# Multiagent Systems (BE4M36MAS)

## Solving Normal-Form Games

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Previously ... on multi-agent systems.

1. Formal definition of a game $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$
   - $\mathcal{N}$ – a set of players
   - $\mathcal{A}$ – a set of actions
   - $u$ – outcome for each combination of actions
2. Pure strategies
3. Dominance of strategies
4. Nash equilibrium

... and now we continue ...

Please, bookmark this page `https://bit.ly/3dpuxm7`. There will be (anonymous) online quizzes!

# Rock Paper Scissors

| | **R** | **P** | **S** |
|---|---|---|---|
| **R** | $(0, 0)$ | $(-1, 1)$ | $(1, -1)$ |
| **P** | $(1, -1)$ | $(0, 0)$ | $(-1, 1)$ |
| **S** | $(-1, 1)$ | $(1, -1)$ | $(0, 0)$ |

What is the best strategy to play in Rock-Paper-Scissors?

Every time we are about to play we randomly select an action we are going to use.

The concept of pure strategies is not sufficient.

# Mixed Strategies

**Definition (Mixed Strategies)**

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Then the set of *mixed strategies* $\mathcal{S}_i$ for player $i$ is the set of all probability distributions over $\mathcal{A}_i$; $\mathcal{S}_i = \Delta(\mathcal{A}_i)$.

Player selects a pure strategy according to the probability distribution.

We extend the utility function to correspond to *expected utility*:

$$u_i(s) = \sum_{a \in A} u_i(a) \prod_{j \in \mathcal{N}} s_j(a_j)$$

We can extend existing concepts (dominance, best response, ...) to mixed strategies.

# Dominance

**Definition (Strong Dominance)**

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that $s_i$ *strongly dominates* $s'_i$ if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

**Definition (Weak Dominance)**

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that $s_i$ *weakly dominates* $s'_i$ if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ and $\exists s_{-i} \in \mathcal{S}_{-i}$ such that $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$.

**Definition (Very Weak Dominance)**

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. We say that $s_i$ *very weakly dominates* $s'_i$ if $\forall s_{-i} \in \mathcal{S}_{-i}, u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$.

# Best Response and Equilibria

## Definition (Best Response)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $BR_i(s_{-i}) \subseteq \mathcal{S}_i$ such that $s_i^* \in BR_i(s_{-i})$ iff $\forall s_i \in \mathcal{S}_i, u_i(s_i^*, s_{-i}) \geq u_i(s_i, s_{-i})$.

## Definition (Nash Equilibrium)

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game. Strategy profile $s = \langle s_1, \ldots, s_n \rangle$ is a Nash equilibrium iff $\forall i \in \mathcal{N}, s_i \in BR_i(s_{-i})$.

# Existence of Nash equilibria?

| | C | D |
|---|---|---|
| **C** | $(-1, -1)$ | $(-5, 0)$ |
| **D** | $(0, -5)$ | $(-3, -3)$ |

| | R | P | S |
|---|---|---|---|
| **R** | $(0, 0)$ | $(-1, 1)$ | $(1, -1)$ |
| **P** | $(1, -1)$ | $(0, 0)$ | $(-1, 1)$ |
| **S** | $(-1, 1)$ | $(1, -1)$ | $(0, 0)$ |

**Theorem (Nash)**

*Every game with a finite number of players and action profiles has at least one Nash equilibrium in mixed strategies.*

# Support of Nash Equilibria

## Definition (Support)

The *support* of a mixed strategy $s_i$ for a player $i$ is the set of pure strategies $\mathrm{Supp}(s_i) = \{a_i | s_i(a_i) > 0\}$.

## Question

Assume Nash equilibrium $(s_i, s_{-i})$ and let $a_i \in \mathrm{Supp}(s_i)$ be an (arbitrary) pure strategy from the support of $s_i$. Which of the following possibilities can hold?

