# COALITIONAL GAMES

**Exercises**

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## ABOUT THIS DOCUMENT

The purpose of these exercises is to make students of *Multiagent Systems* think actively about the concepts introduced in the lectures on coalitional games. Although all the solutions are included in the document, you are encouraged to work out your approach. Almost all exercises can be solved using pen & paper – write your notes and calculations on the wide margins of this document. Please see the lecture materials available online for the basic results and notation. Most of the exercises are adopted from [1, 2, 3, 4]. Some questions are more difficult or require more extended mathematical arguments. Such items are marked with ★ .

We will often adopt the standard practice of omitting curly brackets and commas in applications of coalitional function, if clarity is not impaired. For example, we may write $v(237)$ in place of $v(\{2,3,7\})$, when this improves readability. We also use the following notation for the sum of coordinates of vector $\mathbf{x} = (x_1, \dots, x_n)$:

$$\mathbf{x}(A) := \sum_{i \in A} x_i, \qquad A \subseteq \{1, \dots, n\}. \tag{1}$$

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# 1 COALITIONAL GAMES AND THE CORE

**Exercise 1.**
Let $v$ be a superadditive game: $v(A \cup B) \geq v(A) + v(B)$ for every disjoint pair of coalitions $A$ and $B$. Show that this implies the following condition:

$$v(N) \geq v(A_1) + \cdots + v(A_k),$$

for every coalitional structure $\{A_1, \ldots, A_k\}$.

**Exercise 2.**
Five friends want to sell gin & tonic cocktails at a party. Three of them have a bottle of gin apiece and each of the other two friends has five bottles of tonic. A price of cocktails made from one gin bottle and five tonic bottles is 2000 CZK. Model this situation as a coalitional game, decide if it is superadditive, and compute its core.

**Exercise 3.**
A coalitional game $v\colon \mathcal{P}(N) \to \mathbb{R}$ over the player set $N = \{1, 2, 3\}$ is

$$v(A) = \begin{cases} 0 & A = \emptyset, \\ 1 & A = \{1\}, \{2\}, \\ 2 & A = \{3\}, \\ 4 & |A| = 2, \\ 5 & A = N. \end{cases}$$

Is $v$ superadditive? What is the core of $v$?

**Exercise 4.**
Let $N = \{1, 2, 3\}$. Describe the core of the game

$$v(A) = \begin{cases} 3 & \text{if } A = N, 13 \\ 1 & \text{if } A = 12, 23 \\ 0 & \text{if } A = 1, 2, 3, \emptyset \end{cases}$$

and decide if $v$ is superadditive.

**Exercise 5.**
Describe the core of a game $v$ over the player set $N = \{1, 2, 3\}$, where

$$v(A) = \begin{cases} 0 & A = \emptyset, \\ |A| - 1 & A \neq \emptyset. \end{cases}$$

**Exercise 6.**
Alice has a left glove. Bob and Cyril have one glove each. The number of pairs of gloves collected by a coalition is its worth. Define the resulting coalitional game, decide if it is superadditive or supermodular, and describe its core.

**Exercise 7.**

A *simple game* is a coalitional game $v\colon \mathcal{P}(N) \to \{0,1\}$ that is monotone and $v(N) = 1$. We call a player $i \in N$ in a simple game $v$ a *veto player*, if $v(A \setminus i) = 0$ holds for each coalition $A \subseteq N$. Show that the following is true for any simple game $v$:

(a) Player $i$ is veto in game $v$ if, and only if, $v(N \setminus i) = 0$.

(b) There is a veto player in game $v$ if, and only if, $\mathcal{C}(v) \neq \emptyset$.

(c) Let $W \neq \emptyset$ be the set of veto players. Then the core of $v$ is

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^n_+ \mid \mathbf{x}(W) = 1 \text{ and } x_i = 0 \text{ for all } i \notin W\}.$$

**Exercise 8.**

Argue that the following game $v$ is superadditive and its core is empty:

$$v(A) = \begin{cases} 0 & |A| \leq 1 \\ 1 & |A| \geq 2 \end{cases} \qquad A \subseteq \{1,2,3\}.$$

**Exercise 9.**

*Minimum spanning tree game.* The costs of connecting the cities denoted as $1, 2$, and $3$ to the supplier of energy $0$ are depicted in Figure 1. The minimum cost spanning tree game is defined as a coalitional game $c$ over player set $N = \{1,2,3\}$, in which the worth of each coalition $A \subseteq N$ is the cost $c(A)$ associated with the minimum spanning tree over vertices $A \cup \{0\}$. Describe the associated savings game

$$v(A) = \sum_{i \in A} c(i) - c(A),$$

show that the core of $v$ is nonempty, and determine at least one corresponding cost distribution.

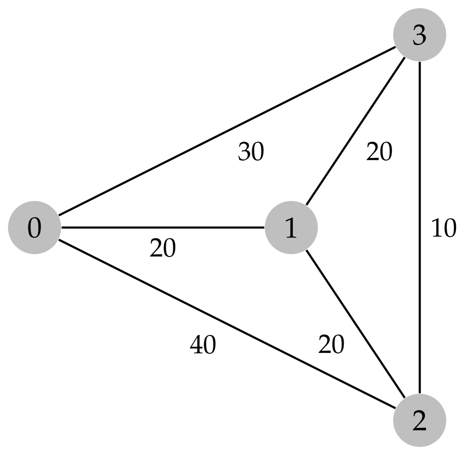


**Figure 1:** Graph from Exercise 9

★ **Exercise 10.**

Coalitional game $v$ is *supermodular* if, for all $A, B \subseteq N$ the inequality

$$v(A) + v(B) \leq v(A \cup B) + v(A \cap B)$$

holds. Prove that the following assertions are equivalent.

(a) $v$ is supermodular.

