

### 9.3.4 Note on homographies that are rotations

First notice that homogeneous coordinates of points and lines constructed as combinations of joins and meets indeed behave under a homography as homogeneous coordinates constructed from affine coordinates of points.

Secondly, when the homography is a rotation and homogeneous coordinates are unit vectors, all  $\lambda$ 's become equal to one, the determinant of  $H$  is one and  $H^{-T} = H$ . Therefore, all homogeneous coordinates in the previous formulas become related just by  $H$ .

## 9.4 Vanishing points

When modeling perspective projection in the affine space with affine projection planes, we meet somewhat unpleasant situations. For instance, imagine a projection of two parallel lines  $K, L$ , which are in a plane  $\tau$  in the space into the projection plane  $\pi$  through the center  $C$ , Figure 9.10

The lines  $K, L$  project to image lines  $k, l$ . As we go with two points  $X, Y$  along the lines  $k, l$  away from the projection plane, their images  $x, y$  get closer and closer to the point  $v$  in the image but they do not reach point  $v$ . We shall call this point of convergence of lines  $K, L$  the *vanishing point*

## 9.5 Vanishing line and horizon

If we take all sets of parallel lines in  $\tau$ , each set with a different direction, then all the points of convergence in the image will fill a complete line  $h$ .

The line  $h$  is called the *vanishing line* or the *horizon*<sup>8</sup> when  $\tau$  is the ground plane.

Now, imagine that we project all points from  $\tau$  to  $\pi$  using the affine geometrical projection model. Then, no point from  $\tau$  will project to  $h$ .

<sup>7</sup>Úběžník in Czech.

<sup>8</sup>Horizont in Czech

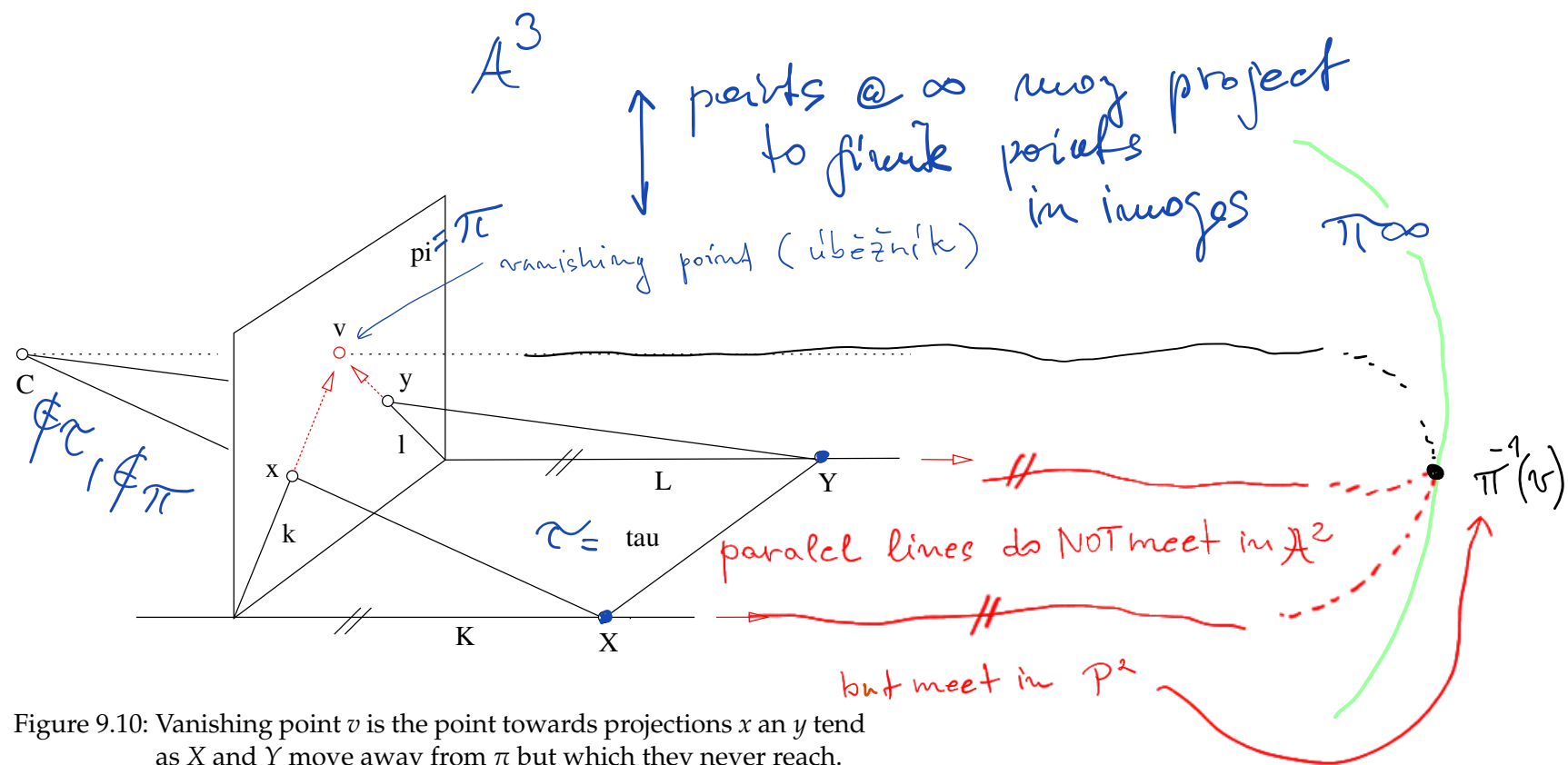


Figure 9.10: Vanishing point  $v$  is the point towards projections  $x$  and  $y$  tend as  $X$  and  $Y$  move away from  $\pi$  but which they never reach.

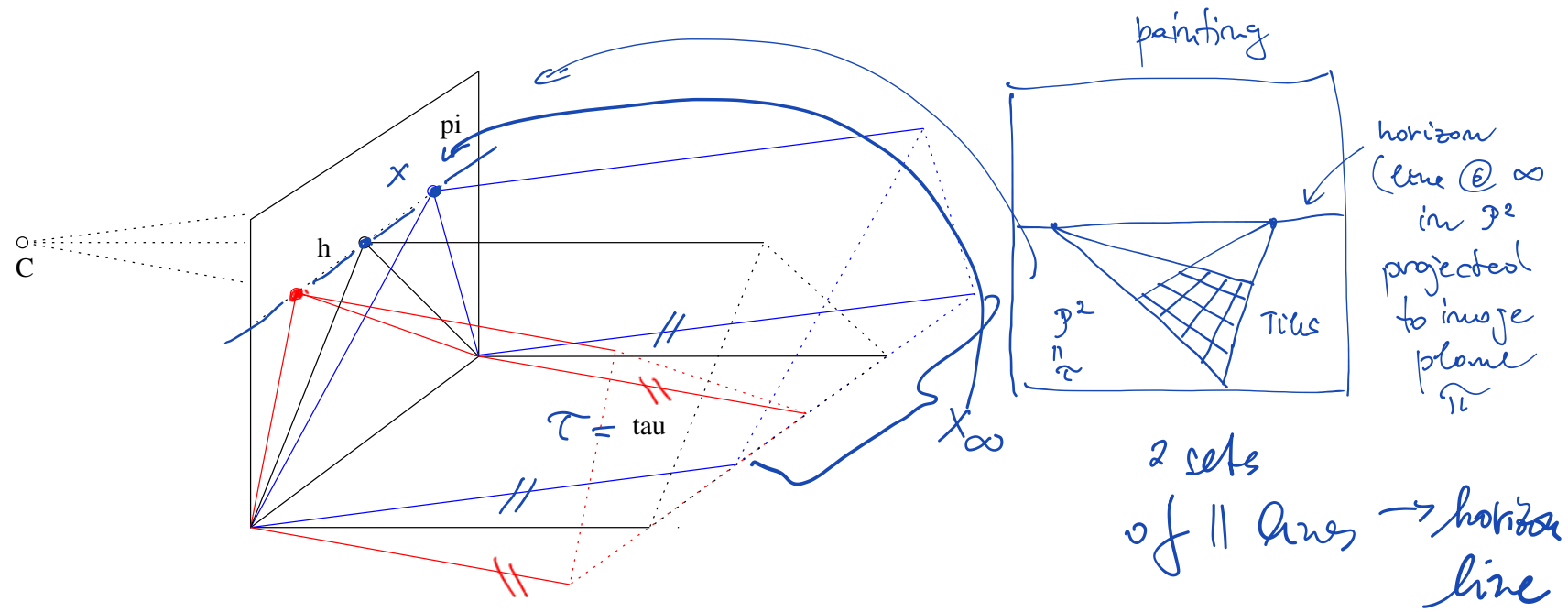


Figure 9.11: Vanishing line (horizon)  $h$  is the line of vanishing points.

Similarly, when projecting in the opposite direction, i.e.  $\pi$  to  $\tau$ , line  $h$  has no image, i.e. it does not project anywhere to  $\tau$ .

