Multiagent Systems

The Nucleolus

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How to divide the estate among claimants?

- After the death of a man, 3 creditors raise claims
- Depending on the estate, 3 variants of division are proposed

Allocations according to the Talmud rule

Estate/Demand	100	200	300
100	$100/3$	$100/3$	$100/3$
200	50	75	75
300	50	100	150

Table 1: Aumann and Maschler (1985)

From bankruptcy problems to bankruptcy games

Let $N = \{1, \ldots, n\}$ be the set of claimants.

Definition

A bankruptcy problem is a pair $(e, \boldsymbol{d})$, where $e \geq 0$ is the estate and $\boldsymbol{d} = (d_1, \ldots, d_n) \in \mathbb{R}^n_+$ are the demands such that

$$e \leq d_1 + \cdots + d_n.$$

Definition

A bankruptcy game associated with a bankruptcy problem $(e, \boldsymbol{d})$ is a coalitional game given by

$$v(A) = \max\left\{e - \boldsymbol{d}(N \setminus A), 0\right\}, \qquad A \subseteq N.$$

Solving bankruptcy games

Every bankruptcy game is supermodular, which implies that

- The core $\mathcal{C}(v)$ is nonempty and
- The Shapley value belongs to $\mathcal{C}(v)$

Example based on Table 1

$$e = 200,\ \boldsymbol{d} = (100, 200, 300), \text{ and } v(A) = \begin{cases} 200 & A = N, \\ 100 & A = 23, \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{C}(v) = \text{conv}\{(100, 100, 0), (100, 0, 100), (0, 200, 0), (0, 0, 200)\}$$

$$\varphi^S(v) = \frac{1}{3} \cdot (100, 250, 250)$$

We will study a division rule different from the Shapley value

- It applies to all coalitional games
- It coincides with the Talmud rule for bankruptcy problems
- The idea is that the maximal dissatisfaction of coalitions with an allocation should be minimized

The nucleolus

Measuring excess of coalitions in game v

The excess of coalition $A \subseteq N$ at allocation $\boldsymbol{x} \in \mathbb{R}^n$ is

$$e(A, \boldsymbol{x}) \coloneqq v(A) - \boldsymbol{x}(A)$$

Definition

Enumerate coalitions $A_1, \ldots, A_{2^n}$ from the highest excess:

$$e(A_1, \boldsymbol{x}) \geq \cdots \geq e(A_{2^n}, \boldsymbol{x}).$$

The excess vector is

$$e(\boldsymbol{x}) \coloneqq \big(e(A_1, \boldsymbol{x}), \ldots, e(A_{2^n}, \boldsymbol{x})\big) \in \mathbb{R}^{2^n}.$$

Lexicographic order

The excess vectors whose maximal excess is minimal are preferred.

Definition

For every $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^m$, define:

- $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ if there is $k = 1, \ldots, m$ such that for each $j < k$, $\alpha_j = \beta_j$ and $\alpha_k < \beta_k$
- $\boldsymbol{\alpha} \preceq \boldsymbol{\beta}$ if $\boldsymbol{\alpha} \prec \boldsymbol{\beta}$ or $\boldsymbol{\alpha} = \boldsymbol{\beta}$

The binary relation $\preceq$ is a *total order* on $\mathbb{R}^m$.

Example

Glove game

$$N = \{1, 2, 3\} \qquad v(A) = \begin{cases} 1 & A = 12, 13, N, \\ 0 & \text{otherwise.} \end{cases}$$

Allocations: $\boldsymbol{x} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, $\boldsymbol{y} = (1, 0, 0)$, $\boldsymbol{z} = \left(\frac{4}{6}, \frac{1}{6}, \frac{1}{6}\right)$

A	$\emptyset$	1	2	3	12	13	23	N
$e(A, \boldsymbol{x})$	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{3}$	$\boxed{\frac{1}{3}}$	$\frac{1}{3}$	$-\frac{2}{3}$	0
$e(A, \boldsymbol{y})$	0	-1	0	0	$\boxed{0}$	0	0	0
$e(A, \boldsymbol{z})$	0	$-\frac{2}{3}$	$-\frac{1}{6}$	$-\frac{1}{6}$	$\boxed{\frac{1}{6}}$	$\frac{1}{6}$	$-\frac{1}{3}$	0

$$e(\boldsymbol{y}) \prec e(\boldsymbol{z}) \prec e(\boldsymbol{x})$$

Imputations

We seek a lexicographic minimizer of excess vectors $e(\boldsymbol{x})$ over a set of allocations $\boldsymbol{x}$ in game v. *But which set to choose?*

- The core? If $\boldsymbol{x} \in \mathcal{C}(v)$ and $\boldsymbol{y} \notin \mathcal{C}(v)$, then $e(\boldsymbol{x}) \prec e(\boldsymbol{y})$
- But it can happen that $\mathcal{C}(v) = \emptyset \ldots$
- We define the set of imputations as

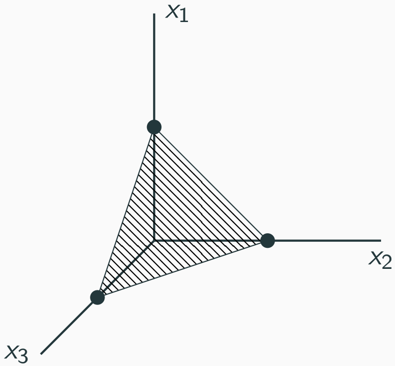
$$\mathcal{I}(v) \coloneqq \{\boldsymbol{x} \in \mathbb{R}^n \mid \underbrace{\boldsymbol{x}(N) = v(N)}_{\text{Efficiency}}, \quad \underbrace{x_i \geq v(i), \; i \in N}_{\text{Individual rationality}}\}$$

