Heuristic (Informed) Search

(Where we try to choose smartly)

R&N: Chap. 4, Sect. 4.1–3

Recall that the ordering of FRINGE defines the search strategy

Search Algorithm #2

SEARCH#2

1. INSERT(initial-node,FRINGE)
2. Repeat:
   a. If empty(FRINGE) then return failure
   b. N ← REMOVE(FRINGE)
   c. s ← STATE(N)
   d. If GOAL?(s) then return path or goal state
   e. For every state s' in SUCCESSORS(s)
      i. Create a node N' as a successor of N
      ii. INSERT(N',FRINGE)

Best-First Search

- It exploits state description to estimate how "good" each search node is
- An evaluation function f maps each node N of the search tree to a real number $f(N) \geq 0$
  [Traditionally, f(N) is an estimated cost; so, the smaller f(N), the more promising N]
- Best-first search sorts the FRINGE in increasing f
  [Arbitrary order is assumed among nodes with equal f]

Best-First Search

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- An evaluation function f maps each node N of the search t
  f(N) ≥ 0
  [Traditionally, f(N)
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"Best" does not refer to the quality of the generated path
Best-first search does not generate optimal paths in general

- Best-first search sorts the FRINGE in increasing f
  [Arbitrary order is assumed among nodes with equal f]

How to construct f?

- Typically, f(N) estimates:
  - either the cost of a solution path through N

    Then $f(N) = g(N) + h(N)$, where
    - $g(N)$ is the cost of the path from the initial node to N
    - $h(N)$ is an estimate of the cost of a path from N to a goal node
  - or the cost of a path from N to a goal node

    Then $f(N) = h(N)$ → Greedy best-search

- But there are no limitations on f. Any function of your choice is acceptable.
  But will it help the search algorithm?

How to construct `f`?

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    Then f(N) = h(N)

Heuristic function

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  But will it help the search algorithm?

Heuristic Function

- The heuristic function $h(N) \geq 0$ estimates the cost to go from STATE(N) to a goal state

  Its value is **independent of the current search tree**; it depends only on STATE(N) and the goal test GOAL?

- Example:

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

$h_1(N)$ = number of misplaced numbered tiles = 6

[Why is it an estimate of the distance to the goal?]

Other Examples

5		8
4	2	1
7	3	6

STATE(N)

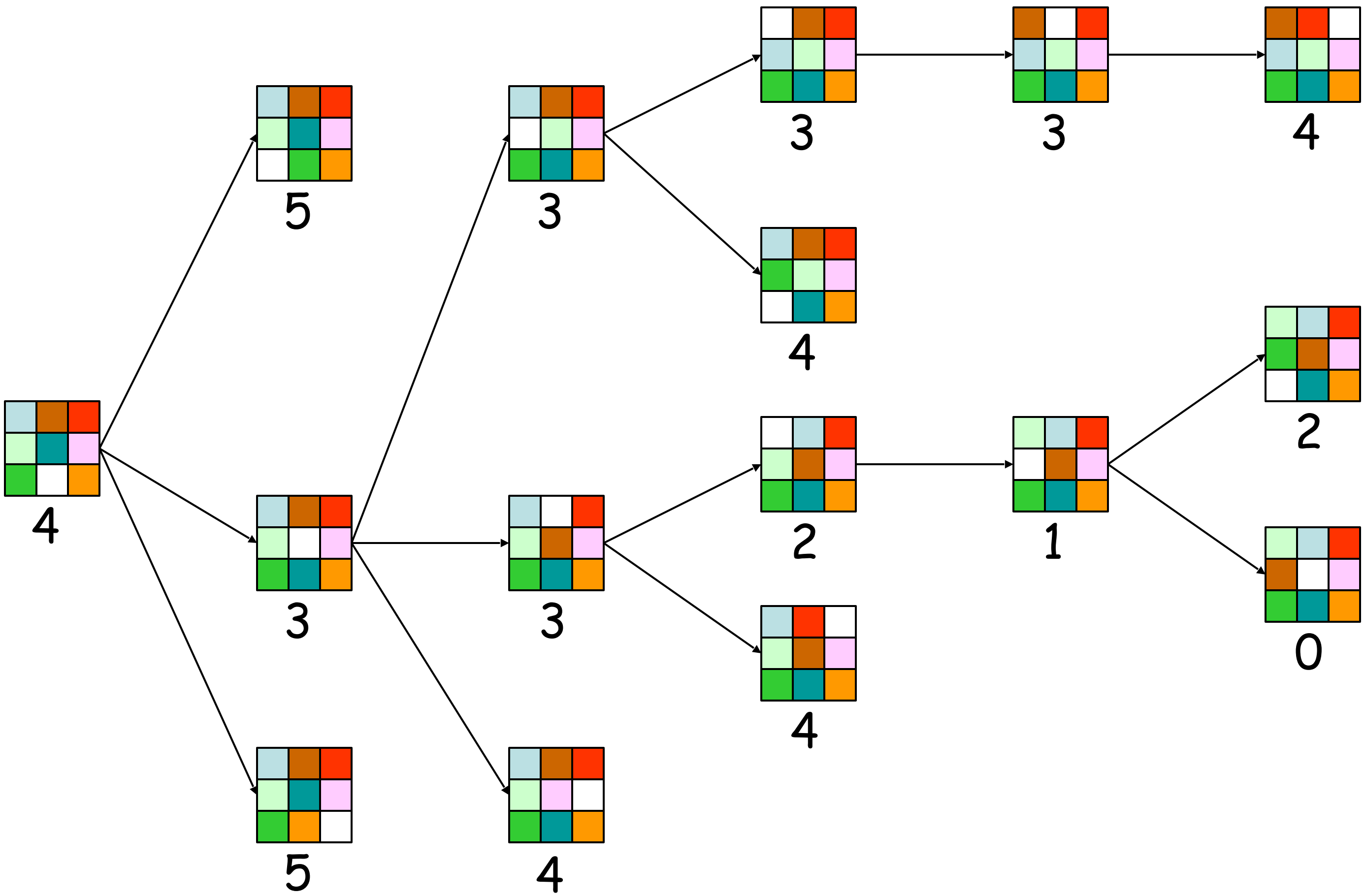
1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced numbered tiles = 6
- $h_2(N)$ = sum of the (Manhattan) distance of every numbered tile to its goal position
  = 2 + 3 + 0 + 1 + 3 + 0 + 3 + 1 = 13
- $h_3(N)$ = sum of permutation inversions
  $= n_5 + n_8 + n_4 + n_2 + n_1 + n_7 + n_3 + n_6$
  = 4 + 6 + 3 + 1 + 0 + 2 + 0 + 0
  = 16