When using the affine geometrical projection model with the real projective plane to model the perspective projection (which is equivalent to the algebraic model in  $\mathbb{R}^3$ ), all points of the projective plane  $\tau$  (obtained as the projective completion of the affine plane  $\tau$ ) will have exactly one image in the projective plane  $\pi$  (obtained as the projective completion of the affine plane  $\pi$ ) and vice versa. This total symmetry is useful and beautiful.

## 10 Projective space

### 10.1 Motivation – the union of ideal points of all affine planes

Figure 10.1(a) shows a perspective image of three sets of parallel lines generated by sides of a cube in the three-dimensional real affine space. The images of the three sets of parallel lines converge to vanishing points  $V_1$ ,  $V_2$  and  $V_3$ . The cube has six faces. Each face generates two pairs of parallel lines and hence two vanishing points. Each face generates an affine plane which can be extended into a projective plane by adding the line of ideal points of that plane. The projection of the three ideal lines are vanishing lines  $l_{12} = V_1 \vee V_2$ ,  $l_{23} = V_2 \vee V_3$  and  $l_{31} = V_3 \vee V_1$ . Imagine now all possible affine planes of the three-dimensional affine space and their corresponding ideal points. Let us take the union  $V$  of the sets of ideal points of all such planes. There is exactly one ideal point for every set of parallel lines in  $V$ , i.e. there is a one-to-one correspondence between elements of  $V$  (ideal points) and directions in the three-dimensional affine space. Notice also that every plane  $\pi$  generates one ideal line  $l_\infty$  of its ideal points and that all other planes parallel with  $\pi$  generate the same  $l_\infty$ , Figure 10.1

It suggests itself to extend the three-dimensional affine space by adding the set  $V$  to it, analogically to how we have extended the affine plane. In this new space, all parallel lines will intersect. We will call this space the *three-dimensional real projective space* and denote it  $\mathbb{P}^3$ . Let us develop an algebraic model of  $\mathbb{P}^3$ . It is practical to require this model to encompass the model of the real projective plane. The real projective plane is modeled

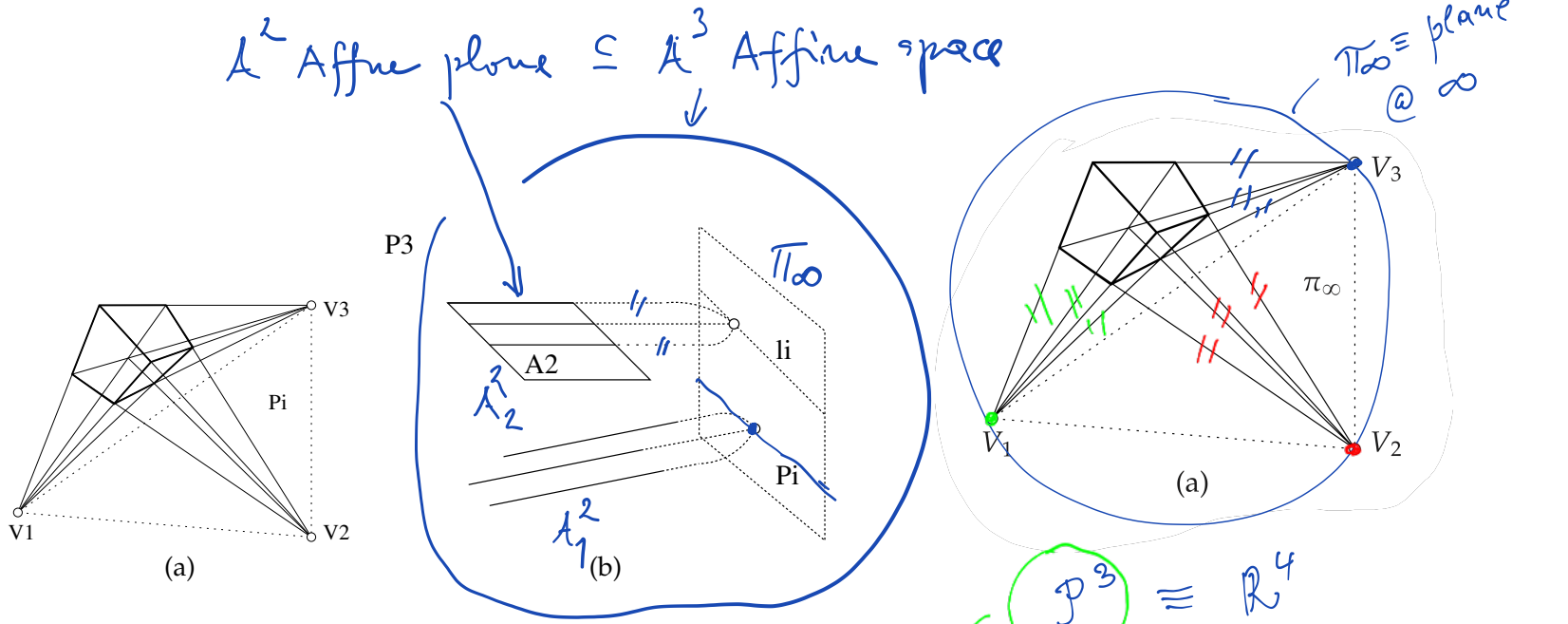


Figure 10.1: (a) A perspective image of a cube generates three vanishing points  $V_1$ ,  $V_2$  and  $V_3$  and hence also three vanishing lines  $l_{12}$ ,  $l_{23}$  and  $l_{31}$ . (b) Every plane adds one line of ideal points to the three-dimensional affine space. Every ideal point corresponds to one direction, i.e. to a set of parallel lines. Each ideal line corresponds to a set of parallel planes.

$P^3 \equiv R^4$

Points  $\equiv$  1D subspaces of  $R^4$

$$P^3 \ni X \equiv \alpha \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \neq \vec{0} \quad \neq 0 \quad \begin{bmatrix} x/w \\ y/w \\ z/w \end{bmatrix} \in P^3$$

$\downarrow$

$$\begin{bmatrix} x/w \\ y/w \\ z/w \\ 1 \end{bmatrix}$$

$= 0 @ \infty$   
 $\begin{bmatrix} x \\ y \\ z \\ 0 \end{bmatrix}$  } direction of line  $\wedge \pi_\infty$

$P^2 \equiv R^3$  (1D subspaces of  $R^3$ )

algebraically by subspaces of  $\mathbb{R}^3$ . Let us observe that subspaces of  $\mathbb{R}^4$  will be a convenient algebraic model of  $\mathbb{P}^3$ .

We start with the three-dimensional real affine space  $\mathbb{A}^3$  and fix a coordinate system  $(O, \delta)$  with  $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$ . An affine plane  $\pi$  is a set of points of  $\mathbb{A}^3$  represented in  $(O, \delta)$  by the set of vectors

$$\pi = \{[x, y, z]^\top \mid ax + by + cz + d = 0, a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0\} \quad (10.1)$$

We see that the point of  $\pi$  represented by vector  $[x, y, z]^\top$  can also be represented by one-dimensional subspace  $\{\lambda [x, y, z, 1]^\top \mid \lambda \in \mathbb{R}\}$  of  $\mathbb{R}^4$  and hence  $\pi$  can be seen as the set

$$\pi = \{ \{ \lambda [x, y, z, 1]^\top \mid \lambda \in \mathbb{R} \} \mid [a, b, c, d] [x, y, z, 1]^\top = 0, a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0 \} \quad (10.2)$$

of one-dimensional subspaces of  $\mathbb{R}^4$ .

Notice that we did not require  $\lambda \neq 0$  in the above definition. This is because we establish the correspondence between a vector  $[x, y, z]$  and the corresponding complete one-dimensional subspace  $\{\lambda [x, y, z, 1]^\top, \lambda \in \mathbb{R}\}$  of  $\mathbb{R}^4$  and since every linear space contains zero vector, we admit zero  $\lambda$ .

Every  $[x, y, z]^\top \in \mathbb{R}^3$  represents in  $(O, \delta)$  a point of  $\mathbb{A}^3$  and hence the subset

$$\mathbb{A}^3 = \{ \{ \lambda [x, y, z, 1]^\top \mid \lambda \in \mathbb{R} \} \mid x, y, z \in \mathbb{R} \} \quad (10.3)$$

of one-dimensional subspaces of  $\mathbb{R}^4$  represents  $\mathbb{A}^3$ .

