ALG 04

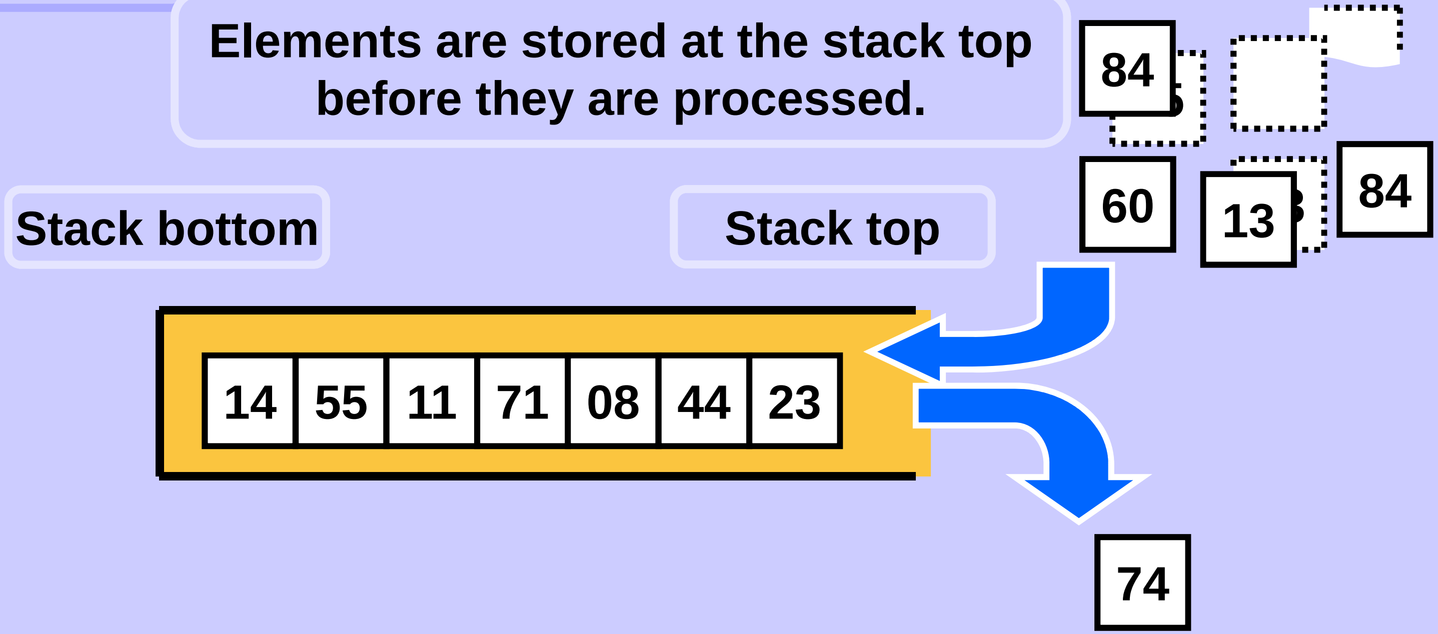
Queue
- Operations Enqueue, Dequeue, Front, Empty....
- Cyclic queue implementation

Graphs
Breadth-first search (BFS) in a tree
Depth-first search (DFS) in a graph
Breadth-first search (BFS) in a graph

Search pruning

Stack

Elements are stored at the stack top before they are processed.



Stack bottom

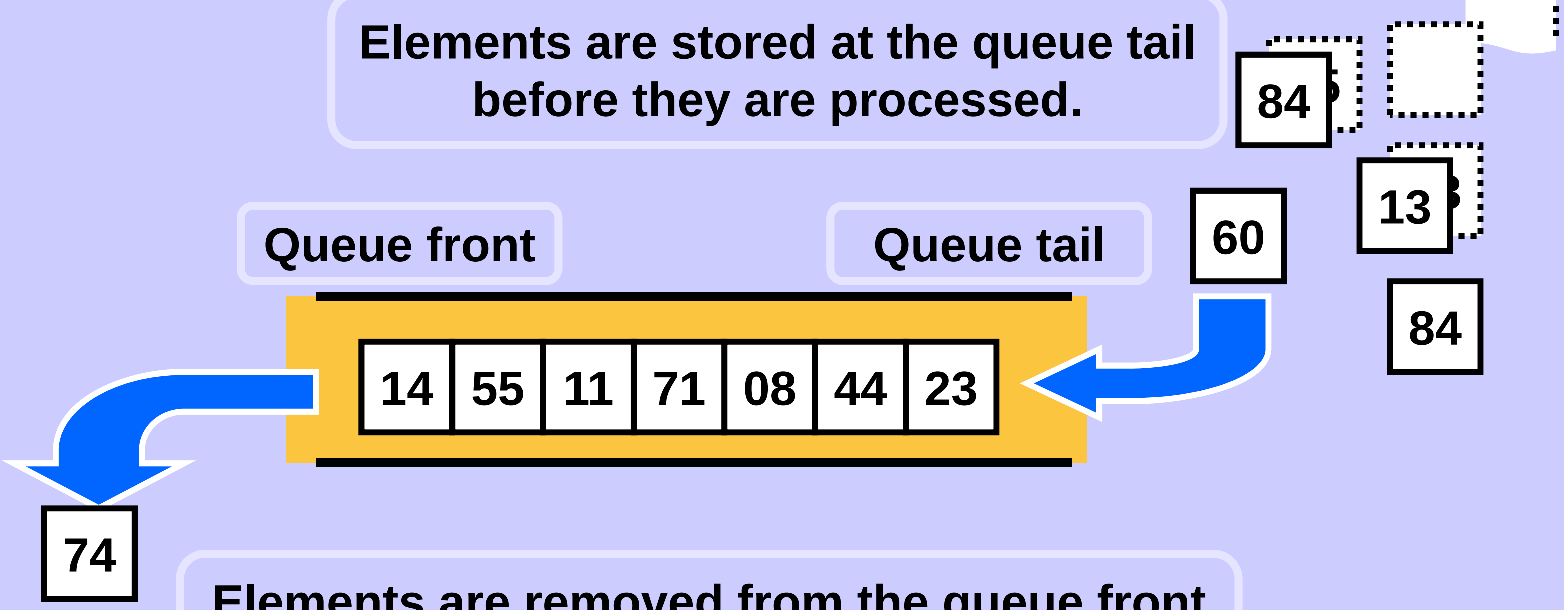
Stack top

Elements are removed from the stack top and then they are processed.

Operation names

Put at the top	Push
Remove from the top	Pop
Read the top	Top
Is the stack empty?	Empty

Queue



Elements are stored at the queue tail before they are processed.

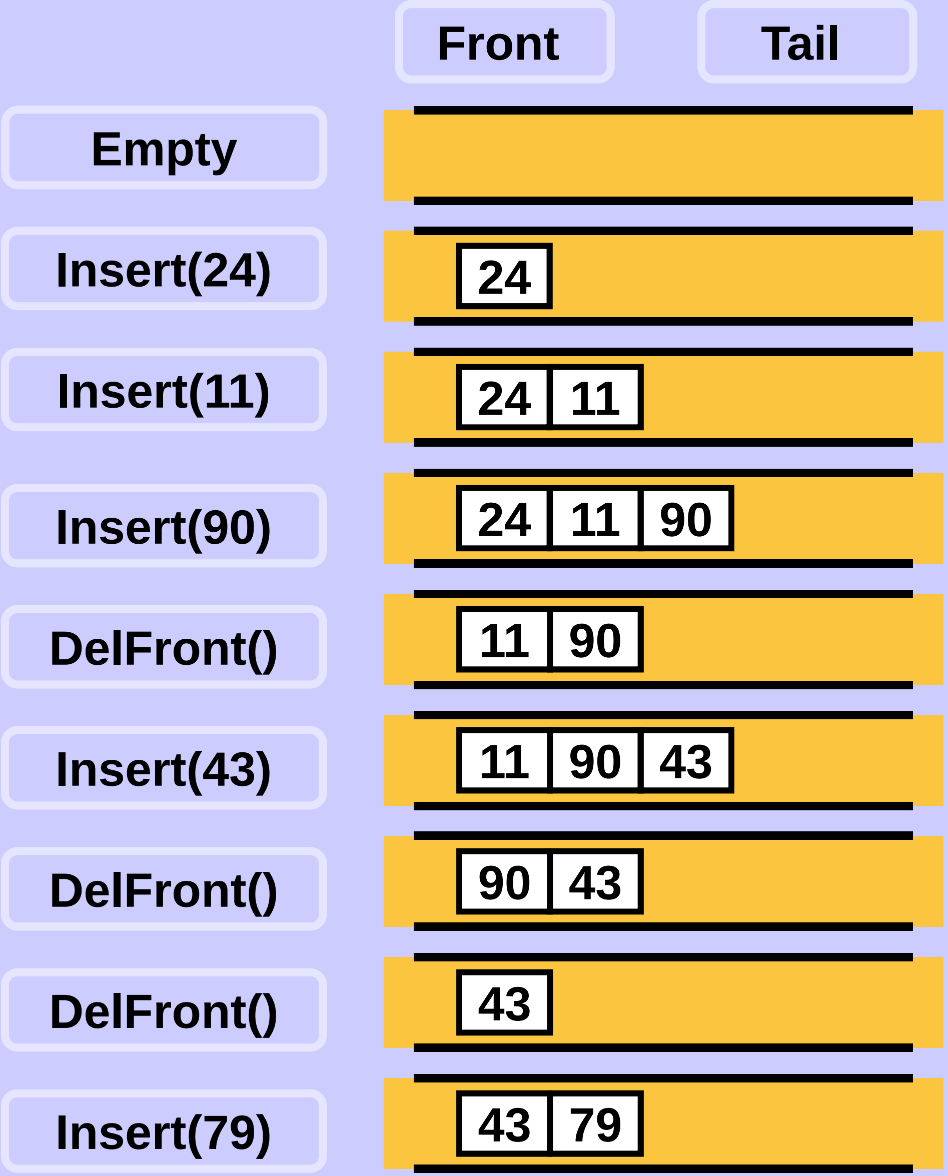
Elements are removed from the queue front and then they are processed.

Operation names

Insert at the tail	Enqueue / InsertLast / Push ...
Remove from the front	Dequeue / delFront / Pop ...
Read the front elem	Front / Peek ...
Is the queue empty?	Empty

Queue

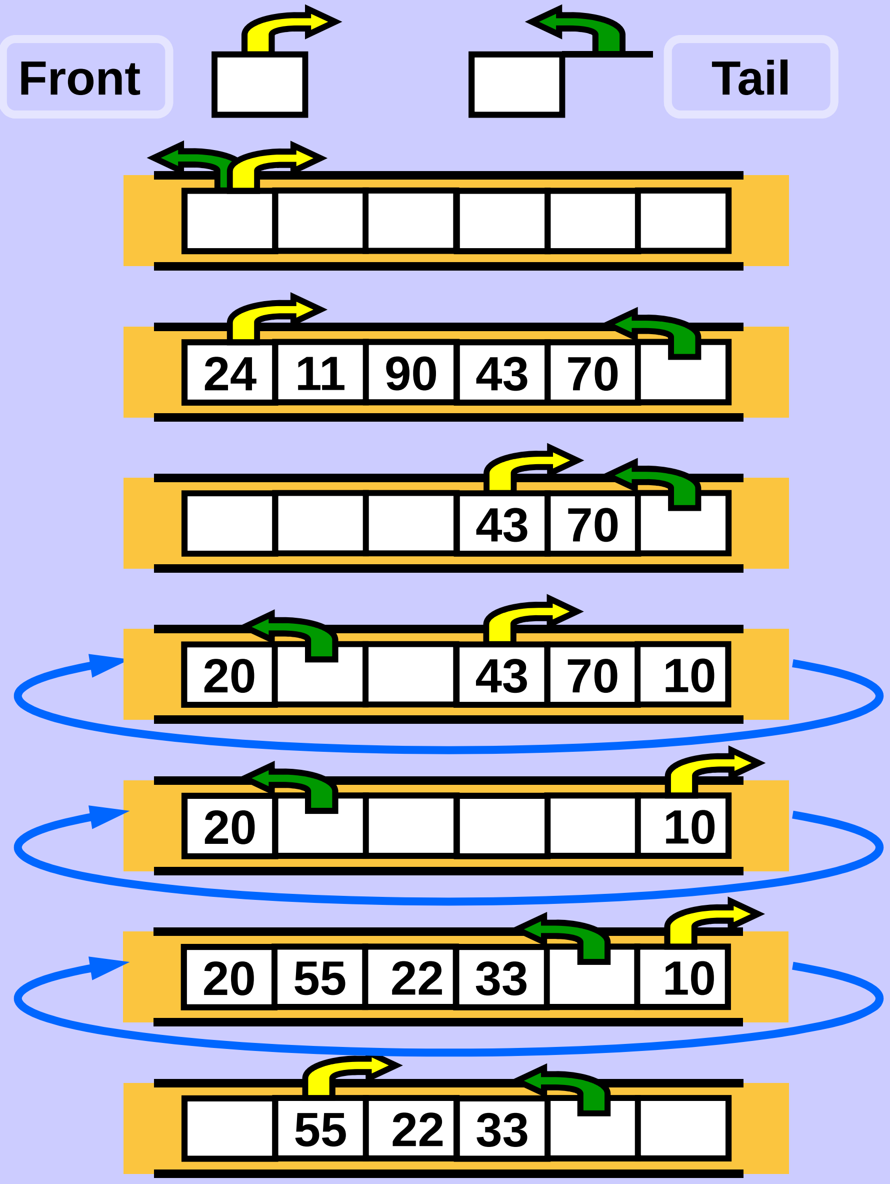
Easy example
of a queue
life cycle.