8-Puzzle

f(N) = h(N) = number of misplaced numbered tiles

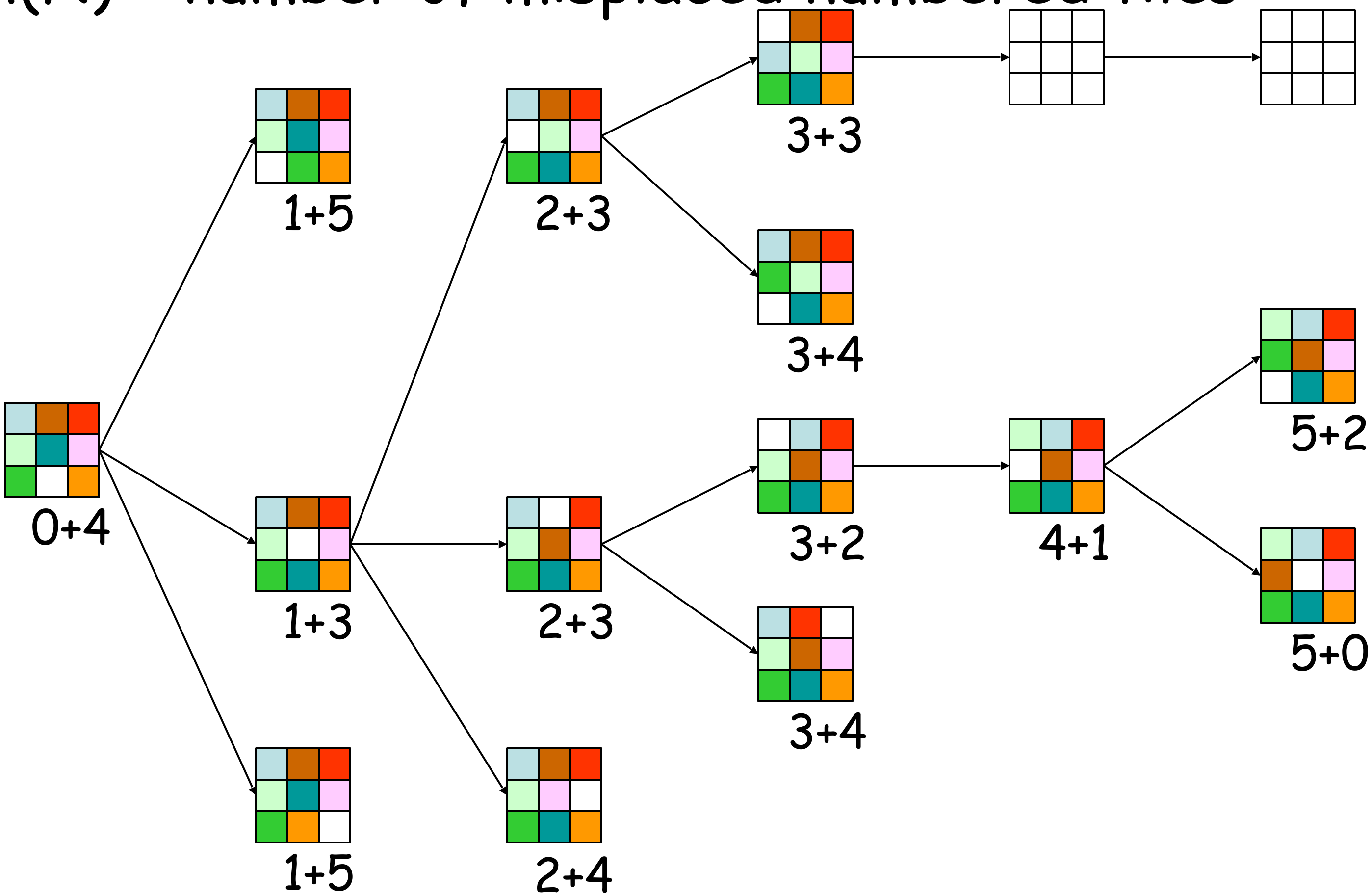


The white tile is the empty tile

8-Puzzle

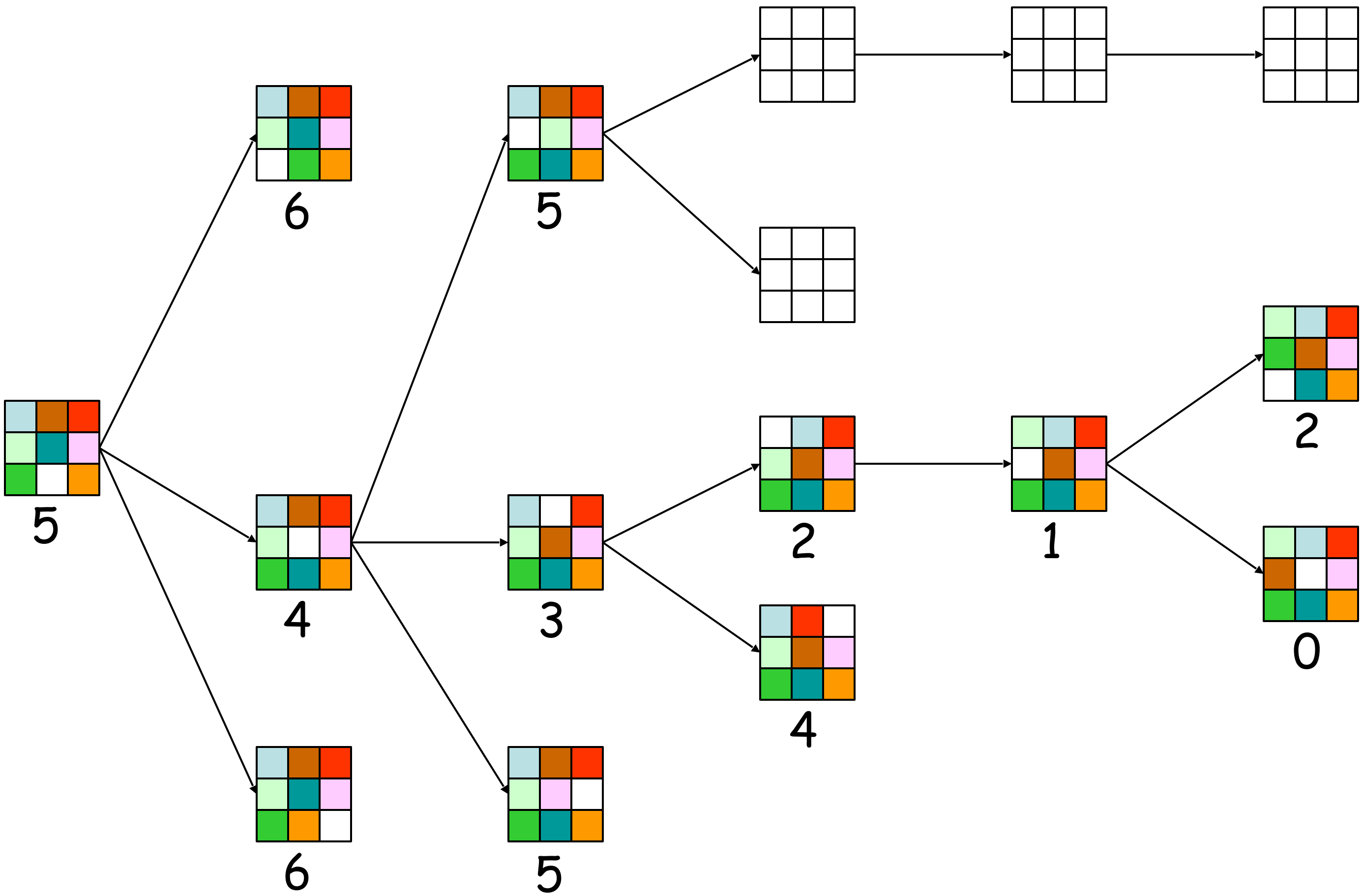
$f(N) = g(N) + h(N)$

with $h(N)$ = number of misplaced numbered tiles

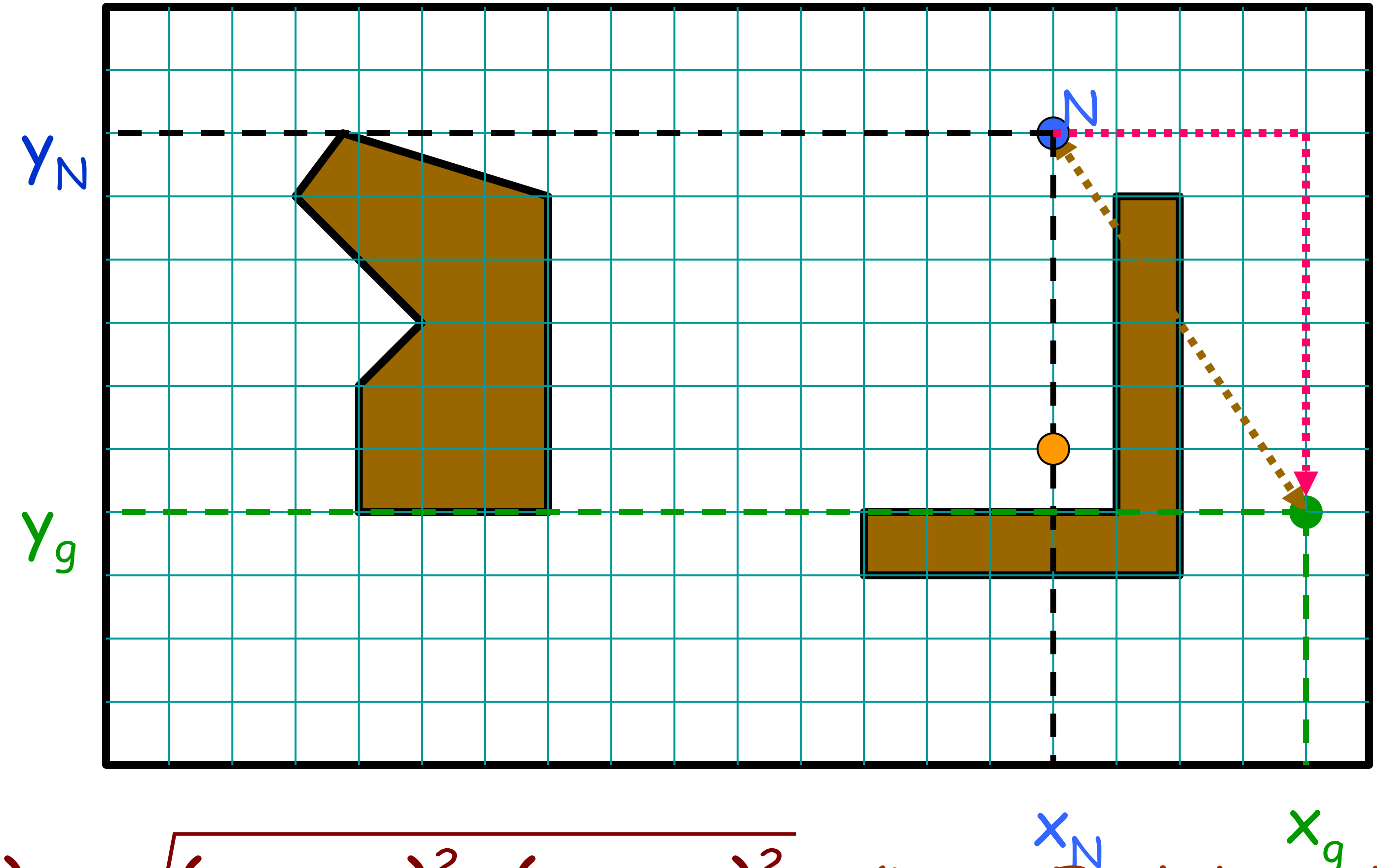


8-Puzzle

f(N) = h(N) = Σ distances of numbered tiles to their goals



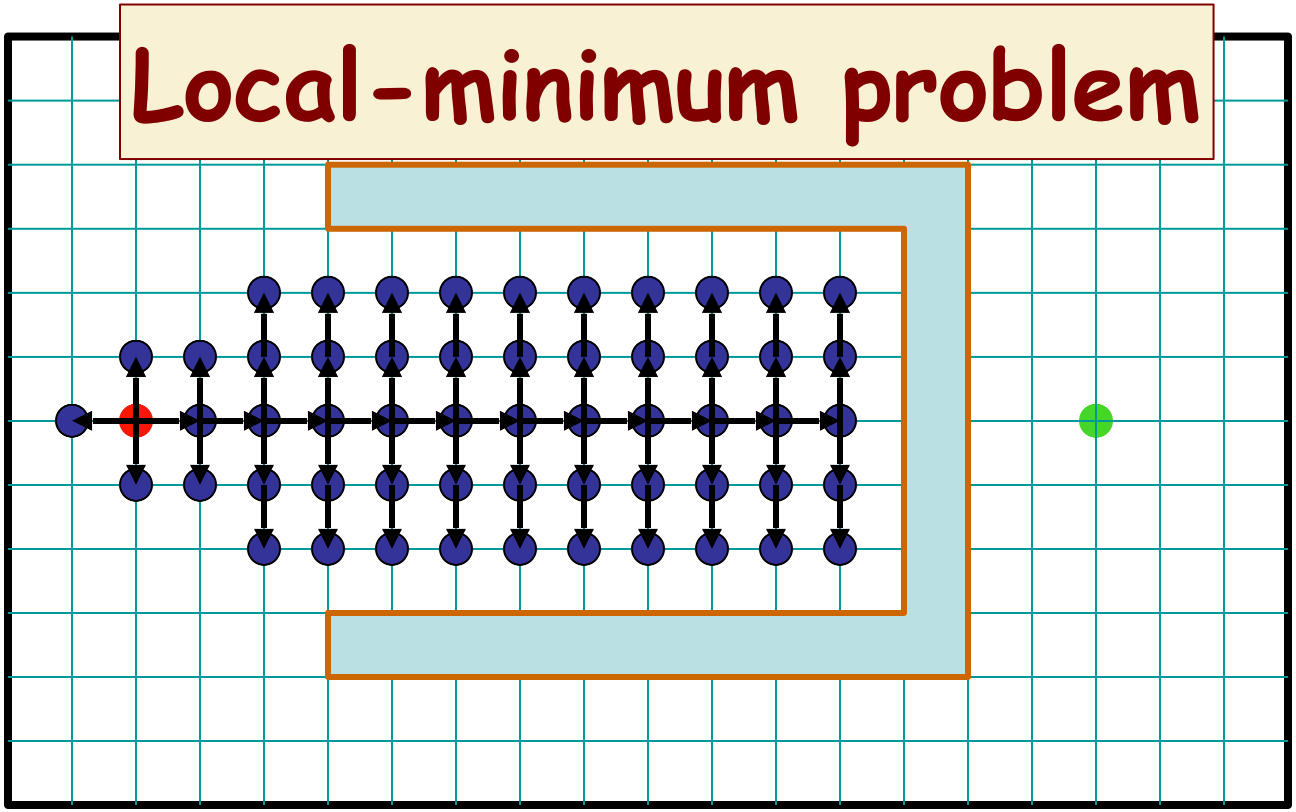
Robot Navigation



$$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$$ (L_2 or Euclidean distance)

$$h_2(N) = |x_N - x_g| + |y_N - y_g|$$ (L_1 or Manhattan distance)

Best-First $\not\rightarrow$ Efficiency



f(N) = h(N) = straight distance to the goal

Can we prove anything?

- If the state space is infinite, in general the search is not complete

- If the state space is finite and we do not discard nodes that revisit states, in general the search is not complete

- If the state space is finite and we discard nodes that revisit states, the search is complete, but in general is not optimal

Admissible Heuristic

- Let $h^*(N)$ be the cost of the optimal path from N to a goal node
- The heuristic function $h(N)$ is admissible if:

$$0 \le h(N) \le h^*(N)$$

- An admissible heuristic function is always optimistic !