We observe that we have not used all one-dimensional subspaces of  $\mathbb{R}^4$  to represent  $\mathbb{A}^3$ . The subset

$$\pi_\infty = \{ \{ \lambda [x, y, z, 0]^\top \mid \lambda \in \mathbb{R} \} \mid x, y, z \in \mathbb{R}, x^2 + y^2 + z^2 \neq 0 \} \quad (10.4)$$

of one-dimensional subspaces of  $\mathbb{R}^4$  is in one-to-one correspondence with all non-zero vectors of  $\mathbb{R}^3$ , i.e. in one-to-one correspondence with the set of directions in  $\mathbb{A}^3$ . This is the set of ideal points which we add to  $\mathbb{A}^3$  to get the three-dimensional real projective space

$$\mathbb{P}^3 = \{ \{ \lambda [x, y, z, w]^\top \mid \lambda \in \mathbb{R} \} \mid x, y, z, w \in \mathbb{R}, x^2 + y^2 + z^2 + w^2 \neq 0 \} \quad (10.5)$$

which is the set of all one-dimensional subspaces of  $\mathbb{R}^4$ . Notice that  $\mathbb{P}^3 = \mathbb{A}^3 \cup \pi_\infty$ .

**§1 Points** Every non-zero vector of  $\mathbb{R}^4$  generates a one-dimensional subspace and thus represents a point of  $\mathbb{P}^3$ . The zero vector  $[0,0,0,0]^\top$  does not represent any point.

**§2 Planes** Affine planes  $\pi_{\mathbb{A}^3}$ , Equation 10.2 are in one-to-one correspondence to the subset

$$\pi_{\mathbb{A}^3} = \{ \{ \lambda [a, b, c, d]^\top \mid \lambda \in \mathbb{R} \} \mid a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 \neq 0 \} \quad (10.6)$$

of the set of one-dimensional subspaces of  $\mathbb{R}^4$ . There is only one one-dimensional subspace of  $\mathbb{R}^4$ ,  $\{ \lambda [0, 0, 0, 1]^\top \mid \lambda \in \mathbb{R} \}$  missing in  $\pi_{\mathbb{A}^3}$ . It is exactly the one-dimensional subspace corresponding to the set  $\pi_\infty$  of ideal points of  $\mathcal{P}^3$

$$\pi_\infty = \{ \{ \lambda [x, y, z, w]^\top \mid \lambda \in \mathbb{R} \} \mid x, y, z, w \in \mathbb{R}, x^2 + y^2 + z^2 \neq 0, [0, 0, 0, 1] [x, y, z, w]^\top = 0 \} \quad (10.7)$$

We can take another view upon planes and observe that affine planes are in one-to-one correspondence with the three-dimensional subspaces of  $\mathbb{R}^4$ . The set  $\pi_\infty$  also corresponds to a three-dimensional subspace of  $\mathbb{R}^4$ . Hence  $\pi_\infty$  can be considered another plane, *the ideal plane* of  $\mathbb{P}^3$ .

The set of planes of  $\mathbb{P}^3$  can be hence represented by the set of one-dimensional subspaces of  $\mathbb{R}^4$

$$\pi_{\mathbb{P}^3} = \{ \{ \lambda [a, b, c, d]^\top \mid \lambda \in \mathbb{R} \} \mid a, b, c, d \in \mathbb{R}, a^2 + b^2 + c^2 + d^2 \neq 0 \} \quad (10.8)$$

but can also be viewed as the set of three-dimensional subspaces of  $\mathbb{R}^4$ .

We see that there is a duality between points and planes of  $\mathbb{P}^3$ . They both are represented by one-dimensional subspaces of  $\mathbb{R}^4$  and we see that point

Planes in  $\mathcal{P}^3$   
Determined by 3 pts

$$\begin{matrix} x & y & z \\ \equiv & | & | \\ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} & \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} & \begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} \\ \vec{x}_\phi & \vec{y}_\phi & \vec{z}_\phi \end{matrix}$$

$$\begin{bmatrix} -\vec{x}_\phi^\top \\ -\vec{y}_\phi^\top \\ -\vec{z}_\phi^\top \end{bmatrix} \vec{\pi}_\phi = 0$$

rank = 3 = 1

Diam of  $\langle \vec{\pi}_\phi \rangle$

Planes in  $\mathcal{P}^3 \equiv$  1D subsp of  $\mathbb{R}^4$

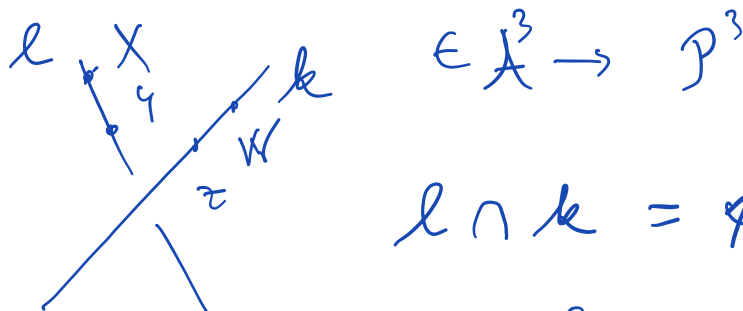


$X$  represented by vector  $\vec{X} = [x, y, z, w]^T$  is incident to plane  $\pi$  represented by vector  $\vec{\pi} = [a, b, c, d]^T$ , i.e.  $X \circ \pi$ , when

$$\vec{\pi}^T \vec{X} = [a \ b \ c \ d] \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = ax + by + cz + dw = 0 \quad (10.9)$$

§3 Lines Lines in  $\mathbb{P}^3$  are represented by two-dimensional subspaces of  $\mathbb{R}^4$ . Unlike in  $\mathbb{P}^2$ , lines are not dual to points.

Lines in  $\mathbb{P}^3$  are a new kind of animals.



Representation.

$$l \equiv \langle \vec{X}_\phi, \vec{Y}_\phi \rangle \rightarrow \langle \vec{p} \rangle \in \mathbb{R}^6$$

Plücker coordinates

lines  $\mathbb{P}^3 \equiv$  2D subspaces of  $\mathbb{R}^4$

$$\langle l \rangle \equiv \langle \vec{X}_\phi, \vec{Y}_\phi \rangle \ni \vec{0}$$

$$\langle k \rangle \equiv \langle \vec{Z}_\phi, \vec{W}_\phi \rangle \ni \vec{0}$$

$\langle l \rangle \cap \langle k \rangle = \vec{0}$   
represents No point

How to find  $K$ ?

1) 6 pts in space  $\rightarrow P \rightarrow [K \ R] \sim K R \vec{C}_\delta$

# 11 Camera auto-calibration

Camera auto-calibration is a process when the parameters of image formation are determined from properties of the observed scene or knowledge of camera motions. We will study camera auto-calibration methods and tasks related to metrology in images. We have seen in Chapter 7 that to measure the angle between projection rays we needed only matrix  $K$ . Actually, it is enough to know matrix  $\omega$

symmetric,  $\omega_{33} = 1$   $\omega = K^{-T} K^{-1} \in \mathbb{R}^{3 \times 3}$ , rank 3

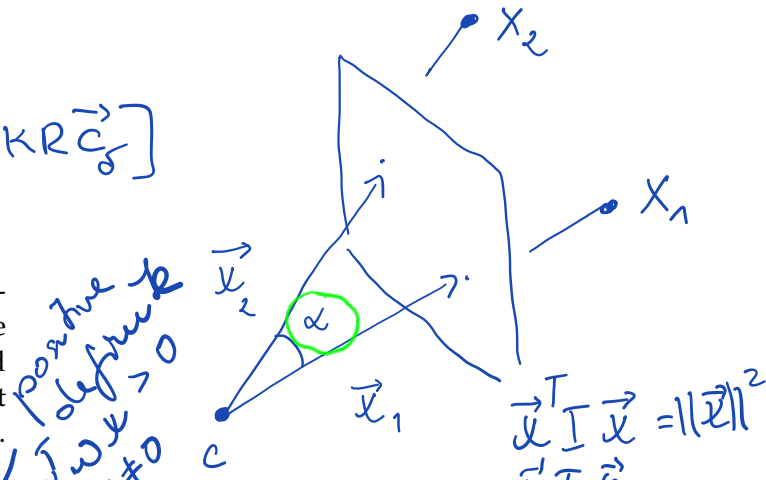
to measure the angle between the rays corresponding to image points  $\vec{x}_{1\beta}, \vec{x}_{2\beta}$  as

$$\cos \angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\beta}^T K^{-T} K^{-1} \vec{x}_{2\beta}}{\|K^{-1} \vec{x}_{1\beta}\| \|K^{-1} \vec{x}_{2\beta}\|} = \frac{\vec{x}_{1\beta}^T \omega \vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^T \omega \vec{x}_{1\beta}} \sqrt{\vec{x}_{2\beta}^T \omega \vec{x}_{2\beta}}} \quad (11.1)$$

Knowing  $\omega$  is however (almost) equivalent to knowing  $K$  since  $K$  can be recovered from  $\omega$  up to two signs as follows.

