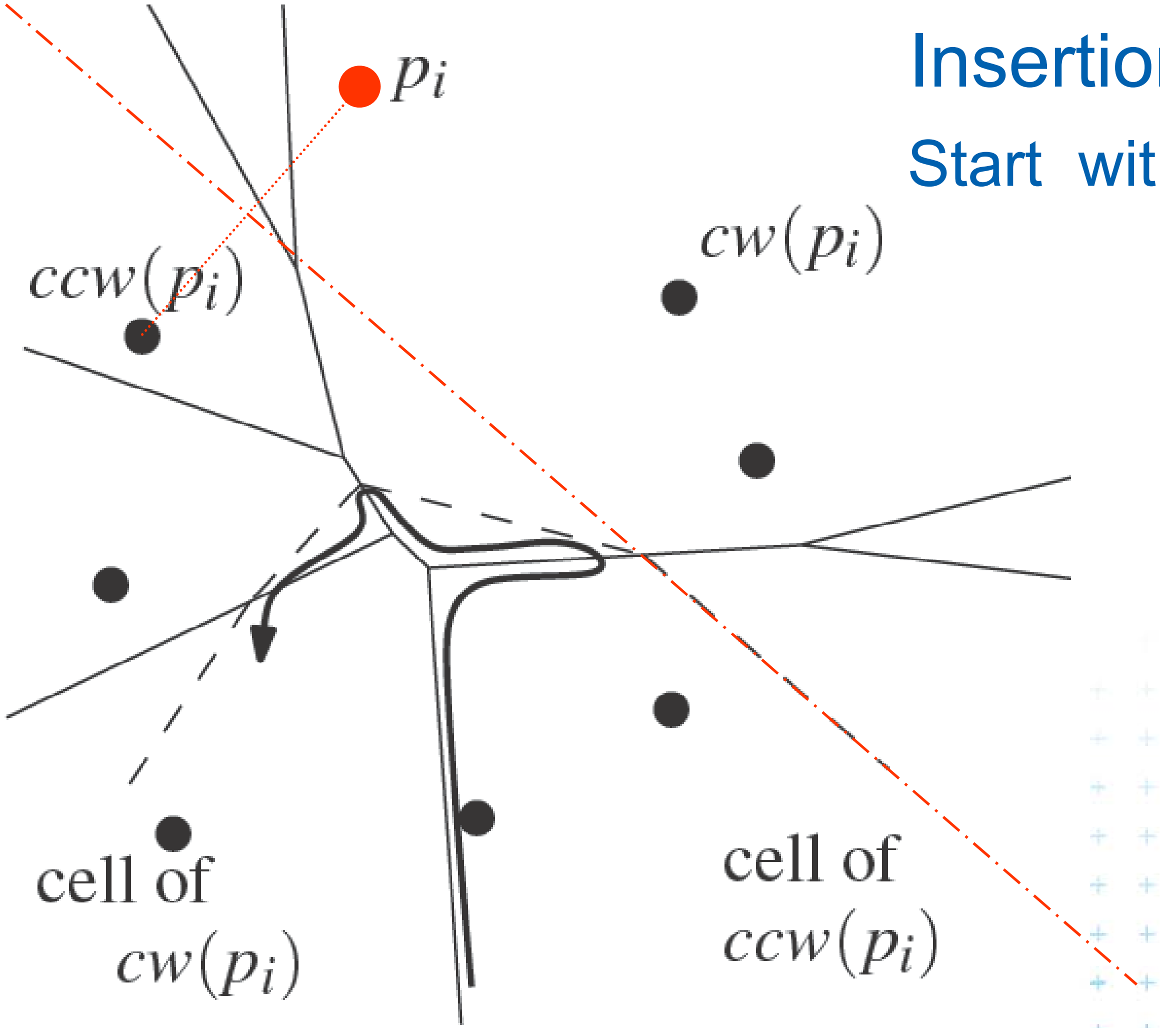
Insertion of site p_i



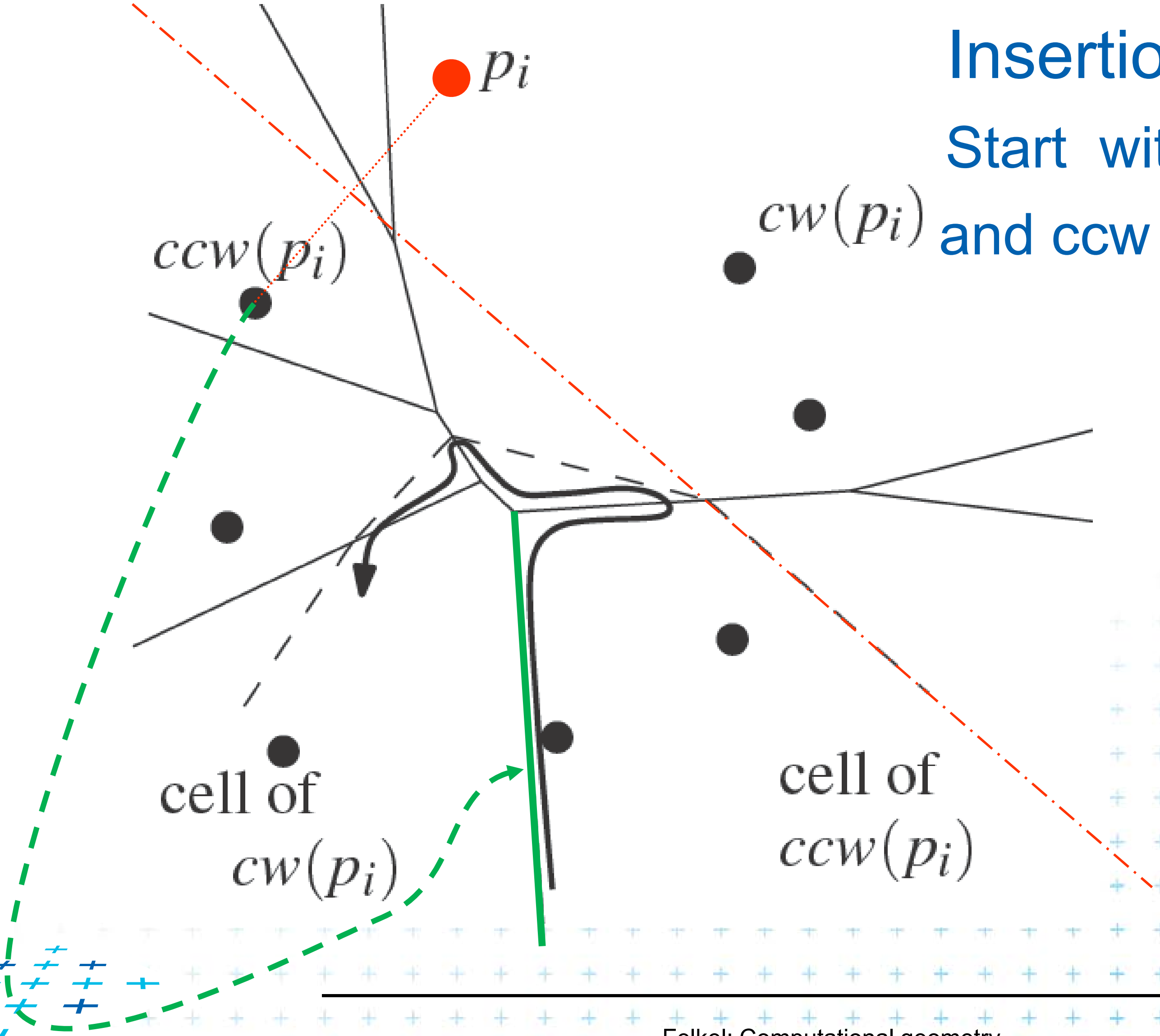
Farthest-point Voronoi d. construction

Insertion of site p_i

Start with site ccw(p_i)



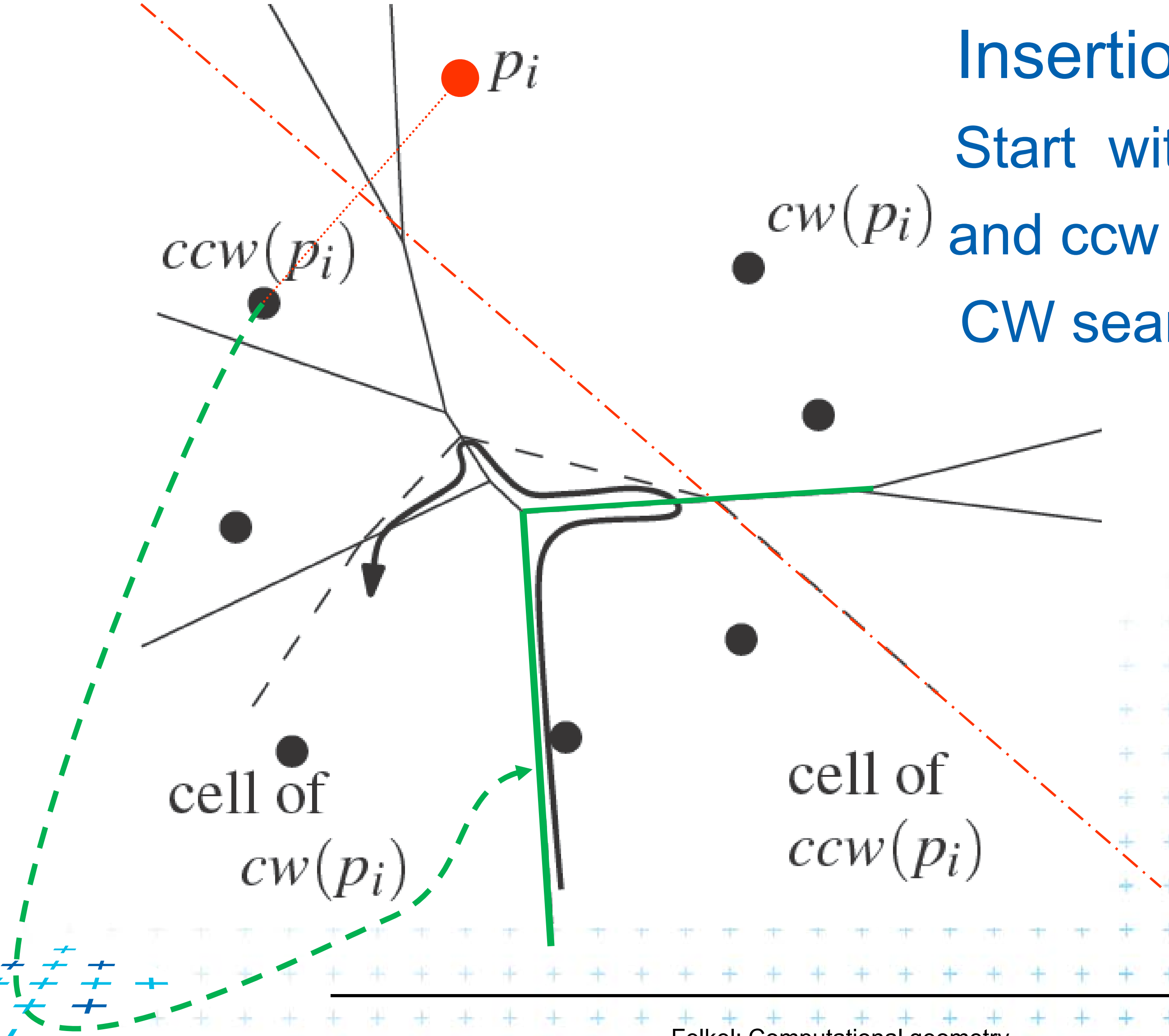
Farthest-point Voronoi d. construction



Insertion of site p_i

Start with site ccw(p_i) and ccw edge of its cell

Farthest-point Voronoi d. construction



Insertion of site p_i

Start with site ccw(p_i)

