Normal-Form Games

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Previously on multi-agent systems (tutorials and lectures).

1. Agent architectures.
2. Reactive agents.
3. BDI, logics.

Now, game theory.

Game Theory – Definition

Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making. These are situations involving several decision makers (called players) with different goals, in which the decision of each affects the outcome for all the decision makers. This interactivity distinguishes game theory from standard decision theory, which involves a single decision maker, and it is its main focus. Game theory tries to predict the behavior of the players and sometimes also provides decision makers with suggestions regarding ways in which they can achieve their goals.

Michael Maschler

Why Game Theory? – Relevance

A wide range of applications in fields such as:

- Economics (markets, auctions).
- Political science (government coalitions, diplomacy, voting methods).
- Military applications (missile pursuit strategies, strategic analysis).
- Cybersecurity (intrusion detection and prevention, privacy preservation and anonymity, cyber attack-defense analysis).
- Philosophy (insights into concepts related to morality and social justice).
- Biology (evolutionary stable strategies).
- Computer science (algorithmic game theory, algorithmic mechanism design, multi-agent systems).

Relevant Literature

Multiagent Systems: Algorithmic, Game-Theoretic, and Logical Foundations, *by Yoav Shoham, Kevin Leyton-Brown*. Available online at `www.masfoundations.org`.

Game Theory, *by Michael Maschler, Eilon Solan, Shmuel Zamir*.

Normal-Form Game

Definition (Normal-Form Game)

A normal-form game (NFG) is a triplet $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$, where:

- $\mathcal{N}$ is a finite set of players.
- $\mathcal{A}_i$ is a finite set of actions ($\mathcal{S}_i$ when speaking of strategy profiles) for player i.
- u_i is a utility function of player i that assigns the reward for joint action $a \in \mathcal{A}$, $a = (a_1, a_2, \ldots, a_{\mathcal{N}})$ to player i.

Consider the set of two players $\mathcal{N} = \{1, 2\}$. We distinguish:

- **Zero-sum games** where $u_1(a) + u_2(a) = 0$, $\forall a \in \mathcal{A}$.
- **General-sum games** where $u_1(a) + u_2(a) = c$, $c \in \mathbb{R}$, $\forall a \in \mathcal{A}$.

Formalizing a game in normal form

A student is at a lecture and he / she wants to go home in tram 22. There are 10 people in the tram, and another 10 people are waiting at the tram stop. The goal of the tram is to carry as many people as possible. We know that no one will be exiting at this stop. The tram stops at either (A), right next to the beginning of the tram stop, or (B), further away (in the middle of the tram stop). The tram can also skip the stop altogether (C). When the tram stops, it opens the door only once. At the same time, the student must decide where at the stop he / she will wait. If the student guesses the position of the tram correctly, he / she will get a utility of 1. If the student guesses the position of the tram incorrectly, he / she will get a utility of $\frac{1}{2}$, as he / she needs to go through the crowd of people to catch the tram. If the student misses the tram, his / her utility will be -1.

Task 1: Formalize the game as a NFG.

Utility Theory

Each agent has his own description of which states of the world he likes. Utility theory deals formally with this aspect.

Definition (Preference relation)

A preference relation of player i over a set of outcomes O is a binary relation denoted by $\succeq$.

Definition (Utility function)

Let O be a set of outcomes and $\succeq$ be a complete, reflexive and transitive preference relation over O. A function $u : O \mapsto \mathbb{R}$ is called a utility function representing $\succeq$ if $\forall x, y \in O$,

$$x \succeq y \iff u(x) \geq u(y).$$

Utility Theory

Considering uncertainty over outcomes (or lotteries), an analogous statement holds.

Theorem (von Neumann and Morgenstern, 1944)

If a preference relation $\succeq$ over lotteries is complete and transitive, and satisfies the four von Neumann-Morgenstern axioms (substitutability, decomposability, monotonicity, and continuity), then the preference relation can be represented by a linear utility function, and it holds that:

- $u(o_1) \geq u(o_2) \iff o_1 \succeq o_2$.
- $u([p_1 : o_1, \cdots, p_k : o_k]) = \sum_{i=1}^{k} p_i \cdot u(o_i)$.
  $\rightarrow$ *The utility over a lottery of outcomes is equal to the expected utility over outcomes.*

Formalizing a game in normal form

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Task 2: Imagine that the number of waiting people is a stochastic variable – there is a 50% chance that 10 people are waiting, and there is a 50% chance that no one else is waiting. How does the formal representation of the game change?