§ 1 Recovering  $K$  from  $\omega$  Let us give a procedure for recovering  $K$  from  $\omega$ . Assuming

$$\omega \rightarrow K = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad (11.2)$$



$\alpha = \angle(\vec{x}_1, \vec{x}_2) \dots$  angle  $\omega$   
 $\vec{x}^T \omega \vec{x} = \|\vec{x}\|^2$   
 $\vec{x}^T \omega \vec{y}$   
 $\omega \rightarrow \omega$   
 $\xi \delta \rightarrow \beta$



$\epsilon \dots$  orthonormal basis  
 $\gamma \dots$  orthogonal basis (f.)

$$\cos \alpha = \frac{\vec{x}_{1\epsilon}^T \vec{x}_{2\epsilon}}{\|\vec{x}_{1\epsilon}\| \|\vec{x}_{2\epsilon}\|} = \frac{\frac{1}{f} \vec{x}_{1\gamma}^T \frac{1}{f} \vec{x}_{2\gamma}}{\|\frac{1}{f} \vec{x}_{1\gamma}\| \|\frac{1}{f} \vec{x}_{2\gamma}\|} = *$$

<sup>1</sup>In [15],  $\omega$  is called the image of the absolute conic.

we get

$$K^{-1} = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ 0 & k_{22} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{k_{11}} & \frac{-k_{12}}{k_{11}k_{22}} & \frac{k_{12}k_{23}-k_{13}k_{22}}{k_{11}k_{22}} \\ 0 & \frac{1}{k_{22}} & \frac{-k_{23}}{k_{22}} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad (11.3)$$

$\omega = K^{-T} K^{-1}$

$M \rightarrow K$

for some real  $m_{11}, m_{12}, m_{13}, m_{22}$  and  $m_{23}$ . Equivalently, we get

$$K = \begin{bmatrix} \frac{1}{m_{11}} & \frac{-m_{12}}{m_{11}m_{22}} & \frac{m_{12}m_{23}-m_{13}m_{22}}{m_{11}m_{22}m_{23}} \\ 0 & \frac{1}{m_{22}} & \frac{-m_{23}}{m_{22}} \\ 0 & 0 & 1 \end{bmatrix} \quad (11.4)$$

Introducing the following notation

$$\omega = K^{-T} K^{-1} = \begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix} \quad (11.5)$$

yields

$$\begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11}m_{12} & m_{11}m_{13} \\ m_{11}m_{12} & m_{12}^2 + m_{22}^2 & m_{12}m_{13} + m_{22}m_{23} \\ m_{11}m_{13} & m_{12}m_{13} + m_{22}m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix} \quad (11.6)$$

$$\omega = M^T M$$

which can be solved for  $K^{-1}$  up to the sign of the rows of  $K^{-1}$  as follows. Equation 11.6 provides equations

From  $\omega$  to  $K$

Equations by comparing elements

$$\begin{bmatrix} \omega_{11} & \omega_{12} & \omega_{13} \\ \omega_{12} & \omega_{22} & \omega_{23} \\ \omega_{13} & \omega_{23} & \omega_{33} \end{bmatrix} = \begin{bmatrix} m_{11}^2 & m_{11} m_{12} & m_{11} m_{13} \\ m_{11} m_{12} & m_{12}^2 + m_{22} & m_{12} m_{13} + m_{22} m_{23} \\ m_{11} m_{13} & m_{12} m_{13} + m_{22} m_{23} & m_{13}^2 + m_{23}^2 + 1 \end{bmatrix}$$

$\omega = M^T M$

$$\begin{aligned} \omega_{11} = m_{11}^2 &\Rightarrow m_{11} = s_1 \sqrt{\omega_{11}} \\ \omega_{12} = m_{11} m_{12} &\Rightarrow m_{12} = \omega_{12} / (s_1 \sqrt{\omega_{11}}) = s_1 \omega_{12} / \sqrt{\omega_{11}} \\ \omega_{13} = m_{11} m_{13} &\Rightarrow m_{13} = \omega_{13} / (s_1 \sqrt{\omega_{11}}) = s_1 \omega_{13} / \sqrt{\omega_{11}} \\ \omega_{22} = m_{12}^2 + m_{22}^2 &\Rightarrow m_{22} = s_2 \sqrt{\omega_{22} - m_{12}^2} = s_2 \sqrt{\omega_{22} - \omega_{12}^2 / \omega_{11}} \\ \omega_{23} = m_{12} m_{13} + m_{22} m_{23} &\Rightarrow m_{23} = s_2 (\omega_{23} - \omega_{12} \omega_{13} / \omega_{11}) / \sqrt{\omega_{22} - \omega_{12}^2 / \omega_{11}} \\ &= s_2 (\omega_{11} \omega_{23} - \omega_{12} \omega_{13}) / \sqrt{\omega_{11}^2 \omega_{22} - \omega_{11} \omega_{12}^2} \end{aligned}$$

3x3 = 9 elements  
but only 6 indep. eqns.  
⇒ 5 unknowns

which can be solved for  $m_{ij}$  with  $s_1 = \pm 1$  and  $s_2 = \pm 1$ . Hence

$$K = \begin{bmatrix} s_1 \sqrt{\omega_{11}} & s_1 \omega_{12} / \sqrt{\omega_{11}} & s_1 \omega_{13} / \sqrt{\omega_{11}} & -1 \\ 0 & s_2 \sqrt{\omega_{22} - \omega_{12}^2 / \omega_{11}} & s_2 (\omega_{23} - \omega_{12} \omega_{13} / \omega_{11}) / \sqrt{\omega_{22} - \omega_{12}^2 / \omega_{11}} & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (11.7)$$

Signs  $s_1, s_2$  are determined by the choice of the image coordinate system. The standard choice is  $s_1 = s_2 = 1$ , which corresponds to  $k_{11} > 0$  and  $k_{22} > 0$ .

Notice that  $\sqrt{\omega_{11}}$  is never zero for a real camera since  $m_{11} = \frac{1}{k_{11}} \neq 0$ . There also holds true

$$\sqrt{\omega_{22} - \omega_{12}^2 / \omega_{11}} = \sqrt{m_{11}^2 - m_{12}^2} = \sqrt{\frac{1}{k_{11}^2} - \frac{k_{12}^2}{k_{11}^2 k_{22}^2}} = \frac{1}{k_{11} k_{22}} \sqrt{k_{22}^2 - k_{12}^2} \neq 0 \quad (11.8)$$

since  $|k_{12}|$  is much smaller than  $|k_{22}|$  for all real cameras.

4 choices  
 $\omega \rightarrow M$   
 $s_1 = \pm 1$   
 $s_2 = \pm 1$   
}  $s_1 = +$   
}  $s_2 = +$

$$K = \begin{bmatrix} > 0 & k_{12} & \cdot \\ 0 & > 0 & \cdot \\ 0 & 0 & 1 \end{bmatrix}$$

$k_{12} \ll k_{22}$

## 11.1 Constraints on $\omega$

Matrix  $\omega$  is a  $3 \times 3$  symmetric matrix and by this it has only six independent elements  $\omega_{11}, \omega_{12}, \omega_{13}, \omega_{22}, \omega_{23}$  and  $\omega_{33}$ . Let us next investigate additional constraints on  $\omega$ , which follow from different choices of  $K$ .

**§ 1 Constraints on  $\omega$  for a general  $K$**  Even a general  $K$  yields a constraint on  $\omega$ . Equation 11.6 relates the six parameters of  $\omega$  to only five parameters  $m_{11}, m_{12}, m_{13}, m_{22}$  and  $m_{23}$  and hence the six parameters of  $\omega$  can't be independent. Indeed, let us see that the following identity holds true

$$\begin{aligned}
 & (\omega_{23}^2 - \frac{\omega_{13}^2 \omega_{12}^2}{\omega_{11}^2} - (\omega_{22} - \frac{\omega_{12}^2}{\omega_{11}}) (\omega_{33} - \frac{\omega_{13}^2}{\omega_{11}} - 1))^2 - 4 \frac{\omega_{13}^2 \omega_{12}^2}{\omega_{11}^2} (\omega_{22} - \frac{\omega_{12}^2}{\omega_{11}}) (\omega_{33} - \frac{\omega_{13}^2}{\omega_{11}} - 1) \\
 = & \left( (m_{12}m_{13} + m_{22}m_{23})^2 - \frac{(m_{11}m_{13})^2(m_{11}m_{12})^2}{m_{11}^4} \right. \\
 & \left. - (m_{12}^2 + m_{22}^2 - \frac{(m_{11}m_{12})^2}{m_{11}^2})(m_{13}^2 + m_{23}^2 + 1 - \frac{(m_{11}m_{13})^2}{m_{11}^4} - 1) \right)^2 \\
 - & 4 \frac{(m_{11}m_{13})^2(m_{11}m_{12})^2}{m_{11}^4} (m_{12}^2 + m_{22}^2 - \frac{(m_{11}m_{12})^2}{m_{11}^2})(m_{13}^2 + m_{23}^2 + 1 - \frac{(m_{11}m_{13})^2}{m_{11}^4} - 1) \\
 = & ((m_{12}m_{13} + m_{22}m_{23})^2 - (m_{12}m_{13})^2 - (m_{22}m_{23})^2)^2 - 4(m_{12}m_{13})^2(m_{22}m_{23})^2 \\
 = & (2(m_{12}m_{13})(m_{22}m_{23}))^2 - 4(m_{12}m_{13})^2(m_{22}m_{23})^2 \\
 = & 0
 \end{aligned}$$

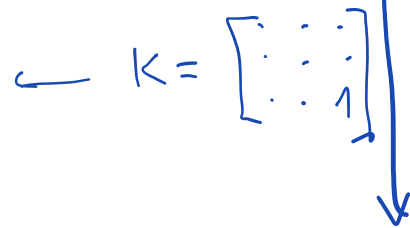
(11.9)

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Since  $\omega_{11} \neq 0$ , we get the following equivalent identity

$$\begin{aligned}
 & (\omega_{11}^2 \omega_{23}^2 - \omega_{13}^2 \omega_{12}^2 - (\omega_{11} \omega_{22} - \omega_{12}^2) (\omega_{11} \omega_{33} - \omega_{13}^2 - \omega_{11}))^2 \\
 & - 4 \omega_{13}^2 \omega_{12}^2 (\omega_{11} \omega_{22} - \omega_{12}^2) (\omega_{11} \omega_{33} - \omega_{13}^2 - \omega_{11}) = 0 \quad (11.10)
 \end{aligned}$$

which is a polynomial equation of degree eight in elements of  $\omega$ .