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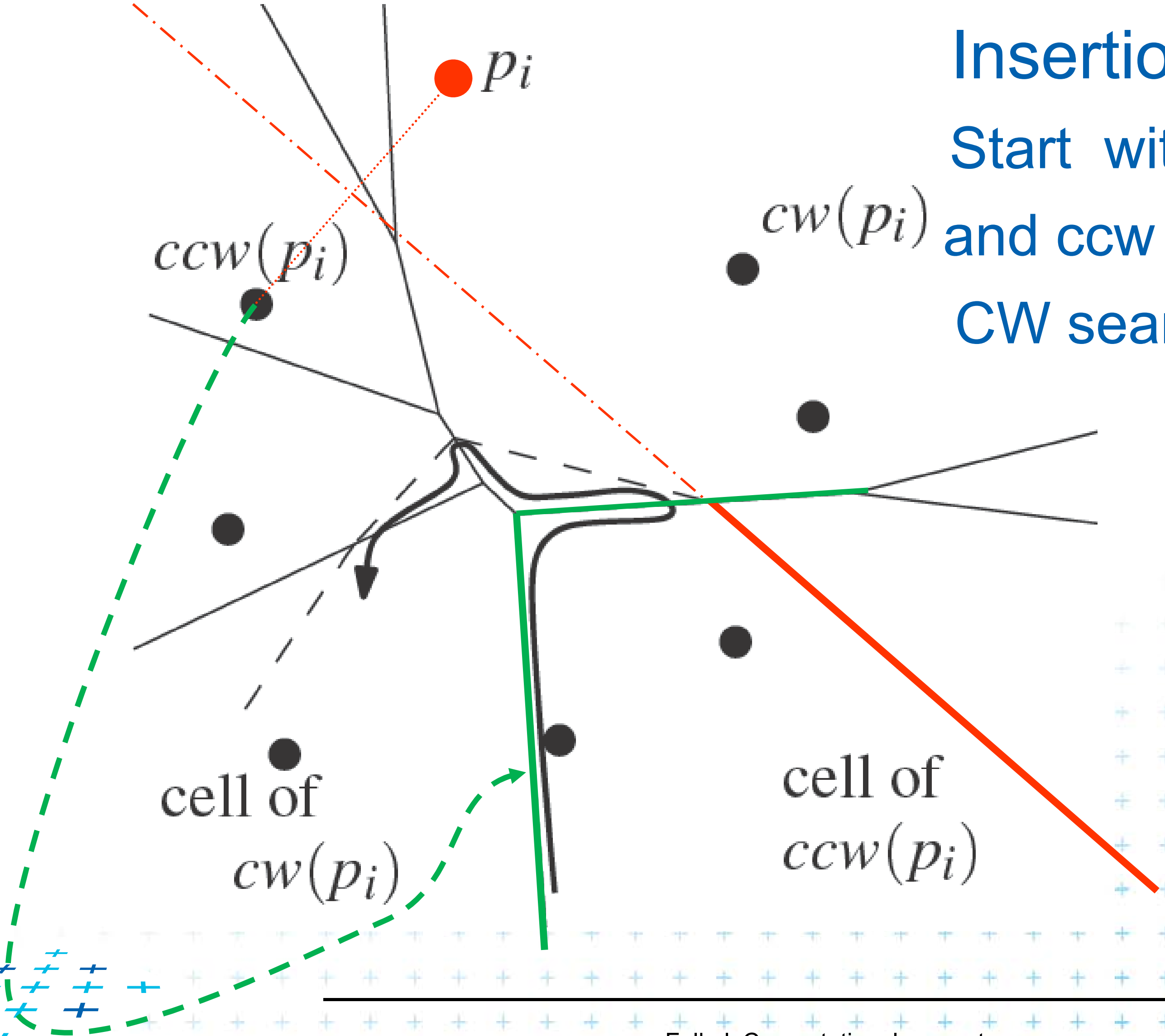
CW search of intersection

Farthest-point Voronoi d. construction



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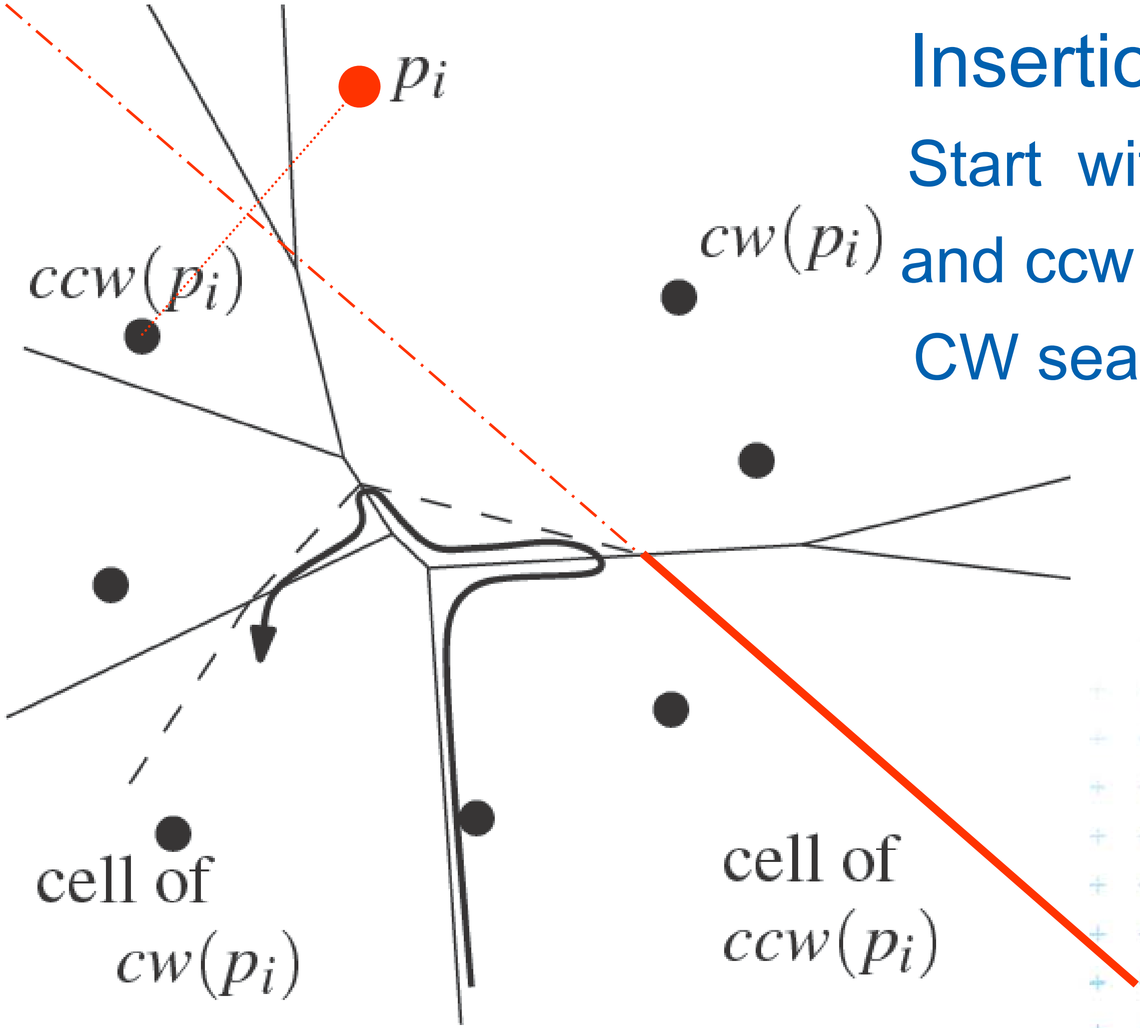

