

Mathematical identities

(Lukas Jelinek, CTU Prague, October 2020)

In the following, the rectangular system is characterized by coordinates (x, y, z) and unit vectors $[\mathbf{x}_0 \ \mathbf{y}_0 \ \mathbf{z}_0]$. The cylindrical system is characterized by coordinates (ρ, φ, z) and unit vectors $[\boldsymbol{\rho}_0 \ \boldsymbol{\varphi}_0 \ \mathbf{z}_0]$. The spherical system is characterized by coordinates (r, θ, φ) and unit vectors $[\mathbf{r}_0 \ \boldsymbol{\theta}_0 \ \boldsymbol{\varphi}_0]$.

0.1 Point Transformations

Rectangular - Cylindrical

$$\begin{aligned}x &= \rho \cos \varphi \\y &= \rho \sin \varphi \\z &= z\end{aligned}\tag{1}$$

Cylindrical - Rectangular

$$\begin{aligned}\rho &= \sqrt{x^2 + y^2} \\ \tan \varphi &= \frac{y}{x} \\ z &= z\end{aligned}\tag{2}$$

Rectangular - Spherical

$$\begin{aligned}x &= r \cos \varphi \sin \theta \\y &= r \sin \varphi \sin \theta \\z &= r \cos \theta\end{aligned}\tag{3}$$

Spherical - Rectangular

$$\begin{aligned}r &= \sqrt{x^2 + y^2 + z^2} \\ \tan \theta &= \frac{\sqrt{x^2 + y^2}}{z} \\ \tan \varphi &= \frac{y}{x}\end{aligned}\tag{4}$$

Cylindrical - Spherical

$$\begin{aligned}\rho &= r \sin \theta \\z &= r \cos \theta \\ \varphi &= \varphi\end{aligned}\tag{5}$$

Spherical - Cylindrical

$$\begin{aligned}r &= \sqrt{\rho^2 + z^2} \\ \tan \theta &= \frac{\rho}{z} \\ \varphi &= \varphi\end{aligned}\tag{6}$$

0.2 Vector Transformations

Rectangular - Cylindrical

$$\begin{aligned}\mathbf{x}_0 &= \boldsymbol{\rho}_0 \cos \varphi - \boldsymbol{\varphi}_0 \sin \varphi \\ \mathbf{y}_0 &= \boldsymbol{\rho}_0 \sin \varphi + \boldsymbol{\varphi}_0 \cos \varphi \\ \mathbf{z}_0 &= \mathbf{z}_0\end{aligned}\tag{7}$$

Cylindrical - Rectangular

$$\begin{aligned}\rho_0 &= \mathbf{x}_0 \cos \varphi + \mathbf{y}_0 \sin \varphi = \frac{\mathbf{x}_0 x + \mathbf{y}_0 y}{\sqrt{x^2 + y^2}} \\ \varphi_0 &= -\mathbf{x}_0 \sin \varphi + \mathbf{y}_0 \cos \varphi = \frac{-\mathbf{x}_0 y + \mathbf{y}_0 x}{\sqrt{x^2 + y^2}} \\ z_0 &= z_0\end{aligned}\tag{8}$$

Rectangular - Spherical

$$\begin{aligned}\mathbf{x}_0 &= \mathbf{r}_0 \sin \theta \cos \varphi + \boldsymbol{\theta}_0 \cos \theta \cos \varphi - \boldsymbol{\varphi}_0 \sin \varphi \\ \mathbf{y}_0 &= \mathbf{r}_0 \sin \theta \sin \varphi + \boldsymbol{\theta}_0 \cos \theta \sin \varphi + \boldsymbol{\varphi}_0 \cos \varphi \\ z_0 &= \mathbf{r}_0 \cos \theta - \boldsymbol{\theta}_0 \sin \theta\end{aligned}\tag{9}$$