- $u_i(a_i, s_{-i}) < u_i(s_i, s_{-i})$
- $u_i(a_i, s_{-i}) = u_i(s_i, s_{-i})$
- $u_i(a_i, s_{-i}) > u_i(s_i, s_{-i})$

Try to answer here: `https://bit.ly/3dpuxm7`

# Support of Nash Equilibria

**Corollary**

*Let $s \in \mathcal{S}$ be a Nash equilibrium and $a_i, a'_i \in \mathcal{A}_i$ are actions from the support of $s_i$. Now, $u_i(a_i, s_{-i}) = u_i(a'_i, s_{-i})$.*

Can we exploit this fact to find a Nash equilibrium?

# Finding Nash Equilibria

| | $\mathbf{L}$ | $\mathbf{R}$ |
|---|---|---|
| $\mathbf{U}$ | $(2,1)$ | $(0,0)$ |
| $\mathbf{D}$ | $(0,0)$ | $(1,2)$ |

Column player (player 2) plays $\mathbf{L}$ with probability $p$ and $\mathbf{R}$ with probability $(1-p)$. In NE it holds

$$\mathbb{E}u_1(\mathbf{U}) = \mathbb{E}u_1(\mathbf{D})$$
$$2p + 0(1-p) = 0p + 1(1-p)$$
$$p = \frac{1}{3}$$

Similarly, we can compute the strategy for player 1 arriving at $(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})$ as Nash equilibrium.

# Finding Nash Equilibria

Can we use the same approach here?

| | **L** | **C** | **R** |
|---|---|---|---|
| **U** | $(2,1)$ | $(0,0)$ | $(0,0)$ |
| **M** | $(0,0)$ | $(1,2)$ | $(0,0)$ |
| **D** | $(0,0)$ | $(0,0)$ | $(-1,-1)$ |

Not really... No strategy $s_i$ of the row player ensures
$u_{-i}(s_i, L) = u_{-i}(s_i, C) = u_{-i}(s_i, R)$ :-(

**Can something help us?**

Iterated removal of dominated strategies.

Search for a possible support (enumeration of all possibilities).

# Maxmin

| | L | R |
|---|---|---|
| **U** | $(2, 1)$ | $(0, 0)$ |
| **D** | $(0, 0)$ | $(1, 2)$ |

Recall that there are multiple Nash equilibria in this game. Which one should a player play? This is a known equilibrium-selection problem.

Playing a Nash strategy does not give any guarantees for the expected payoff. If we want guarantees, we can use a different concept – maxmin strategies.

**Definition (Maxmin)**

The *maxmin strategy* for player $i$ is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player $i$ is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

# Maxmin and Minmax

**Definition (Maxmin)**

The *maxmin strategy* for player $i$ is $\arg\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$ and the *maxmin value* for player $i$ is $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$.

**Definition (Minmax, two-player)**

In a two-player game, the *minmax strategy* for player $i$ against player $-i$ is $\arg\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$ and the *minmax value* for player $-i$ is $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$.

Maxmin strategies are conservative strategies against a worst-case opponent.

Minmax strategies represent punishment strategies for player $-i$.

# Zero-sum case

What about zero-sum case? How do

- player $i$'s maxmin, $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i})$, and
- player $i$'s minmax, $\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$

relate?

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$$\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = -\min_{s_i} \max_{s_{-i}} u_{-i}(s_i, s_{-i})$$

... but we can prove something stronger ...

# Maxmin and Von Neumann's Minimax Theorem

**Theorem (Minimax Theorem (von Neumann, 1928))**

*In any finite, two-player zero-sum game, in any Nash equilibrium each player receives a payoff that is equal to both his maxmin value and the minmax value of his opponent.*

Consequences:

1. $\max_{s_i} \min_{s_{-i}} u_i(s_i, s_{-i}) = \min_{s_{-i}} \max_{s_i} u_i(s_i, s_{-i})$
2. we can safely play Nash strategies in zero-sum games
3. all Nash equilibria have the have the same payoff (by convention, the maxmin value for player 1 is called *value of the game*).

