Normal-Form Games

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October 13, 2020

Previously ... on multi-agent systems (tutorials and lectures).

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1 Introduction to MAS

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2. Agent architectures,BDI
3. Introduction to Game Theory

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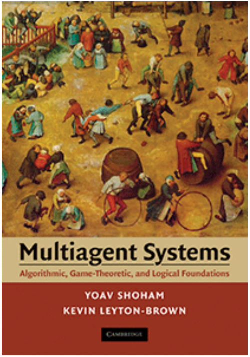
Source for studying: www.masfoundations.org



Why Game Theory – Practical Impact

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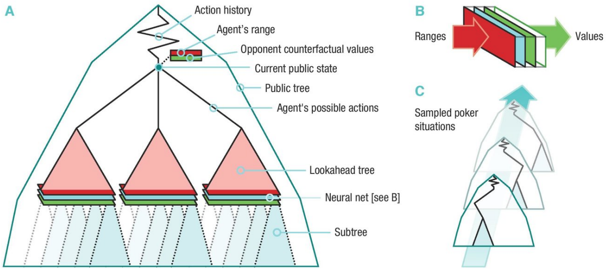
Why Game Theory – Deepstack

DeepStack: essentially solving 2-player heads-up no-limit Texas hold'em poker.

- Finds approximate NE for the game
- Large game (about 10^{160} states), uses neural networks for heuristics



Few Reminders from the Lecture

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Definition (Normal Form Game)

We call a triplet $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ a normal-form game, where

- $\mathcal{N}$ is a finite set of players
- $\mathcal{A}_i$ is a finite set of actions (pure strategies; hence, we also use $\mathcal{S}_i$ in some definitions) for player i
- u_i is a utility function of player i that assigns the reward for joint action $a \in \mathcal{A}$, $a = (a_1, a_2, \ldots, a_{\mathcal{N}})$ to player i

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- **zero-sum games** where $u_1(a_1, a_2) + u_2(a_1, a_2) = 0$ for all $a_i \in \mathcal{A}_i$
- **general-sum games** where $u_1(a_1, a_2) + u_2(a_1, a_2) \neq c$ for any $c \in \mathbb{R}$ and all $a_i \in \mathcal{A}_i$

How to create a game?

Student is at the lecture and he/she wants to go home by tram 22. There are 10 people in the tram and another 10 people are waiting at the tram stop, the goal of the tram is to carry as many people as possible, there is no-one exiting at this stop. The tram stops either (A) right next to the beginning of the tram stop, (B) further away (in the middle of the tram stop), or (C) does not stop at all. The tram stops and opens the door only once. Simultaneously, the student has to decide where at the stop he/she is going to wait. If the student guesses the position of the tram correctly, he/she gets utility 1. If the student guesses the position of the tram incorrectly, he/she gets utility $\frac{1}{2}$ as he/she needs to go through the crowd of people ot catch the tram. If the student misses the tram, his/her reward is -1.

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Task 1: Formalize the game.

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Task 2: Imagine that the number of waiting people is a stochastic variable – there is a 50% chance that 10 people are waiting, and there is a 50% chance that no one else (besides the student) is waiting. How does the formal representation of the game change?

Best Response and Equilibria

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Definition (Best Response)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $BR_i(s_{-i}) \subseteq \mathcal{S}_i$ such that $s_i^* \in BR_i(s_{-i})$ iff $\forall s_i \in \mathcal{S}_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

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Definition (Nash Equilibrium)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Strategy profile $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i \in \mathcal{N}, s_i \in BR_i(s_{-i})$.

$-i$ represents all players except player i ($\mathcal{N} \setminus \{i\}$).

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Definition (Strict NE)

A strategy profile $s = (s_1, \cdots, s_n)$ is a strict NE if, for all agents i and for all strategies, $s_i' \neq s_i$, $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

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Definition (Weak NE)

A strategy profile $s = (s_1, \cdots, s_n)$ is a weak NE if, for all agents i and for all strategies, $s'_i \neq s_i$, $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

Nash Equilibria

Consider the following game:

	L	**C**	**R**
U	$-4, -4$	$3, 2$	$-1, -2$
M	$-4, 1$	$2, -1$	$2, 0$
D	$-6, 2$	$2, 3$	$2, 3$

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Task 3: Find all pure NE, and distinguish between strict and weak.

Domination

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Let s_i and s_i' be two strategies of player i, and $\mathcal{S}_{-i}$ the set of all strategy profiles of the remaining players. Then:

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- s_i weakly dominates s_i' if for all $s_{-i} \in \mathcal{S}_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$, and for at least one $s_{-i} \in \mathcal{S}_{-i}$, it is the case that $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$.

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- s_i very weakly dominates s_i' if for all $s_{-i} \in \mathcal{S}_{-i}$, it is the case that $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$.

Domination

Definition (Dominant strategy)

A strategy is strictly (weakly; very weakly) dominant for an agent if it strictly (weakly; very weakly) dominates other strategy for that agent.

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Definition (Dominated strategy)

A strategy s_i is strictly (weakly; very weakly) dominated for an agent i if for some other strategy s_i' strictly (weakly; very weakly) dominates s_i.

Pareto Optimality

Definition (Pareto domination)

Strategy profle s Pareto dominates strategy profile s' if for all $i \in \mathcal{N}$, $u_i(s) \geq u_i(s')$, and there exists some $j \in \mathcal{N}$ for which $u_j(s) > u_j(s')$.

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Definition (Pareto Optimality)

Strategy profile s is Pareto optimal (PO), or strictly Pareto efficient, if there does not exist another strategy profile $s' \in \mathcal{S}$ that Pareto dominates s.

NE, PO, and dominated strategies

Consider the following game:

	L	**M**	**R**
U	$1, 3$	$4, 2$	$-1, 2$
C	$1, 0$	$2, -2$	$0, -1$
D	$1, 2$	$-1, 1$	$3, 3$

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Task 4: Find all pareto optimal outcomes.

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Task 4: Find all pareto optimal outcomes.

Task 5: Find all pure Nash equilibria.

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Task 4: Find all pareto optimal outcomes.

Task 5: Find all pure Nash equilibria.

Task 6: Find all dominated pure strategies. Apply iterative removal of dominated pure strategies.

NE, PO, and dominated strategies

Task 7: Design a game where a pure Nash equilibrium outcome will be removed in a process of iterative removal of (weakly) dominated pure strategies.

Mixed Strategies

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The player selects a pure strategy according to the probability distribution.

We extend the utility function to correspond to *expected utility*:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in \mathcal{N}} s_j(a_j)$$

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Task 8: Find a mixed NE.