2. The most probable realisation of a GRF

$s = \{s_i \mid i \in V\}$ is a K-valued Gibbs random field w.r.t. an undirected graph (V, E).

$$p(s) = \frac{1}{Z(u)} e^{\langle u, \varphi(s) \rangle} = \frac{1}{Z(u)} \exp \sum_{ij \in E} u_{ij}(s_i, s_j)$$

<u>Task</u>: Find the most probable realisation(s) $s^* \in \mathcal{S} = K^V$

$$s^* \in \underset{s \in \mathcal{S}}{\operatorname{argmax}} \frac{1}{Z(u)} e^{\langle u, \varphi(s) \rangle} = \underset{s \in \mathcal{S}}{\operatorname{argmax}} \sum_{ij} u_{ij}(s_i, s_j) \qquad (1)$$

<u>Remarks</u>

- The task is NP-complete (e.g. reduce the max-clique task)
- The task is polynomial time solvable if (V, E) is a tree
- The task is polynomial time solvable if all functions $u_{ij} : K^2 \to \mathbb{R}$ are <u>supermodular</u> w.r.t. some total ordering of K

<u>Definition 1</u> Let K be a totally ordered finite set.

A function $u : K^n \to \mathbb{R}$ is <u>submodular</u> if

$$u(s) + u(s') \geq u(s \vee s') + u(s \wedge s')$$

holds for any pair $s, s' \in K^n$. A function u is <u>supermodular</u> if $-u$ is submodular.

<u>Remark 1</u> $s \vee s'$ and $s \wedge s'$ denote the elementwise maximum and minimum of the tuples $s, s' \in K^n$

In general, we have to rely on approximation algorithms.

One option: relax the discrete optimisation problem to a <u>linear optimisation</u> problem

LP-relaxation

Let us start from an upper bound for (1) and minimise it w.r.t. re-parametrisations

$$\max_{s \in K^V} \sum_{ij \in E} u_{ij}(s_i, s_j) \leq \sum_{ij \in E} \max_{s_i, s_j} u_{ij}(s_i, s_j)$$

$$\max_{s \in K^V} \sum_{ij \in E} u_{ij}(s_i, s_j) \leq \sum_{ij \in E} \max_{s_i, s_j} \left[\varphi_{ij}(s_i) + u_{ij}(s_i, s_j) + \varphi_{ji}(s_j) \right] \to \min_{\varphi}$$

$$\text{s.t.} \quad \sum_{j \in N_i} \varphi_{ij}(s_i) = 0 \quad \forall i \in V, \ \forall s_i \in K$$

Make it a linear optimisation problem

$$\sum_{ij \in E} c_{ij} \to \min_{\varphi, c}$$

$$\text{s.t.} \quad c_{ij} - \varphi_{ij}(s_i) - \varphi_{ji}(s_j) \geq u_{ij}(s_i, s_j) \quad \forall ij \in E, \ \forall s_i, s_j \in K \qquad (2a)$$

$$\sum_{j \in N_i} \varphi_{ij}(s_i) = 0 \quad \forall i \in V, \ \forall s_i \in K$$

Construct its dual problem

$$\sum_{ij \in E} \sum_{s_i, s_j} u_{ij}(s_i, s_j) \lambda_{ij}(s_i, s_j) \to \max_{\lambda \geq 0}$$

$$\text{s.t.} \quad \sum_{s_i, s_j} \lambda_{ij}(s_i, s_j) = 1 \quad \forall ij \in E \qquad (2b)$$

$$\sum_{s_j} \lambda_{ij}(s_i, s_j) = \lambda_i(s_i) \quad \forall ij \in E, \ \forall s_i \in K$$

Remarks

- The λ_{ij} describe relaxed labellings (weights). They encode a labelling if they are integral, i.e. $\lambda_{ij}(s_i, s_j) = 0,1$
- It might seem that the λ can be interpreted as marginal probabilities. This is not true

$$\lambda \in \operatorname{aff} \mathcal{P}(\mathcal{S}) \cap \mathbb{R}^n_+, \quad \text{whereas} \quad \mu \in \operatorname{conv} \mathcal{P}(\mathcal{S})$$

- If L_* is the optimal value of the LP (2) and M_* is the optimal value of (1), then $L_* \geq M_*$ and, in general, there is an integrality gap, i.e. $L_* > M_*$

Theorem 1 If all functions in (1) are supermodular (w.r.t. some total ordering of K), then there is no integrality gap between the optimal values of (1) and its LP relaxation (2).

Proof (idea)

- Let λ^* be an optimiser of (2b). Find the highest label with non-zero weight in each node

$$k_i^* = \max\{k \in K \mid \lambda_i^*(k) > 0\}$$

- Show, there is another optimiser $\tilde{\lambda}^*$ s.t.

$$\tilde{\lambda}_{ij}^*(k_i^*, k_j^*) > 0 \quad \forall\, ij \in E$$

- Conclude, that the labelling $s^*\colon s_i = k_i^*$ is optimal