(b) For all $A, B \subseteq N$ with $A \subseteq B$ and every $C \subseteq N \setminus B$,

$$v(A \cup C) - v(A) \leq v(B \cup C) - v(B).$$

(c) For all $A, B \subseteq N$ with $A \subseteq B$ and every $i \in N \setminus B$,

$$v(A \cup i) - v(A) \leq v(B \cup i) - v(B).$$

(d) $\mathbf{x}^\pi \in \mathcal{C}(v)$, for every $\pi \in \Pi$, where $x_i^\pi := v(A_i^\pi \cup i) - v(A_i^\pi)$ and $A_i^\pi := \{j \in N \mid \pi(j) < \pi(i)\}$.

(e) The core of $v$ is the convex hull of the marginal vectors in $v$, that is,

$$\mathcal{C}(v) = \text{conv}\{\mathbf{x}^\pi \mid \pi \in \Pi\}.$$

(f) Vertices of $\mathcal{C}(v)$ are precisely marginal vectors $\mathbf{x}^\pi$.

# 2 THE SHAPLEY VALUE

**Exercise 11.**
Prove that the Shapley value $\boldsymbol{\varphi}^S(v)$ of any supermodular game $v$ is the center of gravity of the core of $v$.

**Exercise 12.**
A company has 3 shareholders whose shares are distributed in the following way. The first has 50 % shares and the remaining two have 25 % shares each. The three shareholders vote by using a weighted majority of votes. Define the resulting coalitional game. Compute the Shapley-Shubik index using the random order approach and then calculate the normalized Banzhaf index.

**Exercise 13.**
*Weighted majority game.* Four members of a committee decide on a proposition by weighted majority. The voting weights are $2, 1, 1, 1$, and the decision is approved when the weighted sum of votes is $\geq 3$. Describe this situation as a coalitional game, calculate the Shapley-Shubik index, and discuss how the Shapley-Shubik indices correspond to the individual weights. What is the normalized Banzhaf index?

**Exercise 14.**
Let $\Gamma$ be the set of all coalitional games over the player set $N = \{1, \ldots, n\}$. Consider a solution mapping $\boldsymbol{\psi} \colon \Gamma \to \mathbb{R}^n$ defined by

$$\psi_i(v) := v(1 \ldots i) - v(1 \ldots i-1), \qquad i \in N.$$

Show that $\boldsymbol{\psi}$ is efficient, additive, and it has the null player property, but fails symmetry.

★ **Exercise 15.**
Prove that the Shapley value $\boldsymbol{\varphi}^S$ is efficient, additive, it has the null player property, and it is symmetric.

# 3 THE NUCLEOLUS

**Exercise 16.**
Let $v$ be a bankruptcy game, that is, there exist $e \geq 0$ and a vector $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n_+$ such that $e \leq d_1 + \cdots + d_n$, and

$$v(A) = \max\{e - \boldsymbol{d}(N \setminus A), 0\}, \qquad A \subseteq N.$$

Show that $v$ is supermodular.

**Exercise 17.**
Show that the lexicographic order $\preceq$ on $\mathbb{R}^m$ is a *total order*. Specifically, show that binary relation $\preceq$ is reflexive, transitive, antisymmetric, and complete.

**Exercise 18.**
Show that the set of imputations $\mathcal{I}(v)$ in game $v$ is nonempty if, and only if, this inequality holds:

$$v(1) + \cdots + v(n) \leq v(N).$$

**Exercise 19.**
Prove that the set of imputations in a superadditive game is nonempty.

**Exercise 20.**
Show that the nucleolus of a two-player superadditive game $v$ is an allocation $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$ such that

$$x_1 = v(1) + \frac{v(12) - v(1) - v(2)}{2} \quad \text{and} \quad x_2 = v(2) + \frac{v(12) - v(1) - v(2)}{2}.$$

What is the Shapley value of $v$?

**Exercise 21.**
Decide if the assertions below are true or false.

1. If the core is nonempty, it contains the Shapley value.
2. If marginal contributions of players $i$ and $j$ to every coalition are equal, then their Shapley values coincide.
3. The core of every monotone coalitional game is nonempty.
4. The nucleolus satisfies the properties of efficiency, symmetry, and it has the null player property.
5. The nucleolus is an additive solution concept.
6. The Shapley value $\boldsymbol{\varphi}^S(v) = (\varphi^S_1(v), \ldots, \varphi^S_n(v))$ of every $n$-player coalitional game $v$ is uniquely determined by the Shapley values of the first $n-1$ players $(\varphi^S_1(v), \ldots, \varphi^S_{n-1}(v))$.
7. The Shapley value is *individually rational*, that is, $\varphi^S_i(v) \geq v(i)$ for every game $v$ and each player $i$.
8. The nucleolus is an individually rational allocation.

# A SOLUTIONS TO EXERCISES

**Solution 1.**

Since coalitions $A_1, \dots, A_k$ are pairwise disjoint, we can apply superadditivity to $A_1$ and $A_2$ to obtain

$$v(A_1) + v(A_2) + v(A_3) + \cdots + v(A_k) \le v(A_1 \cup A_2) + v(A_3) + \dots v(A_k).$$

Proceeding analogously in the next $k-2$ steps, we get

$$v(A_1) + \cdots + v(A_k) \le v(A_1 \cup \cdots \cup A_k) = v(N),$$

where the last equality follows from $N = A_1 \cup \cdots \cup A_k$.

