Multiagent Systems (BE4M36MAS)

Extensive-Form Games

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Previously ... on multi-agent systems.

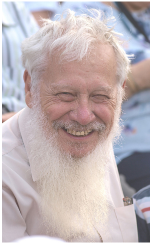
1. Mixed Strategies
2. Minimax Theorem
3. Linear Program for computing NE in zero-sum normal-form games
4. alternative solution concepts

Correlated Equilibrium

Consider again the following game:

	L	**R**
U	$(2, 1)$	$(0, 0)$
D	$(0, 0)$	$(1, 2)$

Wouldn't it be better to coordinate 50:50 between the outcomes (U,L) and (D,R)? Can we achieve this coordination? We can use *a correlation device*—a coin, a streetlight, commonly observed signal—and use this signal to avoid unwanted outcomes.



Robert Aumann

Correlated Equilibrium

Definition (Correlated Equilibrium (simplified))

Let $\mathcal{G} = (\mathcal{N}, \mathcal{A}, u)$ be a normal-form game and let σ be a probability distribution over joint pure strategy profiles $\sigma \in \Delta(\mathcal{A})$. We say that σ is a correlated equilibrium if for every player i and every action $a_i' \in \mathcal{A}_i$ it holds

$$\sum_{a \in \mathcal{A}} \sigma(a) u_i(a_i, a_{-i}) \geq \sum_{a \in \mathcal{A}} \sigma(a) u_i(a_i', a_{-i})$$

Corollary

For every Nash equilibrium there exists a corresponding Correlated Equilibrium.

Computing Correlated Equilibrium

Computing a Correlated equilibrium is easier compared to Nash and can be found by linear programming even in general-sum case:

$$\sum_{a \in \mathcal{A}} \sigma(a) u_i(a_i, a_{-i}) \geq \sum_{a \in \mathcal{A}} \sigma(a) u_i(a'_i, a_{-i}) \qquad \forall i \in \mathcal{N}, \forall a'_i \in \mathcal{A}_i$$

$$\sum_{a \in \mathcal{A}} \sigma(a) = 1 \qquad \sigma(a) \geq 0 \quad \forall a \in \mathcal{A}$$

Stackelberg Equilibrium

Finally, consider a situation where an agent is a central public authority (police, government, etc.) that needs to design and publish a policy that will be observed and reacted to by other agents.

- *the leader* – publicly commits to a strategy
- *the follower(s)* – play a Nash equilibrium with respect to the commitment of the leader

Stackelberg equilibrium is a strategy profile that satisfies the above conditions and maximizes the expected utility value of the leader:

$$\arg\max_{s \in \mathcal{S}; \forall i \in \mathcal{N} \setminus \{1\} s_i \in BR_i(s_{-i})} u_1(s)$$

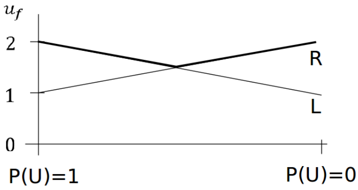
Stackelberg Equilibrium

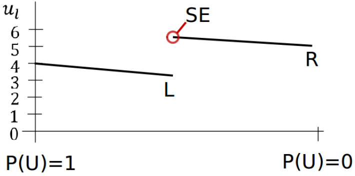
Consider the following game:

	$\mathbf{L}$	$\mathbf{R}$
$\mathbf{U}$	$(4,2)$	$(6,1)$
$\mathbf{D}$	$(3,1)$	$(5,2)$

$(\mathbf{U},\mathbf{L})$ is a Nash equilibrium.

What happens when the row player commits to play strategy $\mathbf{D}$ with probability 1? Can the row player get even more?