Spherical - Rectangular

$$\begin{aligned}\mathbf{r}_0 &= \mathbf{x}_0 \sin \theta \cos \varphi + \mathbf{y}_0 \sin \theta \sin \varphi + \mathbf{z}_0 \cos \theta = \frac{\mathbf{x}_0 x + \mathbf{y}_0 y + \mathbf{z}_0 z}{\sqrt{x^2 + y^2 + z^2}} \\ \boldsymbol{\theta}_0 &= \mathbf{x}_0 \cos \theta \cos \varphi + \mathbf{y}_0 \cos \theta \sin \varphi - \mathbf{z}_0 \sin \theta = \frac{\mathbf{x}_0 z x + \mathbf{y}_0 z y - \mathbf{z}_0 (x^2 + y^2)}{\sqrt{x^2 + y^2} \sqrt{x^2 + y^2 + z^2}} \\ \boldsymbol{\varphi}_0 &= -\mathbf{x}_0 \sin \varphi + \mathbf{y}_0 \cos \varphi = \frac{-\mathbf{x}_0 y + \mathbf{y}_0 x}{\sqrt{x^2 + y^2}}\end{aligned}\tag{10}$$

Cylindrical - Spherical

$$\begin{aligned}\rho_0 &= \mathbf{r}_0 \sin \theta + \boldsymbol{\theta}_0 \cos \theta \\ \varphi_0 &= \varphi_0 \\ z_0 &= \mathbf{r}_0 \cos \theta - \boldsymbol{\theta}_0 \sin \theta\end{aligned}\tag{11}$$

Spherical - Cylindrical

$$\begin{aligned}\mathbf{r}_0 &= \rho_0 \sin \theta + \mathbf{z}_0 \cos \theta = \frac{\rho_0 \rho + \mathbf{z}_0 z}{\sqrt{\rho^2 + z^2}} \\ \boldsymbol{\theta}_0 &= \rho_0 \cos \theta - \mathbf{z}_0 \sin \theta = \frac{\rho_0 z - \mathbf{z}_0 \rho}{\sqrt{\rho^2 + z^2}} \\ \varphi_0 &= \varphi_0\end{aligned}\tag{12}$$

0.3 Differential Operators

Rectangular coordinate system

$$\begin{aligned}\nabla f &= \mathbf{x}_0 \frac{\partial f}{\partial x} + \mathbf{y}_0 \frac{\partial f}{\partial y} + \mathbf{z}_0 \frac{\partial f}{\partial z} \\ \nabla \cdot \mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ \nabla \times \mathbf{F} &= \mathbf{x}_0 \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) + \mathbf{y}_0 \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) + \mathbf{z}_0 \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \\ \nabla^2 f &= \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \\ \nabla^2 \mathbf{F} &= \nabla (\nabla \cdot \mathbf{F}) - \nabla \times \nabla \times \mathbf{F} = \mathbf{x}_0 \nabla^2 F_x + \mathbf{y}_0 \nabla^2 F_y + \mathbf{z}_0 \nabla^2 F_z\end{aligned}\tag{13}$$

Cylindrical coordinate system

$$\begin{aligned}
\nabla f &= \rho_0 \frac{\partial f}{\partial \rho} + \varphi_0 \frac{1}{\rho} \frac{\partial f}{\partial \varphi} + z_0 \frac{\partial f}{\partial z} \\
\nabla \cdot \mathbf{F} &= \frac{1}{\rho} \frac{\partial (\rho F_\rho)}{\partial \rho} + \frac{1}{\rho} \frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial F_z}{\partial z} \\
\nabla \times \mathbf{F} &= \rho_0 \left(\frac{1}{\rho} \frac{\partial F_z}{\partial \varphi} - \frac{\partial F_\varphi}{\partial z} \right) + \varphi_0 \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) + z_0 \left(\frac{1}{\rho} \frac{\partial (\rho F_\varphi)}{\partial \rho} - \frac{1}{\rho} \frac{\partial F_\rho}{\partial \varphi} \right) \\
\nabla^2 f &= \nabla \cdot (\nabla f) = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2} \\
\nabla^2 \mathbf{F} &= \nabla (\nabla \cdot \mathbf{F}) - \nabla \times \nabla \times \mathbf{F} = \\
&\rho_0 \left(\frac{\partial^2 F_\rho}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F_\rho}{\partial \rho} - \frac{F_\rho}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 F_\rho}{\partial \varphi^2} - \frac{2}{\rho^2} \frac{\partial F_\varphi}{\partial \varphi} + \frac{\partial^2 F_\rho}{\partial z^2} \right) + \\
&\varphi_0 \left(\frac{\partial^2 F_\varphi}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F_\varphi}{\partial \rho} - \frac{F_\varphi}{\rho^2} + \frac{1}{\rho^2} \frac{\partial^2 F_\varphi}{\partial \varphi^2} + \frac{2}{\rho^2} \frac{\partial F_\rho}{\partial \varphi} + \frac{\partial^2 F_\varphi}{\partial z^2} \right) + \\
&z_0 \left(\frac{\partial^2 F_z}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial F_z}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 F_z}{\partial \varphi^2} + \frac{\partial^2 F_z}{\partial z^2} \right)
\end{aligned} \tag{14}$$