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G is a goal node → $h(G) = 0$

8-Puzzle Heuristics

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  is admissible
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- $h_3(N)$ = sum of permutation inversions
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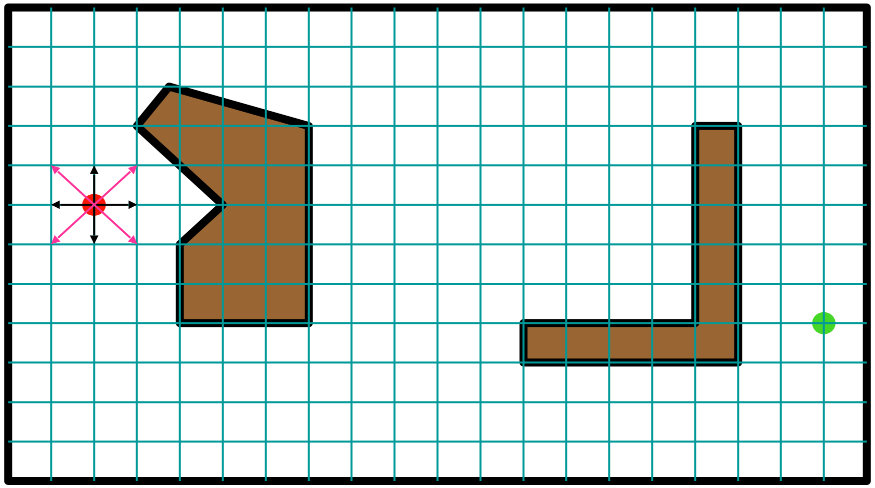
  is not admissible

Robot Navigation Heuristics

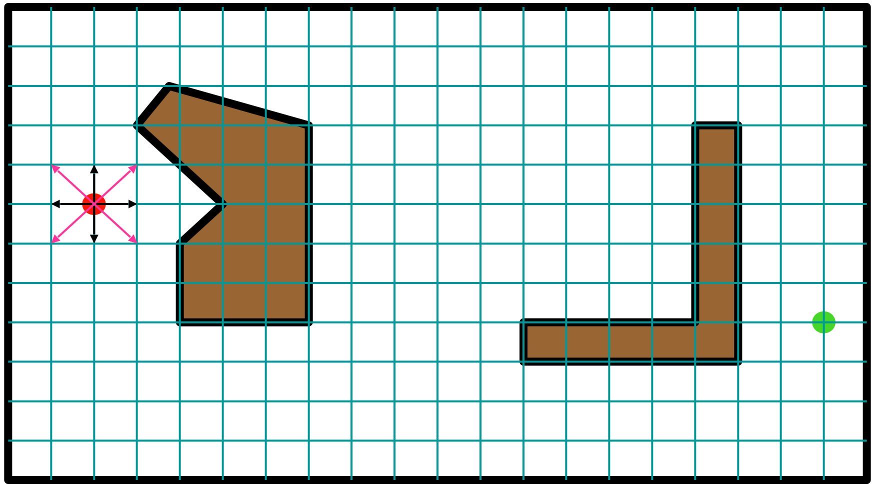


Cost of one horizontal/vertical step = 1

Cost of one diagonal step = $\sqrt{2}$

$h_1(N) = \sqrt{(x_N - x_g)^2 + (y_N - y_g)^2}$ is admissible

Robot Navigation Heuristics



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Cost of one diagonal step = $\sqrt{2}$

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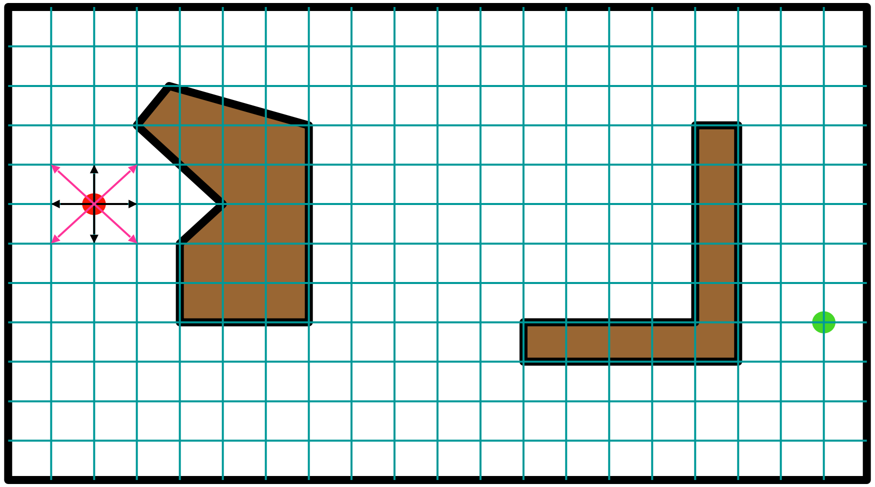
Robot Navigation Heuristics



Cost of one horizontal/vertical step = 1

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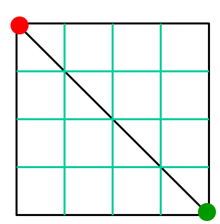
$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is admissible if moving along diagonals is not allowed, and not admissible otherwise

$h^*(I) = 4\sqrt{2}$

$h_2(I) = 8$



How to create an admissible h?