Solution concepts

Definition (Solution concept)

Let Γ be the class of all games. For each $G \in \Gamma$, let $\mathcal{S}_G$ be the set of strategy profiles of G. A solution concept is a function $F : \Gamma \mapsto \bigcup_{G \in \Gamma} 2^{\mathcal{S}_G}$, such that $F(G) \subseteq \mathcal{S}_G, \forall G \in \Gamma$.

Definition (Best response)

Player i 's best response (BR) to the strategy profile s_{-i} is a (mixed) strategy $s_i^* \in \mathcal{S}_i$ such that $u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$, for all strategies $s_i \in \mathcal{S}_i$.

Nash Equilibria

Definition (Nash Equilibrium)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a NFG. Strategy profile $s = (s_1, \cdots, s_n)$ is a Nash equilibrium (NE) if and only if $\forall i \in \mathcal{N}, s_i \in BR_i(s_{-i})$.

Definition (Strict NE)

A strategy profile $s = (s_1, \cdots, s_n)$ is a strict NE if, for all agents i and for all strategies, $s_i' \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

Definition (Weak NE)

A strategy profile $s = (s_1, \cdots, s_n)$ is a weak NE if, for all agents i and for all strategies, $s_i' \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$.

Nash Equilibria

Consider the following game:

	L	**C**	**R**
U	$-4, -4$	$3, 2$	$-1, -2$
M	$-4, 1$	$2, -1$	$2, 0$
D	$-6, 2$	$2, 3$	$2, 3$

Task 3: Find all pure NE, and distinguish between strict NE and weak NE.

Domination

Definition (Domination)

Let s_i and s_i' be two strategies of player i, and $\mathcal{S}_{-i}$ the set of all strategy profiles of the remaining players. Then:

- s_i strictly dominates s_i' if for all $s_{-i} \in \mathcal{S}_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.
- s_i weakly dominates s_i' if for all $s_{-i} \in \mathcal{S}_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, and for at least one $s_{-i} \in \mathcal{S}_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

Domination

Definition (Dominant strategy)

A strategy is strictly (weakly) dominant for an agent if it strictly (weakly) dominates other strategy for that agent.

Definition (Dominated strategy)

A strategy s_i is strictly (weakly) dominated for an agent i if for some other strategy s'_i strictly (weakly) dominates s_i.

Pareto Optimality

Definition (Pareto domination)

Strategy profile s Pareto dominates strategy profile s' if for all $i \in \mathcal{N}$, $u_i(s) \geq u_i(s')$, and there exists some $j \in \mathcal{N}$ for which $u_j(s) > u_j(s')$.

Definition (Pareto Optimality)

Strategy profile s is Pareto optimal (PO), or strictly Pareto efficient, if there does not exist another strategy profile $s' \in \mathcal{S}$ that Pareto dominates s.

NE, PO, and dominated strategies

Consider the following game:

	L	**M**	**R**
U	$1,3$	$4,2$	$-1,2$
C	$1,0$	$2,-2$	$0,-1$
D	$1,2$	$-1,1$	$3,3$

Task 4: Find all pareto optimal outcomes.

Task 5: Find all pure Nash equilibria.

Task 6: Find all dominated pure strategies. Apply iterative removal of dominated pure strategies.

NE, PO, and dominated strategies

Task 7: Design a game where a pure Nash equilibrium outcome will be removed in a process of iterative removal of (weakly) dominated pure strategies.

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a NFG. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; namely, $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

The player selects a pure strategy according to the probability distribution.

We extend the utility function to correspond to the expected utility:

$$u_i(s) = \sum_{a \in A} u_i(a) \cdot \prod_{j \in \mathcal{N}} s_j(a_j)$$

In mixed NE, no player is better off by playing a pure strategy.

Mixed Strategies

Consider the following game

	L	**R**
U	$3, 2$	$1, 3$
D	$1, 0$	$2, -2$

Task 8: Find a mixed NE.