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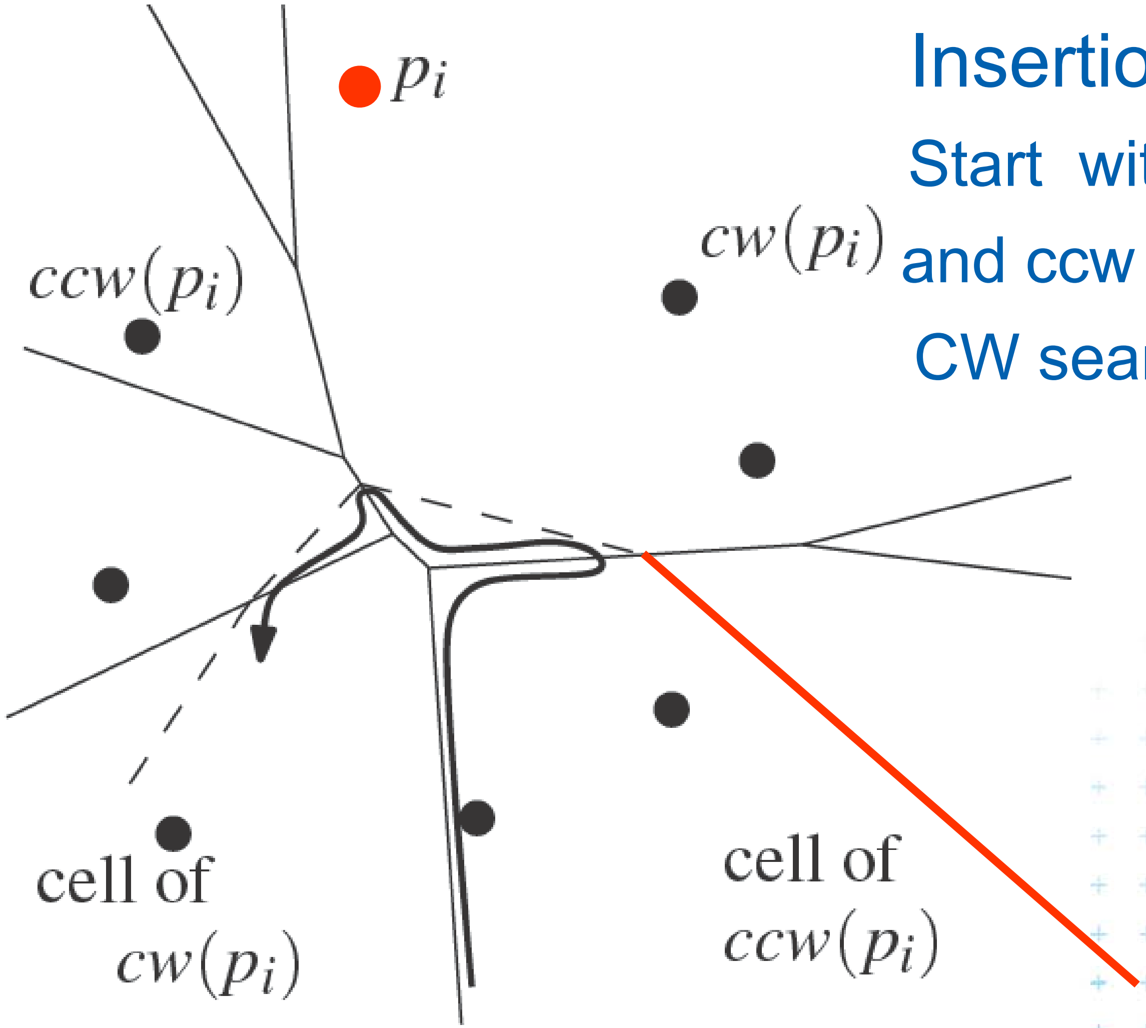

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CW search of intersection

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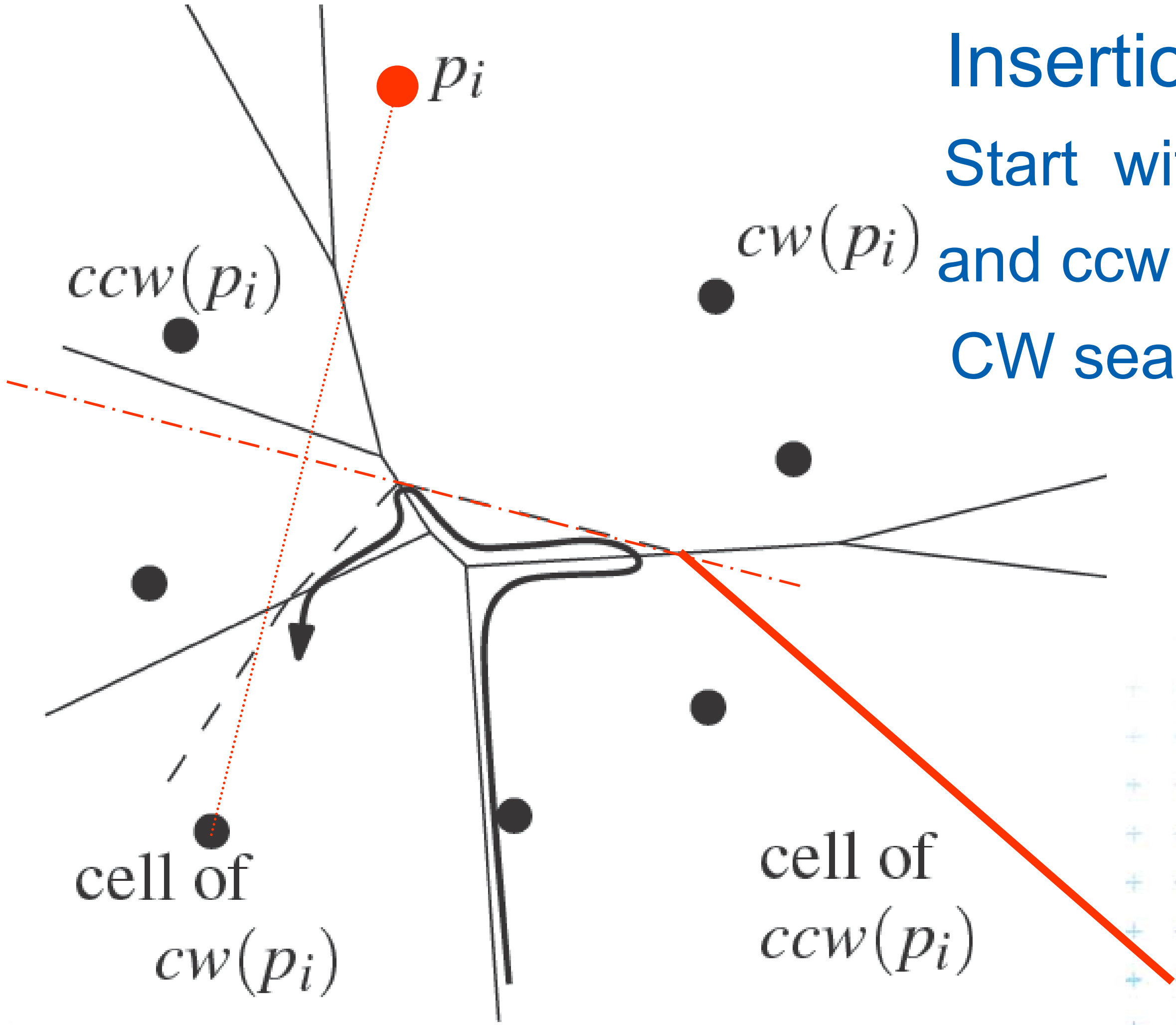

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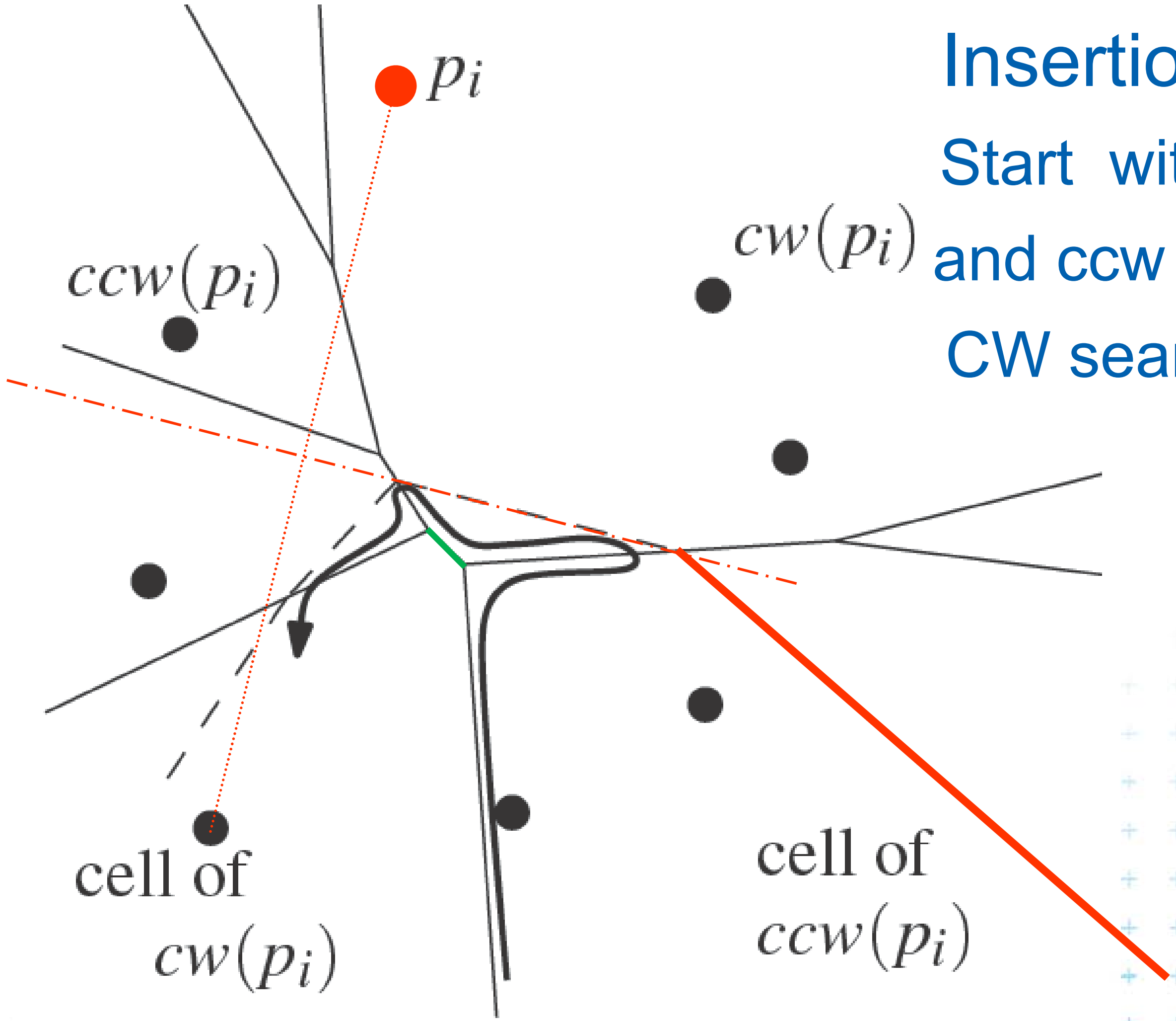