Front Tail

Empty			
Insert(24)	24		
Insert(11)	24	11	
Insert(90)	24	11	90
DelFront()	11	90	
Insert(43)	11	90	43
DelFront()	90	43	
DelFront()	43		
Insert(79)	43	79	

Cyclic queue implementation in an array



An empty queue in
a fixed length array

Insert 24, 11, 90, 43, 70.

DelFront, DelFront, DelFront .

Insert 10, 20.

DelFront, DelFront .

Insert 55, 22, 33.

DelFront, DelFront .

Cyclic queue implementation in an array

**Tail index points to the first free position behind the last queue element.
Front index points to the first position occupied by a queue element.
When both indices point to the same position the queue is empty.**

```
class Queue:
    def __init__(self, sizeOfQ):
        self.size = sizeOfQ
        self.q = [None] * sizeOfQ
        self.front = 0
        self.tail = 0

def isEmpty(self):
    return (self.tail == self.front)

def Enqueue(self, node):
    if self.tail+1 == self.front or \
        self.tail - self.front == self.size-1:
            pass                  # implement overflow fix here
    self.q[self.tail] = node
    self.tail = (self.tail + 1) % self.size
```

Continue...

Cyclic queue implementation in an array

Tail index points to the first free position behind the last queue element.
Front index points to the first position occupied by a queue element.
When both indices point to the same position the queue is empty.

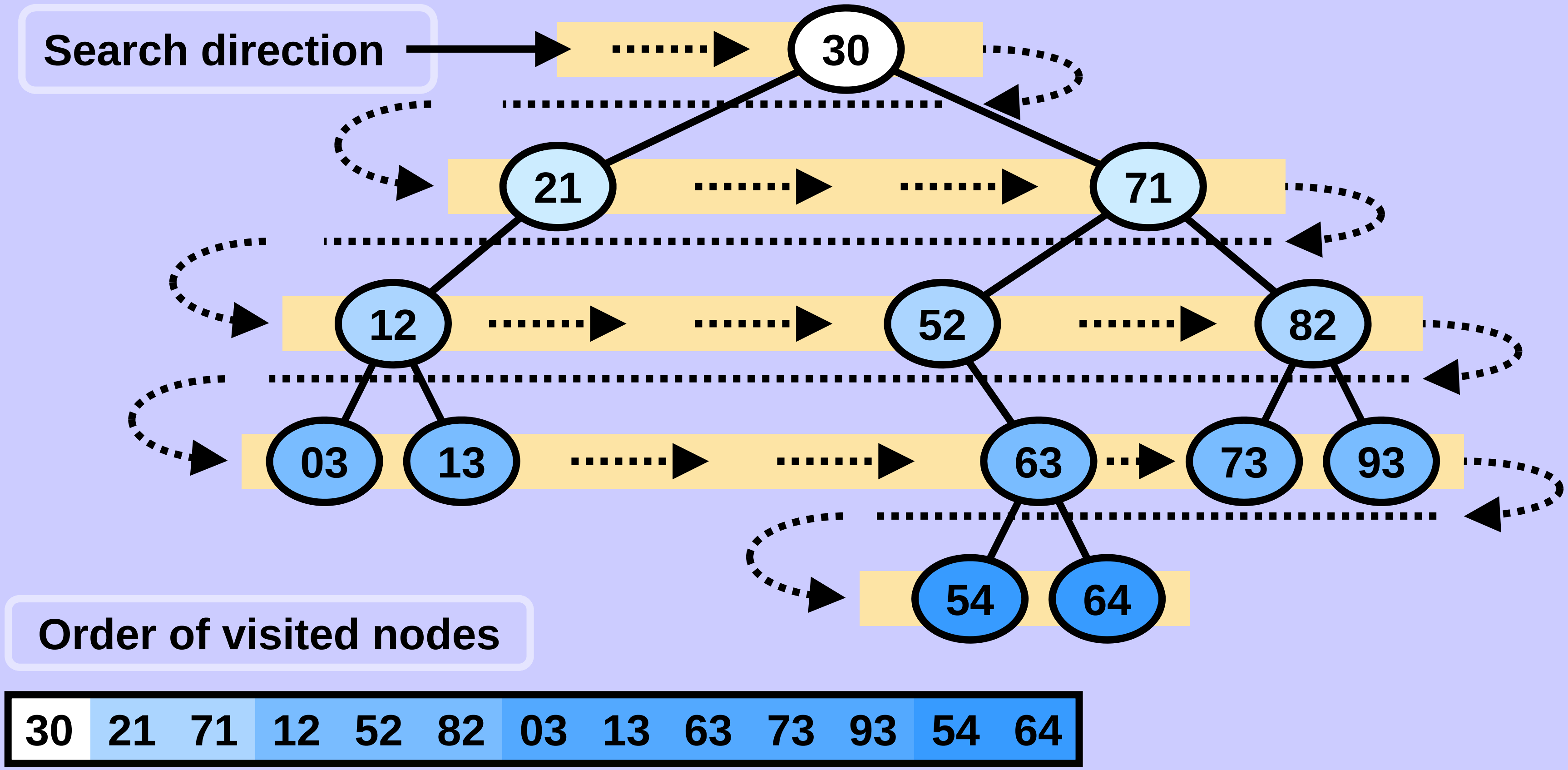
... continued

```
def Dequeue(self):
    node = self.q[self.front]
    self.front = (self.front + 1) % self.size
    return node

def pop(self):
    return self.Dequeue()

def push(self, node):
    self.Enqueue(node)
```

Breadth-first search (BFS) in a tree


Search direction

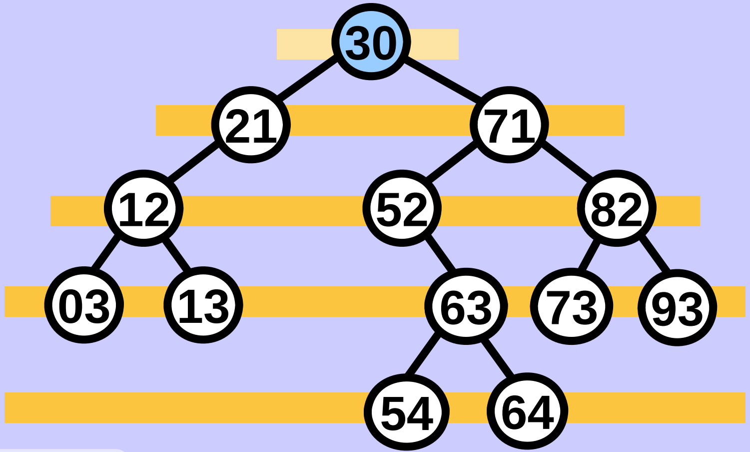
Order of visited nodes

30 21 71 12 52 82 03 13 63 73 93 54 64

Neither the tree structure nor the recursion support this approach directly.

Breadth-first search (BFS) in a tree

Initialization



Output

Create an empty queue.

Enqueue the tree root.

Front

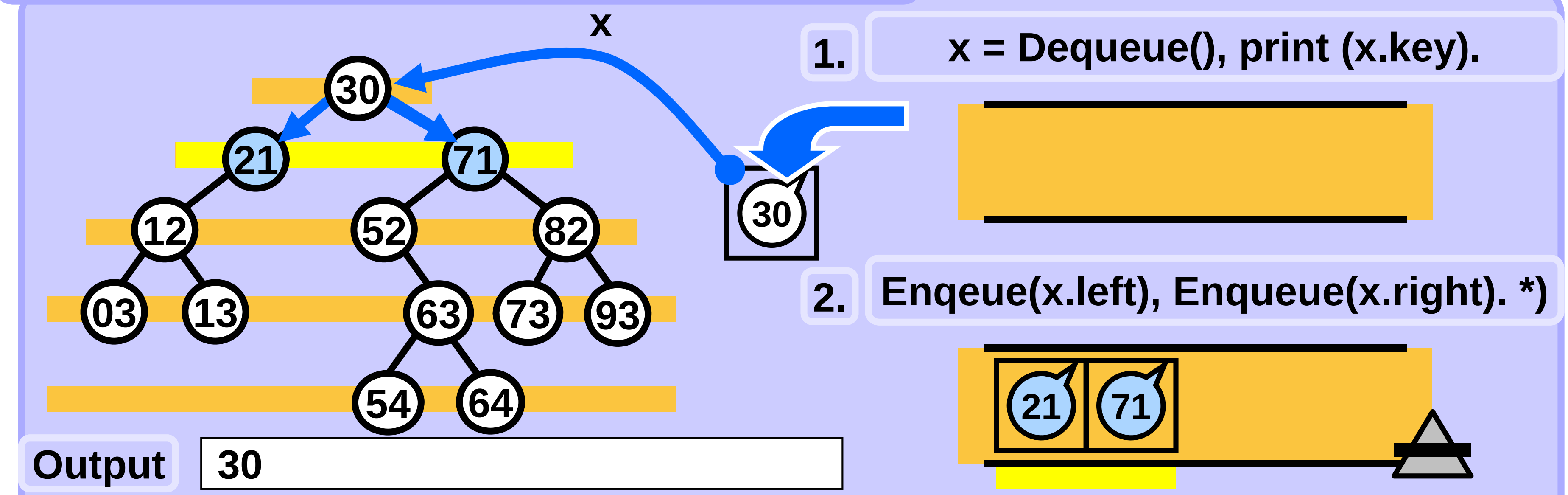
Tail

Main loop

While the queue is not empty do:

1. Remove the first element from the queue and process it.
2. Enqueue the children of removed element.

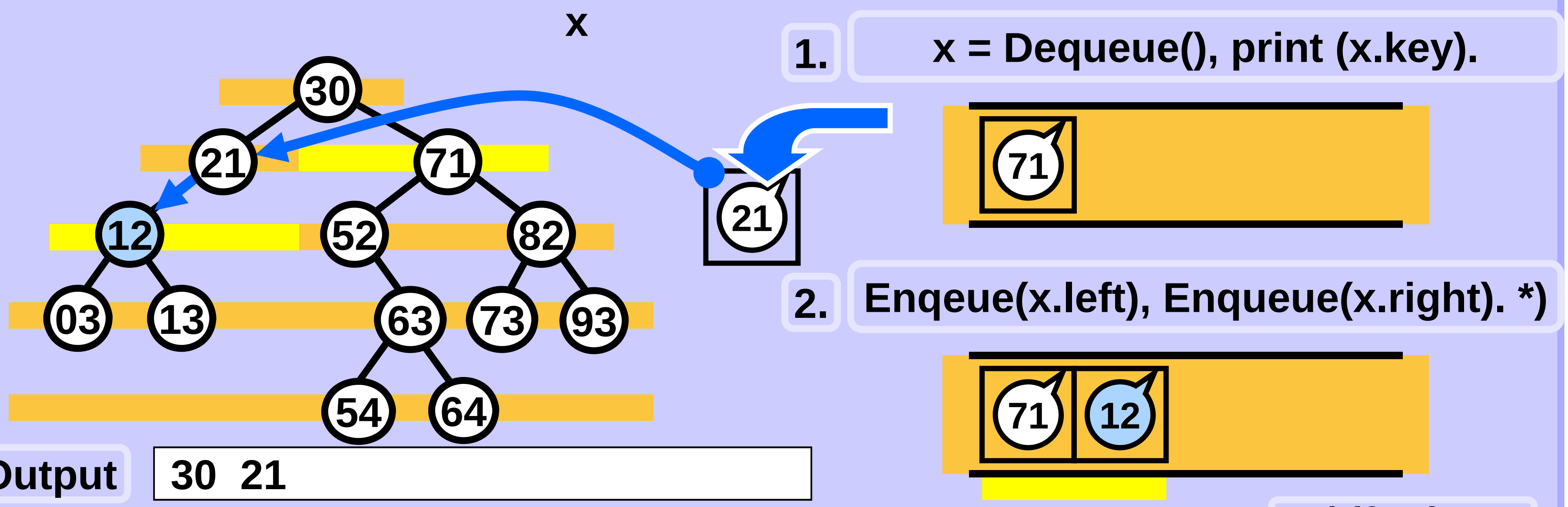
Breadth-first search (BFS) in a tree



1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30



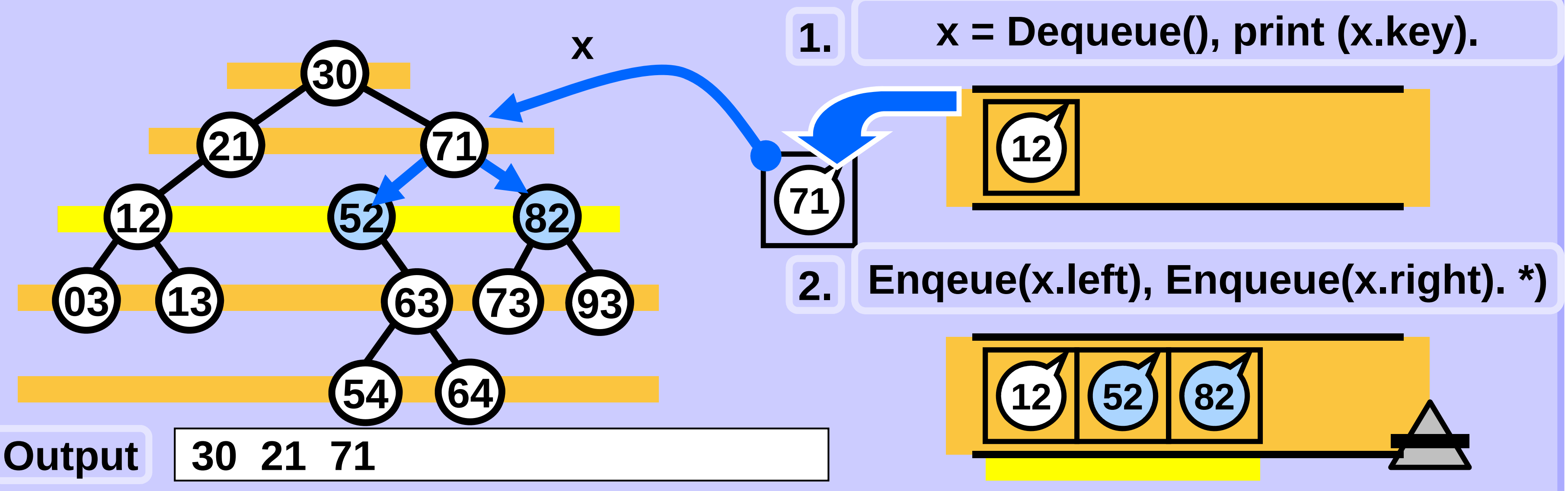
1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21

*) if exists

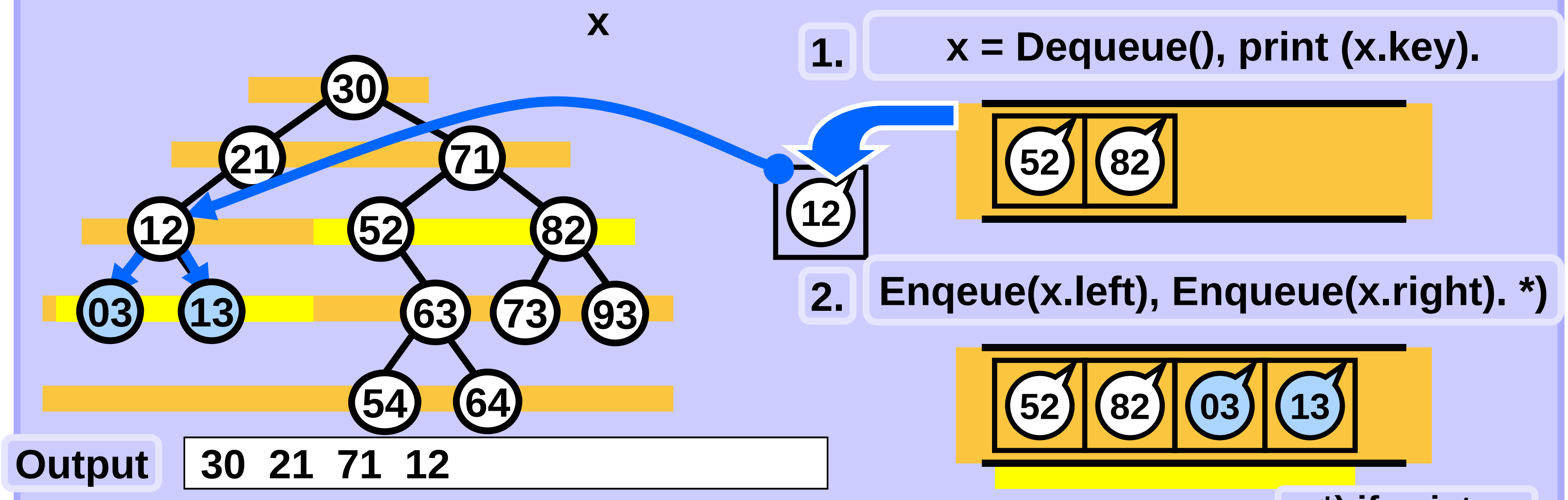
Breadth-first search (BFS) in a tree



1\. x = Dequeue(), print (x.key).

2\. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71



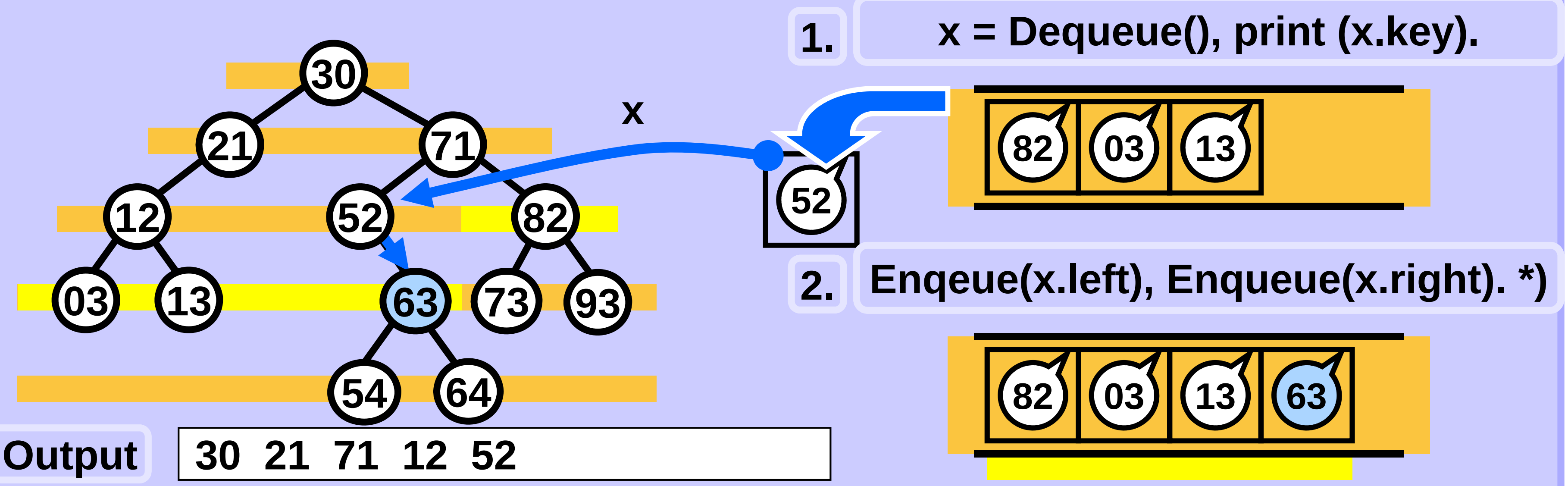
1\. x = Dequeue(), print (x.key).

2\. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12

*) if exists

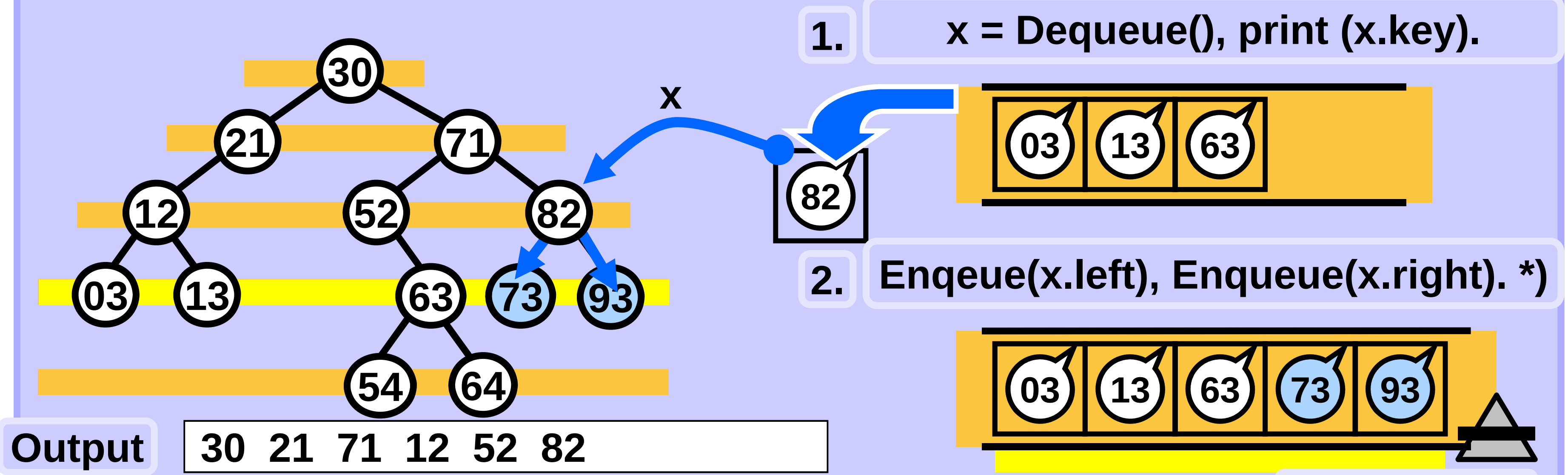
Breadth-first search (BFS) in a tree



1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52



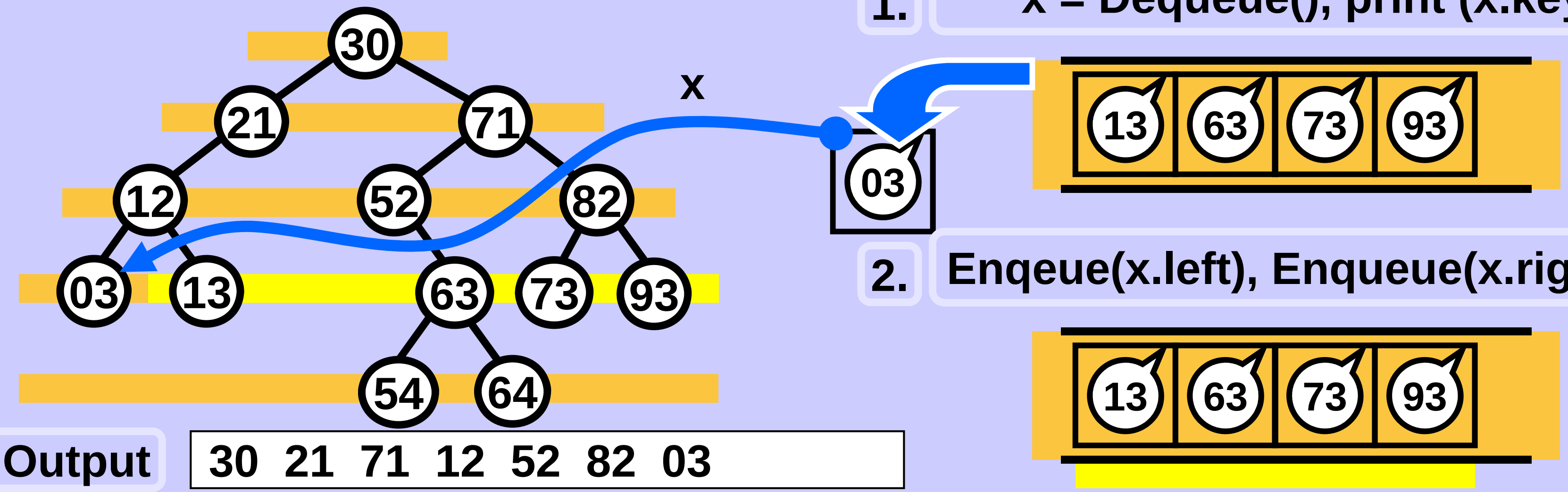
1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82