- An admissible heuristic can usually be seen as the cost of an optimal solution to a relaxed problem (one obtained by removing constraints)

- In robot navigation:
  - The Manhattan distance corresponds to removing the obstacles
  - The Euclidean distance corresponds to removing both the obstacles and the constraint that the robot moves on a grid

- More on this topic later

A* Search
(most popular algorithm in AI)

1) $f(N) = g(N) + h(N)$, where:
   - $g(N)$ = cost of best path found so far to N
   - $h(N)$ = **admissible** heuristic function

2) for all arcs: $c(N,N') \geq \varepsilon > 0$

3) SEARCH#2 algorithm is used

→ Best-first search is then called *A* search*

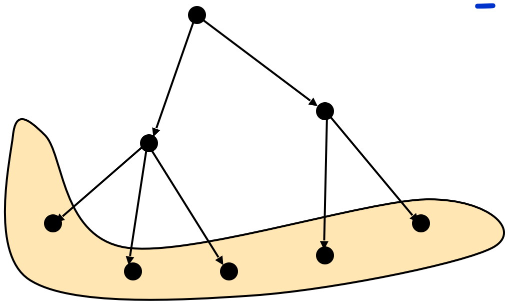
Result #1

A* is complete and optimal

[This result holds if nodes revisiting states are not discarded]

Proof (1/2)

1) **If a solution exists, A* terminates and returns a solution**



- For each node N on the OpenList, $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times\varepsilon$, where d(N) is the depth of N in the tree

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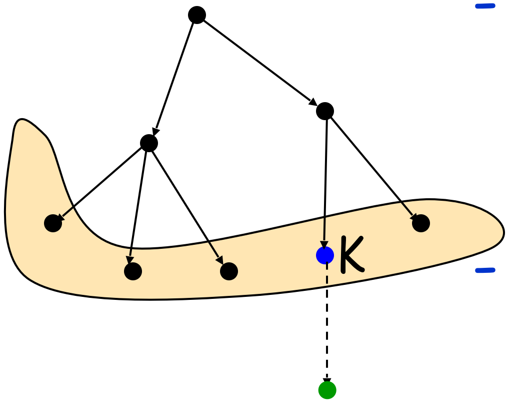

- For each node N on the OpenList,
  $f(N) = g(N)+h(N) \geq g(N) \geq d(N)\times\varepsilon$,
  where d(N) is the depth of N in the tree
- As long as A* hasn't terminated, a node K on the OpenList lies on a solution path

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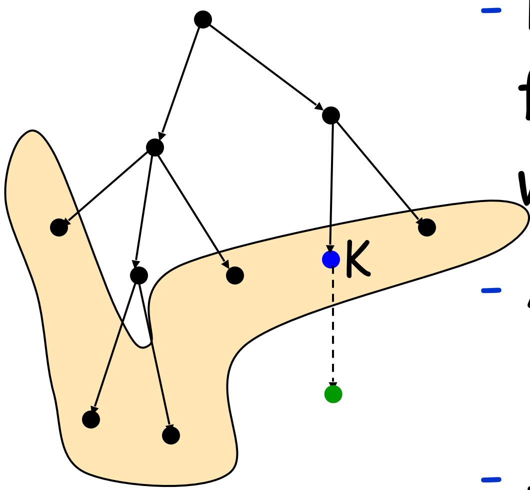
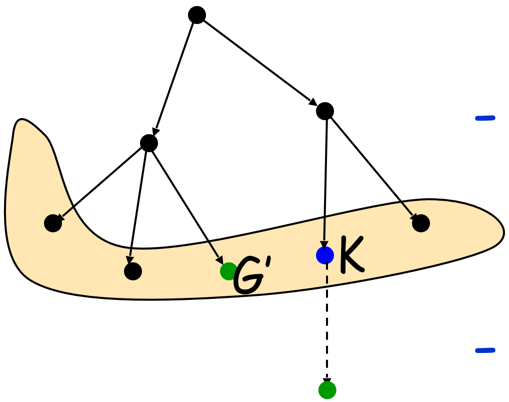


- For each node N on the OpenList, $f(N) = g(N)+h(N) \geq g(N) \geq d(N) \times \varepsilon$, where $d(N)$ is the depth of N in the tree
- As long as A* hasn't terminated, a node K on the OpenList lies on a solution path
- Since each node expansion increases the length of one path, K will eventually be selected for expansion, unless a solution is found along another path

Proof (2/2)

2) Whenever A* chooses to expand a goal node, the path to this node is optimal



- C^*= cost of the optimal solution path
- G': non-optimal goal node in the fringe
  $f(G') = g(G') + h(G') = g(G') > C^*$
- A node K in the fringe lies on an optimal path:
  $$f(K) = g(K) + h(K) \leq C^*$$
- So, G' will not be selected for expansion

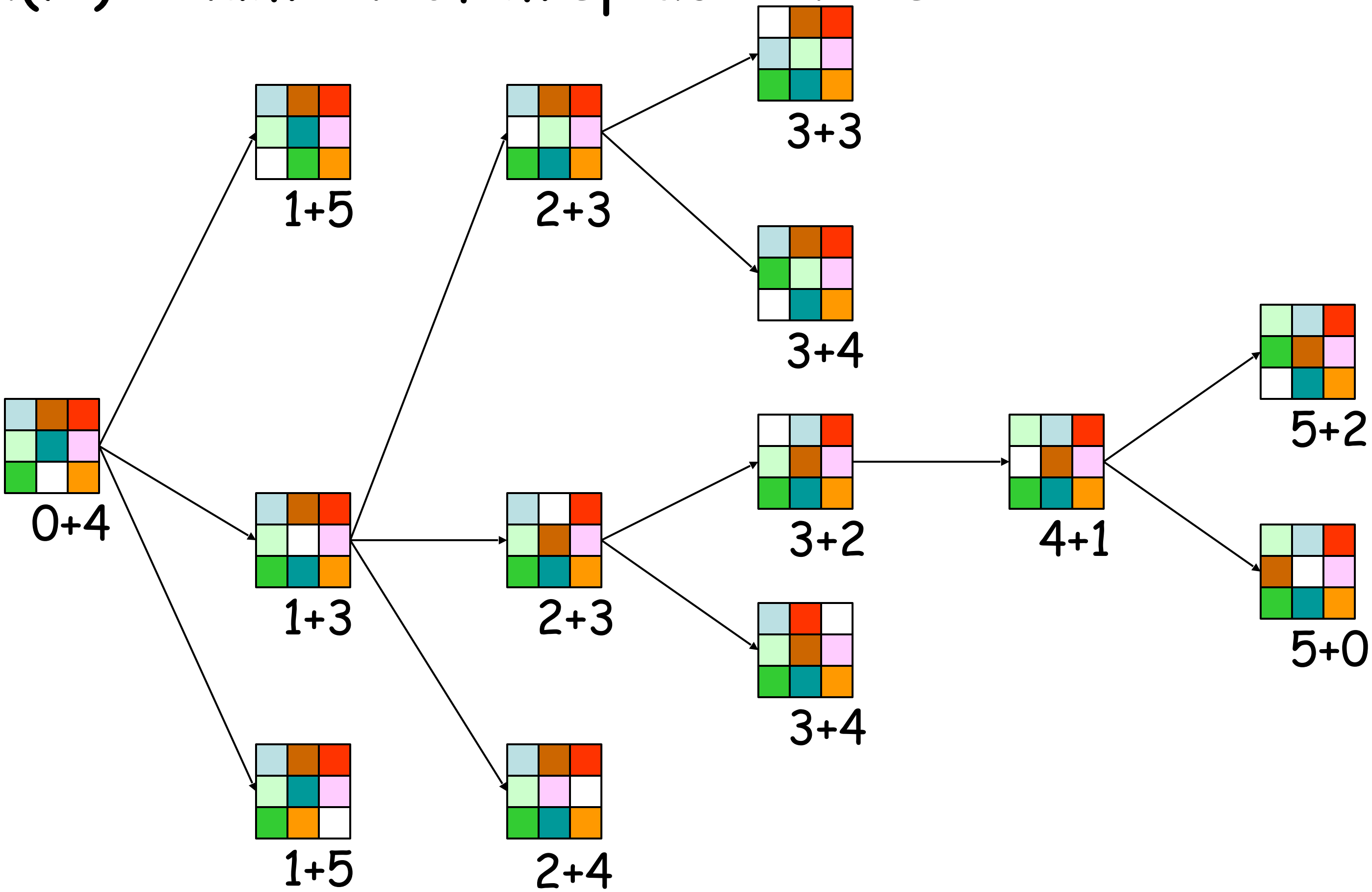
Time Limit Issue

- When a problem has no solution, A* runs for ever if the state space is infinite. In other cases, it may take a huge amount of time to terminate
- So, in practice, A* is given a time limit. If it has not found a solution within this limit, it stops. Then there is no way to know if the problem has no solution, or if more time was needed to find it
- When AI systems are "small" and solving a single search problem at a time, this is not too much of a concern.
- When AI systems become larger, they solve many search problems concurrently, some with no solution. **What should be the time limit for each of them?** More on this in the lecture on Motion Planning ...