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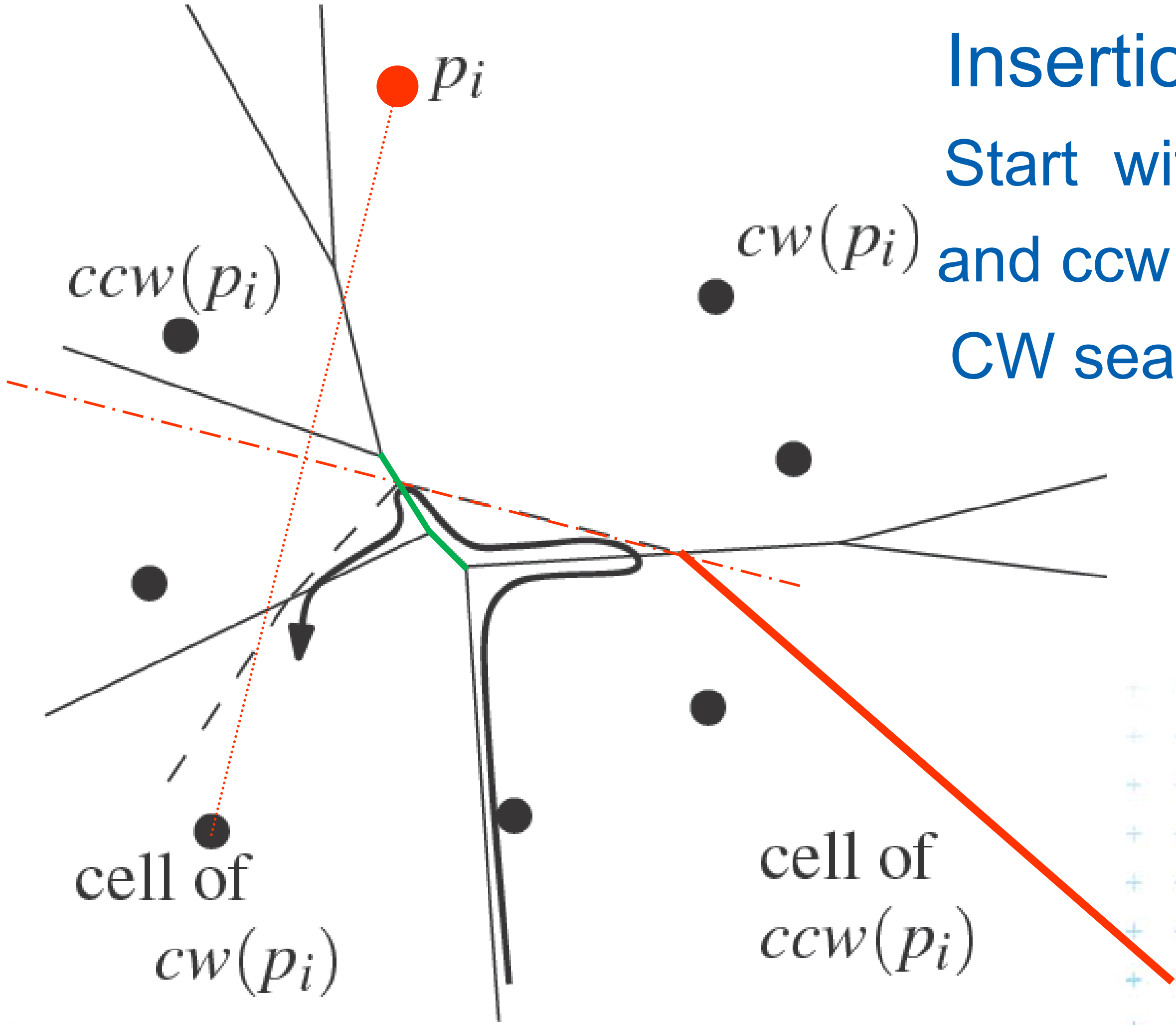

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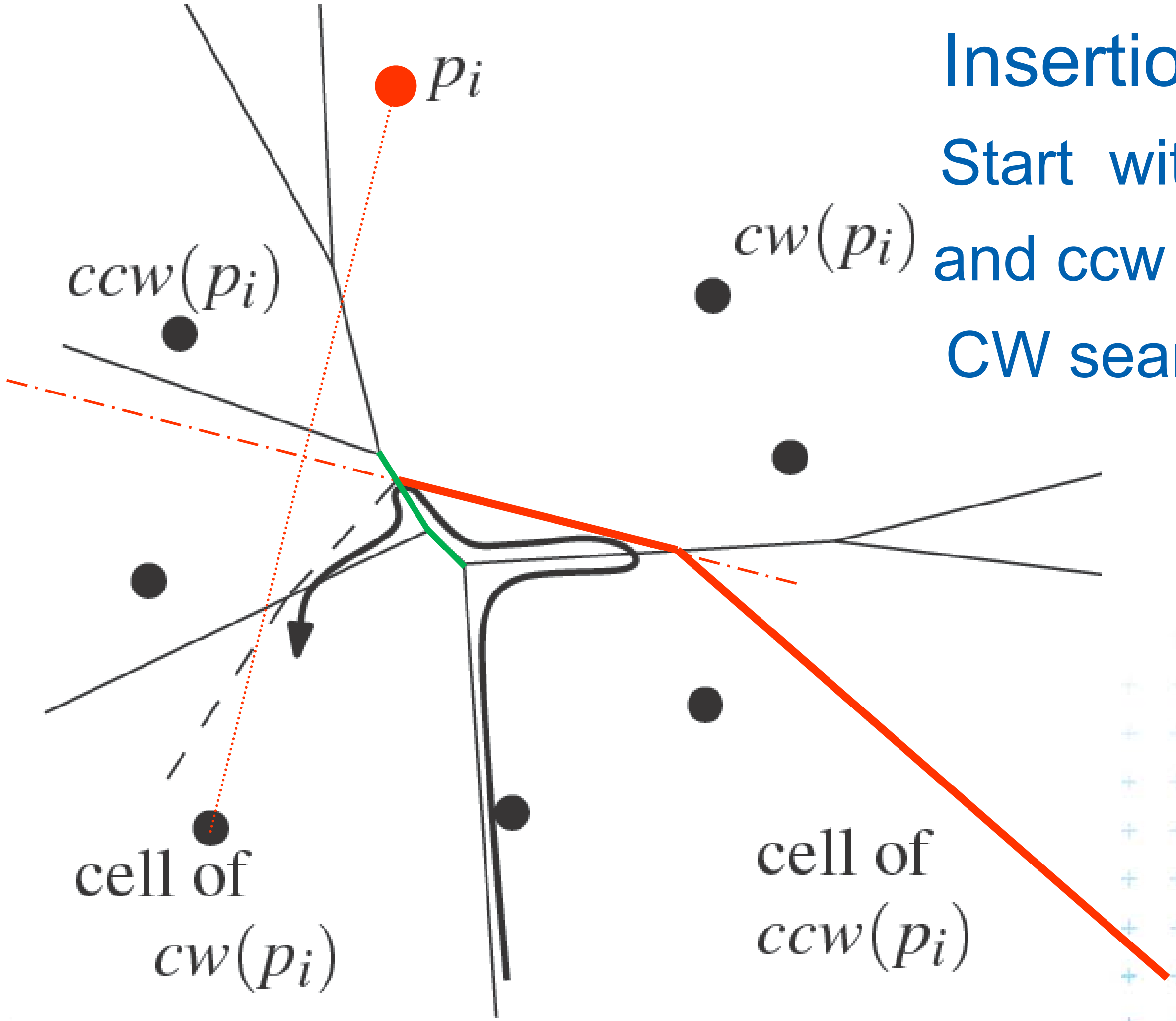
CW search of intersection