There may be Multiple Nash Equilibria

The followers need to break ties in case there are multiple NE:

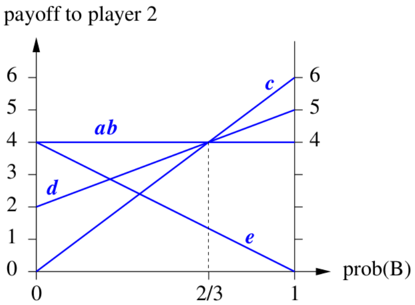
- arbitrary but fixed tie breaking rule
- *Strong SE* – the followers select such NE that maximizes the outcome of the leader (when the tie-braking is not specified we mean SSE),
- *Weak SE* – the followers select such NE that minimizes the outcome of the leader.

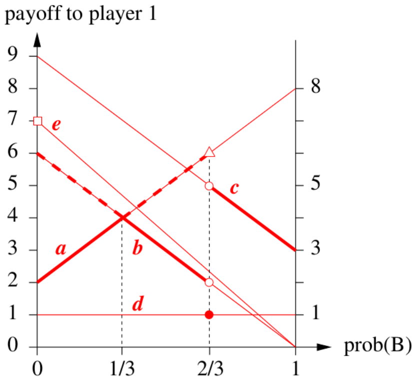
Exact Weak Stackelberg equilibrium does not have to exist.

Different Stackelberg Equilibria

Exact Weak Stackelberg equilibrium does not have to exist.

$1 \setminus 2$	a	b	c	d	e
T	$(2,4)$	$(6,4)$	$(9,0)$	$(1,2)$	$(7,4)$
B	$(8,4)$	$(0,4)$	$(3,6)$	$(1,5)$	$(0,0)$





Computing a Stackelberg equilibrium in NFGs

The problem is polynomial for two-players normal-form games; 1 is the leader, 2 is the follower.

Baseline polynomial algorithm requires solving $|\mathcal{A}_2|$ linear programs:

$$\max_{s_1 \in \mathcal{S}_1} \sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_1(a_1, a_2)$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_2(a_1, a_2) \geq \sum_{a_1 \in \mathcal{A}_1} s_1(a_1) u_2(a_1, a_2') \quad \forall a_2' \in \mathcal{A}_2$$

$$\sum_{a_1 \in \mathcal{A}_1} s_1(a_1) = 1$$

one for each $a_2 \in \mathcal{A}_2$ assuming a_2 is the best response of the follower.

Beyond Normal-Form Representations

One representation does not rule them all

Beyond Normal-Form Representations

Beyond Normal-Form Representations

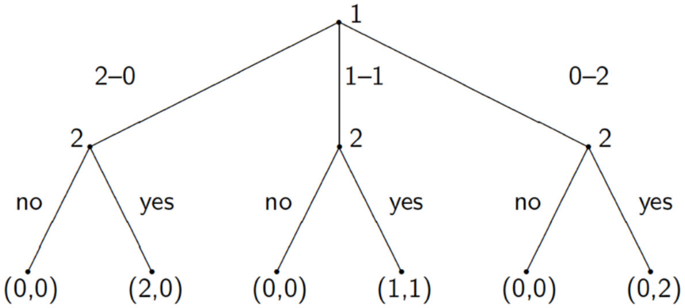
We can represent such sequential scenarios using the normal-form representation.

A strategy in such games have to reflect all possible situations we can encounter in a game (including due to the moves by the opponent and/or stochastic events). Therefore, we need to have an action prescribed to be played in each situation that can happen.

The obvious drawback of using this representation is that there is exponentially many possible strategies given a description of the game.

Extensive-Form Representation

We can use a more compact representation that is suitable for finite games termed *extensive-form games*.