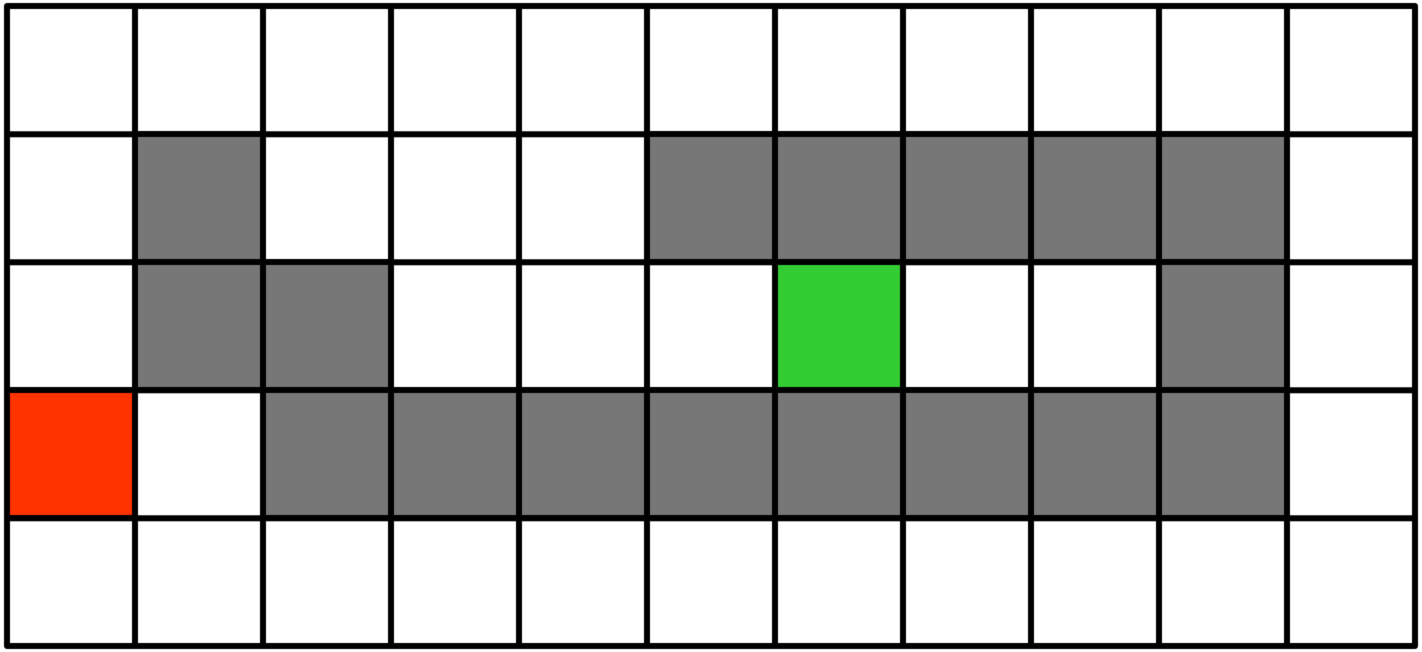
8-Puzzle

$f(N) = g(N) + h(N)$

with $h(N)$ = number of misplaced tiles

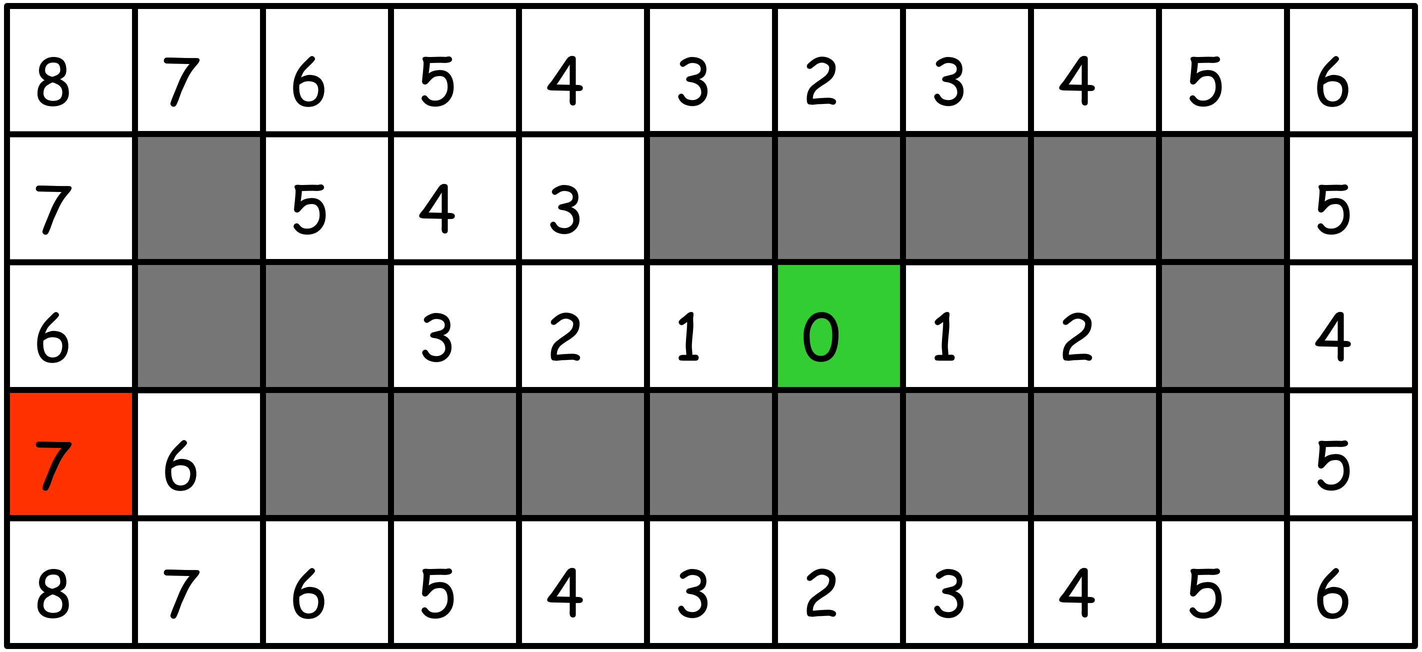


Robot Navigation



Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal
(not A*)



Robot Navigation

f(N) = h(N), with h(N) = Manhattan distance to the goal
(not A*)

8	7	6	5	4	3	2	3	4	5	6
7		5	4	3						5
6			3	2	1	0	1	2		4
7	6									5
8	7	6	5	4	3	2	3	4	5	6

Robot Navigation

f(N) = g(N)+h(N), with h(N) = Manhattan distance to goal

(A*)

8+3	7+4	6+3	5+6	4+7	3+8	2+9	3+10	4	5	6
7+2		5+6	4+7	3+8						5
6+1			3	2+9	1+10	0+11	1	2		4
7+0	6+1									5
8+1	7+2	6+3	5+4	4+5	3+6	2+7	3+8	4	5	6

Best-First Search

- An evaluation function f maps each node N of the search tree to a real number
  $f(N) \geq 0$
- Best-first search sorts the OpenList in increasing f

A* Search

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2) for all arcs: $c(N,N') \geq \varepsilon > 0$

3) SEARCH#2 algorithm is used

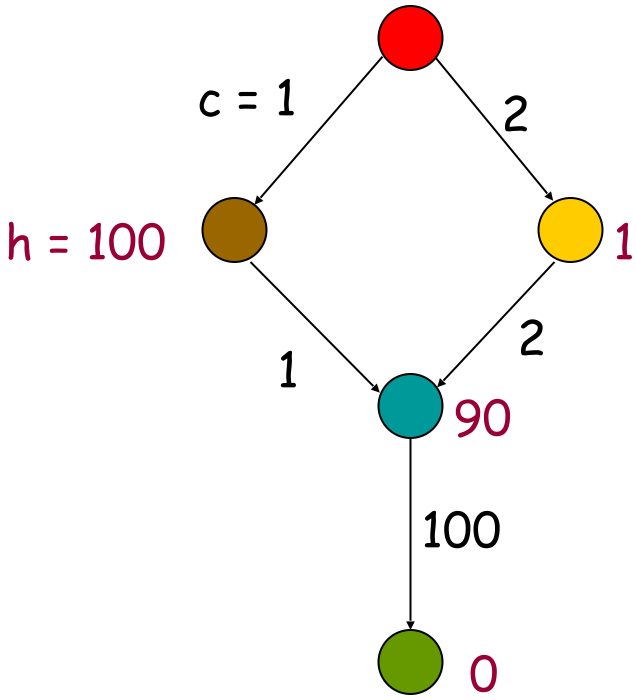
→ Best-first search is then called *A* search*

Result #1

A* is complete and optimal

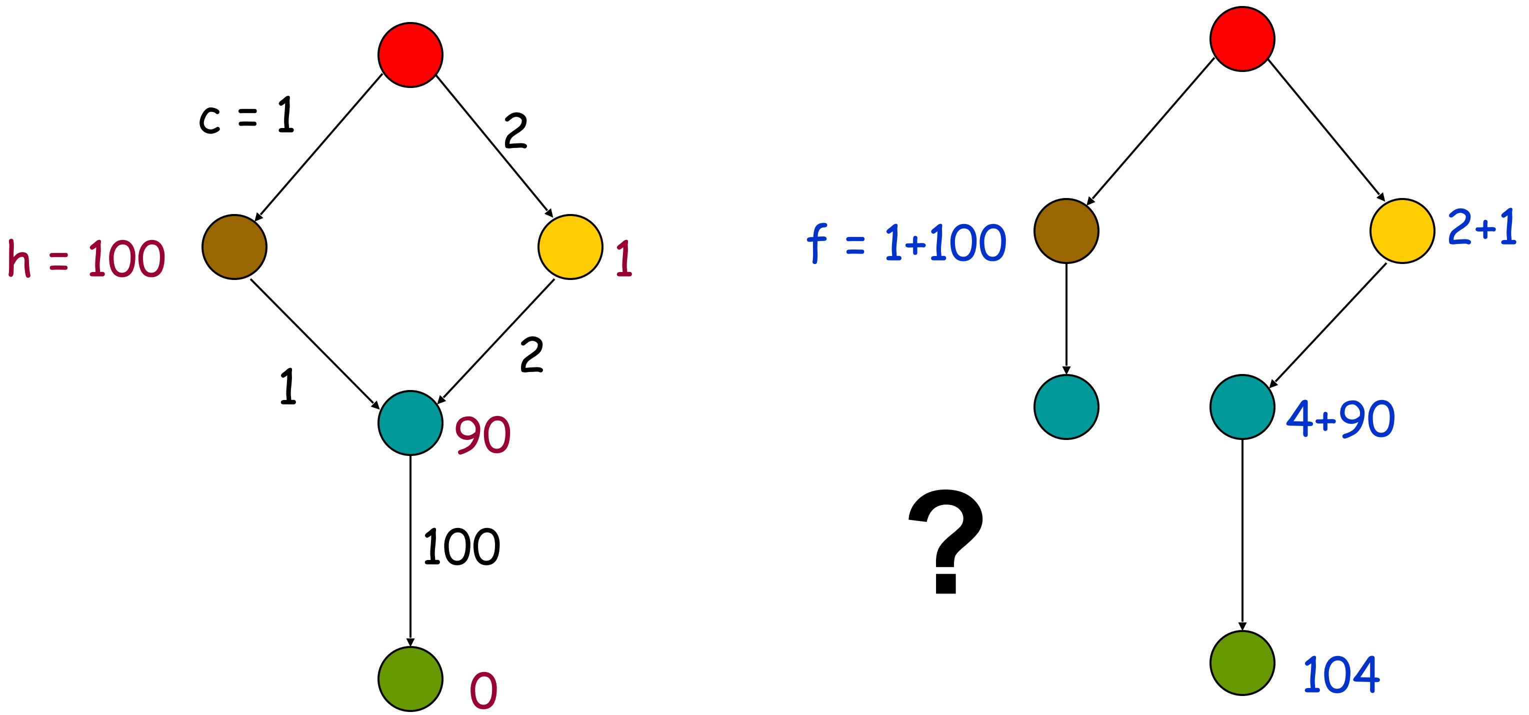
[This result holds if nodes revisiting states are not discarded]

What to do with revisited states?