Farthest-point Voronoi d. construction

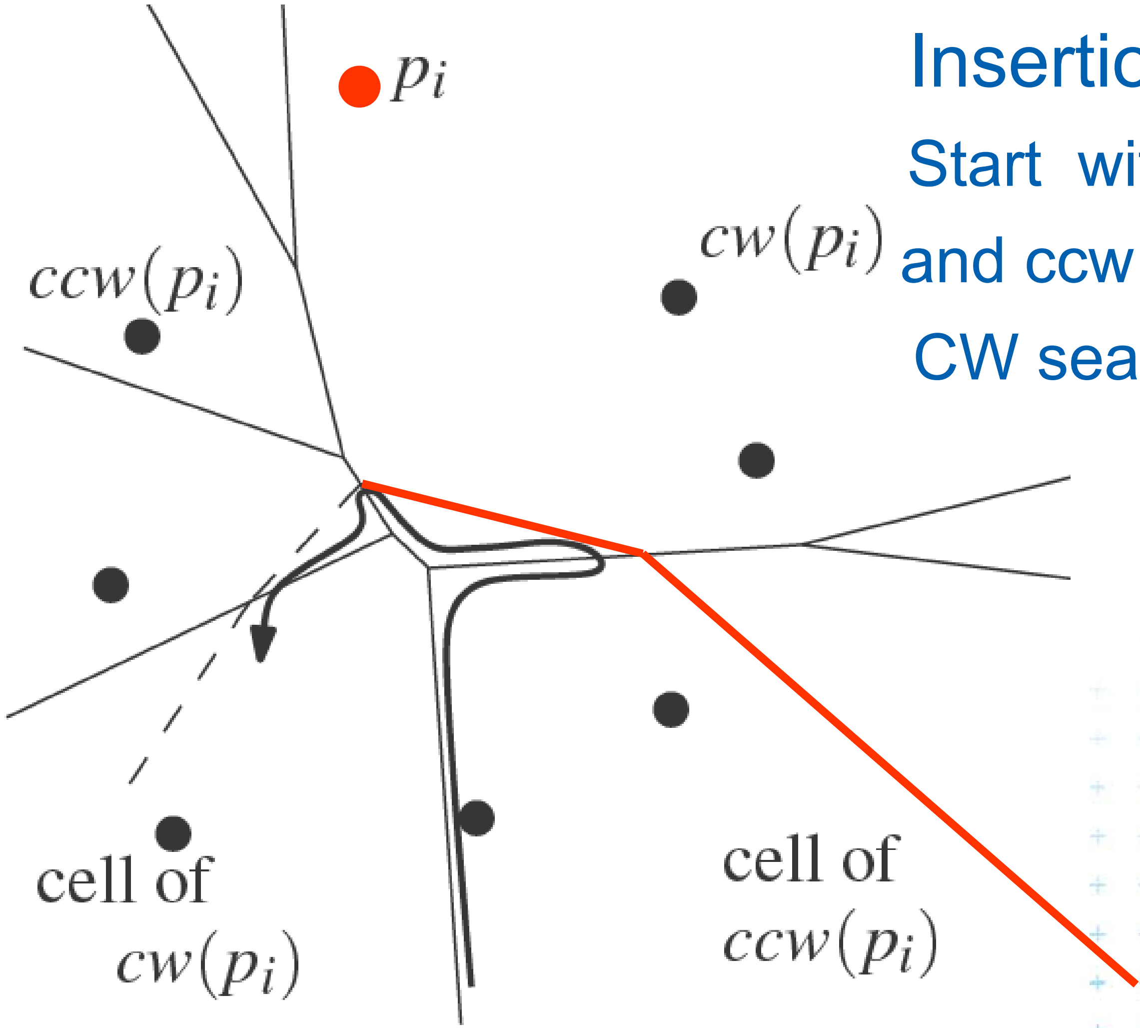


Insertion of site p_i

Start with site ccw(p_i) and ccw edge of its cell

CW search of intersection

Farthest-point Voronoi d. construction



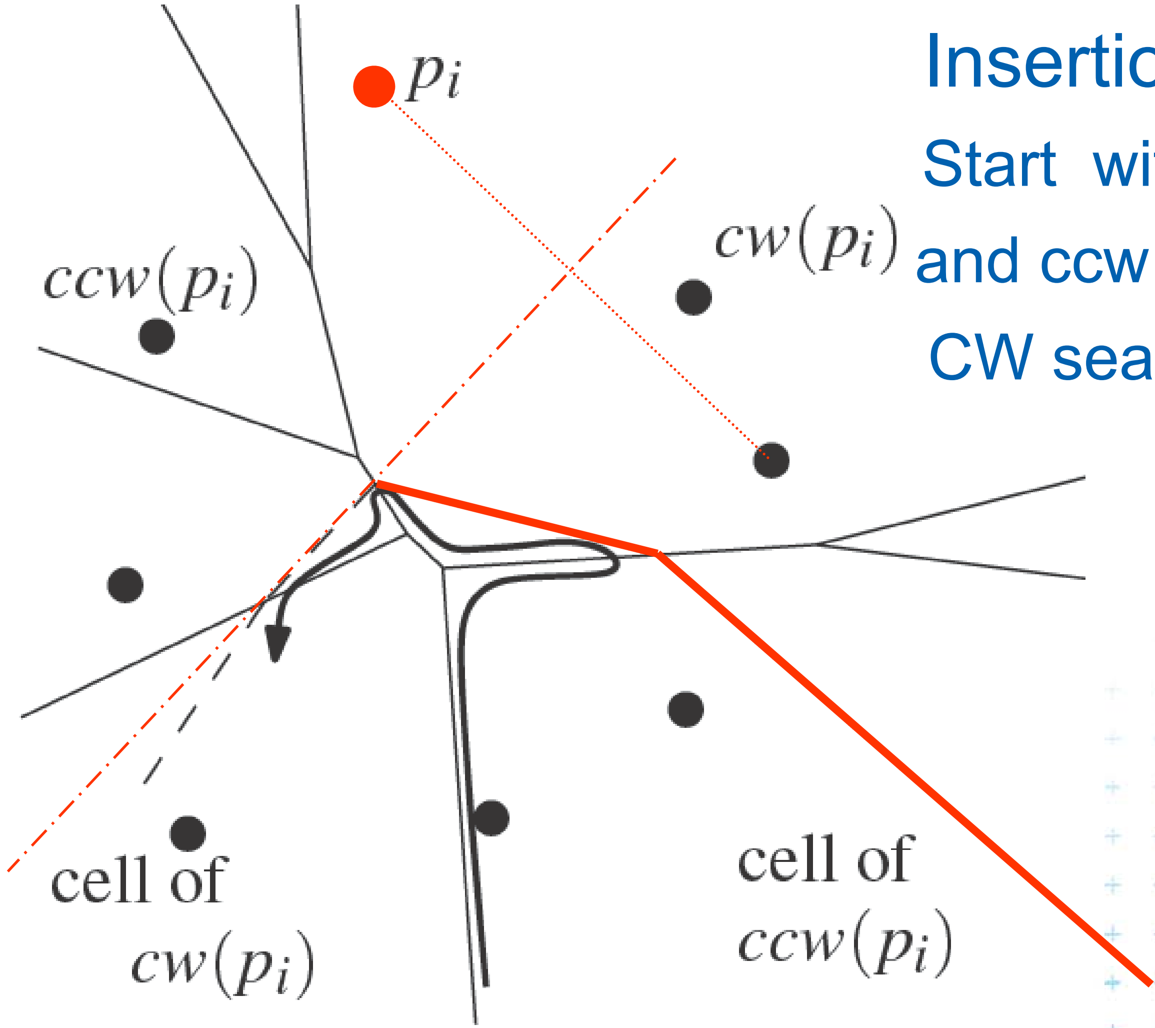
Insertion of site p_i

Start with site ccw(p_i)

and ccw edge of its cell

CW search of intersection

Farthest-point Voronoi d. construction



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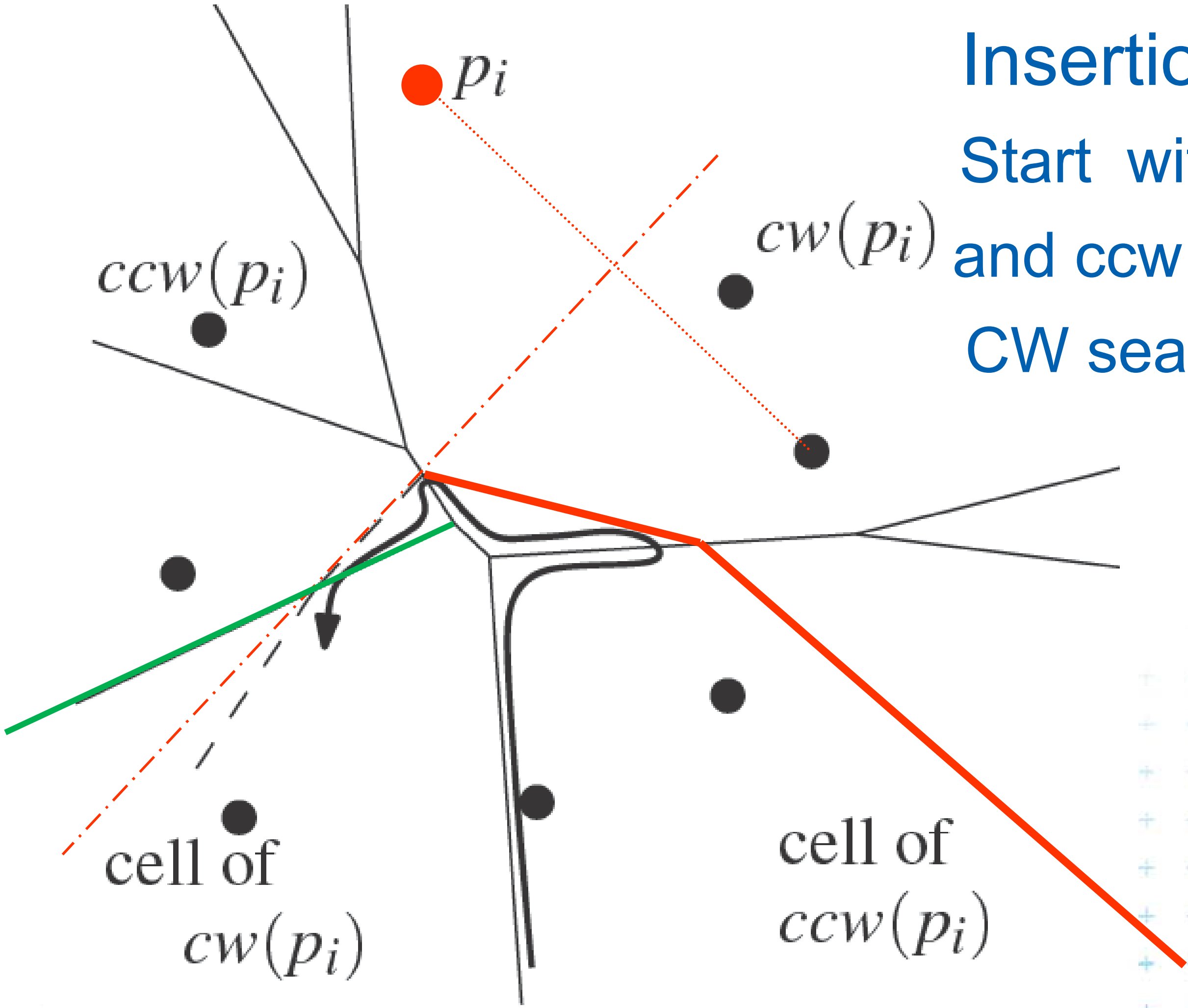
CW search of intersection