*) if exists

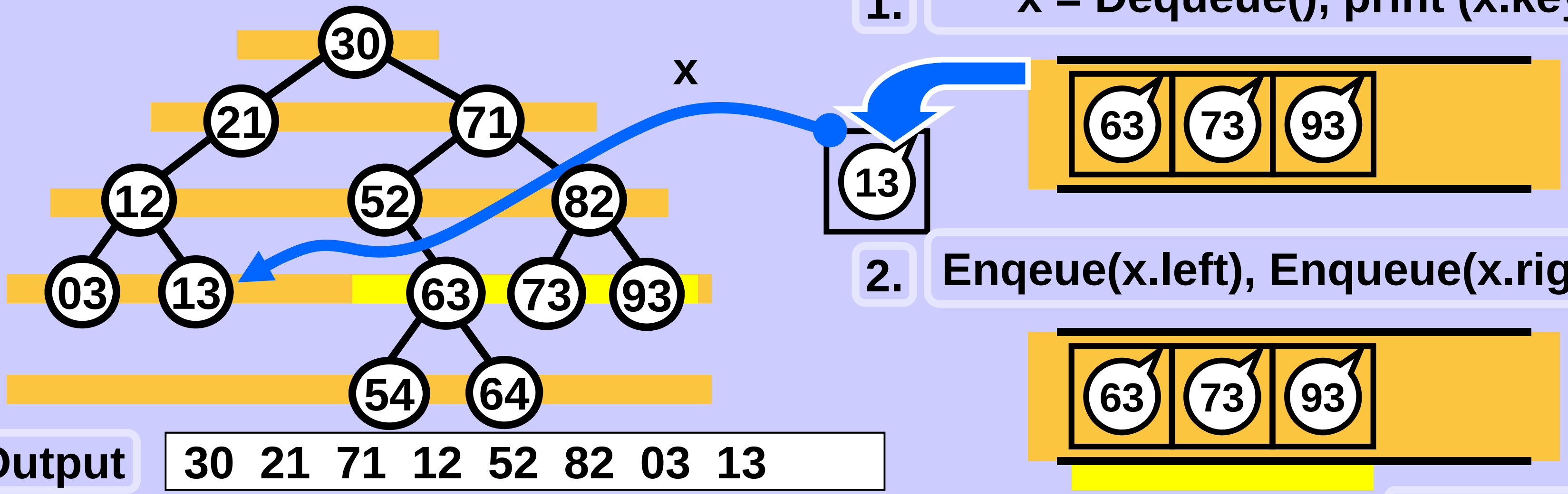
Breadth-first search (BFS) in a tree



1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03



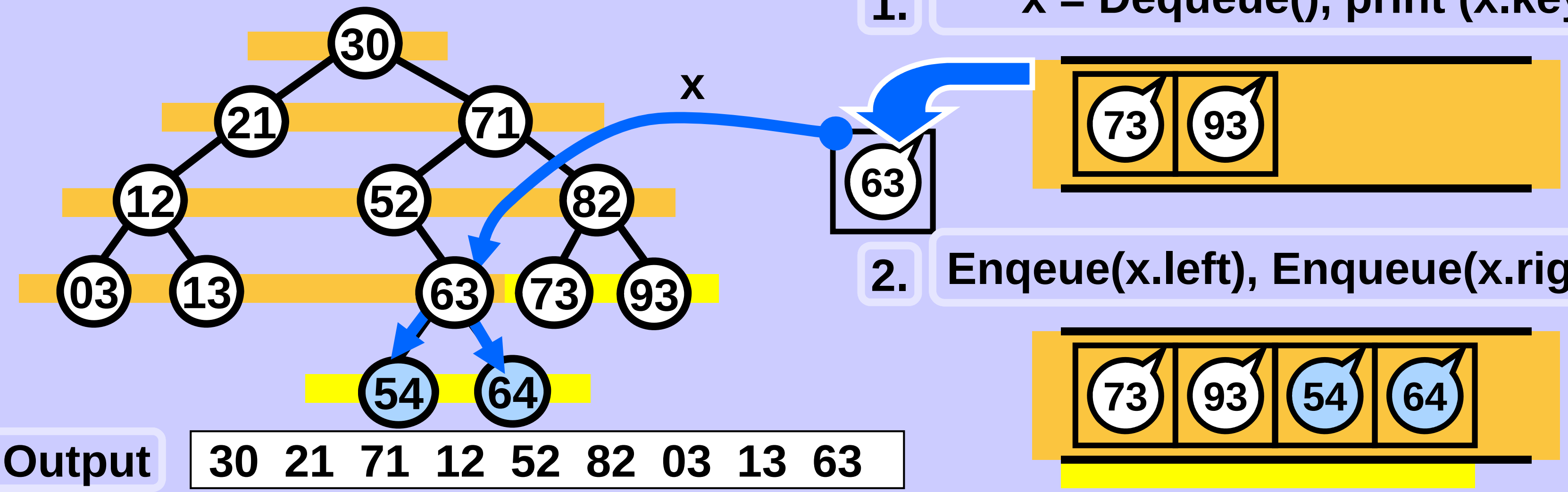
1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03 13

*) if exists

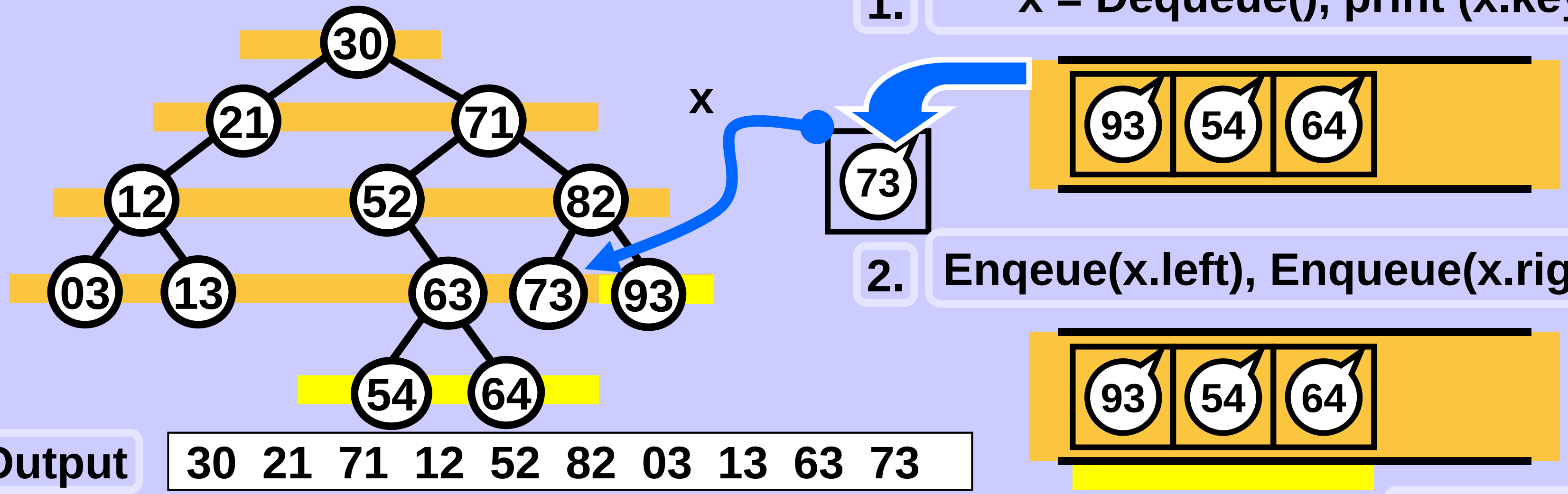
Breadth-first search (BFS) in a tree



1\. x = Dequeue(), print (x.key).

2\. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03 13 63



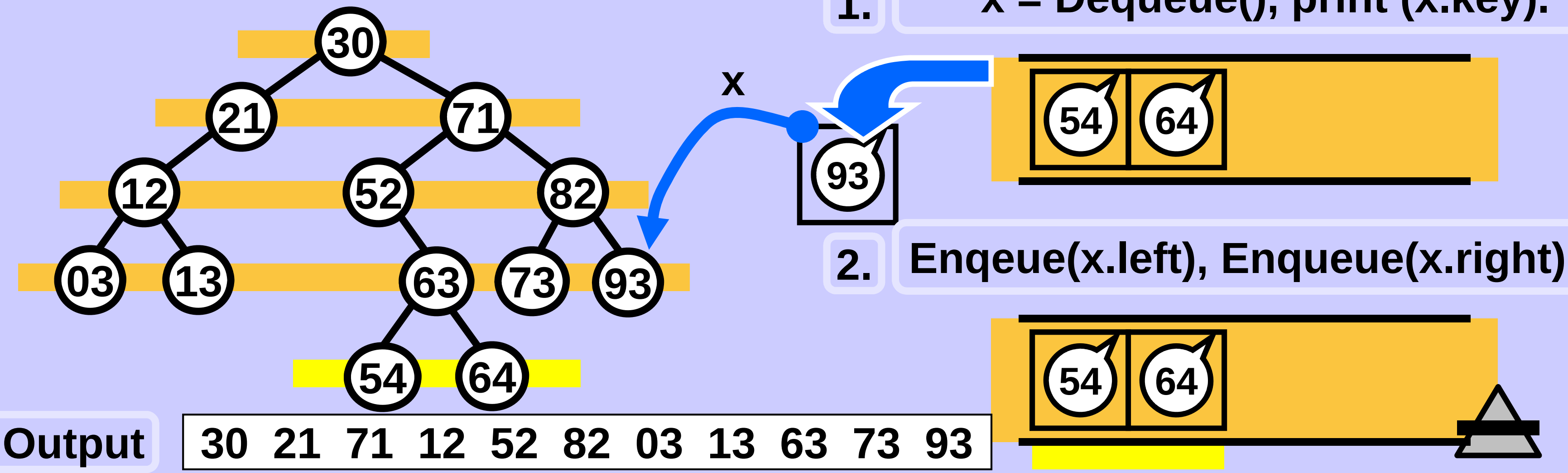
1\. x = Dequeue(), print (x.key).

2\. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03 13 63 73

*) if exists

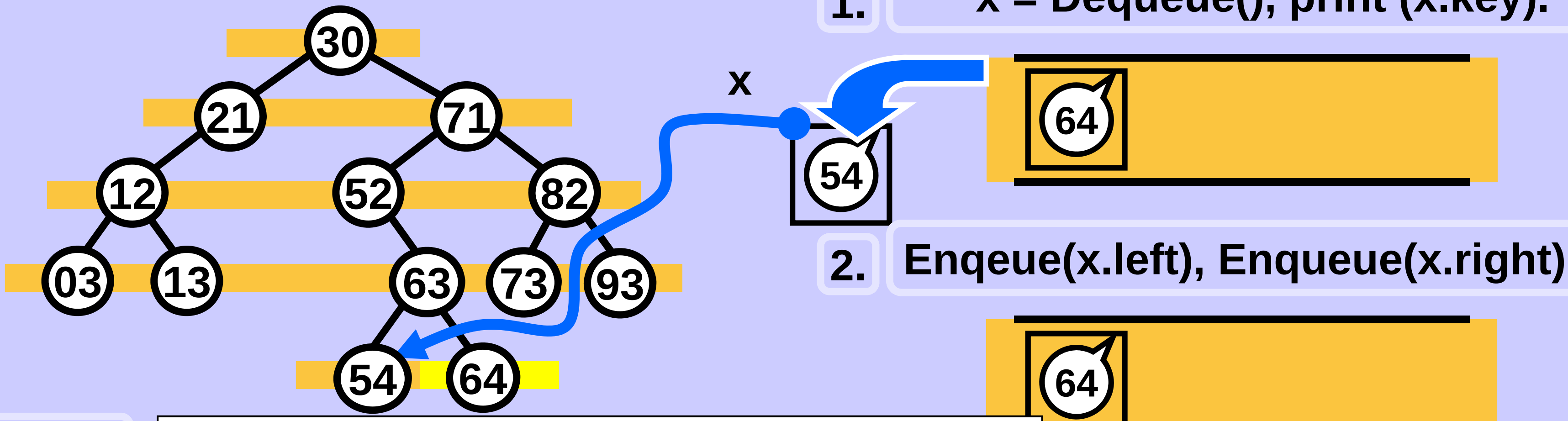
Breadth-first search (BFS) in a tree



1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03 13 63 73 93



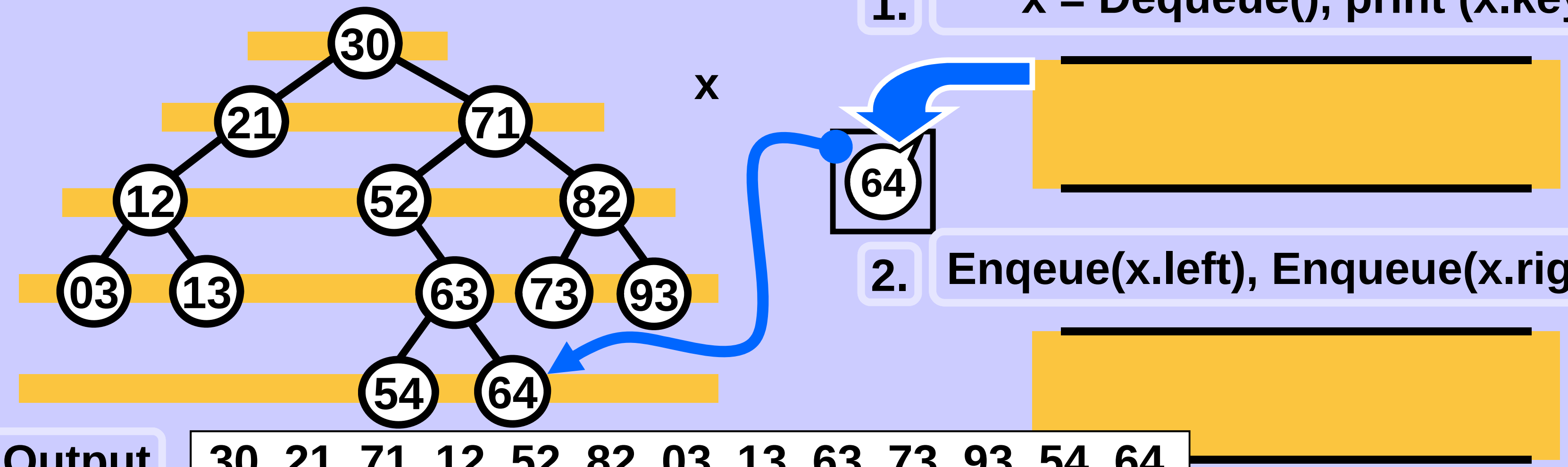
1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03 13 63 73 93 54

*) if exists

Breadth-first search (BFS) in a tree



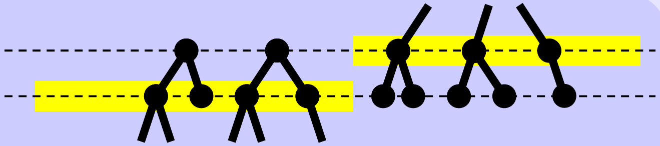
1. x = Dequeue(), print (x.key).