Spherical coordinate system

$$\begin{aligned}
\nabla f &= r_0 \frac{\partial f}{\partial r} + \theta_0 \frac{1}{r} \frac{\partial f}{\partial \theta} + \varphi_0 \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \\
\nabla \cdot \mathbf{F} &= \frac{1}{r^2} \frac{\partial (r^2 F_r)}{\partial r} + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (F_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial F_\varphi}{\partial \varphi} \\
\nabla \times \mathbf{F} &= \frac{r_0}{r \sin \theta} \left(\frac{\partial}{\partial \theta} (F_\varphi \sin \theta) - \frac{\partial F_\theta}{\partial \varphi} \right) + \frac{\theta_0}{r} \left(\frac{1}{\sin \theta} \frac{\partial F_r}{\partial \varphi} - \frac{\partial (r F_\varphi)}{\partial r} \right) + \\
&\frac{\varphi_0}{r} \left(\frac{\partial (r F_\theta)}{\partial r} - \frac{\partial F_r}{\partial \theta} \right) \\
\nabla^2 f &= \nabla \cdot (\nabla f) = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2} \\
\nabla^2 \mathbf{F} &= \nabla (\nabla \cdot \mathbf{F}) - \nabla \times \nabla \times \mathbf{F} = \\
&r_0 \left(\frac{\partial^2 F_r}{\partial r^2} + \frac{2}{\rho} \frac{\partial F_r}{\partial r} - \frac{2}{r^2} F_r + \frac{1}{r^2} \frac{\partial^2 F_r}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial F_r}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F_r}{\partial \varphi^2} \right) + \\
&\theta_0 \left(\frac{\partial^2 F_\theta}{\partial r^2} + \frac{2}{\rho} \frac{\partial F_\theta}{\partial r} - \frac{1}{r^2 \sin^2 \theta} F_\theta + \frac{1}{r^2} \frac{\partial^2 F_\theta}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial F_\theta}{\partial \theta} \right) + \\
&\varphi_0 \left(\frac{\partial^2 F_\varphi}{\partial r^2} + \frac{2}{\rho} \frac{\partial F_\varphi}{\partial r} - \frac{1}{r^2 \sin^2 \theta} F_\varphi + \frac{1}{r^2} \frac{\partial^2 F_\varphi}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial F_\varphi}{\partial \theta} \right) + \\
&\left(\frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 F_\varphi}{\partial \varphi^2} + \frac{2}{r^2 \sin \theta} \frac{\partial F_r}{\partial \varphi} + \frac{2 \cot \theta}{r^2 \sin \theta} \frac{\partial F_\theta}{\partial \varphi} \right)
\end{aligned} \tag{15}$$

