Statistical Machine Learning (BE4M33SSU)
Lecture 11: Markov Random Fields

Czech Technical University in Prague

- Markov Random Fields & Gibbs Random Fields
- Approximated Inference for MRFs
- (Generative) Parameter learning for MRFs

1. Motivation: Two Examples from Computer Vision

Example 1 (Image segmentation)**.** Recall the segmentation model used in the EM-Algorithm lab, where $x\colon D \to \mathbb{R}^3$ denotes an image and $s\colon D \to K$ denotes its segmentation (K – set of segment labels)

$$p(s) = \prod_{i\in D} p(s_i) = \frac{1}{Z(u)} \exp\Big[\sum_{i\in D} u_i(s_i)\Big] \quad \text{and} \quad p(x\,|\,s) = \prod_{i\in D} p(x_i\,|\,s_i)$$

This model is pixelwise independent and, consequently, so is the inference.

We want to take into account that:

- neighbouring pixels belong more often than not to the same segment,
- the segment boundaries are in most places smooth, . . .

We may consider e.g. a prior model for segmentations

$$p(s) = \frac{1}{Z(u)} \exp\Big[\sum_{i\in D} u_i(s_i) + \sum_{\{i,j\}\in E} u_{ij}(s_i, s_j)\Big],$$

where E are edges connecting neighbouring pixels in D.

1. Motivation: Two Examples from Computer Vision

Example 2 (Motion Flow)**.** Given two images $x, x' \colon D \to \mathbb{R}^3$ from a video, determine the motion flow, i.e. find a displacement vector v_i for each pixel $i \in D$.

- projections of the same 3D points look similar in x and x'.
- 3D points projected onto neighbouring image pixels move more often than not coherently.
- Assume a discriminative model $p(v \,|\, x, x')$ since the method does not intend to model the image appearance.

$$p(v \,|\, x, x') = \frac{1}{Z(x, x')} \exp\Big[-\sum_{i \in D} \|x_i - x'_{i+v_i}\|^2 - \alpha \sum_{\{i,j\} \in E} \|v_i - v_j\|^2\Big]$$

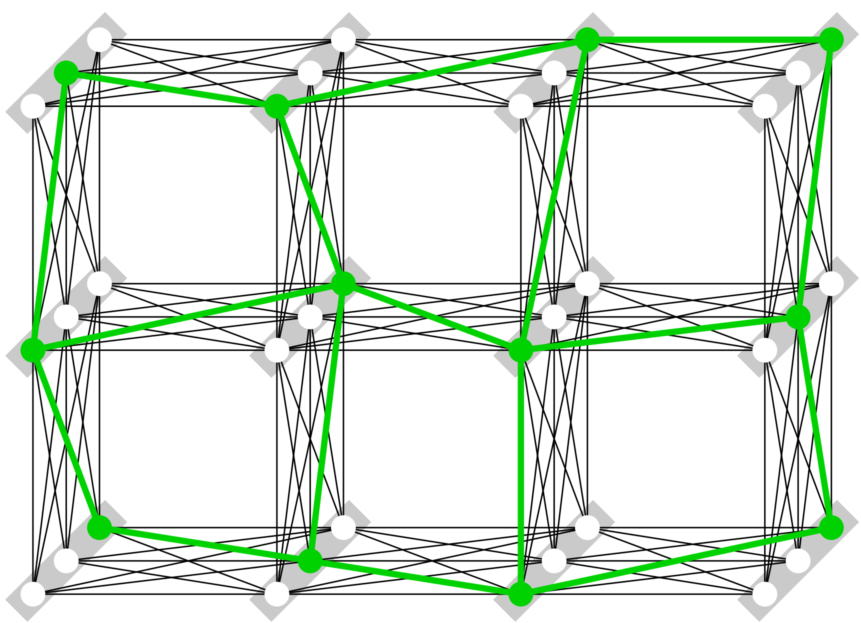
Such models can be generalised for stereo cameras and combined with segmentation approaches.

2. Markov Random Fields & Gibbs Random Fields

Let (V, E) denote an undirected graph and let $S = \{S_i \mid i \in V\}$ be a field of random variables indexed by the nodes of the graph and taking values from a finite set K.