# Computing NE in Zero-Sum Games

We can now compute Nash equilibrium for two-player, zero-sum games using a linear programming:

$$
\begin{aligned}
& \max_{s,U} \quad U && && (1) \\
\text{s.t.} \quad & \sum_{a_1 \in \mathcal{A}_1} s(a_1) u_1(a_1, a_2) \geq U && \forall a_2 \in \mathcal{A}_2 && (2) \\
& \sum_{a_1 \in \mathcal{A}_1} s(a_1) = 1 && && (3) \\
& s(a_1) \geq 0 && \forall a_1 \in \mathcal{A}_1 && (4)
\end{aligned}
$$

Computing a Nash equilibrium in zero-sum normal-form games can be done in polynomial time.

# Computing NE in General-Sum Games

The problem is more complex for general-sum games (LCP program):

$$\sum_{a_2 \in \mathcal{A}_2} u_1(a_1, a_2) s_2(a_2) + q(a_1) = U_1 \qquad \forall a_1 \in \mathcal{A}_1$$

$$\sum_{a_1 \in \mathcal{A}_1} u_2(a_1, a_2) s_1(a_1) + w(a_2) = U_2 \qquad \forall a_2 \in \mathcal{A}_2$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1 \qquad \sum_{a_2 \in \mathcal{A}_2} s_2(a_2) = 1$$

$$q(a_1) \geq 0,\ w(a_2) \geq 0,\ s_1(a_1) \geq 0,\ s_2(a_2) \geq 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

$$s_1(a_1) \cdot q(a_1) = 0,\ s_2(a_2) \cdot w(a_2) = 0 \qquad \forall a_1 \in \mathcal{A}_1, \forall a_2 \in \mathcal{A}_2$$

Computing a Nash equilibrium in two-player general-sum normal-form game is a PPAD-complete problem. The problem gets even more complex (FIXP-hard) when moving to $n \geq 3$ players.

# Regret

The concept of regret is useful when the other players are not completely malicious.

| | **L** | **R** |
|---|---|---|
| **U** | $(100, a)$ | $(1-\varepsilon, b)$ |
| **D** | $(2, c)$ | $(1, d)$ |

## Definition (Regret)

A player $i$'s *regret* for playing an action $a_i$ if the other agents adopt action profile $a_{-i}$ is defined as

$$\left[\max_{a_i' \in \mathcal{A}_i} u_i(a_i', a_{-i})\right] - u_i(a_i, a_{-i})$$

# Regret

## Definition (MaxRegret)

A player $i$s *maximum regret* for playing an action $a_i$ is defined as

$$\max_{a_{-i} \in \mathcal{A}_{-i}} \left( \left[ \max_{a'_i \in \mathcal{A}_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

## Definition (MinimaxRegret)

Minimax regret actions for player $i$ are defined as

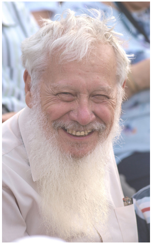
$$\arg\min_{a_i \in \mathcal{A}_i} \max_{a_{-i} \in \mathcal{A}_{-i}} \left( \left[ \max_{a'_i \in \mathcal{A}_i} u_i(a'_i, a_{-i}) \right] - u_i(a_i, a_{-i}) \right)$$

# Correlated Equilibrium

Consider again the following game:

| | **L** | **R** |
|---|---|---|
| **U** | $(2,1)$ | $(0,0)$ |
| **D** | $(0,0)$ | $(1,2)$ |

Wouldn't it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination? We can use *a correlation device*—a coin, a streetlight, commonly observed signal—and use this signal to avoid unwanted outcomes.



