1. Random fields (undirected graphical models)

- A K-valued random field is a collection $\{S_i \mid i \in V\}$ of K-valued random variables
- $i \in V$ can be: pixels, object parts, etc.
- $k \in K$ can be: colours, segment labels, depth values, poses of object parts, etc.
- $s \in K^V \triangleq \mathcal{S}$ denotes realisations $s_i \in K$, $i \in V$
- $p \in \mathcal{P}$ denotes a p.d. or p.m. $p : \mathcal{S} \to \mathbb{R}_+$

1A. Exponential families

- $S = \{S_i \mid i \in V\}$ is a K-valued random field
- $\varphi : \mathcal{S} = K^V \to \mathbb{R}^n$ is a random vector

Consider the task

$$\inf_{p \in \mathcal{P}} \Big\{ \sum_{s \in \mathcal{S}} p(s) \log p(s) \;\Big|\; \mathbb{E}_p(\varphi) = \mu, \; \sum_{s \in \mathcal{S}} p(s) = 1 \Big\}$$

Lagrange function

$$L(p) = \sum_{s \in \mathcal{S}} p(s) \log p(s) - \langle u, \mathbb{E}_p(\varphi) - \mu \rangle - \lambda \Big[\sum_{s \in \mathcal{S}} p(s) - 1 \Big]$$

$$\cdots\to \qquad p_u(s) = \exp\big[\langle u, \varphi(s) \rangle - \log Z(u) \big],$$

where $u \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ are Lagrange ~~mul~~ multipliers and $Z(u)$ is the normalisation constant

$$Z(u) = \sum_{s \in \mathcal{S}} e^{\langle u, \varphi(s) \rangle}$$

Questions

(1) Which $\mu \in \mathbb{R}^n$ qualify?

$$\mathbb{E}_p(\varphi) = \sum_{s \in \mathcal{S}} p(s) \varphi(s) = \mu \qquad p(s) > 0 \;\; \forall s \in \mathcal{S}$$

$$\sum_{s \in \mathcal{S}} p(s) = 1$$

$\Rightarrow \mu$ is in the relative interior of $\operatorname{conv} \varphi(\mathcal{S})$, where

$$\varphi(\mathcal{S}) = \{\varphi(s) \mid s \in \mathcal{S}\} \subset \mathbb{R}^n$$

(2) $u \mapsto \mu$? Yes, because

$$\nabla \log Z(u) = \nabla \log \sum_{s \in \mathcal{S}}' e^{\langle u, \varphi(s) \rangle}$$

$$= \frac{1}{Z(u)} \sum_{s \in \mathcal{S}}' e^{\langle u, \varphi(s) \rangle} \varphi(s) = \mathbb{E}_u(\varphi) = \mu$$

(3) $\mu \mapsto u$? (provided that $\mu \in \operatorname{ri}(\operatorname{conv} \varphi(\mathcal{S}))$)

No, not always. p_u is unique, but not u!

$$\langle u, \varphi(s) \rangle - \log Z(u) = \langle \tilde{u}, \varphi(s) \rangle - \log Z(\tilde{u}) \quad \forall s \in \mathcal{S}$$

i.e.

$$\langle u - \tilde{u}, \varphi(s) \rangle = \text{const}_s \quad \forall s \in \mathcal{S}$$

$$\{u \in \mathbb{R}^n \mid \langle u, \varphi(s) \rangle = \text{const}_s, \ \forall s \in \mathcal{S}\} = L^\perp$$

$$\operatorname{aff} \varphi(\mathcal{S}) = \varphi(s_0) - \underbrace{\operatorname{span}(\varphi(\mathcal{S}) - \varphi(s_0))}_{L} \quad \text{for any } s_0 \in \mathcal{S}$$

Suppose we write

$\operatorname{aff} \varphi(\mathcal{S}) = \{u \in \mathbb{R}^n \mid Au = b\}$ with some $A : \mathbb{R}^n \to \mathbb{R}^m$,

then $L = \operatorname{Ker} A$ and $L^\perp = \operatorname{Im} A^T$.

<u>Answer:</u>

- if $\operatorname{aff} \varphi(\mathcal{S}) = \mathbb{R}^n \Rightarrow \mu \mapsto u$ is a mapping
- if $\operatorname{aff} \varphi(\mathcal{S}) = \{u \in \mathbb{R}^n \mid Au = b\} \neq \mathbb{R}^n \Rightarrow$

  we can re-parametrise u by

$$u \to u + A^T \psi, \quad \psi \in \mathbb{R}^m$$

and have

$$\langle u + A^T\psi, \varphi(s)\rangle = \langle u, \varphi(s)\rangle + \langle \psi, b\rangle$$

$$\log Z(u + A^T\psi) = \log Z(u) + \langle \psi, b\rangle$$

B. Graphical models on undirected graphs

- $S = \{S_i \mid i \in V\}$ is a K-valued random field
- (V, E) is an undirected graph
- Let us fix marginal distributions $p(s_i, s_j) = \mu_{ij}(s_i, s_j)$ for all edges $\{i,j\} \in E$ and all $s_i, s_j \in K$

We search the p.d. with maximal entropy (i.e. minimal information) under the above constraints. Everything said in A. applies because any marginal prob. $p(s_i = k, s_j = k')$ can be seen as expectation of a random variable $\varphi_{ijkk'}(s) = \delta_{s_i k}\,\delta_{s_j k'}$.