We shall see next that it makes sense to introduce a new matrix

$$\Omega = \begin{bmatrix} 1 & o_{12} & o_{13} \\ o_{12} & o_{22} & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = \begin{bmatrix} 1 & \frac{\omega_{12}}{\omega_{11}} & \frac{\omega_{13}}{\omega_{11}} \\ \frac{\omega_{12}}{\omega_{11}} & \frac{\omega_{22}}{\omega_{11}} & \frac{\omega_{23}}{\omega_{11}} \\ \frac{\omega_{13}}{\omega_{11}} & \frac{\omega_{23}}{\omega_{11}} & \frac{\omega_{33}}{\omega_{11}} \end{bmatrix} = \frac{\omega}{\omega_{11}} \quad (11.11)$$

which contains only five unknowns, and use Equation 11.10 to get the positive  $\omega_{11}$  from  $\Omega$  by solving the following quadratic equation

$$a_2 \omega_{11}^2 + a_1 \omega_{11} + a_0 = 0 \quad (11.12)$$

with

$$a_2 = -4 o_{23}^2 o_{13}^2 o_{12}^2 + o_{23}^4 - 2 o_{23}^2 o_{22} o_{33} + 2 o_{13}^2 o_{12}^2 o_{22} o_{33} - 2 o_{22}^2 o_{33} o_{13}^2 + o_{12}^4 o_{33}^2 + 2 o_{23}^2 o_{22} o_{13}^2 + 2 o_{23}^2 o_{12}^2 o_{33} + o_{22}^2 o_{13}^4 + o_{22}^2 o_{33}^2 - 2 o_{22} o_{33}^2 o_{12}^2 \quad (11.13)$$

$$a_1 = 2 o_{13}^2 o_{12}^2 o_{22} + 2 o_{23}^2 o_{22} - 2 o_{22}^2 o_{33} - 2 o_{12}^4 o_{33} + 4 o_{22} o_{33} o_{12}^2 - 2 o_{23}^2 o_{12}^2 + 2 o_{22}^2 o_{13}^2 \quad (11.14)$$

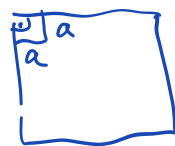
$$a_0 = -2 o_{22} o_{12}^2 + o_{22}^2 + o_{12}^4 \quad (11.15)$$

general, i.e. full K

**§2 Constraints on  $\omega$  for K from square pixels** Cameras have often square pixels, i.e.  $\|\vec{b}_1\| = \|\vec{b}_2\| = 1$  and  $\angle(\vec{b}_1, \vec{b}_2) = \pi/2$ , which implies, Equations 7.13, 7.15, 7.16, a simplified

$$K = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{11} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad (11.16)$$

Square pixels



This gives also simpler

$$\omega = \frac{1}{k_{11}^2} \begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix} \quad (11.17)$$

$$= \frac{1}{k_{11}^2} \Omega \quad \begin{matrix} \Omega \rightarrow K \\ \omega \rightarrow K \end{matrix}$$

We see that we get the following three identities

$$\begin{aligned} \omega_{12} &= 0 & (11.18) \\ \omega_{22} - \omega_{11} &= 0 & (11.19) \\ \omega_{13}^2 + \omega_{23}^2 - \omega_{11}\omega_{33} + \omega_{11} &= 0 & (11.20) \end{aligned}$$

identities for  $\square$  pixels  
on  $\omega$  g.g.y.

We also get simpler

$$\Omega = \begin{bmatrix} 1 & 0 & o_{13} \\ 0 & 1 & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} = k_{11}^2 \omega = \begin{bmatrix} 1 & 0 & -k_{13} \\ 0 & 1 & -k_{23} \\ -k_{13} & -k_{23} & k_{11}^2 + k_{13}^2 + k_{23}^2 \end{bmatrix} \quad (11.21)$$

of real  
off the shelf  
cameras

and use Equation 11.21 to get

$$k_{11}^2 = o_{33} - o_{13}^2 - o_{23}^2 \quad (11.22)$$

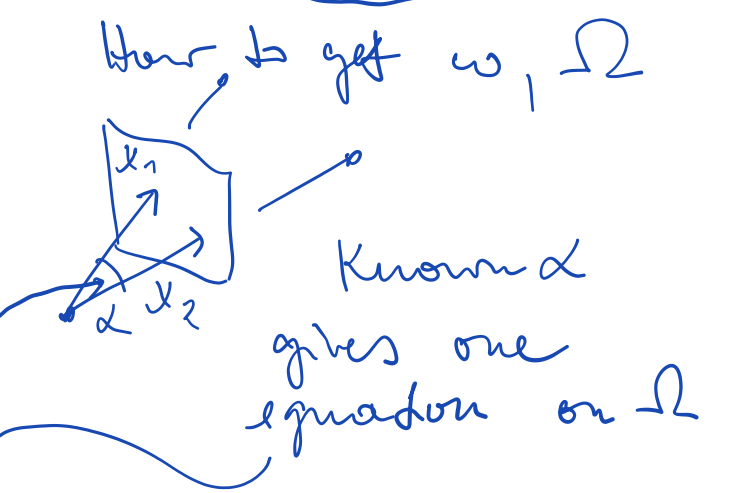
$$k_{13} = -o_{13} \quad (11.23)$$

$$k_{23} = -o_{23} \quad (11.24)$$

## 11.2 Camera calibration from angles between projection rays

We will now show how to calibrate a camera by finding the matrix  $\omega = K^{-T}K^{-1}$ .

In general, matrix  $\omega$  is constrained by knowing angles contained between pairs of projection rays. Consider two projection rays with direction vectors  $\vec{x}_1, \vec{x}_2$ . Then the angle between them is related to  $\omega$  and  $\Omega$  by



$$\cos \angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\beta}^T \omega \vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^T \omega \vec{x}_{1\beta}} \sqrt{\vec{x}_{2\beta}^T \omega \vec{x}_{2\beta}}} = \frac{\vec{x}_{1\beta}^T \Omega \vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^T \Omega \vec{x}_{1\beta}} \sqrt{\vec{x}_{2\beta}^T \Omega \vec{x}_{2\beta}}} \quad (11.25)$$

Squaring the above and clearing the denominators gives *unknown*

$$(\cos \angle(\vec{x}_1, \vec{x}_2))^2 (\vec{x}_{1\beta}^T \Omega \vec{x}_{1\beta}) (\vec{x}_{2\beta}^T \Omega \vec{x}_{2\beta}) = (\vec{x}_{1\beta}^T \Omega \vec{x}_{2\beta})^2 \quad (11.26)$$

which is a second order equation in elements of  $\Omega$ . To find  $\Omega$ , which has five independent parameters for a general K, we need to be able to establish five pairs of rays with known angles and solve a system of five quadratic equations 11.26 above.

**§1 Camera with square pixels** A simpler situation arises when the camera has square pixels. Then, we can use constraints from §2 to recover  $\omega$  and K from three pairs of rays containing known angles. That amounts to solving three second order equations 11.26 in  $o_{13}, o_{23}, o_{33}$ .