0.4 Differential Identities

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{b} \cdot (\mathbf{c} \times \mathbf{a}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c}) \mathbf{b} - (\mathbf{a} \cdot \mathbf{b}) \mathbf{c}$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c})$$

$$\nabla \times (\nabla f) = 0$$

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

$$\nabla (f \cdot g) = g \nabla f + f \nabla g$$

$$\nabla \cdot (\mathbf{F} f) = \mathbf{F} \cdot (\nabla f) + f (\nabla \cdot \mathbf{F})$$

$$\nabla \times (\mathbf{F} f) = (\nabla f) \times \mathbf{F} + f (\nabla \times \mathbf{F})$$

$$\nabla \left(\frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2}$$

$$\nabla \cdot \left(\frac{\mathbf{F}}{f} \right) = \frac{f (\nabla \cdot \mathbf{F}) - \mathbf{F} \cdot (\nabla f)}{f^2}$$

$$\nabla \times \left(\frac{\mathbf{F}}{f} \right) = \frac{f (\nabla \times \mathbf{F}) - (\nabla f) \times \mathbf{F}}{f^2}$$

$$\nabla (\mathbf{F} \cdot \mathbf{G}) = (\mathbf{F} \cdot \nabla) \mathbf{G} + (\mathbf{G} \cdot \nabla) \mathbf{F} + \mathbf{F} \times (\nabla \times \mathbf{G}) + \mathbf{G} \times (\nabla \times \mathbf{F})$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G})$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = \mathbf{F} (\nabla \cdot \mathbf{G}) - \mathbf{G} (\nabla \cdot \mathbf{F}) + (\mathbf{G} \cdot \nabla) \mathbf{F} - (\mathbf{F} \cdot \nabla) \mathbf{G}$$

$$\nabla (f \circ g) = (f' \circ g) \nabla g$$

$$\nabla \cdot (\mathbf{F} \circ g) = (\mathbf{F}' \circ g) \cdot \nabla g$$

$$\nabla \times (\mathbf{F} \circ g) = -(\mathbf{F}' \circ g) \times \nabla g$$

(16)

0.5 Integration Identities

$$\begin{aligned}
\int_V (\nabla \cdot \mathbf{F}) dV &= \oint_S \mathbf{F} \cdot d\mathbf{S} \\
\int_V (\nabla f) dV &= \oint_S f d\mathbf{S} \\
\int_V (\nabla \times \mathbf{F}) dV &= - \oint_S \mathbf{F} \times d\mathbf{S} \\
\int_V [f (\nabla^2 g)] dV &= \oint_S f (\nabla g) \cdot d\mathbf{S} - \int_V [(\nabla f) \cdot (\nabla g)] dV \\
\int_V [\mathbf{F} \cdot (\nabla \times \nabla \times \mathbf{G})] dV &= - \oint_S [\mathbf{F} \times (\nabla \times \mathbf{G})] \cdot d\mathbf{S} + \int_V [(\nabla \times \mathbf{F}) \cdot (\nabla \times \mathbf{G})] dV \\
\int_V [f (\nabla^2 g) - g (\nabla^2 f)] dV &= \oint_S [f (\nabla g) - g (\nabla f)] \cdot d\mathbf{S} \\
\int_V [\mathbf{G} \cdot (\nabla \times \nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \nabla \times \mathbf{G})] dV &= \oint_S [\mathbf{F} \times (\nabla \times \mathbf{G}) - \mathbf{G} \times (\nabla \times \mathbf{F})] \cdot d\mathbf{S} \\
\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} &= \oint_l \mathbf{F} \cdot d\mathbf{l} \\
\int_S (\nabla f) \times d\mathbf{S} &= - \oint_l f d\mathbf{l}
\end{aligned} \tag{17}$$

0.6 3D Fourier's Transformation

Definition

$$\text{FT} \{f(\mathbf{r})\} = \tilde{f}(\mathbf{k}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{r}) e^{-j\mathbf{k} \cdot \mathbf{r}} dx dy dz \tag{18}$$

$$\text{FT}^{-1} \{\tilde{f}(\mathbf{k})\} = f(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(\mathbf{k}) e^{j\mathbf{k} \cdot \mathbf{r}} dk_x dk_y dk_z$$

Convolution and its Fourier's transformation

$$f(\mathbf{r}) * g(\mathbf{r}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\mathbf{r} - \mathbf{r}') g(\mathbf{r}') dx' dy' dz' \tag{19}$$

$$\text{FT} \{f(\mathbf{r}) * g(\mathbf{r})\} = \tilde{f}(\mathbf{k}) \tilde{g}(\mathbf{k})$$