Farthest-point Voronoi d. construction



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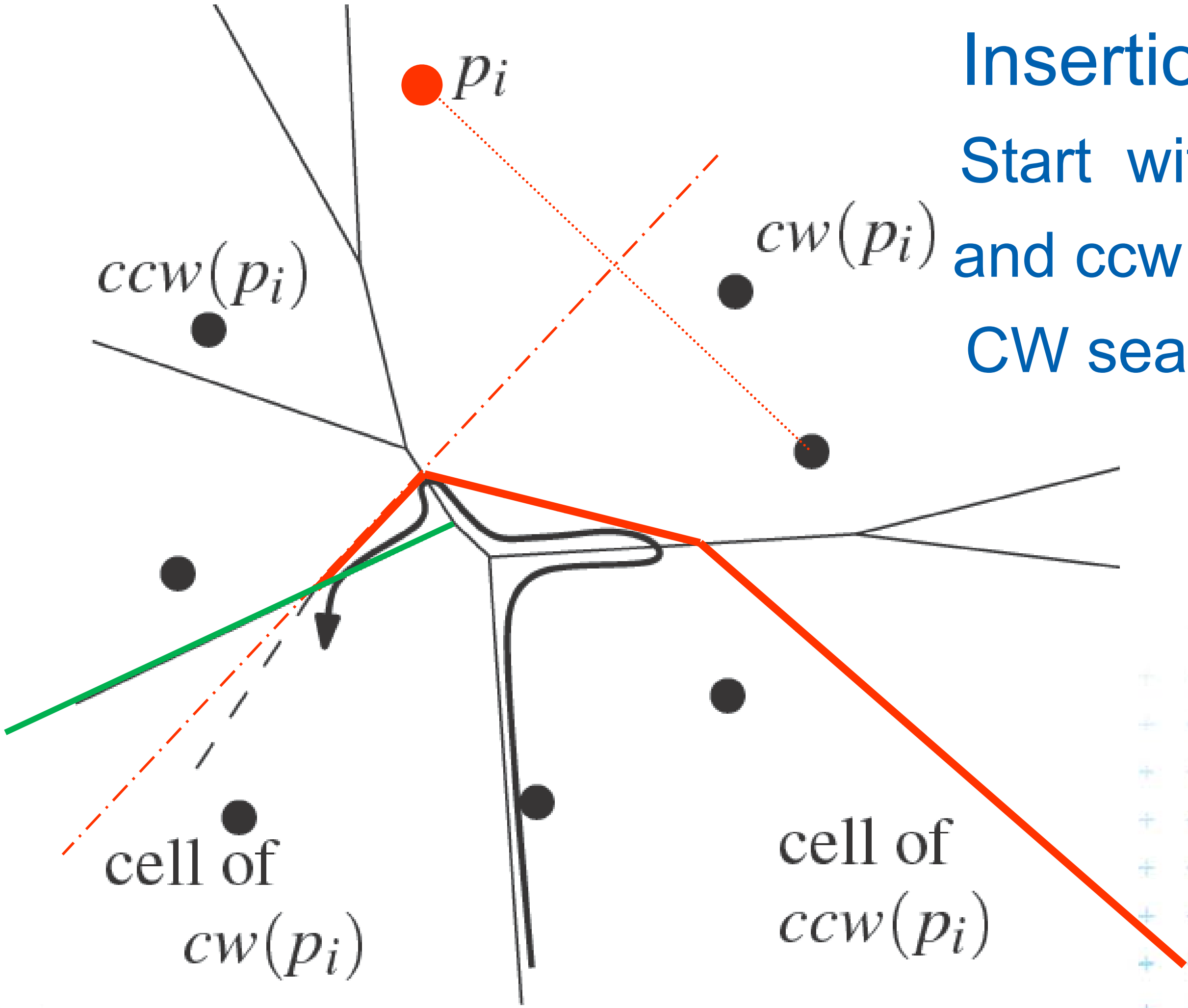
CW search of intersection

Farthest-point Voronoi d. construction



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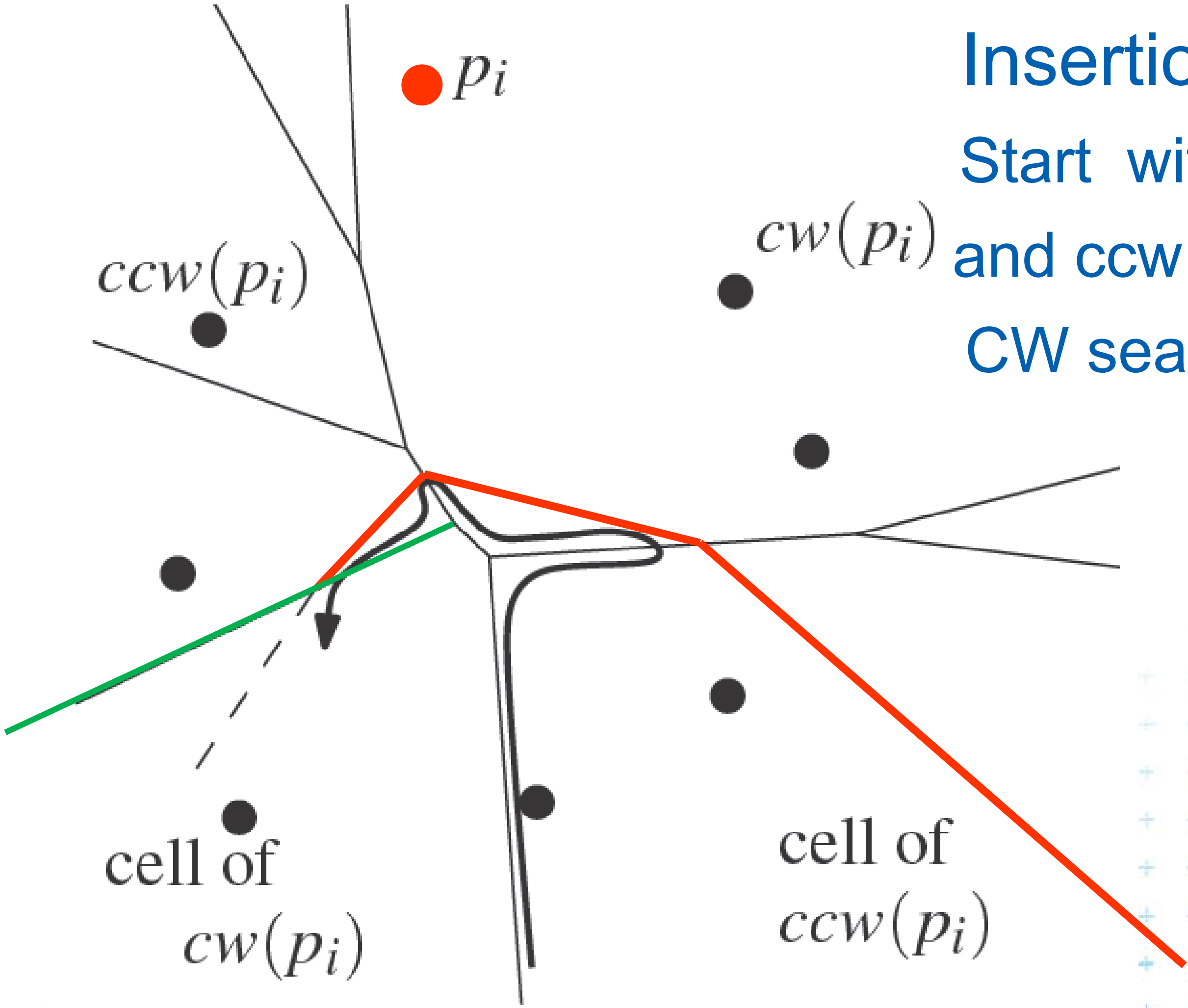
Farthest-point Voronoi d. construction



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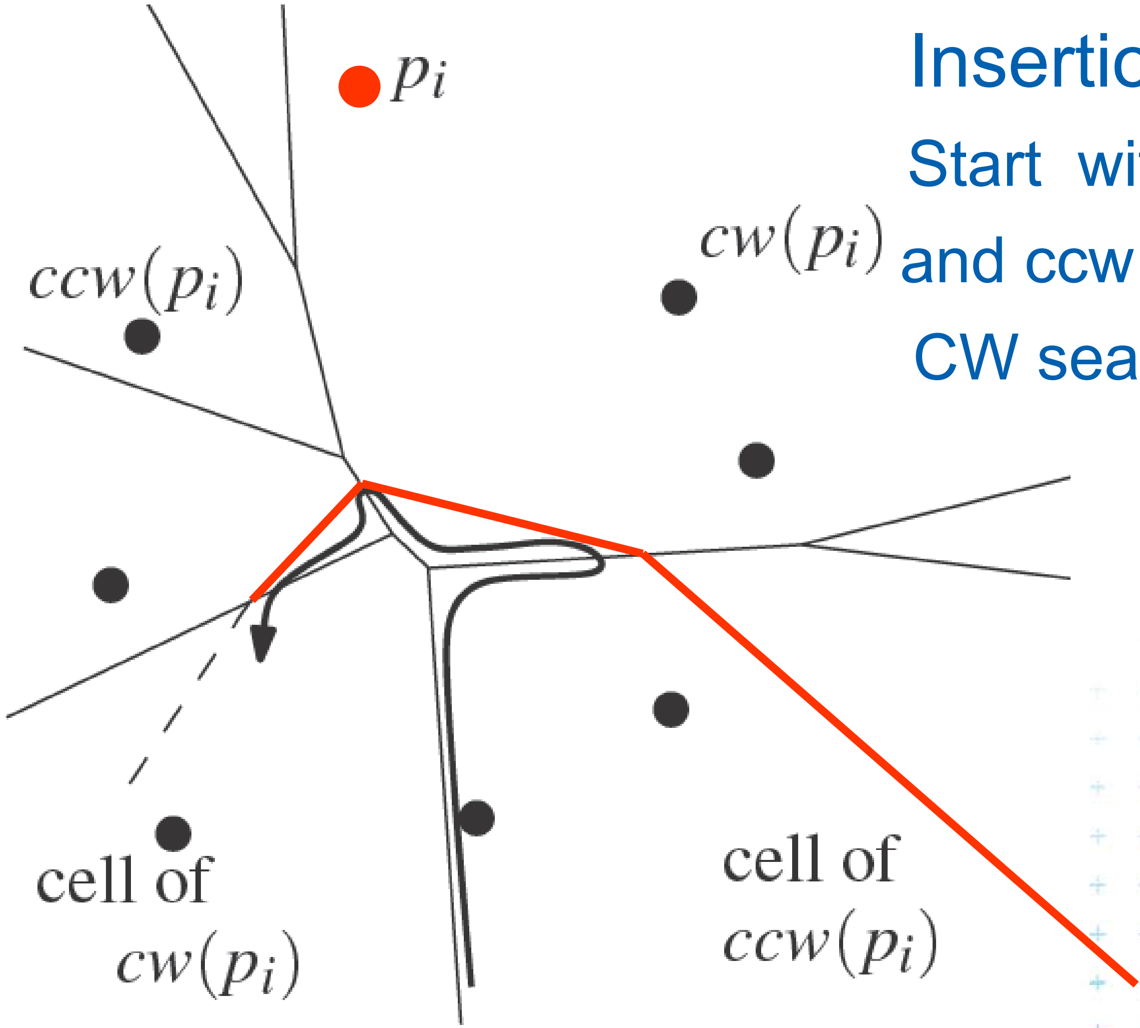
Farthest-point Voronoi d. construction