Hence, if $\mu_{ij}(s_i, s_j)$ define a valid set of marginal distributions, then the task has a unique solution

$$p_u(s) = \frac{1}{Z(u)} \exp \sum_{ij \in E} u_{ij}(s_i, s_j) = \frac{1}{Z(u)} \exp \langle u, \varphi(s)\rangle,$$

where

$$\varphi(s) = \left(\varphi_{ijkk'}(s)\right)_{ij \in E,\, k,k' \in K} \in \mathbb{R}^{|E||K|^2}$$

However, the potentials $u_{ij} : K^2 \to \mathbb{R}$ are defined up to re-parametrisations only.

The distribution p_u is a Gibbs random field and factorises over the edges of the graph (V, E).

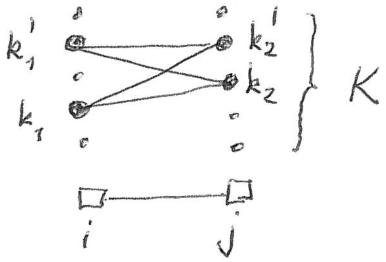
<u>Remark 1</u> All this can be generalised to (undirected) hypergraphs

Let us now analyse the possible re-parametrisations

$$u \to u + v \quad \text{s.t.} \quad \langle v, \Phi(s) \rangle = \mathrm{const}_s$$

a) Fix an edge $\{ij\}$ and consider four labellings s^k, $k = 1,2,3,4$ which coincide on $V \setminus \{ij\}$ and differ on $\{ij\}$ as follows

$$(s_i^1, s_j^1) = (k_1, k_2)$$
$$(s_i^2, s_j^2) = (k_1, k_2')$$
$$(s_i^3, s_j^3) = (k_1', k_2)$$
$$(s_i^4, s_j^4) = (k_1', k_2')$$



We have

$$\langle v, \Phi(s^1) + \Phi(s^4) - \Phi(s^2) - \Phi(s^3) \rangle =$$

$$= v_{ij}(k_1, k_2) + v_{ij}(k_1', k_2') - v_{ij}(k_1, k_2') - v_{ij}(k_1', k_2) \overset{!}{=} 0$$

This holds for any edge $\{ij\}$ and any k_1, k_1', k_2, k_2'.
It follows that

$$v_{ij}(s_i, s_j) = \varphi_{ij}(s_i) + \varphi_{ji}(s_j)$$

with some functions $\varphi_{ij}(s_i)$ for <u>directed</u> edges

b) Fix a node $i \in V$ and consider two labellings s^1, s^2 which differ on i only: $s_i^1 = k$, $s_i^2 = k'$.
We have

$$\langle v, \Phi(s^1) \rangle - \langle v, \Phi(s^2) \rangle =$$

$$= \sum_{j \in N_i} \varphi_{ij}(s_i = k) - \sum_{j \in N_i} \varphi_{ij}(s_i = k') \overset{!}{=} 0$$

This holds for any node $i \in V$ and any $k, k' \in K$.
Hence, it follows that

$$\sum_{j \in \mathcal{N}_i}' \psi_{ij}(s_i) = const_s$$

Finally, we have all possible re-parametrisations given by

$$u_{ij}(s_i, s_j) \longrightarrow \psi_{ij}(s_i) + u_{ij}(s_i, s_j) + \psi_{ji}(s_j)$$

$$\text{s.t.} \quad \sum_{j \in \mathcal{N}_i}' \psi_{ij}(s_i) = c_i$$

Problems: All tasks

a) given potentials $u_{ij}: K^2 \to \mathbb{R} \ \forall \{ij\} \in E$, compute $Z(u)$ and/or marginal probabilities $p_u(s_i)$, $p_u(s_i, s_j)$

b) check whether $\mu_{ij}: K^2 \to \mathbb{R}_+ \ \forall \{ij\} \in E$ represent a valid system of pairwise marginal prob's

c) given a valid system of pairwise marginals $\mu_{ij}: K^2 \to \mathbb{R}_+ \ \forall \{ij\} \in E$, compute the potentials u_{ij}

d) given the potentials $u_{ij}: K^2 \to \mathbb{R} \ \forall \{ij\} \in E$, find the most probable realisations (labellings)

$$\operatorname*{argmax}_{s \in \mathcal{S}} p_u(s) = \\ = \operatorname*{argmax}_{s \in \mathcal{S}} \sum_{ij \in E}' u_{ij}(s_i, s_j)$$

are NP-hard. Polynomial time complexity algorithms exist if (V, E) is acyclic.