**Solution 2.**

$$G = \{1,2,3\}, \;\; T = \{4,5\}, \;\; N = G \cup T$$
$$v(A) = 2000 \cdot \min\{|A \cap G|, |A \cap T|\}, \qquad A \subseteq N$$

It is easy to prove that $v$ is superadditive. Let $A \cap B = \emptyset$. We can neglect the multiplicative constant 2000. Then we want to show that

$$v(A \cup B) \ge v(A) + v(B).$$

Using the definition of $v$, we get

$$\min\{|(A \cup B) \cap G|, |(A \cup B) \cap T|\} \ge \min\{|A \cap G|, |A \cap T|\} + \min\{|B \cap G|, |B \cap T|\},$$

which is the same as

$$\min\{\underbrace{|A \cap G| + |B \cap G|}_{a_1+b_1}, \underbrace{|A \cap T| + |B \cap T|}_{a_2+b_2}\} \ge$$
$$\min\{\underbrace{|A \cap G|}_{a_1}, \underbrace{|A \cap T|}_{a_2}\} + \min\{\underbrace{|B \cap G|}_{b_1}, \underbrace{|B \cap T|}_{b_2}\}.$$

This can be written as $a_i + b_i \ge a_j + b_k$, where $i, j, k$ are the indices of the corresponding minimizers. The definition of minimum yields $a_i \ge a_j$ and $b_i \ge b_k$, from which the superadditive inequality follows.

The core contains a unique allocation,

$$\mathcal{C}(v) = \{(0, 0, 0, 2\,000, 2\,000)\}.$$

We can again neglect the multiplicative constant 2000. From

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + x_5 &= 2 \\ x_1 + x_2 + x_4 + x_5 &\ge 2 \\ x_1 + x_3 + x_4 + x_5 &\ge 2 \\ x_2 + x_3 + x_4 + x_5 &\ge 2 \end{aligned}$$

derive $x_4 + x_5 = 2 - x_1 - x_2 - x_3$, which implies together with nonnegativity constraints that $x_1 = x_2 = x_3 = 0$, and also that $x_4 \ge 1$ and $x_5 \ge 1$. Hence, necessarily $x_4 = x_5 = 1$.

**Solution 3.**
Game $v$ is superadditive, if the inequality $v(A \cup B) \geq v(A) + v(B)$ holds for all $A, B \subseteq N$, $A \cap B = \emptyset$. Since $v(N) < v(12) + v(3)$, game $v$ is not superadditive. It is easy to see that $\mathcal{C}(v)$ is empty. Indeed, every allocation $\mathbf{x} \in \mathcal{C}(v)$ must satisfy the conditions $x_1 + x_2 + x_3 = 5$, $x_1 + x_2 \geq 4$, and $x_3 \geq 2$. But adding the last two inequalities together yields $5 = x_1 + x_2 + x_3 \geq 6$, a contradiction.

**Solution 4.**
By the definition,

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^3_+ \mid x_1 + x_2 + x_3 = 3, \quad x_1 + x_2 \geq 1, \quad x_1 + x_3 \geq 3, \quad x_2 + x_3 \geq 1\}.$$

We will show that $\mathcal{C}(v) = \operatorname{conv}\{(1,0,2),(2,0,1)\}$. Since the coalition $\{1,3\}$ accepts only total payoffs $\geq 3$ and since $x_1 + x_2 + x_3 = 3$, it is reasonable to think that player 2's payoff should be 0. Indeed, from $x_1 + x_3 \geq 3$ and from $x_1 + x_3 = 3 - x_2$, we get $3 - x_2 \geq 3$, which gives $0 \geq x_2$, hence $x_2 = 0$. Then

$$\mathcal{C}(v) = \{\mathbf{x} \in \mathbb{R}^3 \mid x_1 \geq 1, x_2 = 0, x_3 \geq 1, x_1 + x_3 = 3\},$$

which is the line segment with vertices $(1,0,2),(2,0,1)$. It can be checked that this is a superadditive game. Observe that Player 2 cannot block the formation of grand coalition, since he accepts payoff 0. However, if Player 2's worth is changed to $v(\{2\}) = 1$, then $\mathcal{C}(v) = \emptyset$.

**Solution 5.**
Using the identity $|A \cup B| = |A| + |B| - |A \cap B|$ we can easily verify that $v$ is supermodular, that is, $v(A \cup B) + v(A \cap B) \geq v(A) + v(B)$. This implies that its core $\mathcal{C}(v)$ coincides with the convex hull of its marginal vectors $\mathbf{x}^\pi$, where $\pi$ is a permutation of $N$. For example, permutation $\pi(1) = 3, \pi(2) = 1$, $\pi(3) = 2$ determine the order of players

$$2\,3\,1$$

together with the marginal vector $\mathbf{x}^\pi$ whose coordinates are

$$\begin{aligned} x_2^\pi &= v(2) - v(\emptyset) = 0, \\ x_3^\pi &= v(23) - v(2) = 1, \\ x_1^\pi &= v(123) - v(23) = 1. \end{aligned}$$

The remaining marginal vectors are computed analogously. This shows that the core is a triangle with vertices $(0,1,1),(1,0,1)$, and $(1,1,0)$, which is located in the plane given by the equation $x_1 + x_2 + x_3 = 2$.

**Solution 6.**
The glove game over the player set $N = \{1,2,3\}$ is

$$v(A) = \begin{cases} 1 & A = \{1,2\}, \{1,3\}, N, \\ 0 & \text{otherwise.} \end{cases}$$

The glove game $v$ is monotone and superadditive, but not supermodular. The core of $v$ is

$$\mathcal{C}(v) = \{(1,0,0)\}.$$

Indeed, we can argue as follows. The inequalities $x_1 + x_2 \geq 1$ and $x_1 + x_3 \geq 1$ combined together give $2x_1 + x_2 + x_3 \geq 2$. Since $x_2 + x_3 = 1 - x_1$, the last inequality says that $x_1 \geq 1$. Since $x_2$ and $x_3$ must be nonnegative, we get $x_1 = 1$. Thus, the only core allocation is $(1, 0, 0)$.

**Solution 7.**

(a) The first implication is trivial. Assume that $v(N \setminus i) = 0$. Then monotonicity gives $v(A \setminus i) = 0$ for every $A \subseteq N$.