However, this is actually exactly the same problem as we have already solved in Section 7.3. Figure 11.2 shows an image plane  $\pi$  with a coordinate system  $(o, \delta')$  with  $\delta' = (\vec{b}_1, \vec{b}_2, \vec{b}'_3)$  derived from the image coordinate system  $(o, \alpha)$ . Having square pixels, vectors  $\vec{b}_1, \vec{b}_2$  can be complemented with  $\vec{b}'_3$  to form an orthogonal coordinates system  $(O = o, \delta')$ . Next, we choose the global orthonormal coordinate system,  $(O = o, \delta)$ ,  $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$ , such that

$$\vec{d}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|}, \quad \vec{d}_2 = \frac{\vec{b}_2}{\|\vec{b}_2\|}, \quad \text{and} \quad \vec{d}_3 = \frac{\vec{b}'_3}{\|\vec{b}'_3\|} \quad (11.27)$$

and hence

$$\vec{x}_\delta = \begin{bmatrix} \|\vec{b}_1\| & 0 & 0 \\ 0 & \|\vec{b}_1\| & 0 \\ 0 & 0 & \|\vec{b}_1\| \end{bmatrix} \vec{x}_{\delta'} \quad (11.28)$$

We know angles  $\angle(\vec{x}_1, \vec{x}_2)$ ,  $\angle(\vec{x}_2, \vec{x}_3)$  and  $\angle(\vec{x}_3, \vec{x}_1)$ . We also know image points  $\vec{u}_{1\alpha} = \vec{X}_{1\delta'}$ ,  $\vec{u}_{2\alpha} = \vec{X}_{2\delta'}$ ,  $\vec{u}_{3\alpha} = \vec{X}_{3\delta'}$  and thus we can compute

$$\cos \angle(\vec{x}_1, \vec{x}_2) = \frac{\vec{x}_{1\beta}^T \Omega \vec{x}_{2\beta}}{\sqrt{\vec{x}_{1\beta}^T \Omega \vec{x}_{1\beta}} \sqrt{\vec{x}_{2\beta}^T \Omega \vec{x}_{2\beta}}}$$

algebraic eqns.  $\rightarrow$  <sup>12</sup> elements of  $\Omega$

$\Omega \dots$  deg 2 in  $\Omega$

5 unknowns in  $\Omega \rightarrow$  5 eqns of deg 2

In general solvable but too hard for us

Absolute camera  $\rightarrow$  pose

Square pixels  $\downarrow$  3 eqns in 3 unknowns



distances  $d_{12} = \|\vec{X}_{2\delta'} - \vec{X}_{1\delta'}\|$ ,  $d_{23} = \|\vec{X}_{3\delta'} - \vec{X}_{2\delta'}\|$  and  $d_{31} = \|\vec{X}_{3\delta'} - \vec{X}_{1\delta'}\|$ . Having that, we can find the pose  $\vec{C}_{\delta'} = [c_1, c_2, c_3]^T$  of the camera center  $C$  in  $(O, \delta')$  by solving the absolute pose problem from Chapter 7.3. We will select a solution with  $c_3 < 0$  and, if necessary, use a fourth point in  $\pi$  to choose the right solution among them. To find  $K$ , we can form the following equation

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{f} [KR | -KR\vec{C}_{\delta'}] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad (11.29)$$

since point  $o$  is represented by  $[0, 0, 1]^T$  in  $\beta$  and by  $[0, 0, 0]^T$  in  $\delta$ . Coordinate system  $(O, \delta)$  is chosen such that  $R = I$  and  $\vec{C}_{\delta} = \|\vec{b}_1\| \vec{C}_{\delta'}$  and thus we get

$$K^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = -\frac{\|\vec{b}_1\|}{f} \vec{C}_{\delta'} \quad (11.30)$$

Now, let us consider matrix  $K$  as in Equation 11.16 and use the interpretation of elements of  $K$  from Chapter 7, Equations 7.16, 7.17. We can write

skip

$$K = \begin{bmatrix} \frac{f}{\|\vec{b}_1\|} & 0 & k_{13} \\ 0 & \frac{f}{\|\vec{b}_1\|} & k_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and thus} \quad K^{-1} = \begin{bmatrix} \frac{\|\vec{b}_1\|}{f} & 0 & -\frac{\|\vec{b}_1\|}{f} k_{13} \\ 0 & \frac{\|\vec{b}_1\|}{f} & -\frac{\|\vec{b}_1\|}{f} k_{23} \\ 0 & 0 & 1 \end{bmatrix} \quad (11.31)$$

and use it in Equation 11.30 to get

$$\begin{bmatrix} k_{13} \\ k_{23} \\ -\frac{f}{\|\vec{b}_1\|} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \quad (11.32)$$



and thus

$$K = \begin{bmatrix} -c_3 & 0 & c_1 \\ 0 & -c_3 & c_2 \\ 0 & 0 & 1 \end{bmatrix} \quad (11.33)$$

### 11.3 Camera calibration from vanishing points

Let us first make an interesting observation about parallel lines in space and its corresponding vanishing point in an image. Let us consider a pair of parallel lines  $K, L$  in space as shown in Figure 11.1 (a). There is an affine plane  $\sigma$  containing the lines. The lines  $K, L$  are projected to image plane  $\pi$  into lines  $k, l$ , respectively.

Now, first extend affine plane  $\sigma$  to a projective plane  $\Sigma$  using the camera center  $C$ . Then, define a coordinate system  $(C, \delta)$  with orthonormal basis  $\delta = (\vec{d}_1, \vec{d}_2, \vec{d}_3)$  such that vectors  $\vec{d}_1, \vec{d}_2$  span affine plane  $\sigma$ .

Let  $\vec{K}_\delta, \vec{L}_\delta$  be homogeneous coordinates of lines  $K, L$  w.r.t.  $\delta$ . Then

$$\vec{w}_\delta = \vec{K}_\delta \times \vec{L}_\delta \quad (11.34)$$

are homogeneous coordinates of the intersection of lines  $K, L$  in  $\Sigma$ .

Next, extend the affine plane  $\pi$  to a projective plane  $\Pi$  using the camera center  $C$  with the (camera) coordinate system  $(C, \beta)$ .

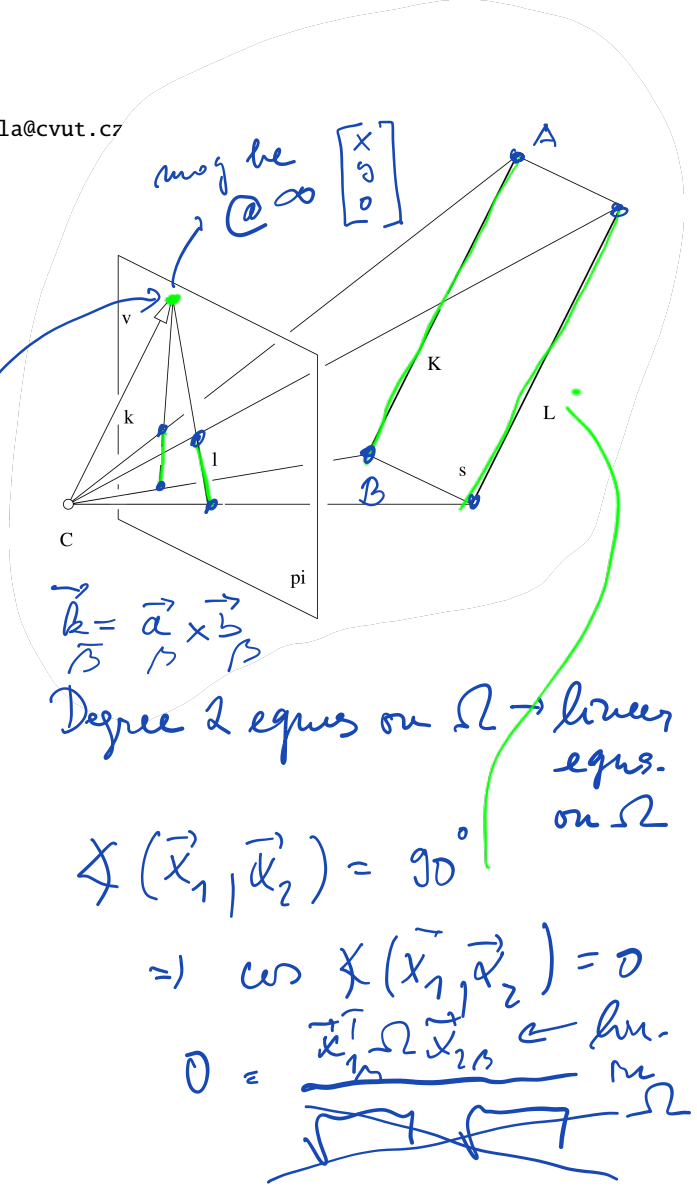
Let  $\vec{k}_\beta, \vec{l}_\beta$  be homogeneous coordinates of lines  $k, l$  w.r.t.  $\beta$ . Then

$$\vec{v}_\beta = \vec{k}_\beta \times \vec{l}_\beta \quad (11.35)$$

are homogeneous coordinates of the intersection of lines  $k, l$  in  $\Pi$ .

Now, consider Equation 8.14 for planes  $\Sigma$  and  $\Pi$ . Since  $\delta$  is orthonormal, we have  $K' = I$  and thus that there is a homography

$$H = KR \quad (11.36)$$



which maps plane  $\Sigma$  to plane  $\Pi$ . Matrices  $K$  and  $R$  of the camera are here w.r.t. the world coordinate system  $(C, \delta)$ .