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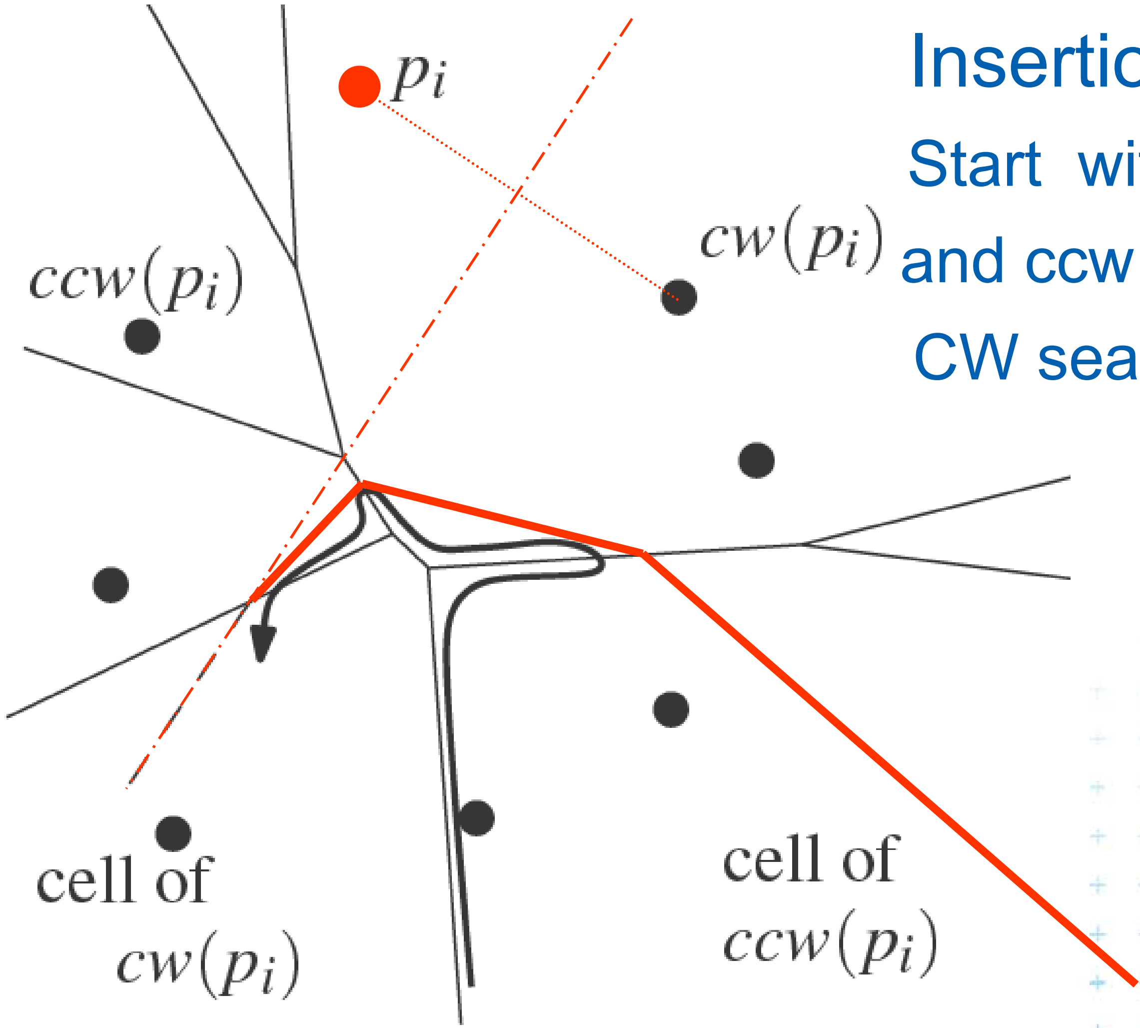
CW search of intersection

Farthest-point Voronoi d. construction



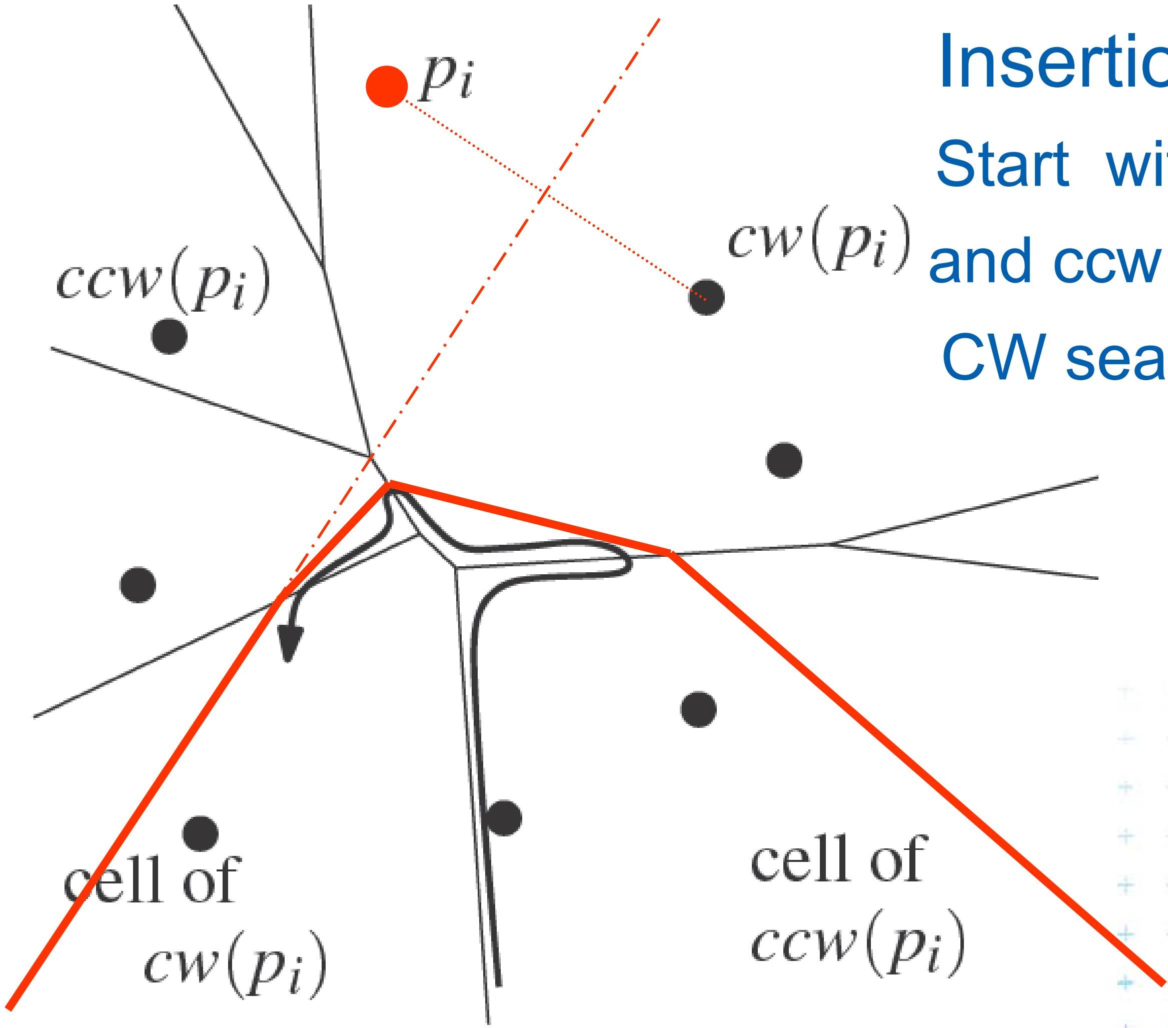
Insertion of site p_i

Start with site ccw(p_i)

and ccw edge of its cell

CW search of intersection

Farthest-point Voronoi d. construction



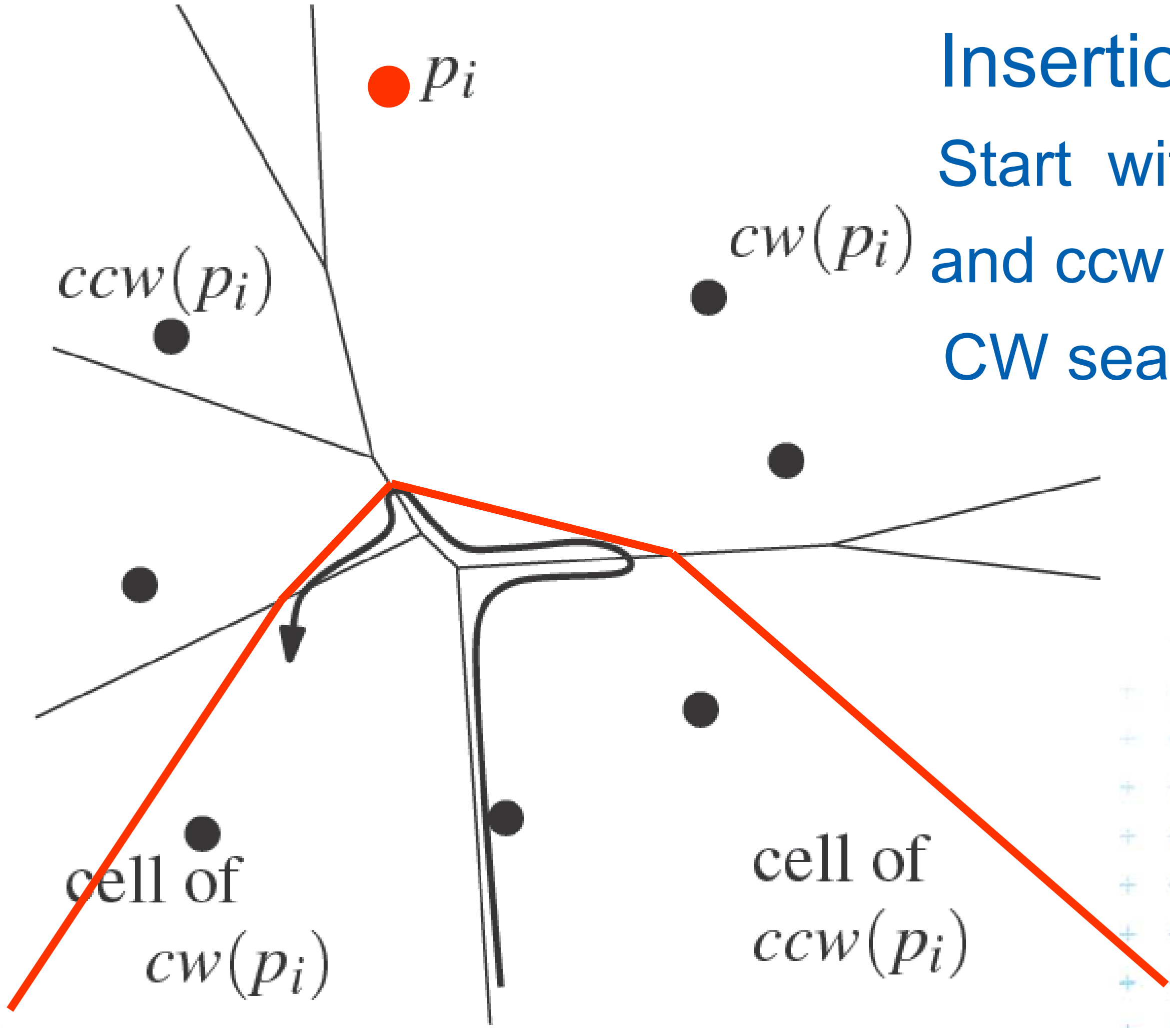
Insertion of site p_i

Start with site ccw(p_i)