Extensive-Form Games (EFGs)

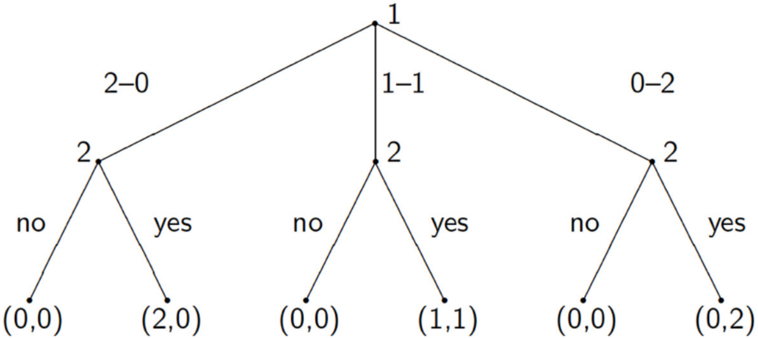
Formal Definition:

- players $\mathcal{N} = \{1, 2, \ldots, n\}$
- actions $\mathcal{A}$
- choice nodes (histories) $\mathcal{H}$
- action function $\chi : \mathcal{H} \to 2^{\mathcal{A}}$
- player function $\rho : \mathcal{H} \to \mathcal{N}$
- terminal nodes $\mathcal{Z}$
- successor function $\varphi : \mathcal{H} \times \mathcal{A} \to \mathcal{H} \cup \mathcal{Z}$
- utility function $u = (u_1, u_2, \ldots, u_n)\,;\; u_i : \mathcal{Z} \to \mathbb{R}$

A pure strategy of player i in an EFG is an assignment of an action for each state where player i acts

$$S_i := \prod_{h \in \mathcal{H}, \rho(h) = i} \chi(h)$$

Strategies in EFGs

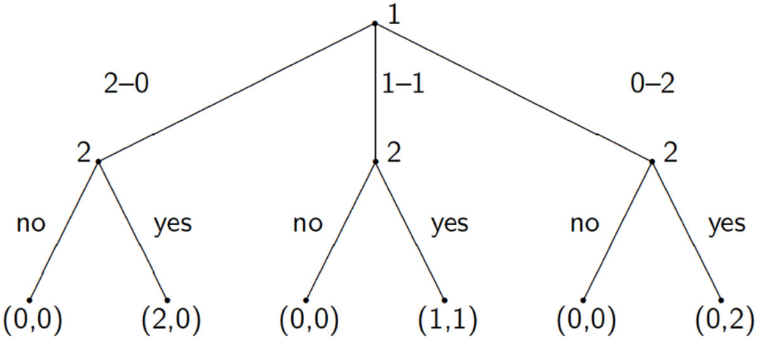


What are actions and strategies in this game?

$$\mathcal{A}_1 = \{2-0, 1-1, 0-2\};\ \mathcal{S}_1 = \{2-0, 1-1, 0-2\}$$

$$\mathcal{A}_2 = \{no, yes\};\ \mathcal{S}_2 = \{(no, no, no), (no, no, yes), \ldots, (yes, yes, yes)\}$$

Strategies in EFGs

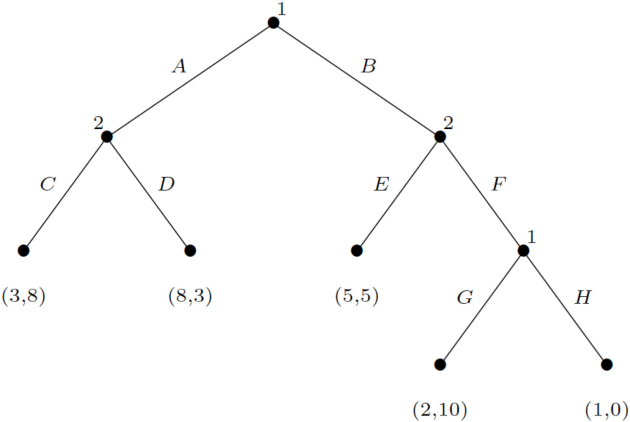


We can replace the function χ by multiplying actions so that an action can be applied only in a single state.

$$\mathcal{A}_2 = \{no_{\{2-0\}}, yes_{\{2-0\}}, no_{\{1-1\}}, yes_{\{1-1\}}, no_{\{0-2\}}, yes_{\{0-2\}}\};$$

$$\mathcal{S}_2 = \{(no_{\{2-0\}}, no_{\{1-1\}}, no_{\{0-2\}}), \ldots, (yes_{\{2-0\}}, yes_{\{1-1\}}, yes_{\{0-2\}})\}$$