Definition 1. A joint probability distribution $p(s)$ is a Gibbs Random Field on the graph (V, E) if it factorises over the the nodes and edges, i.e.

$$p(s) = \frac{1}{Z(u)} \exp\Big[\sum_{i \in V} u_i(s_i) + \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j)\Big].$$



Remark 1. This can be generalised to Gibbs random fields on hypergraphs.

2. Markov Random Fields & Gibbs Random Fields

Definition 2. A probability distribution $p(s)$ is a Markov Random Field w.r.t. graph (V, E) if

$$p(s_A, s_B \,|\, s_C) = p(\boldsymbol{s}_A \mid \boldsymbol{s}_C)\, p(\boldsymbol{s}_B \mid \boldsymbol{s}_C)$$

holds for any subsets $A, B \subset V$ and a separating set C.

Theorem 1 (Hammersley, Clifford, 1971)**.** *If the distribution $p(s)$ is an MRF w.r.t. graph (V, E) and strictly positive, then it is a GRF on the hypergraph defined by all cliques of (V, E) and vice versa.*

The following tasks for MRFs/GRFs are NP-complete

- Computing the most probable labelling $s^* \in \operatorname*{arg\,max}_{s \in K^V} p(s)$.
- Computing the normalisation constant

$$Z(u) = \sum_{\boldsymbol{s} \in K^V} \exp\Big[\sum_{i \in V} u_i(s_i) + \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j)\Big].$$

The same holds for computing marginal probabilities of $p(s)$.

3. Computing the most probable labelling of an MRF (Boolean case)

Consider $\log p(s)$, replace $u \to -u$. The task reads then

$$\sum_{i\in V} u_i(s_i) + \sum_{\{i,j\}\in E} u_{ij}(s_i, s_j) \to \min_{\boldsymbol{s}\in K^V}$$

The variables s_i, $i \in V$ are boolean: the functions u_i, u_{ij} can be written as polynomials in $s_i = 0, 1$, and, any function $u_{ij}(s_i, s_j)$ can be written as

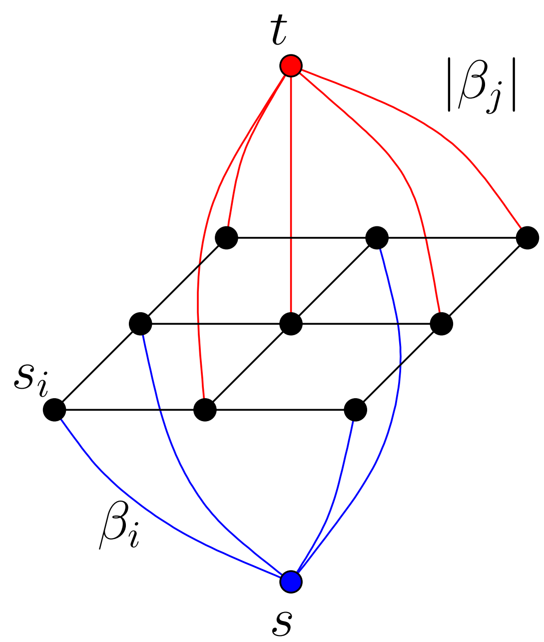
$$u_{ij}(s_i, s_j) = \alpha_{ij}|s_i - s_j| + g_i(s_i) + g_j(s_j).$$

Thus, after re-defining the unary functions $u_i(s_i)$, the task reads as

$$\begin{aligned} \boldsymbol{s}^* &= \arg\min_{\boldsymbol{s}\in K^V} \sum_{\{i,j\}\in E} \alpha_{ij}|s_i - s_j| + \sum_{i\in V} \beta_i s_i \\ &= \arg\min_{\boldsymbol{s}\in K^V} \sum_{\{i,j\}\in E} \alpha_{ij}|s_i - s_j| + \sum_{i\in V_+} \beta_i s_i + \sum_{i\in V_-} |\beta_i|(1 - s_i), \end{aligned}$$

where $V_+ = \{i \in V \mid \beta_i \geqslant 0\}$ and $V_- = V \setminus V_+$. This is a **MinCut-problem!**