2. Enqeue(x.left), Enqueue(x.right). *)

Output 30 21 71 12 52 82 03 13 63 73 93 54 64

*) if exists.

The queue is empty,
BFS is complete.



An unempty **queue** always contains exactly
-- some (or all) nodes of one level and
-- all children of those nodes of this level which have already left the queue.

Sometimes the queue contains just nodes of one level. See above: ▲

Breadth-first search (BFS) in a tree

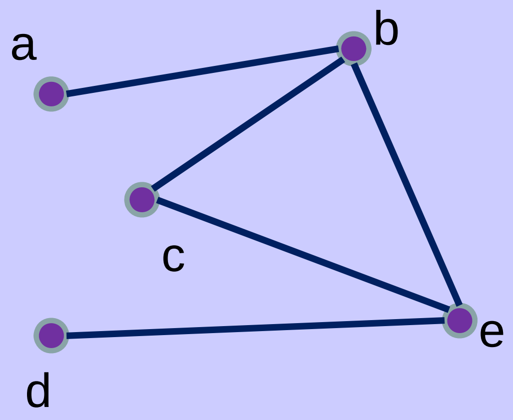
```
def binaryTreeBFS(node):
    if node == None: return
    q = Queue(100)                    # init
    q.Enqueue(node)                   # root into queue
    while (not q.isEmpty()):
        node = q.Dequeue()
        print(node.key, end = ' ')    # process node
        if node.left != None: q.Enqueue(node.left)
        if node.right != None: q.Enqueue(node.right)
```

Graphs

- **Graph is an ordered pair of**
- **set of vertices (nodes) $\mathcal{V}$ and**

  set of pairs of vertices $\mathcal{E}$.

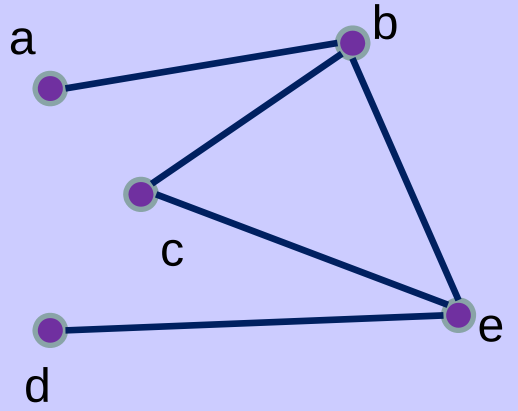
  Each pair is an edge.
- $\mathcal{G} = (\mathcal{V}, \mathcal{E})$
- **Example:**
  - **V = {a, b, c, d, e}**
  - **E = {{a,b},{b,e},{b,c},**
    {c,e},{e,d}}

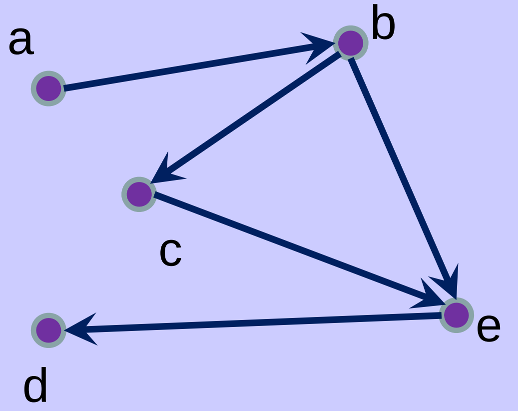


Graphs - directed/undirected

- **Undirected graph**
  - **An edge is an <u>unordered</u> pair of vertices.**
- **E = {{a,b},{b,e},{b,c}, {c,e},{e,d}}**

- **Directed graph**
  - **An edge is an <u>ordered</u> pair of vertices.**
- **E = {{a,b},{b,e},{b,c}, {c,e},{e,d}}**





Graph – adjacency matrix

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be graph with n vertices
- Denote vertices $v_1, \ldots, v_n$ (in an arbitrary order)
- Adjacency matrix of G is a matrix of order n

$$A_G = (a_{i,j})_{i,j=1}^{n}$$

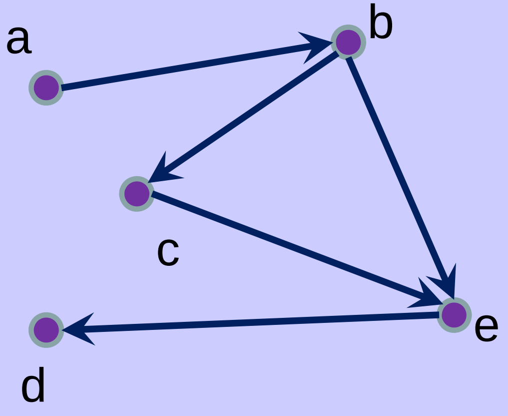
defined by the relation

$$a_{i,j} = \begin{cases} 1 & \mathit{for}\ \{v_i, v_j\} \in E \\ 0 & \mathit{otherwise} \end{cases}$$

Graph – adjacency matrix

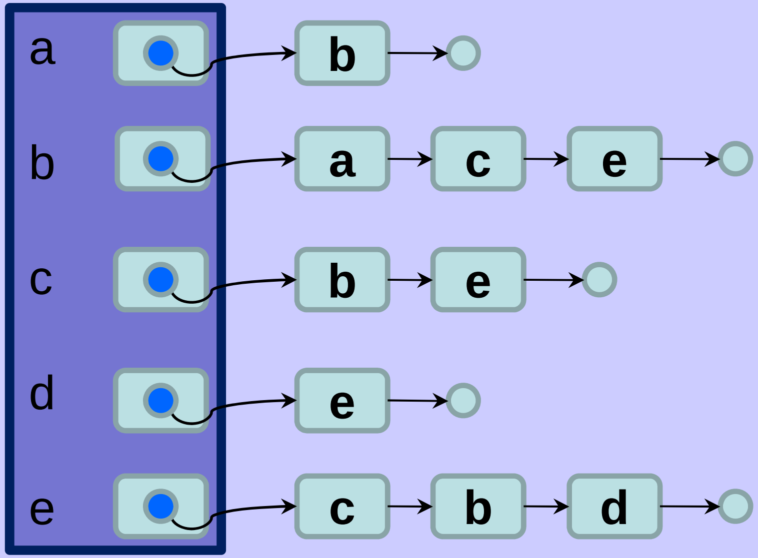
- **Directed graph example**

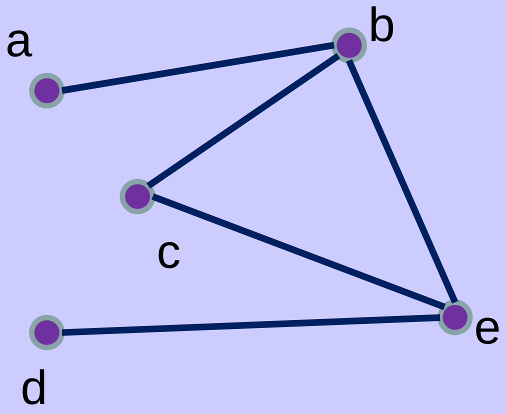
	a	**b**	**c**	**d**	**e**
a	0	1	0	0	0
b	0	0	1	0	1
c	0	0	0	0	1
d	0	0	0	0	0
e	0	0	0	1	0



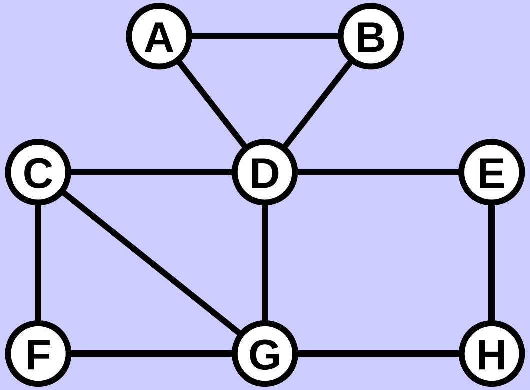
Graph – list of neighbours

- Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an (un)directed graph with n vertices.
- Denote vertices $v_1, \ldots, v_n$ (in an arbitrary order).
- <u>List of neighbours of</u> $\mathcal{G}$ is an array $\mathcal{P}$ of size n of pointers.
  - $\mathcal{P}[i]$ points to the list of all vertices which are adjacent to v_i.

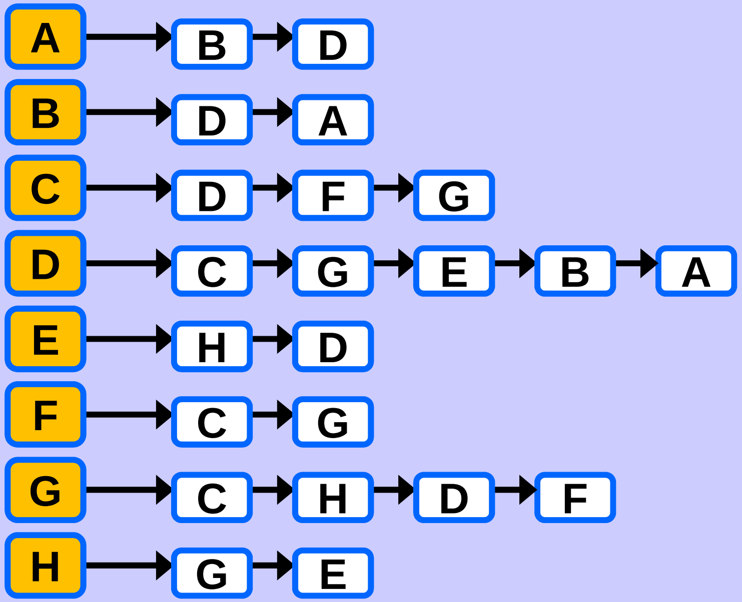




Graph most ususal representations



Linked list representation

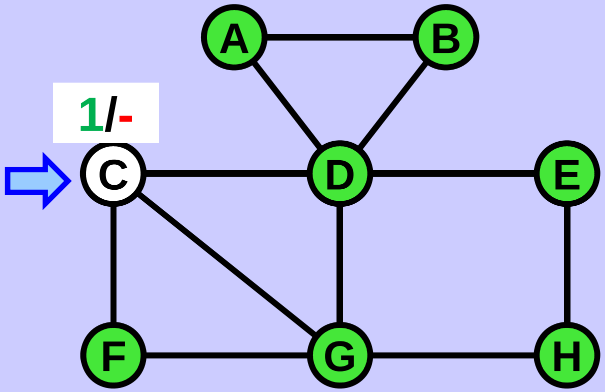


A → B → D
B → D → A
C → D → F → G
D → C → G → E → B → A
E → H → D
F → C → G
G → C → H → D → F
H → G → E

Adjacency matrix

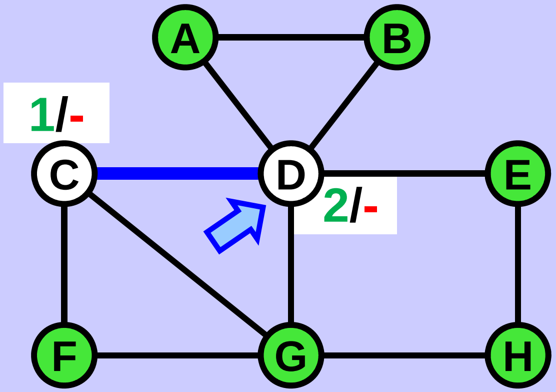
	A	B	C	D	E	F	G	H
A	0	**1**	0	**1**	0	0	0	0
B	**1**	0	0	**1**	0	0	0	0
C	0	0	0	**1**	0	**1**	**1**	0
D	**1**	**1**	**1**	0	**1**	0	**1**	0
E	0	0	0	**1**	0	0	0	**1**
F	0	0	**1**	0	0	0	**1**	0
G	0	0	**1**	**1**	0	**1**	0	**1**
H	0	0	0	0	**1**	0	**1**	0

