STRUCTURED MODEL LEARNING (SS2019)
6. SEMINAR

Assignment 1. Let F be a $n \times n$ matrix with matrix elements F_{ij} and $a, b \in \mathbb{R}^n_+$ be two non-negative n-dimensional vectors such that their respective components sum up to 1. Consider the following linear optimisation problem

$$\begin{aligned} &\sum_{ij} F_{ij}\lambda_{ij} \to \min_{\lambda \geq 0} \\ \text{s.t. } &\sum_{j} \lambda_{ij} = a_i \;\; \forall i \\ &\sum_{i} \lambda_{ij} = b_j \;\; \forall j \end{aligned} \tag{1}$$

This problem can be interpreted as a transportation problem, where a is the vector of supplies, b is the vector of demands and F_{ij} is the unit transportation cost from supplier i to consumer j.

a) Construct the dual problem for (1).

b) (submodular polyhedron) Assume that the matrix F is submodular, i.e.

$$F_\tau + F_\sigma \geq F_{\tau \vee \sigma} + F_{\tau \wedge \sigma},$$

where τ and σ denote multi-indices (i, j) and the operations $\vee$ and $\wedge$ denote the element-wise maximum and minimum respectively. Prove that the transportation problem has an optimiser λ^* such that its non-zero (matrix) elements $\lambda^*_\tau > 0$ form a chain in the partially ordered multi-index space.

c) (Wasserstein distance) Consider the cost matrix $F_{ij} = |i - j|$. Prove that it is submodular. Show that the dual problem for this cost matrix can be further simplified to the task

$$\begin{aligned} &\sum_i a_i x_i - \sum_j b_j x_j \to \max_x \\ \text{s.t. } &x_i - x_j \leq |i - j| \;\; \forall i, j. \end{aligned}$$

Assignment 2. Prove consistency of the maximum likelihood estimator for the exponential family

$$p_u(x) = \frac{1}{Z} e^{\langle \Phi(x), u \rangle}$$

under the assumptions $\|\Phi(x)\| \leq 1 \; \forall x$ and $u \in \mathcal{B}_r$, where $\mathcal{B}_r$ denotes the closed ball with radius r centered at the origin.