3. Computing the most probable labelling of an MRF (Boolean case)



- If all edge weights are non-negative, i.e. $\alpha_{ij} \geqslant 0$, $\forall \{i, j\} \in E$: the task can be solved via MinCut – MaxFlow duality,
- If some of the α-s are negative: apply approximation algorithms, e.g. relax the discrete variables to $s_i \in [0, 1]$, consider an LP-relaxation of the task and solve the LP task e.g. by Tree-Reweighted Message Passing (Kolmogorov, 2006)

4. Computing the most probable labelling (general case)

Approximation algorithms for the general case, when $s_i \in K$

$$u(\boldsymbol{s}) = \sum_{i \in V} u_i(s_i) + \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j) \to \min_{s \in K^V}$$

Move making algorithms: Construct a sequence of labellings $\boldsymbol{s}^{(t)}$ with decreasing values of the objective function:

- Define neighbourhoods $\mathcal{N}(s) \subset K^V$ such that the task

  $$\operatorname*{arg\,min}_{s \in \mathcal{N}(s')} \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j) + \sum_{i \in V} u_i(s_i)$$

  is tractable for every s'.
- Iterate

  $$s^{(t+1)} \in \operatorname*{arg\,min}_{s \in \mathcal{N}(s^{(t)})} \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j) + \sum_{i \in V} u_i(s_i)$$

  until no further improvement possible.

4. Computing the most probable labelling (general case)

α-**Expansions** (Boykov et al., 2001)

- Define the neighbourhoods by choosing a label $\alpha \in K$ and setting

$$\mathcal{N}_\alpha(s) = \left\{ s' \in K^V \;\middle|\; s'_i = \alpha \text{ if } s'_i \neq s_i \right\}.$$

  Notice that $|\mathcal{N}_\alpha(s)| \sim 2^V$.

- The task

$$\operatorname*{arg\,min}_{s \in \mathcal{N}_\alpha(s')} \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j) + \sum_{i \in V} u_i(s_i)$$

  can be encoded as labelling problem with boolean variables.

- It can be solved by MinCut-MaxFlow if

$$u_{ij}(k, k') + u_{ij}(\alpha, \alpha) \leqslant u_{ij}(\alpha, k') + u_{ij}(k, \alpha)$$

  holds for all pairwise functions u_{ij} and all $k, k' \in K$.

Learning parameters of MRFs

Learning task: Given i.i.d. training data $\mathcal{T}^m = \{s^\ell \in K^V \mid \ell = 1, \ldots, m\}$, estimate the parameters u_i, u_{ij} of the MRF.

The maximum likelihood estimator reads

$$\log p_u(\mathcal{T}^m) = \frac{1}{m} \sum_{\ell=1}^{m} \Big[\sum_{\{i,j\} \in E} u_{ij}(s_i^\ell, s_j^\ell) + \sum_{i \in V} u_i(s_i^\ell) \Big] - \log Z(u) \rightarrow \max_{u_i, u_{ij}} .$$

It is intractable: the objective function is concave in u, but we can compute neither $\log Z(u)$ nor its gradient (in polynomial time).

We may use the **pseudo-likelihood** estimator (Besag, 1975) instead. It is based on the following observation

- Let $\mathcal{N}_i$ denote the neighbouring nodes of $i \in V$.
- We can compute the conditional distributions

$$p(s_i \mid s_{V \setminus i}) \stackrel{!}{=} p(s_i \mid s_{\mathcal{N}_i}) \sim e^{u_i(s_i)} \prod_{j \in \mathcal{N}_i} e^{u_{ij}(s_i, s_j)}$$

Learning parameters of MRFs

The pseudo-likelihood of an single example $\boldsymbol{s} \in \mathcal{T}^m$ is defined by

$$
\begin{aligned}
L_p(u) &= \sum_{i \in V} \log p_u(s_i \mid s_{\mathcal{N}_i}) \\
&= 2 \sum_{\{i,j\} \in E} u_{ij}(s_i, s_j) + \sum_{i \in V} u_i(s_i) - \sum_{i \in V} \log \sum_{s_i \in K} \exp\Big[u_i(s_i) + \sum_{j \in \mathcal{N}_i} u_{ij}(s_i, s_j) \Big]
\end{aligned}
$$

The pseudo-likelihood estimator is

- a concave function of the parameters u,
- tractable, i.e. both $L_p(u, \mathcal{T}^m)$ and its gradient are easy to compute,
- consistent.