Depth-first search (DFS) in a graph



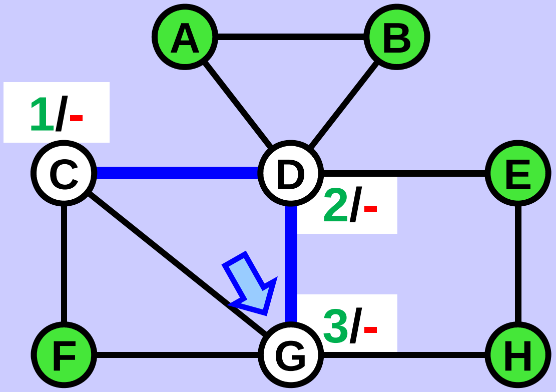
Stack C

Output C



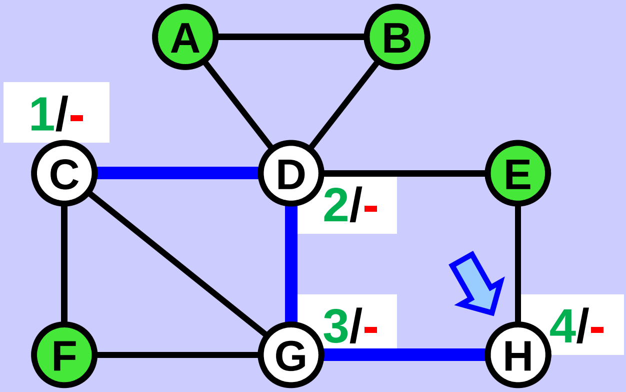
Stack C D

Output C D



Stack C D G

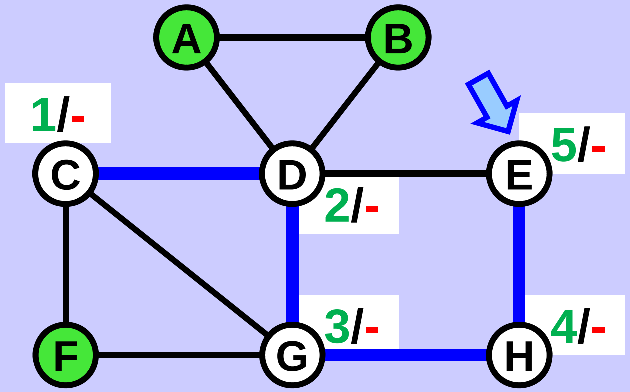
Output C D G



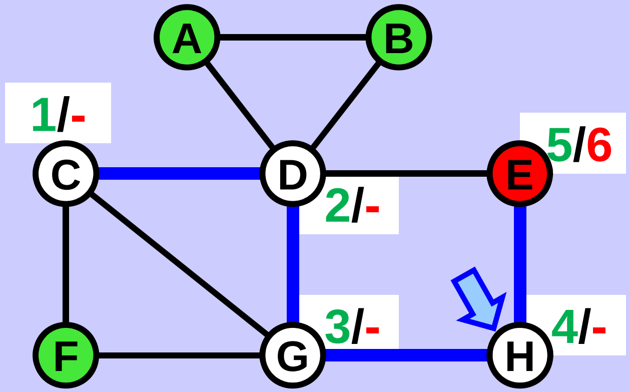
Stack C D G H

Output C D G H

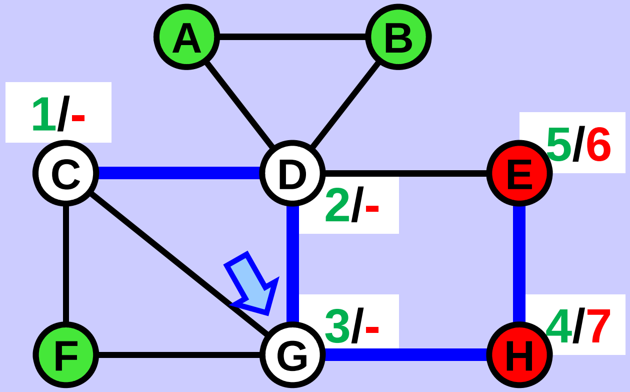
Depth-first search (DFS) in a graph



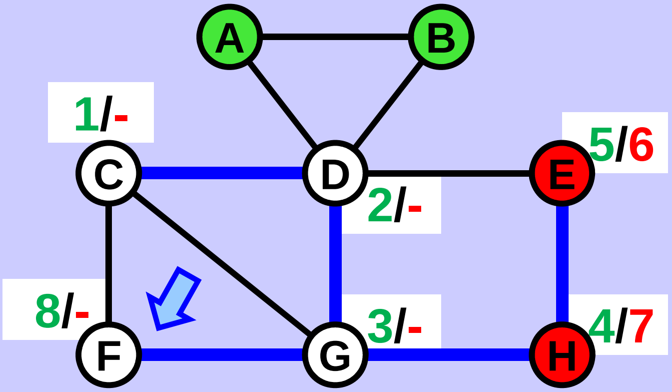
Stack	C D G H E
Output	C D G H E



Stack	C D G H
Output	C D G H E

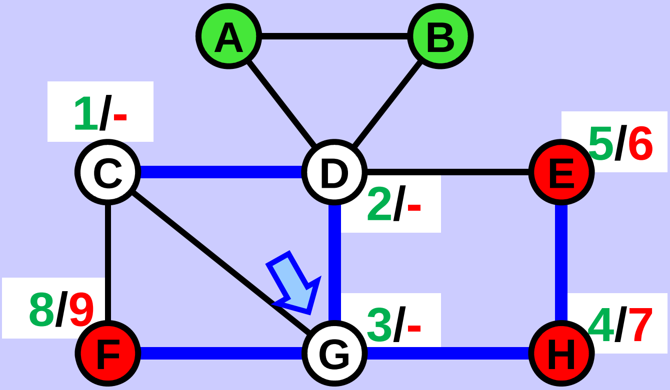


Stack	C D G
Output	C D G H E

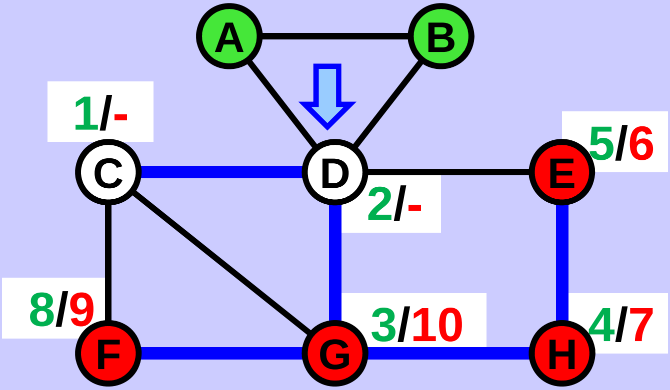


Stack	C D G F
Output	C D G H E F

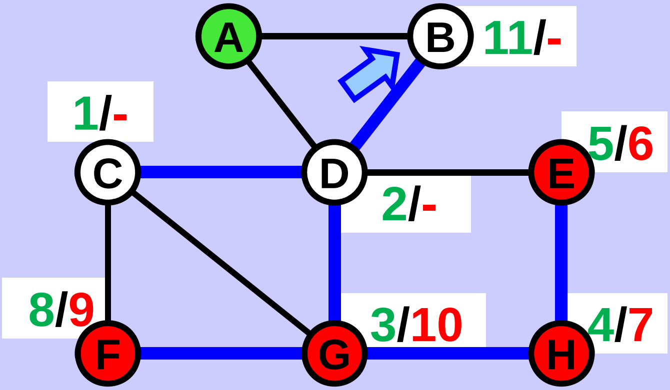
Depth-first search (DFS) in a graph



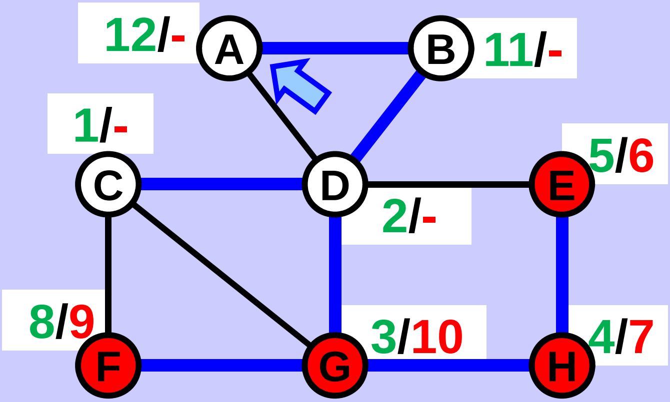
Stack	C D G
Output	C D G H E F



Stack	C D
Output	C D G H E F

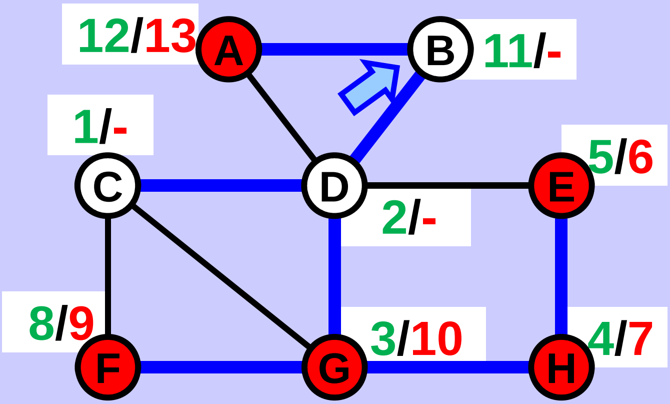


Stack	C D B
Output	C D G H E F B

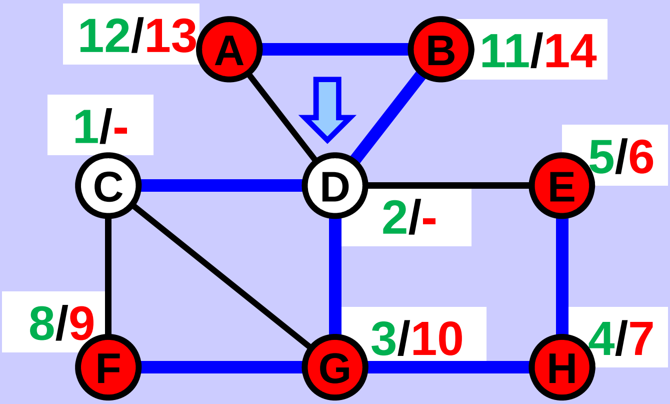


Stack	C D B A
Output	C D G H E F B A

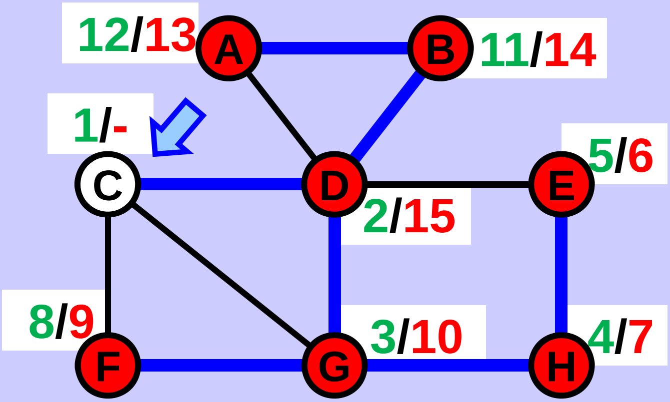
Depth-first search (DFS) in a graph



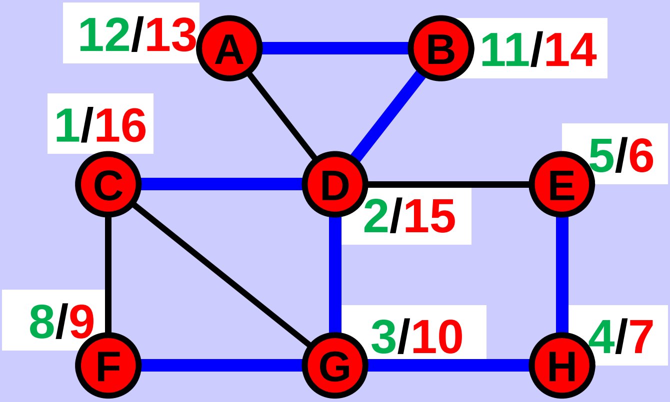
Stack	C D B
Output	C D G H E F B A



Stack	C D
Output	C D G H E F B A



Stack	C
Output	C D G H E F B A



Stack	
Output	C D G H E F B A

Depth-first search (DFS) in a graph

Life cycle of a node during the DFS
Fresh - open - closed

Fresh

Fresh nodes are those nodes which have not been visited yet.
Before the search starts, all nodes are fresh.
A fresh node becomes open when it it visited for the first time.
The set of fresh nodes shrinks or remains the same during the search.

Open

Open nodes are those nodes which have been already visited but were not closed yet.
The set of open nodes may grow and shrink during the search.

Closed

Closed nodes are those nodes which will not be visited any more.
When each neighbour of a current node in the search is either open or closed current node becomes closed.
The set of closed does only grow during the search.
When the search terminates all nodes are closed.

Depth-first search (DFS) in a graph

Implementation remark

Fresh: A fresh node is assigned no time (neither open nor closed).
Open: An open node is assigned open time and no close time.
Closed: A closed node is assigned both open and close times.

In some implementations, it is not necessary to produce the open and close times.