Differential operators and their Fourier's transformation

$$\text{FT} \{\nabla \cdot \mathbf{F}(\mathbf{r})\} = j\mathbf{k} \cdot \tilde{\mathbf{F}}(\mathbf{k}) \tag{20}$$

$$\text{FT} \{\nabla \times \mathbf{F}(\mathbf{r})\} = j\mathbf{k} \times \tilde{\mathbf{F}}(\mathbf{k})$$

0.7 Trigonometric identities

$$\sin^2(\alpha) = \frac{1 - \cos(2\alpha)}{2}$$

$$\cos^2(\alpha) = \frac{1 + \cos(2\alpha)}{2}$$

$$\sin(\alpha + \beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\sin(\alpha - \beta) = \sin(\alpha)\cos(\beta) - \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha + \beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$$

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

(21)

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\sin(\alpha) - \sin(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha + \beta}{2}\right)\cos\left(\frac{\alpha - \beta}{2}\right)$$

$$\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha + \beta}{2}\right)\sin\left(\frac{\alpha - \beta}{2}\right)$$

0.8 Identities involving separation vector

$$\mathbf{R} = \mathbf{r} - \mathbf{r}', \quad R = |\mathbf{R}|, \quad \mathbf{R}_0 = \frac{\mathbf{R}}{R}$$

$$\nabla R = \mathbf{R}_0$$

$$\nabla R^n = nR^{n-1}\nabla R = nR^{n-1}\mathbf{R}_0$$

$$\nabla \cdot \mathbf{R} = 3$$

$$\nabla \times \mathbf{R} = 0$$

$$\nabla \cdot \mathbf{R}_0 = \frac{2}{R}$$

(22)

$$\nabla(e^{-jkR}) = -jke^{-jkR}\mathbf{R}_0$$

$$\nabla^2\left(\frac{1}{R}\right) = -4\pi\delta(\mathbf{R})$$

$$(\nabla^2 + k^2)\frac{e^{-jkR}}{R} = -4\pi\delta(\mathbf{R})$$

$$\frac{\partial^2}{\partial r_i \partial r_j} \frac{1}{R} = \frac{3R_i R_j - R^2 \delta_{ij}}{R^5}, \quad R \neq 0$$

0.9 Useful integrals

$$\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \arctan\left(\frac{x}{a}\right)$$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin(x)$$

$$\int \frac{dx}{\sqrt{x^2-1}} = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \ln\left(x + \sqrt{x^2+a^2}\right)$$

$$\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{x}{a^2\sqrt{x^2+a^2}}$$

$$\int x^n \ln\left(\frac{a}{x}\right) dx = \frac{x^{n+1}}{(n+1)^2} + \frac{x^{n+1}}{n+1} \ln\left(\frac{a}{x}\right)$$

$$\int x e^{ax} dx = \frac{ax-1}{a^2} e^{ax}$$

$$\int x^2 e^{ax} dx = \frac{a^2 x^2 - 2ax + 2}{a^3} e^{ax}$$

(23)

$$\int \sin^3(x) dx = -\cos(x) + \frac{\cos^3(x)}{3}$$

$$\int \cos^3(x) dx = \sin(x) - \frac{\sin^3(x)}{3}$$

$$\int \frac{dx}{\cos(x)} = \ln\left(\frac{1+\sin(x)}{\cos(x)}\right)$$

$$\int \frac{dx}{\sin(x)} = \ln\left(\frac{1-\cos(x)}{\sin(x)}\right)$$

$$\int \frac{\cos(x) dx}{(1-a^2\cos^2(x))^{3/2}} = \frac{\sin(x)}{(1-a^2)\sqrt{1-a^2\cos^2(x)}}$$

$$\int \frac{\sin(x) dx}{(1-a^2\sin^2(x))^{3/2}} = \frac{-\cos(x)}{(1-a^2)\sqrt{1-a^2\sin^2(x)}}$$