and ccw edge of its cell

CW search of intersection

Farthest-point Voronoi d. construction



Insertion of site p_i

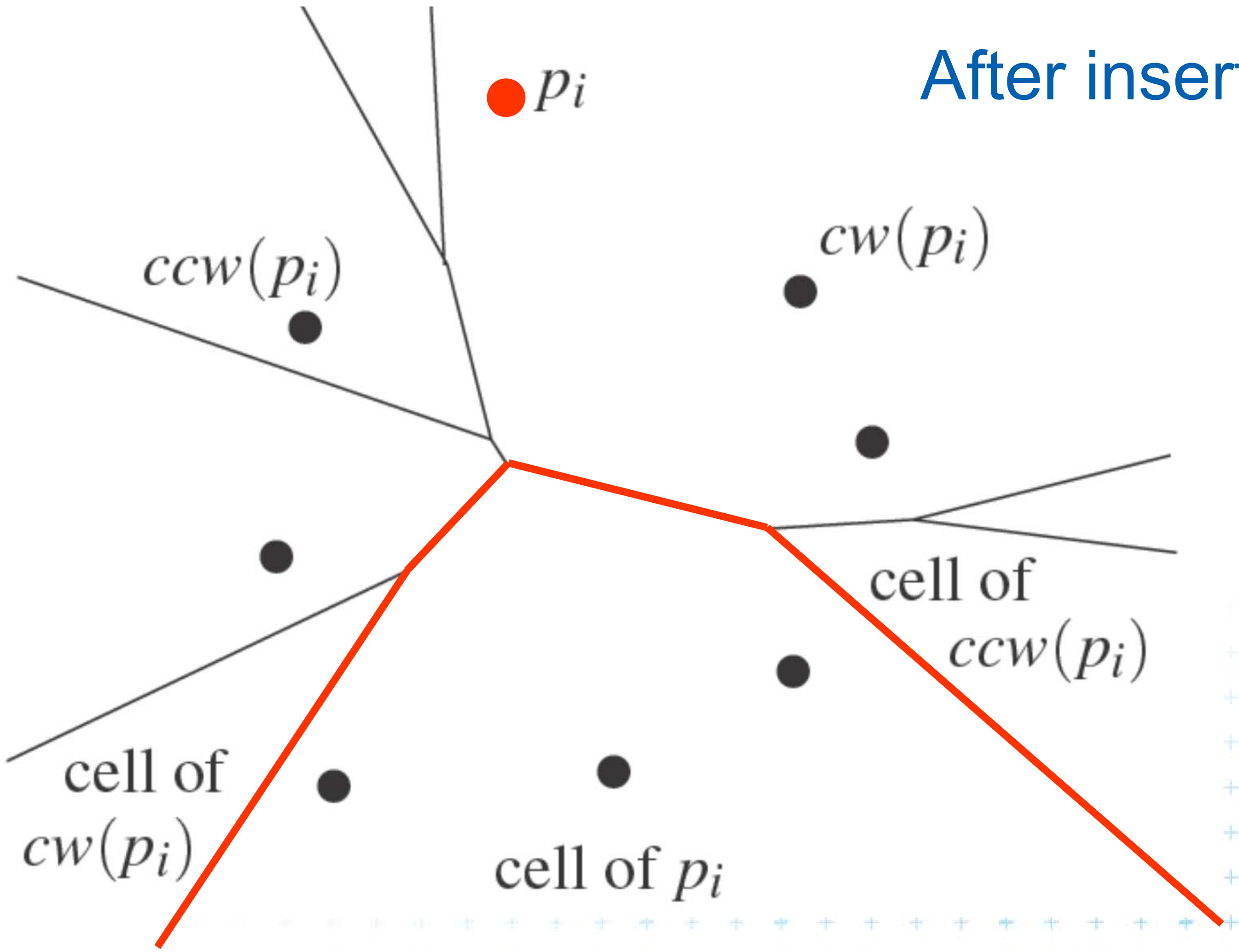
Start with site ccw(p_i)

and ccw edge of its cell

CW search of intersection

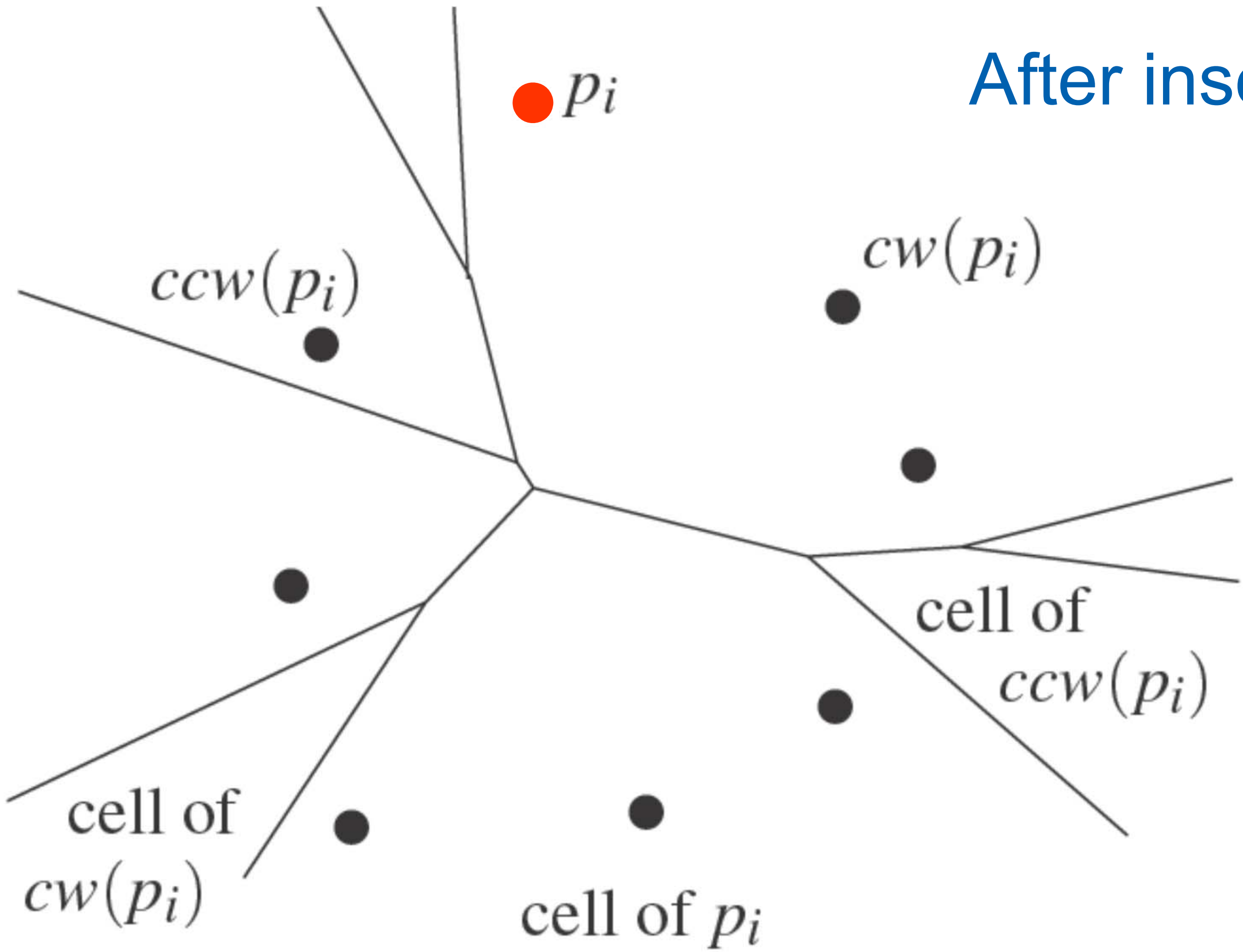
Farthest-point Voronoi d. construction

After insertion of site p_i



Farthest-point Voronoi d. construction

After insertion of site p_i



References

[Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapter 7, http://www.cs.uu.nl/geobook/

[Preparata] Preperata, F.P., Shamos, M.I.: *Computational Geometry. An Introduction.* Berlin, Springer-Verlag,1985. Chapters 5 and 6

[Reiberg] Reiberg, J: Implementierung Geometrischer Algoritmen. Berechnung von Voronoi Diagrammen fuer Liniensegmente. http://www.reiberg.net/project/voronoi/avortrag.ps.gz

[Nandy] Subhas C. Nandy: Voronoi Diagram – presentation. Advanced Computing and Microelectronics Unit. Indian Statistical Institute. Kolkata 700108 http://www.tcs.tifr.res.in/~igga/lectureslides/vor-July-08-2009.ppt

[CGAL] http://www.cgal.org/Manual/3.1/doc_html/cgal_manual/Segment_Voronoi_diagram_2/Chapter_main.html

[applets] http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/Fortune/fortune.htm a http://www.liefke.com/hartmut/cis677/