(b) Let $k \in N$ be a veto player in game $v$. We define an allocation vector $\mathbf{x} \in \mathbb{R}^n$ as follows:

$$x_i = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

Since $v$ is non-constant, $v(N) = 1 = \sum_{i \in N} x_i = \mathbf{x}(N)$. Choose $A \subseteq N$. If $k \in A$, then $\mathbf{x}(A) = 1 \geq v(A)$. If $k \notin A$, then $\mathbf{x}(A) = 0 = v(A)$, since $k$ is veto. We have shown that $\mathbf{x} \in \mathcal{C}(v)$.

Conversely, assume that $v$ has no veto players. We want to conclude that $v$ has empty core. By way of contradiction, let $\mathbf{x} \in \mathcal{C}(v)$. Then the condition $\mathbf{x}(N) = 1$ implies that there exists $i \in N$ such that $x_i > 0$, hence $\mathbf{x}(N \setminus i) = 1 - x_i < 1$. Since $i$ is not veto, $v(N \setminus i) = 1 > \mathbf{x}(N \setminus i)$, which contradicts our assumption $\mathbf{x} \in \mathcal{C}(v)$.

(c) Let $\mathbf{x} \in \mathbb{R}^n_+$ be such that $\mathbf{x}(W) = 1$ and $x_i = 0$ for all $i \notin W$. We want to show that $\mathbf{x} \in \mathcal{C}(v)$. Clearly, $\mathbf{x}(N) = \mathbf{x}(W) = 1$. If $A \subseteq N$ is loosing, that is, $v(A) = 0$, then $\mathbf{x}(A) \geq 0$. Let $v(A) = 1$. This implies that $A \supseteq W$, which gives

$$\mathbf{x}(A) \geq \mathbf{x}(W) = 1 = v(A).$$

Therefore, $\mathbf{x} \in \mathcal{C}(v)$.

Conversely, let $\mathbf{x} \in \mathcal{C}(v)$. Then $x_i \geq 0$ for all $i \in N$ and $\mathbf{x}(N) = 1$. We need to show that $x_i = 0$ for all $i \in N \setminus W$. Pick $i \in N \setminus W$. Player $i$ is not veto and, hence,

$$1 = \mathbf{x}(N) \geq \mathbf{x}(N \setminus i) \geq v(N \setminus i) = 1,$$

which implies $\mathbf{x}(N) = \mathbf{x}(N \setminus i)$, so that $x_i = 0$.

**Solution 8.**

The game $v$ is in fact a simple majority game. It is easy to see that $v$ is superadditive. We show that $\mathcal{C}(v) = \emptyset$. By contradiction, assume that $\mathbf{x} \in \mathcal{C}(v)$. Then $x_1 + x_2 + x_3 = 1$ and $x_1 + x_2 \geq 1$, $x_1 + x_3 \geq 1$, $x_2 + x_3 \geq 1$. Combining the last three inequalities together gives

$$2 \cdot \underbrace{(x_1 + x_2 + x_3)}_{1} \geq 3,$$

a contradiction. Hence, $\mathcal{C}(v) = \emptyset$. The same conclusion follows immediately from Exercise 7, since there are no veto players in the game.

**Solution 9.**
We can easily compute

$$c(A) = \begin{cases} 0 & A = \emptyset, \\ 20 & A = 1, \\ 30 & A = 3, \\ 50 & A = 123, \\ 40 & \text{otherwise,} \end{cases} \qquad v(A) = \begin{cases} 0 & A = \emptyset, 1, 2, 3, \\ 10 & A = 13, \\ 20 & A = 12, \\ 30 & A = 23, \\ 40 & A = 123. \end{cases}$$

The core of $v$ is the set of allocations $\mathbf{x} \in \mathbb{R}^3_+$ such that $x_1 + x_2 + x_3 = 40$ and

$$x_1 + x_2 \geq 20, \quad x_1 + x_3 \geq 10, \quad x_2 + x_3 \geq 30.$$

We can easily see that the savings vector $\mathbf{x} = (0, 20, 20)$ is in the core of $v$, since it measures the savings of individual players with respect to the minimum spanning tree over the full player set $N$. This corresponds to the cost distribution $\mathbf{y} \in \mathbb{R}^3$, where $y_i = c(i) - x_i$. Hence, $\mathbf{y} = (20, 20, 10)$. Observe that $\mathbf{y}$ is just the vector of costs corresponding to the minimum spanning tree.

★ **Solution 10.**
First, we prove (a) ⇒ (b). Let (a) be true. Choose $A, B \subseteq N$, where $A \subseteq B$ and $C \subseteq N \setminus B$. Then

$$v(A \cup C) + v(B) \leq v(\underbrace{(A \cup C) \cup B}_{B \cup C}) + v(\underbrace{(A \cup C) \cap B}_{A}).$$

Implication (b) ⇒ (c) is trivial. We show that (c) ⇒ (a). Select $A, B \subseteq N$. If $A \subseteq B$, then (a) is true. Therefore, assume that $A \not\subseteq B$ and let $P := A \cap B$, $R := A \setminus B$. The assumption implies $R \neq \emptyset$ and we may write $R = \{i_1, \ldots, i_k\}$, where $k = |R|$. Since $B \supseteq P$ and for any $\ell = 1, \ldots, k-1$,

$$B \cup \{i_1 \ldots, i_\ell\} \supseteq P \cup \{i_1 \ldots, i_\ell\},$$

item (c) gives the following inequalities:

$$v(B \cup i_1) - v(B) \geq v(P \cup i_1) - v(P)$$
$$v(B \cup i_1 \ldots i_{\ell+1}) - v(B \cup i_1 \ldots i_\ell) \geq v(P \cup i_1 \ldots i_{\ell+1}) - v(P \cup i_1 \ldots i_\ell)$$

Summing all the inequalities, we get

$$v(\underbrace{B \cup R}_{A \cup B}) - v(B) \geq v(\underbrace{P \cup R}_{A}) - v(P),$$

which proves (a).