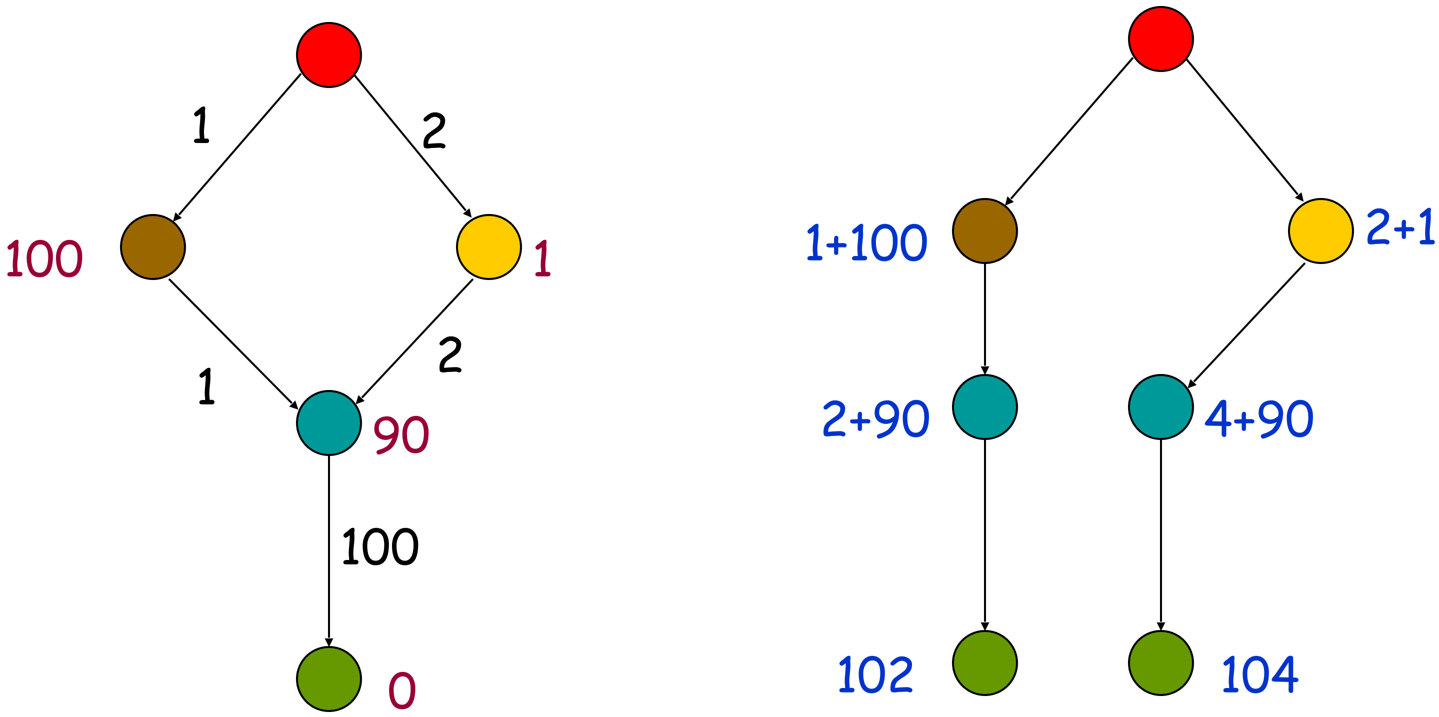
The heuristic h is clearly admissible

What to do with revisited states?



If we discard this new node, then the search algorithm expands the goal node next and returns a non-optimal solution

What to do with revisited states?



Instead, if we do not discard nodes revisiting states, the search terminates with an optimal solution

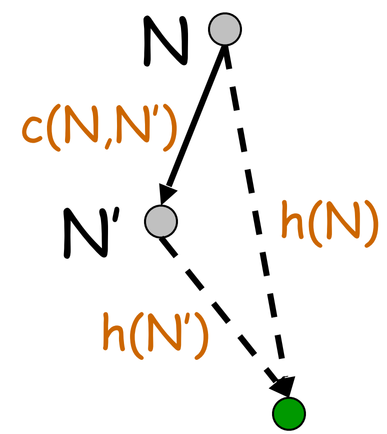
- It is not harmful to discard a node revisiting a state if the cost of the new path to this state is ≥ cost of the previous path
  [so, in particular, one can discard a node if it re-visits a state already visited by one of its ancestors]
- A* remains optimal, but states can still be re-visited multiple times
  [the size of the search tree can still be exponential in the number of visited states]
- Fortunately, for a large family of admissible heuristics – consistent heuristics – there is a much more efficient way to handle revisited states

Consistent Heuristic

An <u>admissible heuristic</u> h is consistent (or monotone) if for each node N and each child N' of N:

$$h(N) \leq c(N,N') + h(N')$$

$h(N) \leq C^*(N) \leq c(N,N') + h^*(N')$
$h(N) - c(N,N') \leq h^*(N')$
$h(N) - c(N,N') \leq h(N') \leq h^*(N')$



(triangle inequality)

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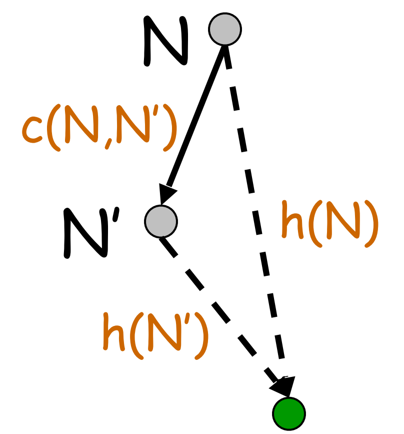


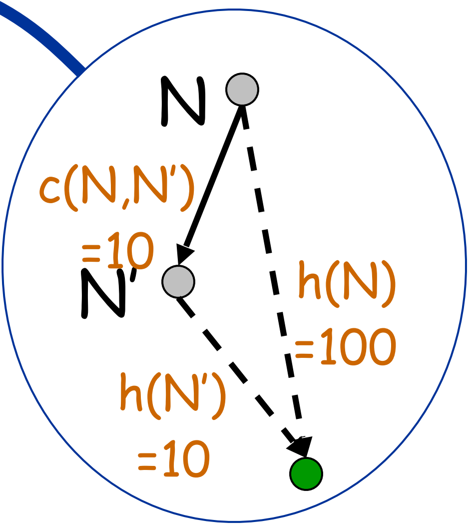
(triangle inequality)

$$h(N) \leq C^*(N) \leq c(N,N') + h^*(N')$$
$$h(N) - c(N,N') \leq h^*(N')$$
$$h(N) - c(N,N') \leq h(N') \leq h^*(N')$$

→ Intuition: a consistent heuristics becomes more precise as we get deeper in the search tree

Consistency Violation

If h tells that N is 100 units from the goal, then moving from N along an arc costing 10 units should **not** lead to a node N' that h estimates to be 10 units away from the goal



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Consistent Heuristic

(alternative definition)

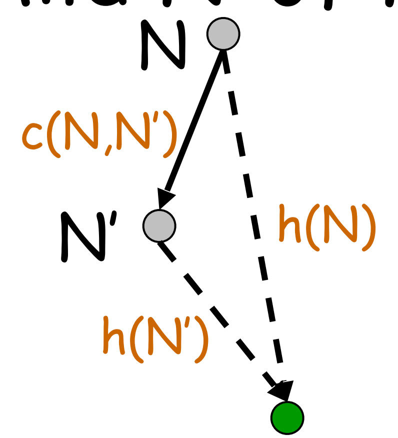
Any heuristic h is consistent (or monotone) if

1) for each node N and each child N' of N:

$$h(N) \leq c(N,N') + h(N')$$

2) for each goal node G:

$$h(G) = 0$$



(triangle inequality)

A consistent heuristic is also admissible

Admissibility and Consistency

- A consistent heuristic is also admissible
- An admissible heuristic may not be consistent, but many admissible heuristics are consistent
- Proof?

