Multiagent Systems (BE4M36MAS)

Solving Normal-Form Games

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Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions
   - u – outcome for each combination of actions
2. Pure strategies
3. Dominance of strategies
4. Nash equilibrium

... and now we continue ...

Rock Paper Scissors

	R	**P**	**S**
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

What is the best strategy to play in Rock-Paper-Scissors?

Every time we are about to play we randomly select an action we are going to use.

The concept of pure strategies is not sufficient.

Mixed Strategies

Definition (Mixed Strategies)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player i is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

We extend the utility function to correspond to *expected utility*:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in \mathcal{N}} s_j(a_j)$$

We can extend existing concepts (dominance, best response, ...) to mixed strategies.

Dominance

Definition (Strong Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *strongly dominates* s'_i if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Definition (Weak Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *weakly dominates* s'_i if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in \mathcal{S}_{-i}$ such that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

Definition (Very Weak Dominance)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that s_i *very weakly dominates* s'_i if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

Best Response and Equilibria

Definition (Best Response)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $BR_i(s_{-i}) \subseteq \mathcal{S}_i$ such that $s_i^* \in BR_i(s_{-i})$ iff $\forall s_i \in \mathcal{S}_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

Definition (Nash Equilibrium)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Strategy profile $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i \in \mathcal{N}, s_i \in BR_i(s_{-i})$.

Existence of Nash equilibria?

	C	**D**
C	$(-1, -1)$	$(-5, 0)$
D	$(0, -5)$	$(-3, -3)$

	R	**P**	**S**
R	$(0, 0)$	$(-1, 1)$	$(1, -1)$
P	$(1, -1)$	$(0, 0)$	$(-1, 1)$
S	$(-1, 1)$	$(1, -1)$	$(0, 0)$

Theorem (Nash)

Every game with a finite number of players and action profiles has at least one Nash equilibrium in mixed strategies.

Support of Nash Equilibria

Definition (Support)

The *support* of a mixed strategy s_i for a player i is the set of pure strategies $\mathrm{Supp}(s_i) = \{a_i | s_i(a_i) > 0\}$.

Question

Assume Nash equilibrium (s_i, s_{-i}) and let $a_i \in \mathrm{Supp}(s_i)$ be an (arbitrary) pure strategy from the support of s_i. Which of the following possibilities can hold?

- $u_i(a_i, s_{-i}) < u_i(s_i, s_{-i})$
- $u_i(a_i, s_{-i}) = u_i(s_i, s_{-i})$
- $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i})$

Support of Nash Equilibria

Corollary

Let $s \in \mathcal{S}$ be a Nash equilibrium and $a_i, a_i' \in \mathcal{A}_i$ are actions from the support of s_i. Now, $u_i(a_i, s_{-i}) = u_i(a_i', s_{-i})$.

Can we exploit this fact to find a Nash equilibrium?

Finding Nash Equilibria

	$\mathbf{L}$	$\mathbf{R}$
$\mathbf{U}$	$(2, 1)$	$(0, 0)$
$\mathbf{D}$	$(0, 0)$	$(1, 2)$

Column player (player 2) plays $\mathbf{L}$ with probability p and $\mathbf{R}$ with probability $(1 - p)$. In NE it holds

$$\mathbb{E}u_1(\mathbf{U}) = \mathbb{E}u_1(\mathbf{D})$$
$$2p + 0(1 - p) = 0p + 1(1 - p)$$
$$p = \frac{1}{3}$$

Similarly, we can compute the strategy for player 1 arriving at $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$ as Nash equilibrium.

Finding Nash Equilibria

Can we use the same approach here?

	L	C	R
U	$(2, 1)$	$(0, 0)$	$(0, 0)$
M	$(0, 0)$	$(1, 2)$	$(0, 0)$
D	$(0, 0)$	$(0, 0)$	$(-1, -1)$

Not really... No strategy s_i of the row player ensures
$u_{-i}(s_i, L) = u_{-i}(s_i, C) = u_{-i}(s_i, R)$:-(

Can something help us?
Iterated removal of dominated strategies.
Search for a possible support (enumeration of all possibilities).

Maxmin

	L	R
U	$(2,1)$	$(0,0)$
D	$(0,0)$	$(1,2)$

Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Playing a Nash strategy does not give any guarantees for the expected payoff. If we want guarantees, we can use a different concept – maxmin strategies.

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Maxmin and Minmax

Definition (Maxmin)

The *maxmin strategy* for player i is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player i is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

Definition (Minmax, two-player)

In a two-player game, the *minmax strategy* for player i against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the *minmax value* for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Maxmin strategies are conservative strategies against a worst-case opponent.

Minmax strategies represent punishment strategies for player $-i$.

Zero-sum case

What about zero-sum case? How do

- player i's maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
- player i's minmax, $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

relate?

$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = -\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

... but we can prove something stronger ...

Maxmin and Von Neumann's Minimax Theorem

Theorem (Minimax Theorem (von Neumann, 1928))

In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.

Consequences:

1. $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
2. we can safely play Nash strategies in zero-sum games
3. all Nash equilibria have the have the same payoff (by convention, the maxmin value for player 1 is called *value of the game*).

Computing NE in Zero-Sum Games

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

$$\max_{s,U} \quad U \tag{1}$$

$$\text{s.t.} \quad \sum_{a_1 \in \mathcal{A}_1} s(a_1) u_1(a_1, a_2) \geq U \qquad \forall a_2 \in \mathcal{A}_2 \tag{2}$$

$$\sum_{a_1 \in \mathcal{A}_1} s(a_1) = 1 \tag{3}$$

$$s(a_1) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1 \tag{4}$$

Computing a Nash equilibrium in zero-sum normal-form games can be done in polynomial time.