We see that there is a real  $\lambda$  such that there holds

$$\lambda \vec{v}_\beta = KR\vec{w}_\delta \quad (11.37)$$

true.

### §1 Pairs of “orthogonal” vanishing points and camera with square pixels

Let us have two pairs of parallel lines in space, Figure 11.1(b), such that they are also orthogonal, i.e. let  $K_1$  be parallel with  $L_1$  and  $K_2$  be parallel with  $L_2$  and at the same time let  $K_1$  be orthogonal to  $K_2$  and  $L_1$  be orthogonal to  $L_2$ . This, for instance, happens when lines  $K_1, L_1, K_2, L_2$  form a rectangle but they also may be arranged in the three-dimensional space as non-intersecting.

Let lines  $k_1, l_1, k_2, l_2$  be the projections of  $K_1, L_1, K_2, L_2$ , respectively, represented by the corresponding vectors  $\vec{k}_{1\beta}, \vec{l}_{1\beta}, \vec{k}_{2\beta}, \vec{l}_{2\beta}$  in the camera coordinates system with (in general non-orthogonal) basis  $\beta$ . Lines  $k_1$  and  $l_1$ , resp.  $k_2$  and  $l_2$ , generate vanishing points

$$\vec{v}_{1\beta} = \vec{k}_{1\beta} \times \vec{l}_{1\beta}$$

$$\vec{v}_{2\beta} = \vec{k}_{2\beta} \times \vec{l}_{2\beta}$$

The perpendicularity of  $\vec{w}_1$  to  $\vec{w}_2$  is, in the camera orthogonal basis  $\delta$ , modeled by

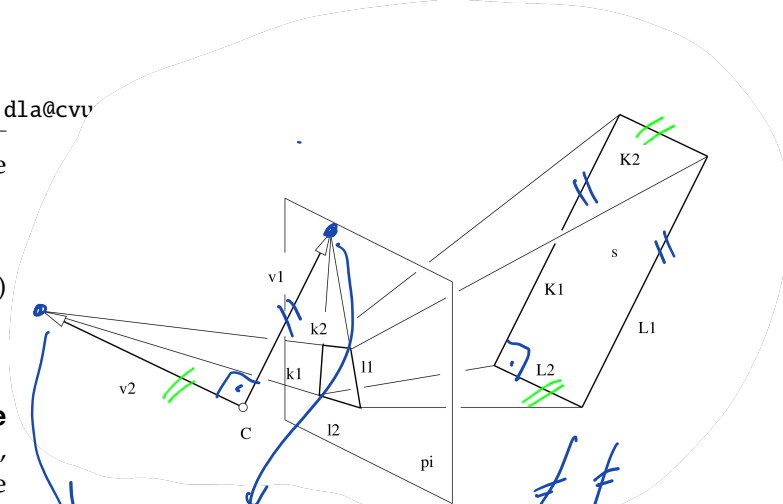
$$\vec{w}_{1\delta}^\top \vec{w}_{2\delta} = 0 \quad (11.38)$$

We therefore get from Equation 11.37

$$\vec{v}_{1\beta}^\top K^{-\top} R^{-\top} K^{-1} \vec{v}_{2\beta} = 0 \quad \omega = \Omega \quad (11.39)$$

$$\vec{v}_{1\beta}^\top K^{-\top} K^{-1} \vec{v}_{2\beta} = 0 \quad (11.40)$$

$$\vec{v}_{1\beta}^\top \omega \vec{v}_{2\beta} = 0 \quad (11.41)$$



Handwritten notes and diagrams:

- Diagram showing two vectors  $\vec{v}_1$  and  $\vec{v}_2$  with a right-angle symbol between them, indicating  $\vec{v}_1 \perp \vec{v}_2$ .
- Equation:  $\vec{v}_1^\top \vec{v}_2 = 0$
- Equation:  $\delta_1 \gamma \quad \delta_1 \delta$
- Equation:  $\neq \beta \quad \neq \beta$
- Equation:  $\vec{x}_\beta = KR \vec{x}_\delta$
- Equation:  $R^{-\top} R^{-1}$
- Equation:  $R R^{-1} = I$
- Equation:  $\beta \rightarrow \gamma$
- Equation:  $\rightarrow \alpha$
- Equation:  $\rightarrow \delta$

which is a linear homogeneous equation in  $\omega$ . Assuming further square pixels, we get, §2,

$$\begin{aligned} \vec{v}_{1\beta}^\top \omega \vec{v}_{2\beta} &= 0 \\ \vec{v}_{1\beta}^\top \Omega \vec{v}_{2\beta} &= 0 \end{aligned}$$

pixels

$$[v_{11} \ v_{12} \ v_{13}] \begin{bmatrix} 1 & 0 & o_{13} \\ 0 & 1 & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$[v_{23} v_{11} + v_{21} v_{13} \quad v_{23} v_{12} + v_{22} v_{13} \quad v_{23} v_{13}] \begin{bmatrix} o_{13} \\ o_{23} \\ o_{33} \end{bmatrix} = -(v_{21} v_{11} + v_{22} v_{12})$$

Now, we need only 3 pairs of perpendicular vanishing points, e.g. to observe 3 rectangles not all in one plane to compute  $o_{13}, o_{23}, o_{33}$  and then

$$\begin{aligned} k_{13} &= -o_{13} \\ k_{23} &= -o_{23} \\ k_{11} &= \sqrt{o_{33} - k_{13}^2 - k_{23}^2} \end{aligned}$$

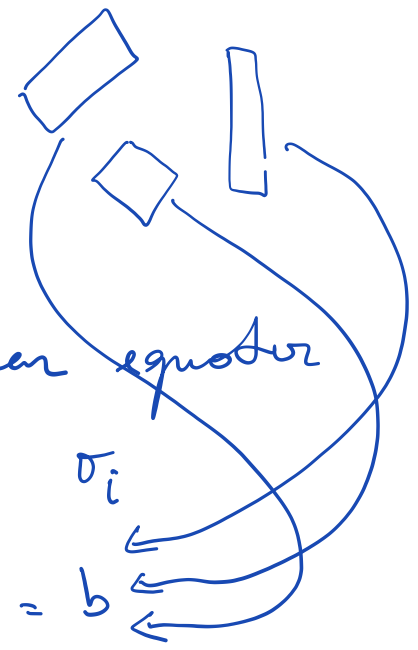
$$K = \begin{bmatrix} k_{11} & 0 & k_{13} \\ 0 & k_{11} & k_{23} \\ 0 & 0 & 1 \end{bmatrix}$$

linear equations

on  $\sigma_i$

$$A \begin{bmatrix} o_1 \\ o_2 \\ o_3 \end{bmatrix} = b$$

3x3                      3x1



which is a linear homogeneous equation in  $\omega$ . Assuming further square pixels, we get, §2

$$\vec{v}_{1\beta}^\top \omega \vec{v}_{2\beta} = 0$$

$$\vec{v}_{1\beta}^\top \Omega \vec{v}_{2\beta} = 0$$

$$\begin{bmatrix} v_{11} & v_{12} & v_{13} \end{bmatrix} \begin{bmatrix} 1 & 0 & o_{13} \\ 0 & 1 & o_{23} \\ o_{13} & o_{23} & o_{33} \end{bmatrix} \begin{bmatrix} v_{21} \\ v_{22} \\ v_{23} \end{bmatrix} = 0$$

$$\begin{bmatrix} v_{23} v_{11} + v_{21} v_{13} & v_{23} v_{12} + v_{22} v_{13} & v_{23} v_{13} \end{bmatrix} \begin{bmatrix} o_{13} \\ o_{23} \\ o_{33} \end{bmatrix} = -(v_{21} v_{11} + v_{22} v_{12})$$