Strategies in EFGs

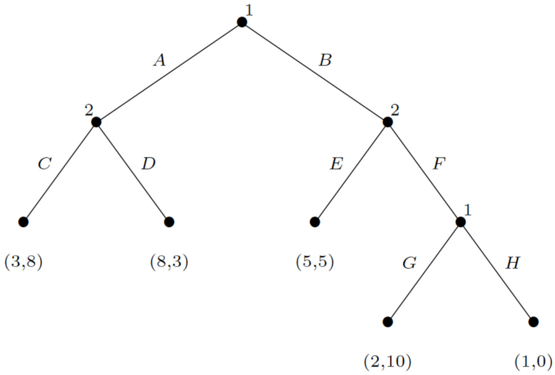


What are actions and strategies in this game?

$$\mathcal{S}_1 = \{(A, G), (A, H), (B, G), (B, H)\}$$

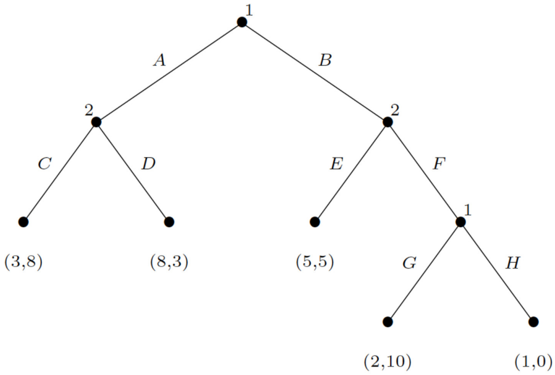
$$\mathcal{S}_2 = \{(C, E), (C, F), (D, E), (D, F)\}$$

Induced Normal Form



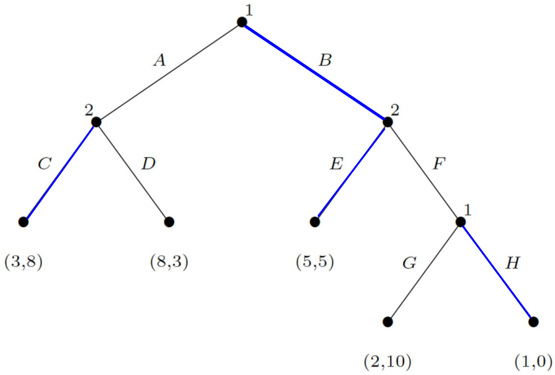
	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	$(3,8)$	$(3,8)$	$(8,3)$	$(8,3)$
(A,H)	$(3,8)$	$(3,8)$	$(8,3)$	$(8,3)$
(B,G)	$(5,5)$	$(2,10)$	$(5,5)$	$(2,10)$
(B,H)	$(5,5)$	$(1,0)$	$(5,5)$	$(1,0)$

Nash Equilibria in EFGs



	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	$(3,8)$	$(\mathbf{3},\mathbf{8})$	$(8,3)$	$(8,3)$
(A,H)	$(3,8)$	$(\mathbf{3},\mathbf{8})$	$(8,3)$	$(8,3)$
(B,G)	$(5,5)$	$(2,10)$	$(5,5)$	$(2,10)$
(B,H)	$(\mathbf{5},\mathbf{5})$	$(1,0)$	$(5,5)$	$(1,0)$

Nash Equilibria in EFGs - threats



	(C,E)	(C,F)	(D,E)	(D,F)
(A,G)	$(3,8)$	$(\mathbf{3},\mathbf{8})$	$(8,3)$	$(8,3)$
(A,H)	$(3,8)$	$(\mathbf{3},\mathbf{8})$	$(8,3)$	$(8,3)$
(B,G)	$(5,5)$	$(2,10)$	$(5,5)$	$(2,10)$
(B,H)	$(\mathbf{5},\mathbf{5})$	$(1,0)$	$(5,5)$	$(1,0)$

Nash Equilibria in EFGs

Not all Nash strategies are entirely “sequentially rational” in EFGs. Off the equilibrium path, the players may use irrational actions.