Further, we need to prove (a) ⇔ (d). We can use already proved equivalences (a) ⇔ (b) ⇔ (c). We will show implication (c) ⇒ (d). Let (c) holds. We want to show that $\mathbf{x}^\pi \in \mathcal{C}(v)$ for any permutation $\pi \in \Pi$. We obtain

$$\mathbf{x}^\pi(N) = \sum_{i \in N} x_i^\pi = \sum_{i \in N} (v(A_i^\pi \cup i) - v(A_i^\pi)) = v(N) - v(\emptyset) = v(N).$$

We show that $\mathbf{x}^\pi$ is coalitionally rational, that is, $\mathbf{x}^\pi(A) \geq v(A)$, for every nonempty coalition $A \subseteq N$. Let $a := |A|$. The players in $A$ can be enumerated as follows: $A = \{i_1, \dots, i_a\}$, where $\pi(i_1) < \cdots < \pi(i_a)$. Write $B_k := \{i_1, \dots, i_k\}$ for each $k = 1, \dots, a$. Then

$$B_k = A \cap \left( A^\pi_{i_k} \cup i_k \right).$$

Define $B_0 := \emptyset$. By assumption (c), this inequality is satisfied for all $k = 1, \dots, a$:

$$v(B_k) - v(B_{k-1}) \leq v\left(A^\pi_{i_k} \cup i_k\right) - v\left(A^\pi_{i_k}\right) = x^\pi_{i_k}.$$

This implies

$$v(A) = v(B_a) = \sum_{k=1}^{a} (v(B_k) - v(B_{k-1})) \leq \sum_{k=1}^{a} x^\pi_{i_k} = \mathbf{x}^\pi(A),$$

which finishes the proof of (d).

In the next step we check that implication (d) $\Rightarrow$ (a) is true. Let $v$ be a coalitional game fulfilling $\mathbf{x}^\pi \in \mathcal{C}(v)$ for all $\pi \in \Pi$. Supermodular inequality (a) holds trivially, when at least one of the sets $A, B \subseteq N$ is empty. Therefore, assume that $A, B \neq \emptyset$. Put $r := |A \cap B|$, $q := |A \cup B|$, $t := |B|$, and write

$$\begin{aligned} A \cap B &= \{i_1, \dots, i_r\}, \\ B \setminus A &= \{i_{r+1}, \dots, i_t\}, \\ A \setminus B &= \{i_{t+1}, \dots, i_q\}, \\ N \setminus (A \cup B) &= \{i_{q+1}, \dots, i_n\}. \end{aligned}$$

Define permutation $\pi$ by $\pi(i_j) := j$, for all $j \in N$. It follows from (d) that

$$v(A) \leq \mathbf{x}^\pi(A) = \sum_{\substack{j \in N \\ i_j \in A}} x^\pi_{i_j} = \sum_{\substack{j \in N \\ i_j \in A}} \left( v(A^\pi_{i_j} \cup i_j) - v(A^\pi_{i_j}) \right).$$

The last sum can be split into two sums,

$$\begin{aligned} &\sum_{j=1}^{r} \left( v(i_1 \dots i_j) - v(i_1 \dots i_{j-1}) \right) \\ &+ \sum_{j=t+1}^{q} \left( v(B \cup i_{t+1} \dots i_j) - v(B \cup i_{t+1} \dots i_{j-1}) \right) = \\ &v(A \cap B) - v(\emptyset) + v(A \cup B) - v(B), \end{aligned}$$

which shows (a).

Clearly, item (e) implies (d) immediately. We show that implication (d) $\Rightarrow$ (e) holds. From (d) we obtain the inclusion $\mathcal{C}(v) \supseteq \operatorname{conv}\{\mathbf{x}^\pi \mid \pi \in \Pi\}$, by convexity of the core. The converse inclusion $\mathcal{C}(v) \subseteq \operatorname{conv}\{\mathbf{x}^\pi \mid \pi \in \Pi\}$ requires an involved proof, for which we refer the reader to [1, Theorem 5.18].

The proof is finished after we show the equivalence (e) $\Leftrightarrow$ (f). The implication (f) $\Rightarrow$ (e) is a direct consequence of the characterization of convex polytope $\mathcal{C}(v)$ as the convex hull of its vertices. Suppose that (e) holds. Then every vertex of $\mathcal{C}(v)$ is necessarily a marginal vector $\mathbf{x}^\pi$, for some $\pi \in \Pi$. It remains to prove that every marginal vector is a vertex of $\mathcal{C}(v)$. Let $\pi \in \Pi$. It follows from the definition of marginal vectors that

$$\mathbf{x}^{\pi}(A_i^{\pi}) = v(A_i^{\pi}), \qquad i = 1, \ldots, n.$$

This is a linear system whose matrix is triangular with nonzero elements on the diagonal. Therefore, the matrix is nonsingular and $\mathbf{x}^{\pi}$ is a vertex of $\mathcal{C}(v)$.

**Solution 11.**
Let $N = \{1, \ldots, n\}$ and $v$ be a supermodular game over $N$. By supermodularity and Exercise 10, the core of $v$ is the convex hull of its marginal vectors,

$$\mathcal{C}(v) = \operatorname{conv}\{\mathbf{x}^{\pi} \mid \pi \in \Pi\},$$

where $\Pi$ is the set of all permutations of the player set $N$. Therefore, it suffices to show that $\boldsymbol{\varphi}^S(v)$ can be written as $\boldsymbol{\varphi}^S(v) = \sum_{\pi \in \Pi} a_{\pi} \cdot \mathbf{x}^{\pi}$, where $a_{\pi} = \frac{1}{n!}$. But this is immediate, since an equivalent formula for the Shapley value is

$$\boldsymbol{\varphi}^S(v) = \sum_{\pi \in \Pi} \frac{1}{n!} \cdot \mathbf{x}^{\pi}.$$

**Solution 12.**
The player set is $N = \{1, 2, 3\}$. The coalitional game is

$$v(A) = \begin{cases} 1 & A = N, \{1,2\}, \{1,3\}, \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