Now, we need only 3 pairs of perpendicular vanishing points, e.g. to observe 3 rectangles not all in one plane to compute  $o_{13}, o_{23}, o_{33}$  and then

$$k_{13} = -o_{13}$$

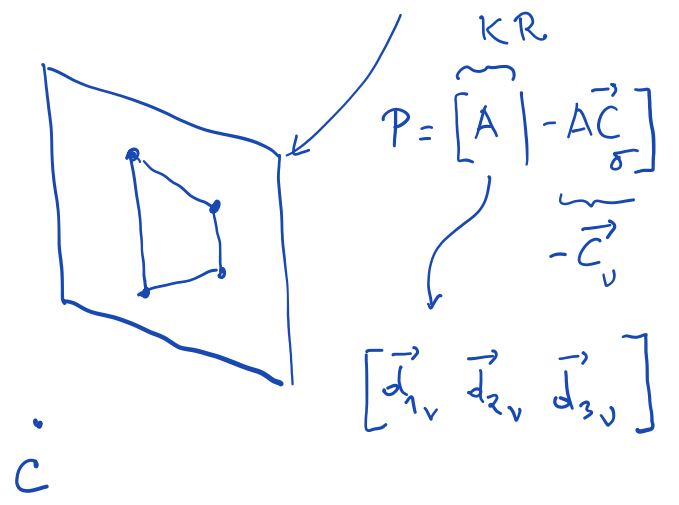
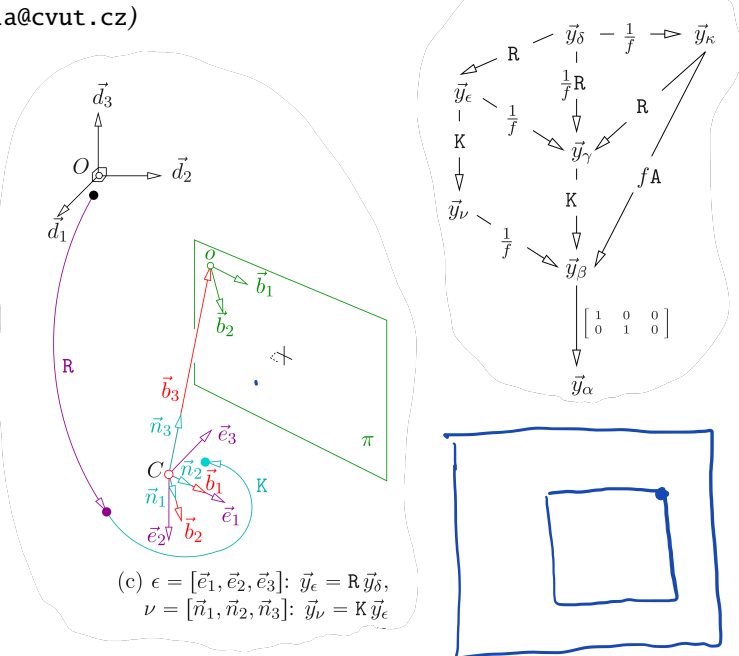
$$k_{23} = -o_{23}$$

$$k_{11} = \sqrt{o_{33} - k_{13}^2 - k_{23}^2}$$

### 11.4 Camera calibration from images of squares

Let us exploit the relationship between the coordinates of points  $X$ , which all lie in a plane  $\sigma$  and are measured in a coordinate system  $(O, \vec{d}_1, \vec{d}_2)$  in  $\sigma$ , Figure 8.2. The points  $X$  are projected by a perspective camera with the camera coordinate system is  $(C, \beta), \beta = (\vec{b}_1, \vec{b}_2, \vec{b}_3)$  and projection matrix  $P$  into image coordinates  $[u \ v]^\top$ , w.r.t. an image coordinate system  $(o, \vec{b}_1, \vec{b}_2)$ , Equation 8.16. See paragraph §1 to recall that the columns of  $P$  can be written as

$$P = \left[ KR \mid -KR\vec{C}_\delta \right] = \begin{bmatrix} \vec{d}_{1v} & \vec{d}_{2v} & \vec{d}_{3v} & -\vec{C}_v \end{bmatrix} \quad (11.42)$$



and therefore we get the columns

$$h_1 = p_1 = \vec{d}_{1v} \quad (11.43)$$

$$h_2 = p_2 = \vec{d}_{2v} \quad (11.44)$$

$$h_3 = p_4 = -\vec{C}_v \quad (11.45)$$

of the homography  $H$  mapping  $\sigma$  to  $\pi$  as defined in Equation 8.17

Now imagine that we are observing a square with 4 corner points  $X_1, X_2, X_3$  and  $X_4$  in the plane  $\sigma$  and we construct the coordinate system in  $\sigma$  by assigning coordinates to the corners as

$$\vec{X}_{1\delta} = [0 \ 0 \ 0] \quad (11.46)$$

$$\vec{d}_{1\delta} = \vec{X}_{2\delta} = [1 \ 0 \ 0] \quad (11.47)$$

$$\vec{d}_{2\delta} = \vec{X}_{3\delta} = [0 \ 1 \ 0] \quad (11.48)$$

$$\vec{X}_{4\delta} = [1 \ 1 \ 0] \quad (11.49)$$

We see that we get two constraints on  $\vec{d}_{1\delta}, \vec{d}_{2\delta}$

$$\vec{d}_{1\delta}^\top \vec{d}_{2\delta} = 0 \quad (11.50)$$

$$\|\vec{d}_{1\delta}\| = \|\vec{d}_{2\delta}\| \implies \vec{d}_{1\delta}^\top \vec{d}_{1\delta} - \vec{d}_{2\delta}^\top \vec{d}_{2\delta} = 0 \quad (11.51)$$

which lead to

$$\vec{d}_{1v}^\top K^{-\top} K^{-1} \vec{d}_{2v} = 0 \quad (11.52)$$

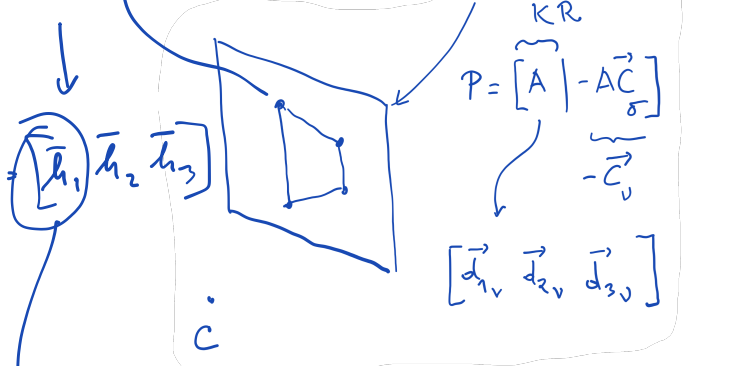
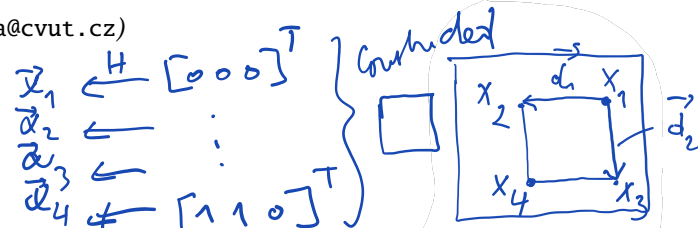
$$\vec{d}_{1v}^\top K^{-\top} K^{-1} \vec{d}_{1v} - \vec{d}_{2v}^\top K^{-\top} K^{-1} \vec{d}_{2v} = 0 \quad (11.53)$$

by using  $\vec{d}_{iv} = KR\vec{d}_{i\delta}$  for  $i = 1, 2$ , and  $R^\top R = I$ .

These are two linear equations on  $\omega$  and hence also, see §1 on  $\Omega$

$$\vec{d}_{1v}^\top \Omega \vec{d}_{2v} = 0 \quad (11.54)$$

$$\vec{d}_{1v}^\top \Omega \vec{d}_{1v} - \vec{d}_{2v}^\top \Omega \vec{d}_{2v} = 0 \quad (11.55)$$



$$\lambda \vec{x}_v = P \begin{bmatrix} \vec{X}_{i\delta} \\ 1 \end{bmatrix} = \begin{bmatrix} \vec{d}_{1v} & \vec{d}_{2v} & \vec{d}_{3v} & -\vec{C}_v \end{bmatrix} \begin{bmatrix} x \\ y \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda \vec{x}_v = \begin{bmatrix} \vec{d}_{1v} & \vec{d}_{2v} & -\vec{C}_v \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\vec{d}_{iv} = KR \vec{d}_{i\delta}$$

$\omega \rightarrow \Omega$

$$\Omega = \frac{1}{K_{11}^2} \omega = \frac{1}{K_{11}^2} K^{-\top} K^{-1}$$

2 lin. eqns.  $\square \rightarrow$  simple formulas.

on  $\omega$  in terms of estimated  $\lambda H$

$$\mathbf{h}_1^\top \Omega \mathbf{h}_2 = 0 \quad (11.56)$$

$$\mathbf{h}_1^\top \Omega \mathbf{h}_1 - \mathbf{h}_2^\top \Omega \mathbf{h}_2 = 0 \quad (11.57)$$

One square provides two equations and therefore three squares in two planes in a general position suffice to calibrate full  $K$ . Actually, such three squares provide one more equations than necessary since  $\Omega$  has only five parameters. Hence, it is enough observe two squares and one rectangle to get five constraints. Similarly, one square and one rectangle in a plane then suffice to calibrate  $K$  when pixels are square.

Notice also that we have never used the special choice of coordinates of  $\vec{X}_\delta$ . Indeed, point  $X_4$  could be anywhere provided that we know how to assign it coordinates in  $(O, \vec{d}_1, \vec{d}_2)$ .

To calibrate the camera, we first assign coordinates to the corners of the square as above, then find the homography  $H$  from the plane to the image

$$\lambda_i \vec{x}_{i\beta} = H \vec{X}_{i\delta} \quad (11.58)$$

for  $\alpha_i = 1, \dots, 4$  and finally use columns of  $H$  the find  $\Omega$ .