We use *refinements of NE* in EFGs to avoid this. The best known (for EFGs with perfect information) is **Subgame-perfect equilibrium**.

Definition (Subgame)

Given a perfect-information extensive-form game G, the subgame of G rooted at node h is the restriction of G to the descendants of h. The set of subgames of G consists of all of subgames of G rooted at some node in G.

Subgame-Perfect Equilibrium (SPE)

Definition (Subgame-perfect equilibrium)

The *subgame-perfect equilibria (SPE)* of a game G are all strategy profiles s such that for any subgame G' of G, the restriction of s to G' is a Nash equilibrium of G'.

function $\textsc{BackwardInduction}$(node h)
    if $h \in \mathcal{Z}$ **then**
        return $u(h)$
    end if
    best_util $\leftarrow \infty$
    for all $a \in \chi(h)$ **do**
        util_at_child $\leftarrow \textsc{BackwardInduction}(\varphi(h, a))$
        if *util_at_child*$_{\rho(h)}$ $>$ *best_util*$_{\rho(h)}$ **then**
            best_util $\leftarrow$ *util_at_child*
        end if
    end for
end function

Subgame-Perfect Equilibrium (SPE)

This is the same algorithm (in principle) that you know as Minimax (or Alpha-Beta pruning, or Negascout) and works (in general) for n-player games.

Corollary

Every extensive-form game with perfect information has at least one Nash equilibria in pure strategies that is also a Subgame-perfect equilibrium.

Is this correct? We have seen examples of games that do not have pure NE.

Not every game can be represented as an EFG with perfect information.

EFGs with Chance

We introduce a new "player" termed chance (or Nature) that plays using a fixed randomized strategy.

Formal Definition:

- players $\mathcal{N} = \{1, 2, \ldots, n\} \cup \{c\}$
- actions $\mathcal{A}$
- choice nodes (histories) $\mathcal{H}$
- action function $\chi : \mathcal{H} \to 2^{\mathcal{A}}$
- player function $\rho : \mathcal{H} \to \mathcal{N}$
- terminal nodes $\mathcal{Z}$
- successor function $\varphi : \mathcal{H} \times \mathcal{A} \to \mathcal{H} \cup \mathcal{Z}$
- stochastic transitions $\gamma : \Delta\{\chi(h) \mid h \in \mathcal{H}, \rho(h) = c\}$
- utility function $u = (u_1, u_2, \ldots, u_n)$; $u_i : \mathcal{Z} \to \mathbb{R}$

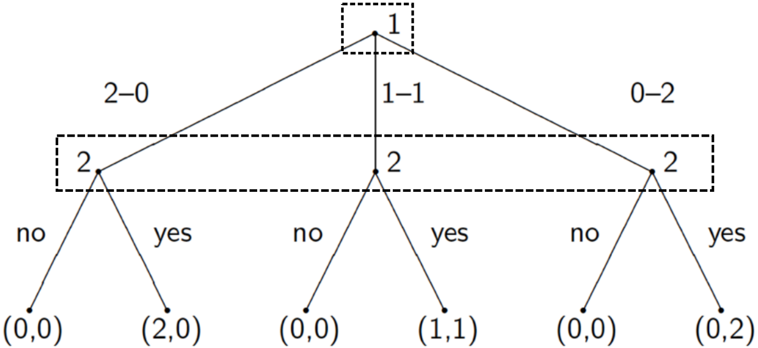
EFGs with Imperfect Information

When players are not able to observe the state of the game perfectly, we talk about *imperfect information games*. The states that are not distinguishable to a player belong to a single *information set*.