For the calculation of the Shapley-Shubik index of $i$ we enumerate all the permutations such that $i$ makes the preceding coalition winning:

$$1\boxed{2}3 \quad 1\boxed{3}2 \quad 2\boxed{1}3 \quad 23\boxed{1} \quad 3\boxed{1}2 \quad 32\boxed{1}$$

Then

$$\varphi_1^S(v) = \tfrac{4}{6}, \quad \varphi_2^S(v) = \varphi_3^S(v) = \tfrac{1}{6}.$$

In order to compute the normalized Banzhaf index $\beta(v)$, we enumerate the number of swings for each player:

$$\boxed{1}\boxed{2} \quad \boxed{1}\boxed{3} \quad \boxed{1}23$$

Hence, $s_1(v) = 3$, $s_2(v) = s_3(v) = 1$. These numbers are divided by the total number of swings:

$$\beta_1(v) = \frac{3}{5}, \quad \beta_1(v) = \beta_1(v) = \frac{1}{5}.$$

**Solution 13.**
This situation is captured by a weighted majority game $v$ defined as follows. Let $w_1 = 2$, $w_2 = w_3 = w_4 = 1$, and $q = 3$. Define

$$v(A) = \begin{cases} 1 & \sum_{i \in A} w_i \geq q, \\ 0 & \text{otherwise}, \end{cases} \qquad \text{for all } A \subseteq \{1,2,3,4\}.$$

The Shapley-Shubik index of player $i$ is

$$\varphi_i^S(v) = \sum_{\substack{A \subseteq N \setminus \{i\} \\ i \text{ pivotal to } A}} \frac{1}{n\binom{n-1}{|A|}}.$$

To compute $\boldsymbol{\varphi}^S(v) = \big(\varphi_1^S(v), \varphi_2^S(v), \varphi_3^S(v), \varphi_4^S(v)\big)$, realize that

$$\varphi_1^S(v) + \varphi_2^S(v) + \varphi_3^S(v) + \varphi_4^S(v) = 1$$

by efficiency. Moreover, players 2, 3, and 4 are symmetric in this game, since their individual contribution to each coalition is equal. By symmetry of the Shapley value, this means that

$$\varphi_2^S(v) = \varphi_3^S(v) = \varphi_4^S(v).$$

Hence, we need to compute only one Shapley-Shubik index, say $\varphi_2^S(v)$. Clearly, player 2 is pivotal to coalitions $\{1\}$ and $\{3,4\}$. Then

$$\varphi_2^S(v) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6},$$

and

$$\varphi_1^S(v) = 1 - 3 \cdot \varphi_2^S(v) = \frac{1}{2}.$$

We obtain the Shapley-Shubik index $\boldsymbol{\varphi}^S(v) = (\frac{1}{2}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6})$, which is different from the vector of relative weights $(\frac{2}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$. In conclusion, the Shapley-Shubik index of player 1 indicates that the voting power of player 1 is slightly higher than the relative voting weight $\frac{2}{5}$. The normalized Banzhaf index is in this case the same as the Shapley-Shubik index, since the swings for players are $s_1(v) = 6$ and $s_2(v) = s_3(v) = s_4(v) = 2$.

**Solution 14.**
First, we check *efficiency*:

$$\sum_{i \in N} \psi_i(v) = \sum_{i \in N} (v(1 \dots i) - v(1 \dots i-1)) = v(N) - v(\emptyset) = v(N).$$

*Aditivity*: For all $v, w \in \Gamma$ we get

$$\begin{aligned}
\psi_i(v+w) &= (v+w)(1 \dots i) - (v+w)(1 \dots i-1) \\
&= v(1 \dots i) - v(1 \dots i-1) \\
&\quad + w(1 \dots i) - w(1 \dots i-1) \\
&= \psi_i(v) + \psi_i(w).
\end{aligned}$$

*Null player property*: Let $i \in N$ be the null player. This means that

$$v(A \cup i) = v(A)$$

for each coalition $A \subseteq N$. Then, putting $A := \{1, \dots, i-1\}$ yields $\psi_i(v) = 0$.

We show that $\boldsymbol{\psi}$ fails symmetry. Letting $N = \{1,2,3\}$ we define a game

$$v(A) = \begin{cases} 1 & A = \{2,3\}, N \\ 0 & \text{otherwise}, \end{cases} \qquad A \subseteq N.$$

Then $\boldsymbol{\psi}(v) = (0,0,1)$. However, players 2 and 3 have the same contributions to one-player coalition $\{1\}$, that is, $v(12) = v(13)$. This implies that $\boldsymbol{\psi}$ fails symmetry.

★ **Solution 15.**

The Shapley value of player $i \in N$ is

$$\varphi_i^S(v) = \sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^\pi.$$

*Efficiency*:

$$\sum_{i\in N} \varphi_i^S(v) = \sum_{i\in N} x_i^\pi = \sum_{i\in N} \big(v(A_i^\pi \cup i) - v(A_i^\pi)\big) = v(N).$$

*Aditivity*: Let $u, v$ be coalitional games. By $\mathbf{x}^{u,\pi}$ and $\mathbf{x}^{v,\pi}$ we denote the corresponding marginal vectors in $\mathbb{R}^n$. It can easily be checked that $\mathbf{x}^{u+v,\pi} = \mathbf{x}^{u,\pi} + \mathbf{x}^{v,\pi}$. Then, for each player $i \in N$,

$$\begin{aligned}\varphi_i^S(u+v) &= \sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^{u+v,\pi} = \sum_{\pi\in\Pi} \frac{1}{n!} \cdot (x_i^{u,\pi} + x_i^{v,\pi}) = \\ &\sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^{u,\pi} + \sum_{\pi\in\Pi} \frac{1}{n!} \cdot x_i^{v,\pi} = \varphi_i^S(u) + \varphi_i^S(v).\end{aligned}$$

*Null player property*: If $i$ is a null player, then necessarily $x_i^\pi = 0$ for any $\pi \in \Pi$. Then $\varphi_i^S(v) = 0$.