Claim

If v is a superadditive game, then $\mathcal{I}(v) \neq \emptyset$

Example: Imputations in a three-player game

$$v(123) > 0, \qquad v(1) = v(2) = v(3) = 0$$



$$\mathcal{I}(v) = \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = v(123), \quad x_1, x_2, x_3 \geq 0 \right\}$$

The nucleolus

Definition

Let v be a game with $\mathcal{I}(v) \neq \emptyset$. The nucleolus of v is the set

$$\mathcal{N}(v) := \left\{ \boldsymbol{x} \in \mathcal{I}(v) \mid e(\boldsymbol{x}) \preceq e(\boldsymbol{y}) \quad \text{for all } \boldsymbol{y} \in \mathcal{I}(v) \right\}$$

1. Is $\mathcal{N}(v)$ nonempty?
2. Is $\mathcal{N}(v)$ single-valued?
3. How to compute $\mathcal{N}(v)$?

Existence of the nucleolus

Theorem (Schmeidler, 1969)

Let v be a game with $\mathcal{I}(v) \neq \emptyset$. Then $|\mathcal{N}(v)| = 1$.

Properties of the nucleolus

- If $\mathcal{C}(v) \neq \emptyset$, then it contains $\mathcal{N}(v)$
- Efficiency
- Symmetry
- Null player property

Example: Solution of the original bankruptcy problem

Example based on Table 1

$$e = 200,\ \boldsymbol{d} = (100, 200, 300),\ \text{and}\ v(A) = \begin{cases} 200 & A = N, \\ 100 & A = 23, \\ 0 & \text{otherwise.} \end{cases}$$

Consider $\boldsymbol{x} = (50, 75, 75)$ and any $\boldsymbol{y} \in \mathcal{C}(v)$ to show $e(\boldsymbol{x}) \preceq e(\boldsymbol{y})$:

A	1	2	3	12	13	23
$e(A, \boldsymbol{x})$	-50	-75	-75	-125	-125	-50
$e(A, \boldsymbol{y})$	$-y_1$	$-y_2$	$-y_3$	$-y_1 - y_2$	$-y_1 - y_3$	$y_1 - 100$

The nucleolus of a two-player game

Example

Consider a superadditive game v with two players:

$$v(12) \geq v(1) + v(2)$$

- The set of imputations is the line segment

$$\mathcal{I}(v) = \{\boldsymbol{x} \in \mathbb{R}^2 \mid x_1 + x_2 = v(12),\ \ x_1 \geq v(1),\ \ x_2 \geq v(2)\}$$

- The nucleolus is allocation

$$\left(v(1) + \frac{v(12) - v(1) - v(2)}{2},\ v(2) + \frac{v(12) - v(1) - v(2)}{2}\right)$$

How to compute the nucleolus?

Computing the nucleolus in many classes of games is NP-hard.

Algorithm

Input: Game v such that $\mathcal{I}(v) \neq \emptyset$

1. Find $X_1 \subseteq \mathcal{I}(v)$ minimizing the maximal excess
2. Find $X_2 \subseteq X_1$ minimizing the second highest excess
3. Continue this procedure...
4. ...until it yields a single imputation, the nucleolus

Minimizing the maximal excess

LP with variables $\boldsymbol{x} = (x_1, \ldots, x_n), t$

$$\begin{aligned} \text{Minimize} \quad & t \\ \text{subject to} \quad & e(A, \boldsymbol{x}) \leq t, \quad \emptyset \neq A \subset N, \\ & \boldsymbol{x} \in \mathcal{I}(v) \end{aligned}$$

$$t_1 := \text{the value of the LP}$$

$$X_1 \times \{t_1\} := \text{the set of optimal solutions}$$

- If X_1 is a singleton, then $X_1 = \mathcal{N}(v)$
- Else put

$$\mathcal{F}_1 := \{A \subset N \mid e(A, \boldsymbol{x}) = t_1, \ \ \boldsymbol{x} \in X_1\}$$

Minimizing the second highest excess

LP with variables $\boldsymbol{x} = (x_1, \ldots, x_n), t$

$$\begin{aligned} \text{Minimize} \quad & t \\ \text{subject to} \quad & e(A, \boldsymbol{x}) \leq t, \quad A \notin \mathcal{F}_1, \ \emptyset \neq A \subset N \\ & \boldsymbol{x} \in X_1 \end{aligned}$$

$$t_2 := \text{the value of the LP}$$

$$X_2 \times \{t_2\} := \text{the set of optimal solutions}$$

- If X_2 is a singleton, then $X_2 = \mathcal{N}(v)$
- Else put

$$\mathcal{F}_2 := \{A \subset N \mid e(A, x) = t_2, \ \boldsymbol{x} \in X_2\}$$

Minimizing the k-th highest excess

The algorithm stops when X_k is a singleton at step $k \leq 2^n$.

- Each t_i is the *i-th highest excess*
- Each $\mathcal{F}_i$ is the collection of coalitions with excess t_i
- At each step, $\mathcal{F}_i$ contains at least one new coalition

Summary: Properties of solution concepts

Property/Solution	*core*	*Shapley value*	*Banzhaf value*	*nucleolus*
Nonemptiness	—	✓	✓	☞
Efficiency	✓	✓	—	✓
Individual rationality	✓	☞	☞	✓
Symmetry	—	✓	✓	✓
Null player property	✓	✓	✓	✓
Additivity	—	✓	✓	—

☞ This property is true for every superadditive game

References

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