8-Puzzle

5		8
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STATE(N)

1	2	3
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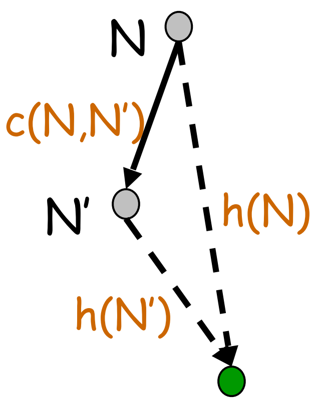
goal

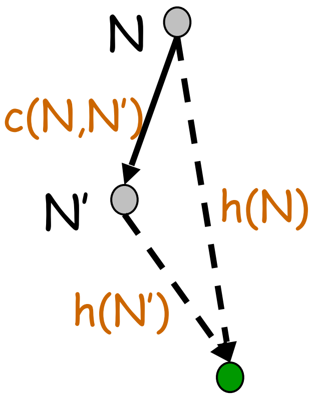


$h(N) \leq c(N,N') + h(N')$

- $h_1(N)$ = number of misplaced tiles
- $h_2(N)$ = sum of the (Manhattan) distances of every tile to its goal position

are both consistent (why?)

Robot Navigation



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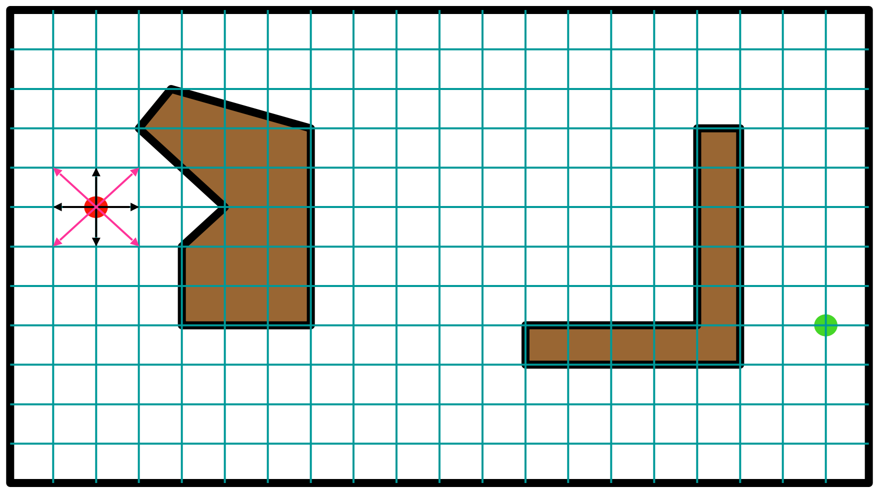

Cost of one horizontal/vertical step = 1

Cost of one diagonal step = $\sqrt{2}$

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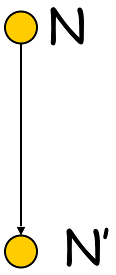
$h_2(N) = |x_N - x_g| + |y_N - y_g|$ is consistent if moving along diagonals is not allowed, and not consistent otherwise

Result #2

If h is consistent, then whenever A* expands a node, it has already found an optimal path to this node's state

Proof (1/2)



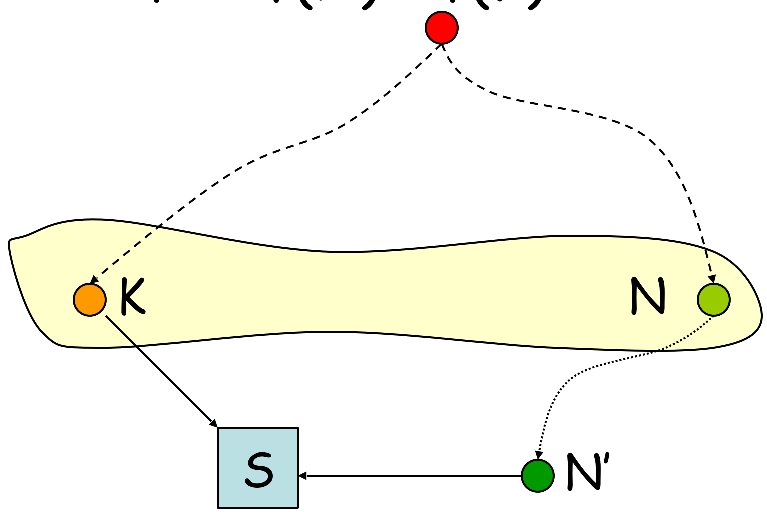
1) Consider a node N and its child N'
   Since h is consistent: $h(N) \leq c(N,N')+h(N')$

   $f(N) = g(N)+h(N) \leq g(N)+c(N,N')+h(N') = f(N')$

   So, f is non-decreasing along any path

Proof (2/2)

2) If a node K is selected for expansion, then any other node N in the OpenList verifies $f(N) \geq f(K)$



If one node N lies on another path to the state of K, the cost of this other path is no smaller than that of the path to K:

$f(N') \geq f(N) \geq f(K)$ and $h(N') = h(K)$

So, $g(N') \geq g(K)$

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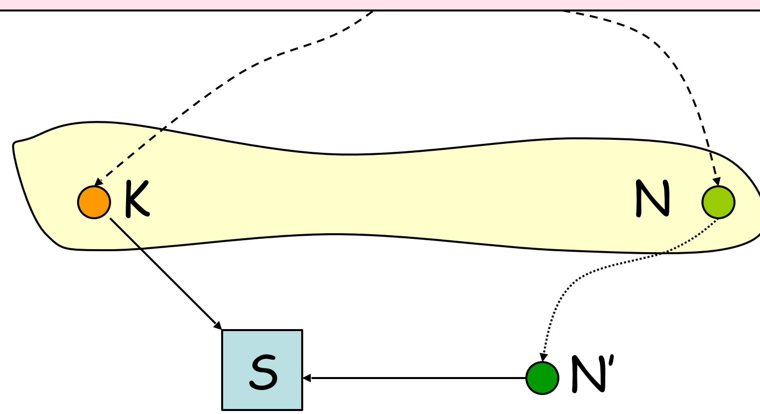

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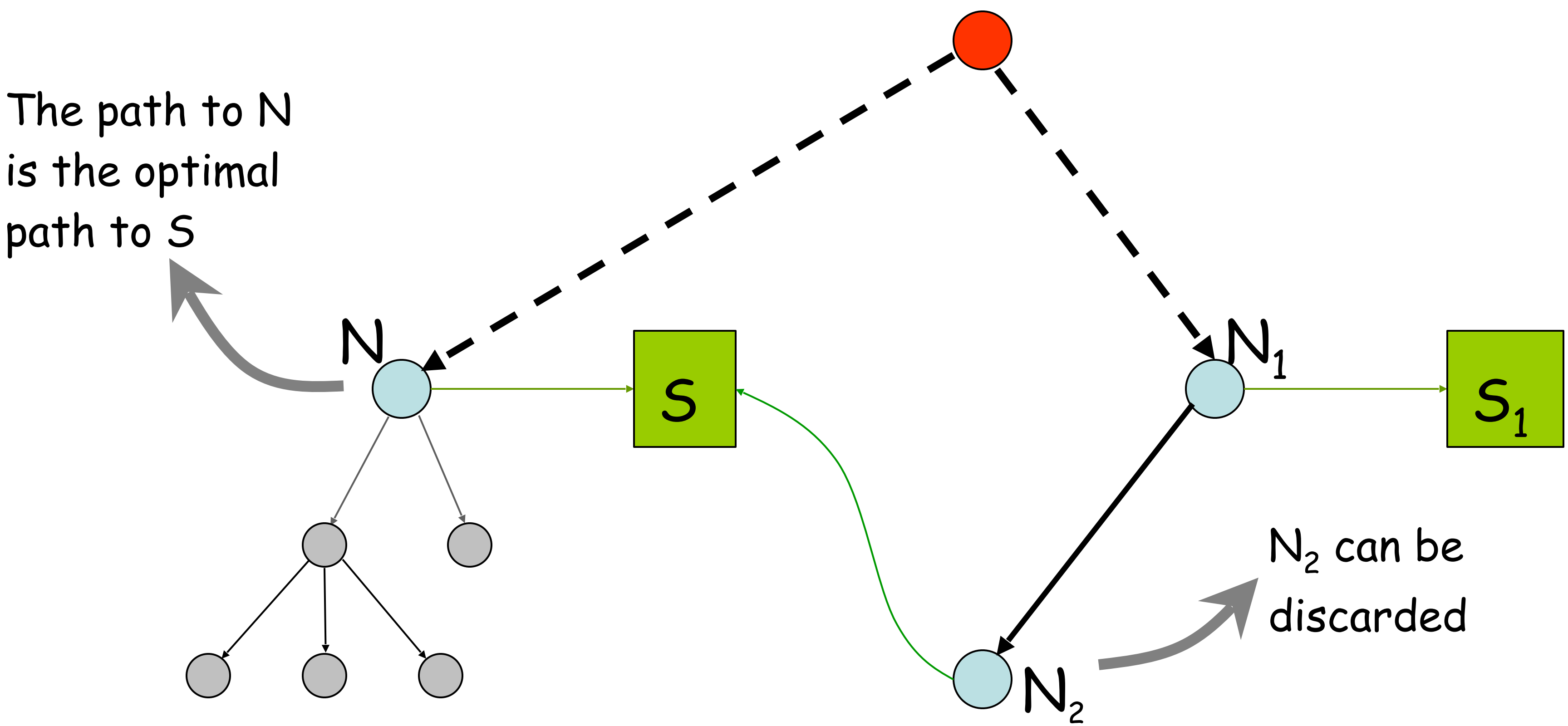
So, $g(N') \geq g(K)$

Implication of Result #2



Revisited States with Consistent Heuristic

- When a node is expanded, store its state into CLOSED
- When a new node N is generated:
  - If STATE(N) is in CLOSED, discard N
  - If there exists a node N' in the OpenList such that STATE(N') = STATE(N), discard the node – N or N' – with the largest f (or, equivalently, g)

Is A* with some consistent heuristic all that we need?