*Symmetry*: Let $i, j \in N$ be players such that

$$v(A \cup i) = v(A \cup j), \qquad \text{for each coalition } A \subseteq N \setminus ij. \tag{2}$$

We want to show that $\varphi_i(v) = \varphi_j(v)$. We will use this formula for the Shapley value:

$$\varphi_i(v) = \sum_{A\subseteq N\setminus i} c_A\big(v(A\cup i) - v(A)\big), \qquad \text{where } c_A := \frac{1}{n\binom{n-1}{|A|}}.$$

We can split the sum into two parts,

$$\varphi_i(v) = \underbrace{\sum_{A\subseteq N\setminus ij} c_A\big(v(A\cup i) - v(A)\big)}_{\text{S1}} + \underbrace{\sum_{\substack{A\subseteq N\setminus i\\ j\in A}} c_A\big(v(A\cup i) - v(A)\big)}_{\text{S2}},$$

and similarly for $\varphi_j(v)$. It follows from (2) that S1 is

$$\sum_{A\subseteq N\setminus ij} c_A\big(v(A\cup i) - v(A)\big) = \sum_{A\subseteq N\setminus ij} c_A\big(v(A\cup j) - v(A)\big).$$

The second sum S2 be further decomposed into

$$\sum_{\substack{A\subseteq N\setminus i\\ j\in A}} c_A\big(v(A\cup i) - v(A)\big) = \sum_{B\subseteq N\setminus ij} c_B\big(v(B\cup ji) - v(B\cup j)\big), \tag{3}$$

and, similarly,

$$\sum_{\substack{A\subseteq N\setminus j\\ i\in A}} c_A\big(v(A\cup j) - v(A)\big) = \sum_{B\subseteq N\setminus ij} c_B\big(v(B\cup ij) - v(B\cup i)\big). \tag{4}$$

By (2), the right-hand sides of (3) and (4) are equal. Hence, $\varphi_i(v) = \varphi_j(v)$.

**Solution 16.**
We want to prove that for all $A, B \subseteq N$,

$$v(A \cup B) + v(A \cap B) \geq v(A) + v(B). \tag{5}$$

Bankruptcy game $v$ is nonnegative by the definition, $v(A) \geq 0$ for all $A \subseteq N$. Game $v$ is monotone:

$$\text{If } A \subseteq B, \text{ then } v(A) \leq v(B).$$

The above inequality holds trivially when $v(A) = 0$, since $v(B) \geq 0$ by nonnegativity of $v$. If $v(A) > 0$ and $B \supseteq A$, then

$$v(A) = e - \boldsymbol{d}(N \setminus A) \leq e - \boldsymbol{d}(N \setminus B) = v(B),$$

since $\boldsymbol{d}(N \setminus A) = \boldsymbol{d}(N) - \boldsymbol{d}(A)$. We will make use of the following identity, which is a direct consequence of definition (1):

$$\boldsymbol{d}(A) + \boldsymbol{d}(B) = \boldsymbol{d}(A \cup B) + \boldsymbol{d}(A \cap B). \tag{6}$$

*Case 1.* First, assume that

$$v(A \cap B) > 0.$$

By monotonicity of $v$, inequality (5) becomes

$$e - \boldsymbol{d}(N \setminus (A \cup B)) + e - \boldsymbol{d}(N \setminus (A \cap B)) \geq e - \boldsymbol{d}(N \setminus A) + e - \boldsymbol{d}(N \setminus B),$$

which is equivalent to

$$\boldsymbol{d}(A \cup B) + \boldsymbol{d}(A \cap B) \geq \boldsymbol{d}(A) + \boldsymbol{d}(B).$$

But we already know that the inequality is in fact an equality by (6).
*Case 2.* Suppose that

$$v(A \cap B) = 0.$$

Observe that this case means that there is no remaining estate left for claimans in $A \cap B$, since it was demanded by the claimants in $N \setminus (A \cap B)$. If $v(A) = v(B) = 0$, then (5) holds by nonnegativity of $v$. If $v(A) > 0$ and $v(B) = 0$, then (5) follows from monotonicity of $v$. To finish the proof, consider the case when $v(A) > 0$ and $v(B) > 0$. Inequality (5) reads as

$$e - \boldsymbol{d}(N \setminus (A \cup B)) \geq e - \boldsymbol{d}(N \setminus A) + e - \boldsymbol{d}(N \setminus B),$$

which is the same as

$$\boldsymbol{d}(N) + \boldsymbol{d}(A \cup B) \geq e + \boldsymbol{d}(A) + \boldsymbol{d}(B). \tag{7}$$

The assumption $v(A \cap B) = 0$ means that

$$\boldsymbol{d}(N \setminus (A \cap B)) \geq e,$$

where the left-hand side is by (6) precisely

$$\boldsymbol{d}(N \setminus (A \cap B)) = \boldsymbol{d}(N) - \boldsymbol{d}(A \cap B) = \boldsymbol{d}(N) + \boldsymbol{d}(A \cup B) - \boldsymbol{d}(A) - \boldsymbol{d}(B),$$

from which follows (7). This finishes the proof of (5).

**Solution 17.**

*Reflexivity*: For any $\boldsymbol{\alpha} \in \mathbb{R}^m$, we have $\boldsymbol{\alpha} \preceq \boldsymbol{\alpha}$ by the definition.

*Transitivity*: Let $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathbb{R}^m$ be such that

$$\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \quad \text{and} \quad \boldsymbol{\beta} \preceq \boldsymbol{\gamma}.$$

We want to conclude that $\boldsymbol{\alpha} \preceq \boldsymbol{\gamma}$. This is trivially true when $\boldsymbol{\alpha} = \boldsymbol{\beta}$ or $\boldsymbol{\beta} = \boldsymbol{\gamma}$. Assume that $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ and $\boldsymbol{\beta} \prec \boldsymbol{\gamma}$. Then, there exists $k$ such that $\alpha_j = \beta_j$ for all $j < k$ and $\alpha_k < \beta_k$. Analogously, for some $\ell$ we have $\beta_j = \gamma_j$ for all $j < \ell$ and $\alpha_\ell < \beta_\ell$. Let $i := \min\{k, \ell\}$. Then $\alpha_j = \beta_j = \gamma_j$ for each $j < i$, and $\alpha_i < \gamma_i$. This proves transitivity.