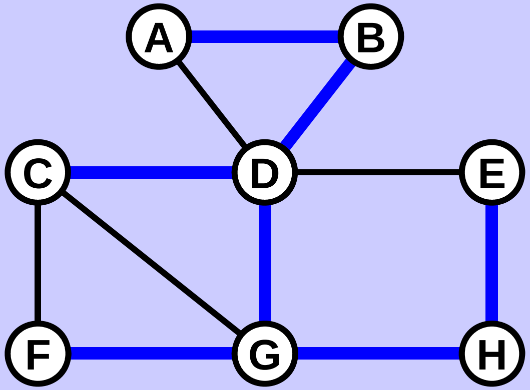
However, it is always necessary to register explicitely the state of each node -- fresh/closed. Open nodes are then those ones which were not closed yet and are still on the stack.

In the recursive variant of DFS, each recursive call corresponds to a single node processing including all visits to this node. The node becomes open when the node is the actual parameter of the current recursive function call. The node becomes closed when the same call terminates.
The neighbours of the node are checked one by one in the body of the function and the fresh ones becomes the parameters of the recursive calls. Therefore, it is enough to register only one-bit information in each node: Fresh or not fresh.

Depth-first search (DFS) in a graph

Stack contents

C
C D
C D G
C D G H
C D G H E
C D G H
C D G
C D G F
C D G
C D
C D B
C D B A
C D B
C D
C
--



Printing the node when the node becomes open results in the sequence

C D G H E F B A

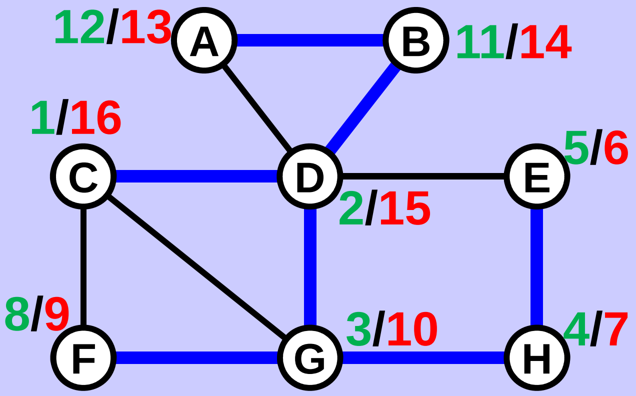
Printing the node when the node becomes closed results in the sequence

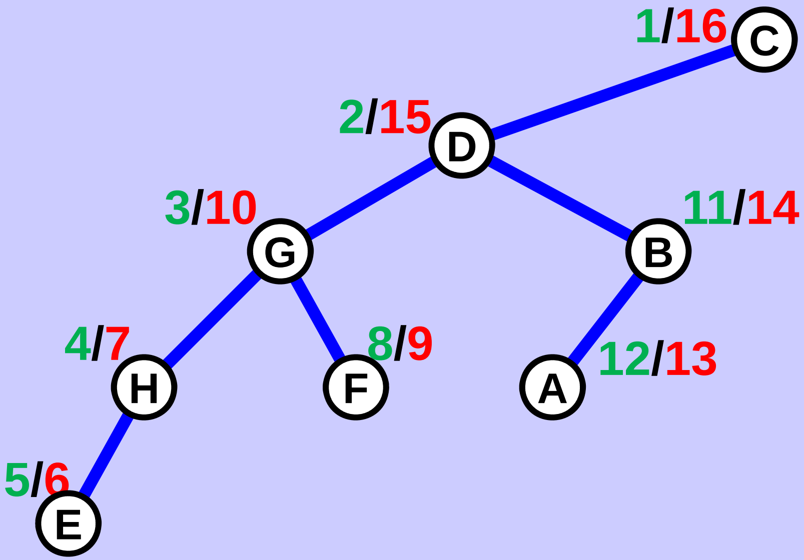
E H F G A B D C

Processing a node when it becomes closed is used in the algorithms of
-- bridges and cutvertices detection in undirected graphs
-- strongly connected components detection in directed graphs.

Depth-first search (DFS) in a graph

DFS-tree
with open and close times of
the nodes





Note, that in a subtree with a root X it holds for each node Y ≠ X:
Open_time(X) < Open_time(Y) < Close_time(Y) < Close_time(X).
On the other hand, when Y is not a part of the subtree rooted in X then
Close_time(X) < Open_time(Y) or Close_time(Y) < Open_time(X)

The nuber of nodes in the subtree rooted in X is always
(Close_time(X) + 1 – Open_time(X)) / 2.

Depth-first search (DFS) in a graph -- iteratively

```
def DFS( graph ):
    visited = [False] * graph.size
    stack = Stack()
    stack.push( graph.nodes[0] ) # start search in node 0
    visited[0] = True
    while not stack.isEmpty():
        node = stack.pop()
        print(node.id, end = " ") # process the node
        for neigh in node.neighbours:
            if not visited[neigh.id]:
                stack.push( neigh )
                visited[neigh.id] = True
```

Depth-first search (DFS) in a graph -- recursively

```
def DFSrec( node, visited ):
    visited[node.id] = True
    print( node.id, end = " " )   # process the node
    for neigh in node.neighbours:
        if visited[neigh.id] == False:
            DFSrec( neigh, visited )

def DFSrecRun( graph ):
    visited =  [False] * graph.size
    DFSrec( graph.nodes[0], visited )
```

Breadth-first search (BFS) in a graph

Life cycle of a node during BFS is conceptually identical to the node lifecycle during DFS.

Fresh

Fresh nodes are those nodes which have not been visited yet.
Before the search starts, all nodes are fresh.
A fresh node becomes open when it it visited for the first time.
The set of fresh nodes shrinks or remains the same during the search.

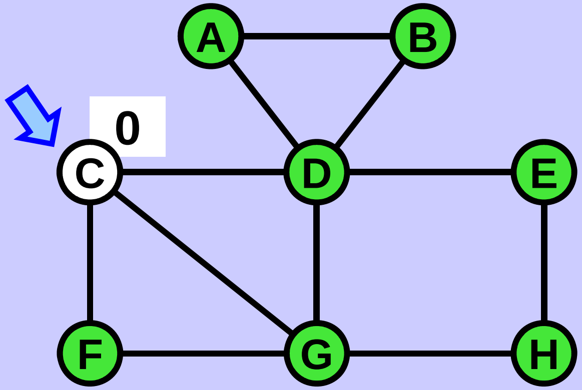
Open

Open nodes are those nodes which have been already visited but were not closed yet.
The set of open nodes may grow and shrink during the search.

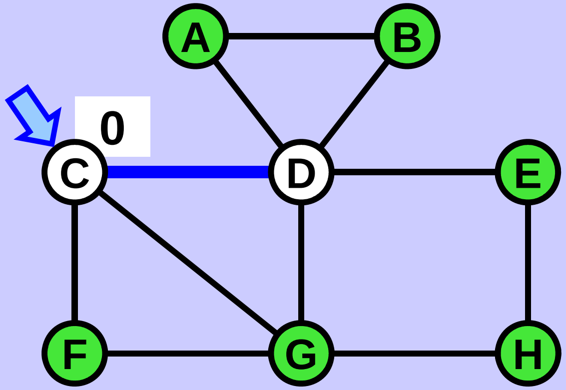
Closed

Closed nodes are those nodes which will not be visited any more.
When each neighbour of a current node in the search is either open or closed current node becomes closed.
The set of closed does only grow during the search.
When the search terminates all nodes are closed.