Formal Definition:

- $\mathcal{G} = (\mathcal{N}, \mathcal{A}, \mathcal{H}, \mathcal{Z}, \chi, \rho, \varphi, \gamma, u)$ is a perfect-information EFG.
- $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \ldots, \mathcal{I}_n)$ where $\mathcal{I}_i$ is a set of equivalence classes on choice nodes of a player i with the property that $\rho(h) = \rho(h') = i$ and $\chi(h) = \chi(h')$, whenever $h, h' \in I$ for some information set $I \in \mathcal{I}_i$
- we can use $\chi(I)$ instead of $\chi(h)$ for some $h \in I$

Strategies in EFGs with Imperfect Information



What are actions and strategies in this game?

$$\mathcal{A}_1 = \{2-0, 1-1, 0-2\};\ \mathcal{S}_1 = \{2-0, 1-1, 0-2\}$$

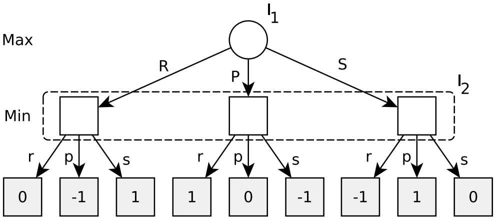
$$\mathcal{A}_2 = \{no, yes\};\ \mathcal{S}_2 = \{no, yes\}$$

Strategies in EFGs with Imperfect Information

There are no guarantees that a pure NE exists in imperfect information games.

Every finite game can be represented as an EFG with imperfect information.

	R	**P**	**S**
R	$(0,0)$	$(-1,1)$	$(1,-1)$
P	$(1,-1)$	$(0,0)$	$(-1,1)$
S	$(-1,1)$	$(1,-1)$	$(0,0)$



Strategies in EFGs with Imperfect Information

Mixed strategies are defined as before as a probability distribution over pure strategies.

There are also other types of strategies in EFGs, namely *behavioral strategies*:

- A *behavioral strategy* of player i is a product of probability distributions over actions in each information set

$$\beta_i : \prod_{I \in \mathcal{I}_I} \Delta(\chi(I))$$

There is a broad class of imperfect-information games in which the expressiveness of mixed and behavioral strategies coincide – *perfect recall games.* Intuitively speaking, in these games no player forgets any information she previously knew.

Perfect Recall in EFGs

Definition

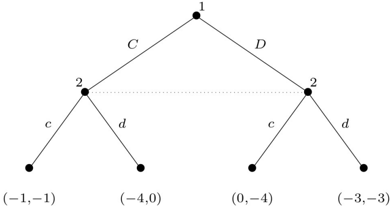
Player i has perfect recall in an imperfect-information game G if for any two nodes h,h' that are in the same information set for player i, for any path $h_0, a_0, \ldots, h_n, a_n, h$ from the root of the game tree to h and for any path $h_0, a'_0, \ldots, h'_m, a'_m, h'$ from the root to h' it must be the case that:

1. $n = m$
2. for all $0 \leq j \leq n$, h_j and h'_j are in the same equivalence class for player i
3. for all $0 \leq j \leq n$, if $\rho(h_j) = i$, then $a_j = a'_j$

Definition

We say that an EFG has a *perfect recall* if all players have perfect recall. Otherwise we say that the game has an *imperfect recall*.

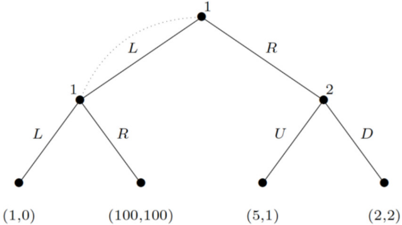
Perfect vs. Imperfect Recall



Conditioning on a complete history induces exponentially large strategies.

They are easier to solve (we will see next week).

Strategies can be compactly represented (we will see next week).



Not necessary information can be forgotten; hence, the strategies can be (exponentially) smaller.

<u>Much</u> harder to solve.

Nash equilibrium (in behavioral strategies) might not exist.

Imperfect Recall Game with no NE