*Antisymmetry*: Let $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$ be such that

$$\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \quad \text{and} \quad \boldsymbol{\alpha} \neq \boldsymbol{\beta}.$$

We need to show that $\boldsymbol{\beta} \not\preceq \boldsymbol{\alpha}$. It follows from the assumptions that there is $k$ such that $\alpha_k < \beta_k$ and $\alpha_j = \beta_j$ for all $j < k$. Hence, $\boldsymbol{\beta} \not\preceq \boldsymbol{\alpha}$.

*Completeness*: For any $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$, we show that

$$\boldsymbol{\alpha} \preceq \boldsymbol{\beta} \quad \text{or} \quad \boldsymbol{\beta} \preceq \boldsymbol{\alpha}.$$

Let $\boldsymbol{\alpha} \not\preceq \boldsymbol{\beta}$. This means that there is $k$ such that $\alpha_j = \beta_j$ for all $j < k$ and $\alpha_k > \beta_k$, which implies $\boldsymbol{\beta} \prec \boldsymbol{\alpha}$.

**Solution 18.**

Let $\mathbf{x} \in \mathcal{I}(v)$. Then

$$v(1) + \cdots + v(n) \leq x_1 + \cdots + x_n = v(N).$$

Conversely, let $v(1) + \cdots + v(n) \leq v(N)$. We will show that $\mathcal{I}(v) \neq \emptyset$. Define an allocation $\mathbf{x} \in \mathbb{R}^n$ with coordinates

$$x_i := v(i) + \frac{v(N) - \sum_{j \in N} v(j)}{n}, \qquad i \in N. \tag{8}$$

Then $x_i \geq v(i)$ and

$$\sum_{i \in N} x_i = \sum_{i \in N} v(i) + n \cdot \frac{v(N) - \sum_{j \in N} v(j)}{n} = v(N).$$

Therefore, $\mathbf{x}$ is an imputation, $\mathbf{x} \in \mathcal{I}(v)$.

**Solution 19.**

We want to show that $\mathcal{I}(v) \neq$ for any superadditive game $v$. But this follows readily from Exercise 18 and Exercise 1.

**Solution 20.**

First, we note that $\mathbf{x}$ so defined is nothing but (8), so it is an imputation. To show that $\mathbf{x}$ is the nucleolus, we need only show that

$$e(\mathbf{x}) \preceq e(\mathbf{y}) \qquad \text{for all } \mathbf{y} \in \mathcal{I}(v) \text{ with } \mathbf{y} \neq \mathbf{x}. \tag{9}$$

It follows from the definition of $\mathbf{x}$ and from the definition of excess vector that $e(\emptyset, \mathbf{x}) = e(12, \mathbf{x}) = 0$, and

$$e(1, \mathbf{x}) = e(2, \mathbf{x}) = \underbrace{\tfrac{1}{2} \cdot (v(1) + v(2) - v(12))}_{\alpha}.$$

Therefore, by superadditivity the excess vector is

$$e(\mathbf{x}) = (0, 0, \alpha, \alpha).$$

Let $\mathbf{y} = (y_1, y_2) \neq \mathbf{x}$ be an imputation. This means that $e(i, \mathbf{y}) = v(i) - y_i \leq 0$ and, without loss of generality, suppose that $e(1, \mathbf{y}) \geq e(2, \mathbf{y})$. Then

$$e(\mathbf{y}) = (0, 0, e(1, \mathbf{y}), e(2, \mathbf{y})).$$

We will show that $\alpha < e(1, \mathbf{y})$, which proves (9). The assumption $\mathbf{y} \neq \mathbf{x}$ makes it possible to discuss two cases.

1. *Case* $x_1 > y_1$. This implies

$$\alpha = e(1, \mathbf{x}) = v(1) - x_1 < v(1) - y_1 = e(1, \mathbf{y}).$$

2. *Case* $x_1 < y_1$. Then $x_2 > y_2$ since $\mathbf{x}$ and $\mathbf{y}$ are imputations, and

$$\alpha = e(2, \mathbf{x}) = v(2) - x_2 < v(2) - y_2 = e(2, \mathbf{y}) \leq e(1, \mathbf{y}).$$

In conclusion, relation (9) was verified.

It can be verified directly that the Shapley value of $v$ is $\mathbf{x}$. Alternatively, it is possible to use Exercise 11 and observe that $v$ is supermodular, $\mathcal{C}(v) = \mathcal{I}(v)$, and $\mathbf{x}$ is the center of gravity of $\mathcal{I}(v)$.

**Solution 21.**

1. False. For example, the Shapley value of the glove game (Exercise 6) is not an element of the core.

2. True. This is exactly the symmetry of Shapley value. Or, it follows immediately from the formula for Shapley value.

3. False. For example, take a two-player game

$$v(1) = v(2) = v(12) = 1.$$

4. True.

5. False. The Shapley value is the only single-valued solution satisfying the four properties – efficiency, symmetry, the null player property, and additivity. Since the nucleolus has the first three properties and it is different from the Shapley value, it cannot be additive.

6. True. By efficiency of the Shapley value,

$$\varphi_n^S(v) = v(N) - \sum_{i=1}^{n-1} \varphi_i^S(v).$$

7. False. Consider a 2-player game $v$ that is not superadditive. Such a game satisfies the inequality $v(1,2) < v(1) + v(2)$, which implies $\varphi_1^S(v) < v(1)$.

8. True. The nucleolus is an imputation, hence the individual rationality.

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