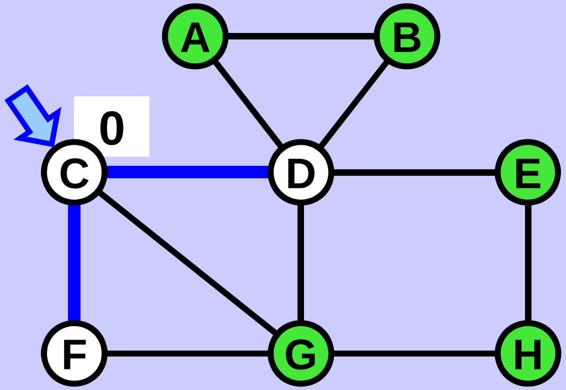
Breadth-first search (BFS) in a graph



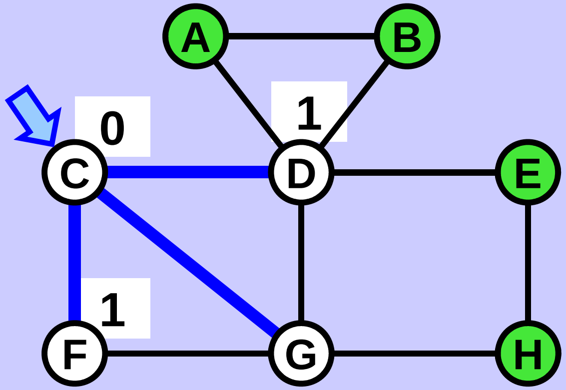
Queue	C
Output	C



Queue	D
Output	C

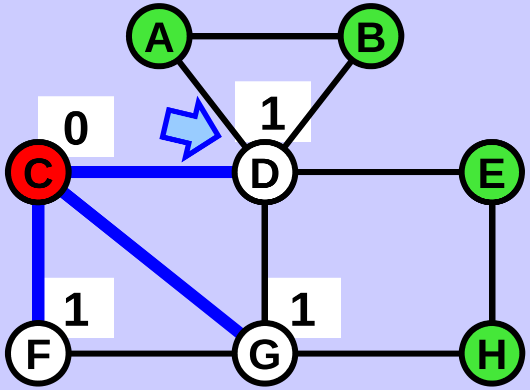


Queue	D F
Output	C

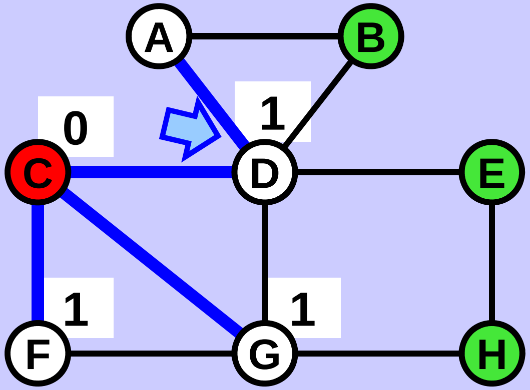


Queue	D F G
Output	C

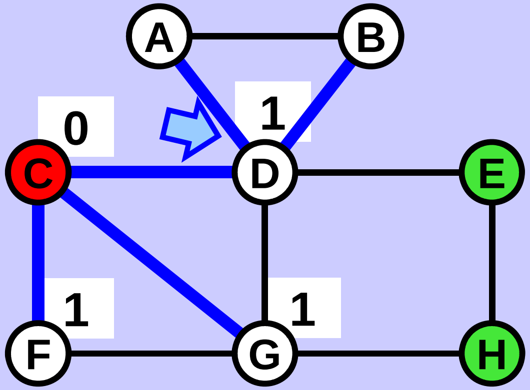
Breadth-first search (BFS) in a graph



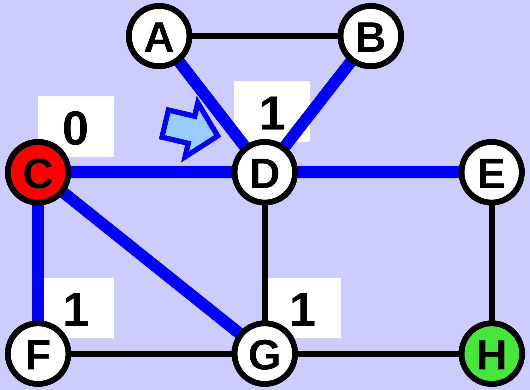
Queue	D F G
Output	C D



Queue	F G A
Output	C D

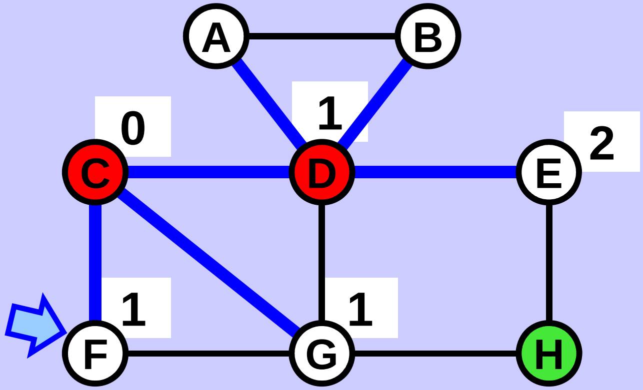


Queue	F G A B
Output	C D



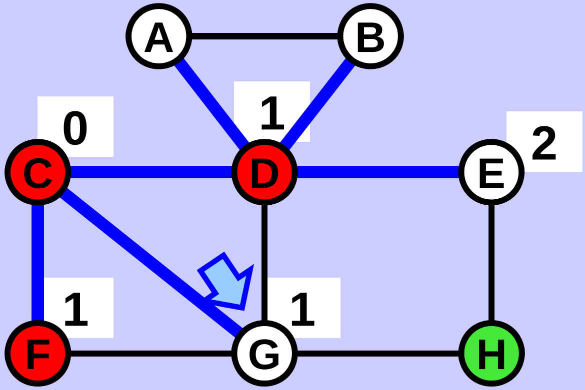
Queue	F G A B E
Output	C D

Breadth-first search (BFS) in a graph



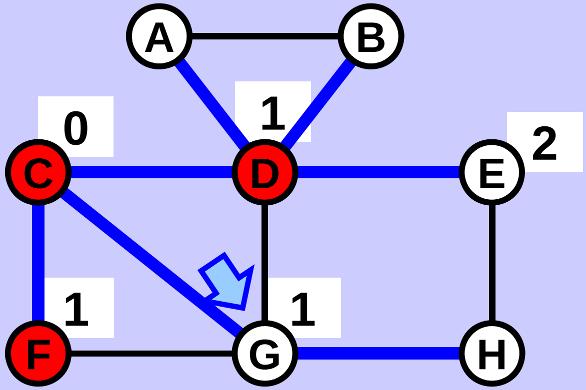
Queue F G A B E

Output C D F



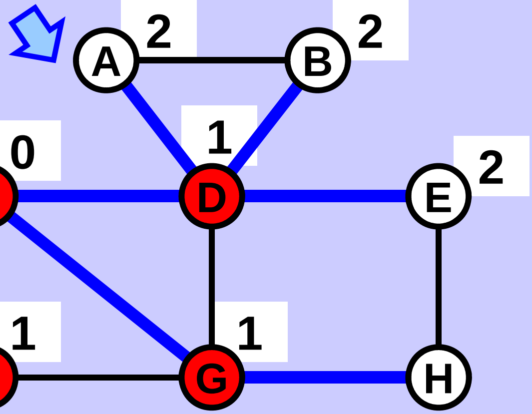
Queue G A B E

Output C D F G



Queue A B E H

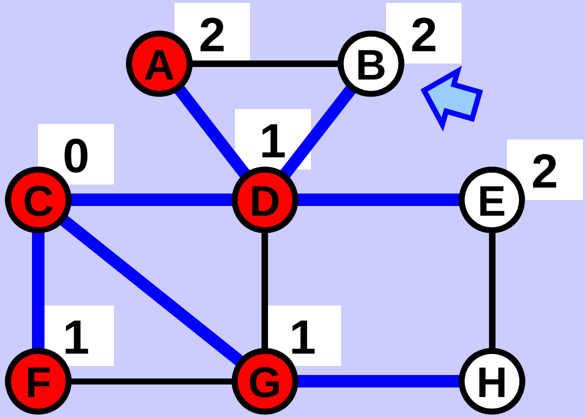
Output C D F G



Queue A B E H

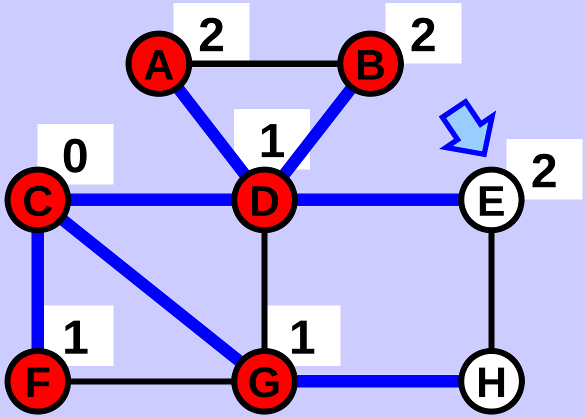
Output C D F G A

Breadth-first search (BFS) in a graph



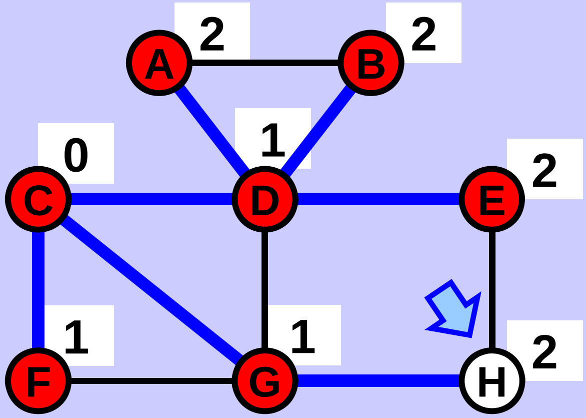
Queue B E H

Output C D F G A B



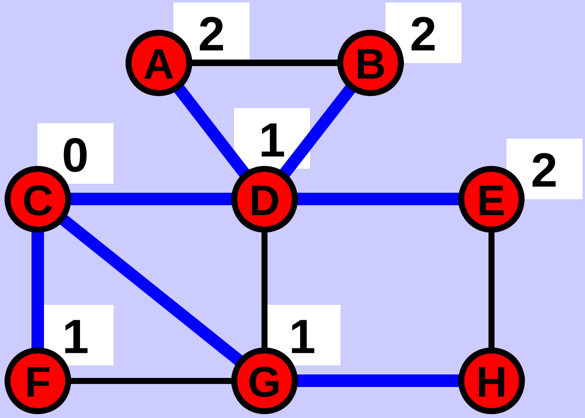
Queue E H

Output C D F G A B E



Queue H

Output C D F G A B E H

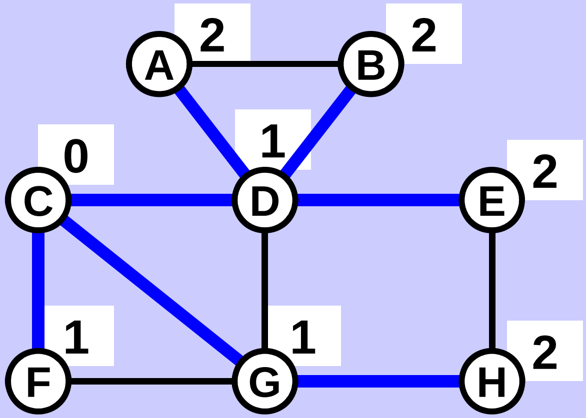


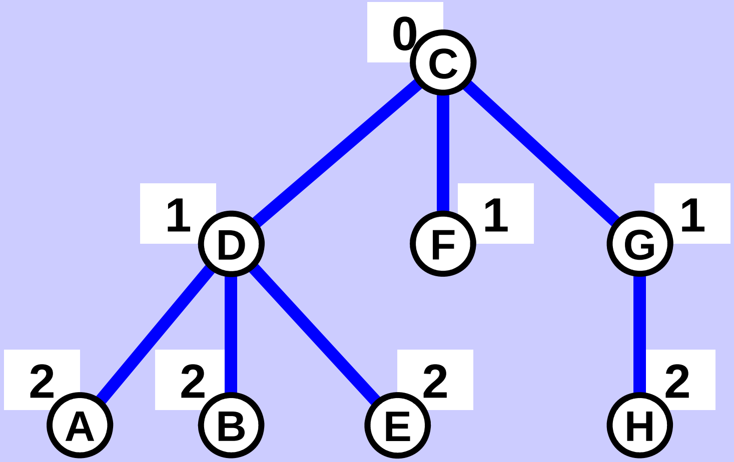
Queue

Output C D F G A B E H

Breadth-first search (BFS) in a graph

BFS-tree
with distances to the start node (root)
of all nodes





The open and close times are not essential in BFS.

The node depth in the BFS tree is equal to its distance from the start node in BFS.

BFS algorithm is exploited in e.g.:
Testing of graph connectivity, testing existence of a cycle in a graph, testing if a graph is bipartite, etc.
Typically BFS is used to compute distance(s) from a given node to either one other node or to all other nodes.

Breadth-first search (BFS) in a graph

Implementation remark

Fresh: A fresh node is assigned no distance from the start node.
Open: An open node is assigned a distance from the start node and it is in the queue.
Closed: A closed node is assigned a distance from the start node and it is not in the queue.

It is not necessary to register explicitely fresh/open/closed state of the nodes.
The contents of the queue and the distance (assigned / not assigned) define unambiguously the node state.

BFS is an iterative process a recursive variant is not used.
(A recursive implementation would be more artificial and less clear.)

Breadth-first search (BFS) in a graph

```
def BFS(graph):
    visited = [False] * graph.size
    queue = Queue(200)
    queue.Enqueue(graph.nodes[0])
    visited[0] = True
    while not queue.isEmpty():
        node = queue.Dequeue()
        print(node.id, end = " ")  # process node
        for neigh in node.neighbours:
            if not visited[neigh.id]:
                queue.Enqueue(neigh)
                visited[neigh.id] = True
```

Node distances by BFS in a graph

```python
def BFSdist( graph ):
  visited = [False] * graph.size
  dist = [9999999999999] * graph.size # infinity == 99...9
  queue = Queue( graph.size )
  queue.Enqueue( graph.nodes[0] )     # start in node 0
  visited[0] = True
  dist[0] = 0
  while not queue.isEmpty():
    node = queue.Dequeue()
    print( node.id, end = " " )       # process node
    for neigh in node.neighbours:
      if not visited[neigh.id]:
        queue.Enqueue( neigh )
        visited[neigh.id] = True
        dist[neigh.id] = dist[node.id]+1

    print ( dist )  # process the distances or return, etc.
```

Breadth-first and Depth-first search (BFS & DFS) in a graph

Asymptotic complexity

Each single operation on the queue/stack and each single operation on additional data structures and nodes/edges is of constant time (and memory) complexity.

Each node enters the queue/stack only once and it leaves the queue/stack only once. The state of the node (fresh/open/closed) is tested more times. The number of these tests is equal to the degree of the node (the search tries to access the node from its neighbours).

The sum of all node degrees is equal to twice the number of edges, in any graph.

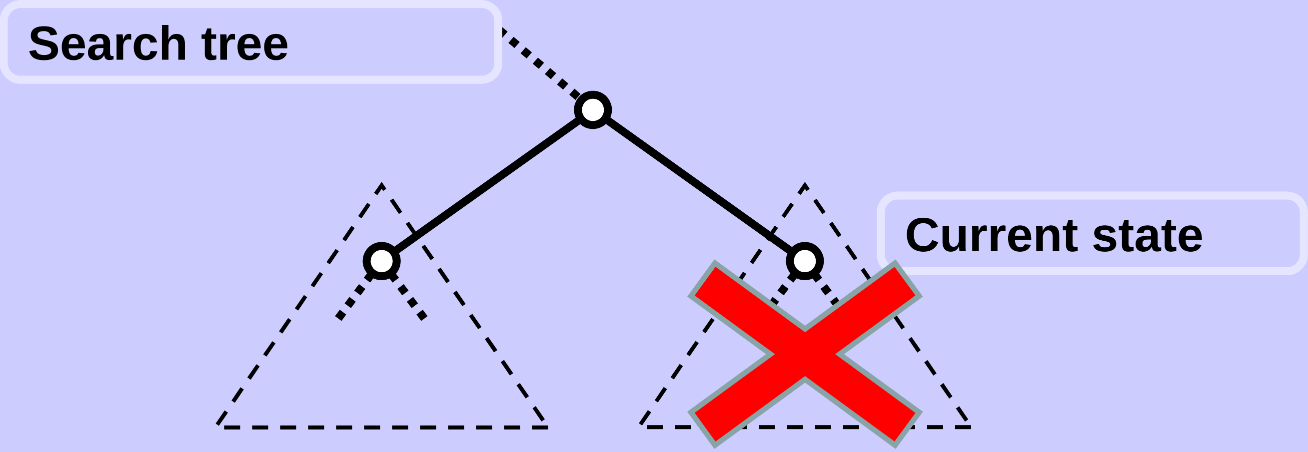
In total

$$\Theta(\ |V| + |E|\ ).$$

Search pruning

- Search speedup
- Pruning (skipping) of unpromising possibilities
- When the analyse of the current state reveals that
  - it is an unpromising state
  - surely it does not lead to the solution

  - we "cut off" (prune) the whole subtree of states of which the current state is the root



Search tree

Current state

Pruning example – magic square

- Magic square of order $\mathcal{N}$
  - square matrix of order $\mathcal{N}$
  - contains exactly once each value from 1 to $\mathcal{N}^2$
  - sum of all rows and all columns is the same
- Example

2	9	4
7	5	3
6	1	8

- Brute force approach: Generate all possible permutations of positions of numbers from 1 to $\mathcal{N}^2$
- Pruning: Whenever the sum of the row or column is not correct:
  - sum of all values in the square is $\frac{1}{2}\mathcal{N}^2(\mathcal{N}^2+1)$
  - sum of all values in a row or column is $\frac{1}{2}\mathcal{N}(\mathcal{N}^2+1)$

Search pruning heuristics

- **<u>Heuristic</u> is a hint which tells us which order of actions is *<u>likely</u>* to produce quickly the solution.**
- **The effectivity of the solution is *<u>not guaranteed</u>*.**
- **Heuristics can be used to asses the order of vertices/edges/paths in which they are processed during the search in large graphs.**

- **Example: Knight tour on an $N \times N$ chessboard (visit all fields).**
- **Good heuristic: Explore first those fields from which there are fewest possibilities of continuing the tour in different directions.**
- **Speedup on the 8×8 chessboard: Almost 100 000 times.**