Robert Aumann

# Correlated Equilibrium

## Definition (Correlated Equilibrium (simplified))

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let $\sigma$ be a probability distribution over joint pure strategy profiles $\sigma \in \Delta(\mathcal{A})$. We say that $\sigma$ is a correlated equilibrium if for every player $i$, every signal $a_i \in \mathcal{A}_i$ and every possible action $a_i' \in \mathcal{A}_i$ it holds

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i', a_{-i})$$

## Corollary

*For every Nash equilibrium there exists a corresponding Correlated Equilibrium.*

# Computing Correlated Equilibrium

Computing a Correlated equilibrium is easier compared to Nash and can be found by linear programming even in general-sum case:

$$\sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i, a_{-i}) \geq \sum_{a_{-i} \in \mathcal{A}_{-i}} \sigma(a_i, a_{-i}) u_i(a_i', a_{-i}) \qquad \forall i \in \mathcal{N}, \forall a_i, a_i' \in \mathcal{A}_i$$

$$\sum_{a \in \mathcal{A}} \sigma(a) = 1 \qquad \sigma(a) \geq 0 \qquad \forall a \in \mathcal{A}$$

# Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

- *the leader* – publicly commits to a strategy
- *the follower(s)* – play a Nash equilibrium with respect to the commitment of the leader

Stackelberg equilibrium is a strategy profile that satisfies the above conditions and maximizes the expected utility value of the leader:

$$\arg\max_{s \in \mathcal{S}; \forall i \in \mathcal{N} \setminus \{1\} s_i \in BR_i(s_{-i})} u_1(s)$$

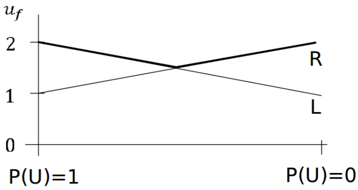
# Stackelberg Equilibrium

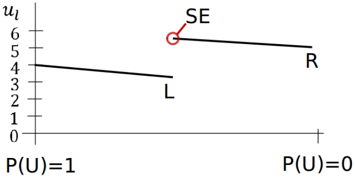
Consider the following game:

| | **L** | **R** |
|---|---|---|
| **U** | $(4, 2)$ | $(6, 1)$ |
| **D** | $(3, 1)$ | $(5, 2)$ |

$(\mathbf{U}, \mathbf{L})$ is a Nash equilibrium.

What happens when the row player commits to play strategy $\mathbf{D}$ with probability $1$? Can the row player get even more?



# There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

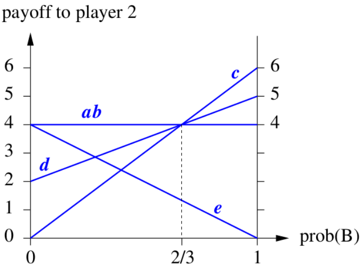
- arbitrary but fixed tie breaking rule
- *Strong SE* – the followers select such NE that maximizes the outcome of the leader (when the tie-braking is not specified we mean SSE),
- *Weak SE* – the followers select such NE that minimizes the outcome of the leader.

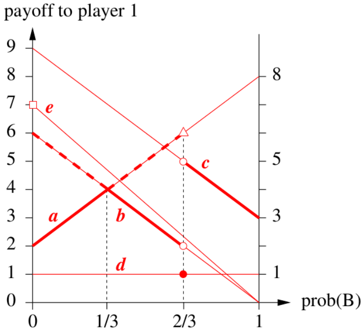
Exact Weak Stackelberg equilibrium does not have to exist.

# Different Stackelberg Equilibria

Exact Weak Stackelberg equilibrium does not have to exist.

| $1 \setminus 2$ | $a$ | $b$ | $c$ | $d$ | $e$ |
|---|---|---|---|---|---|
| $T$ | $(2,4)$ | $(6,4)$ | $(9,0)$ | $(1,2)$ | $(7,4)$ |
| $B$ | $(8,4)$ | $(0,4)$ | $(3,6)$ | $(1,5)$ | $(0,0)$ |



# Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; $1$ is the leader, $2$ is the follower.

Baseline polynomial algorithm requires solving $|\mathcal{A}_2|$ linear programs:

$$\max_{s_1 \in \mathcal{S}_1} \sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_1(a_1, a_2)$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_2(a_1, a_2) \geq \sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_2(a_1, a_2') \quad \forall a_2' \in \mathcal{A}_2$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1$$

one for each $a_2 \in \mathcal{A}_2$ assuming $a_2$ is the best response of the follower.