No !

There are **very dumb** consistent heuristic functions

For example: $h \equiv 0$

- It is consistent (hence, admissible) !
- A* with $h \equiv 0$ is uniform-cost search
- Breadth-first and uniform-cost are particular cases of A*

Heuristic Accuracy

Let h_1 and h_2 be two consistent heuristics such that for all nodes N:

$$h_1(N) \leq h_2(N)$$

h_2 is said to be more accurate (or more informed) than h_1

5		8
4	2	1
7	3	6

STATE(N)

1	2	3
4	5	6
7	8	

Goal state

- $h_1(N)$ = number of misplaced tiles
- $h_2(N)$ = sum of distances of every tile to its goal position

- h_2 is more accurate than h_1

Result #3

- Let h_2 be more accurate than h_1
- Let A_1^* be A^* using h_1
  and A_2^* be A^* using h_2
- Whenever a solution exists, all the nodes expanded by A_2^*, except possibly for some nodes such that
  $f_1(N) = f_2(N) = C^*$ (cost of optimal solution)
  are also expanded by A_1^*

Proof

- $C^* = h^*(\text{initial-node})$ [cost of optimal solution]
- Every node N such that $f(N) < C^*$ is eventually expanded. No node N such that $f(N) > C^*$ is ever expanded
- Every node N such that $h(N) < C^*-g(N)$ is eventually expanded. So, every node N such that $h_2(N) < C^*-g(N)$ is expanded by A_2^*. Since $h_1(N) \leq h_2(N)$, N is also expanded by A_1^*
- If there are several nodes N such that $f_1(N) = f_2(N) = C^*$ (such nodes include the optimal goal nodes, if there exists a solution), A_1^* and A_2^* may or may not expand them in the same order (until one goal node is expanded)

Effective Branching Factor

- It is used as a measure the effectiveness of a heuristic
- Let n be the total number of nodes expanded by A* for a particular problem and d the depth of the solution
- The effective branching factor b* is defined by $n = 1 + b^* + (b^*)^2 + \ldots + (b^*)^d$

Experimental Results

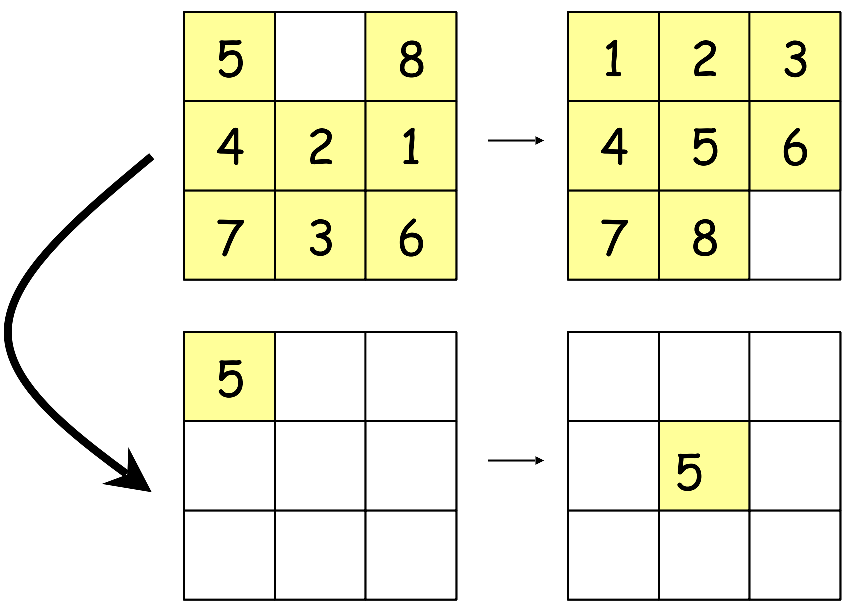
(see R&N for details)

- 8-puzzle with:
  - h_1 = number of misplaced tiles
  - h_2 = sum of distances of tiles to their goal positions
- Random generation of many problem instances
- Average effective branching factors:

d	IDS	A_1^*	A_2^*
2	2.45	1.79	1.79
6	2.73	1.34	1.30
12	2.78 (3,644,035)	1.42 (227)	1.24 (73)
16	--	1.45	1.25
20	--	1.47	1.27
24	--	1.48 (39,135)	1.26 (1,641)

How to create good heuristics?

- By solving relaxed problems at each node
- In the 8-puzzle, the sum of the distances of each tile to its goal position (h_2) corresponds to solving 8 simple problems:



d_i is the length of the shortest path to move tile i to its goal position, ignoring the other tiles, e.g., $d_5 = 2$

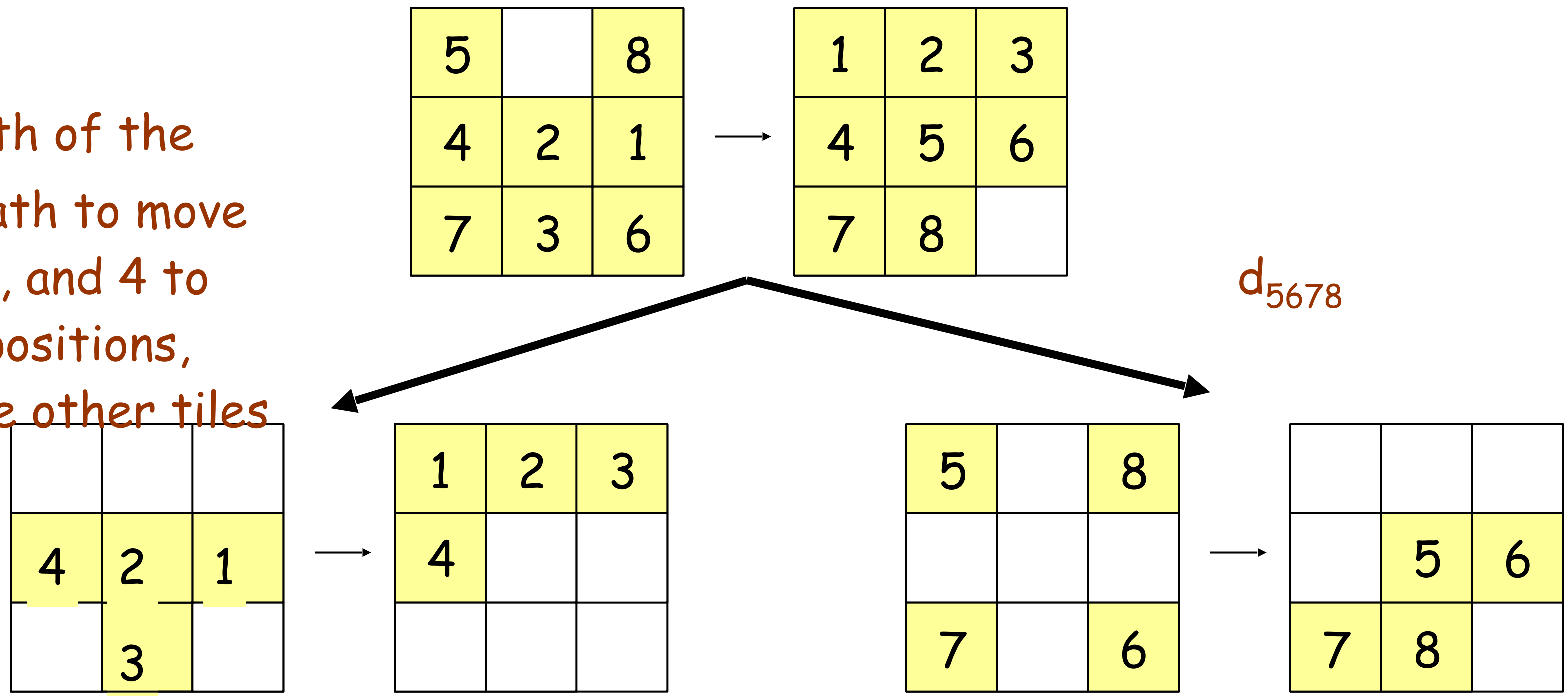
$$h_2 = \Sigma_{i=1,\ldots,8}\, d_i$$

- It ignores negative interactions among tiles

Can we do better?

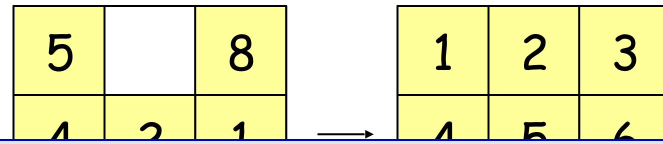
- For example, we could consider two more complex relaxed problems:

d_{1234} = length of the shortest path to move tiles 1, 2, 3, and 4 to their goal positions, ignoring the other tiles



Can we do better?

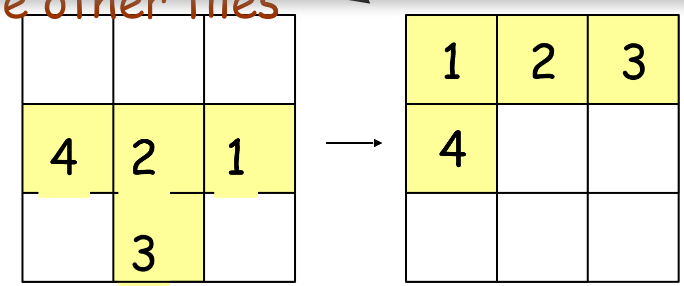
- For example, we could consider two more complex relaxed problems:

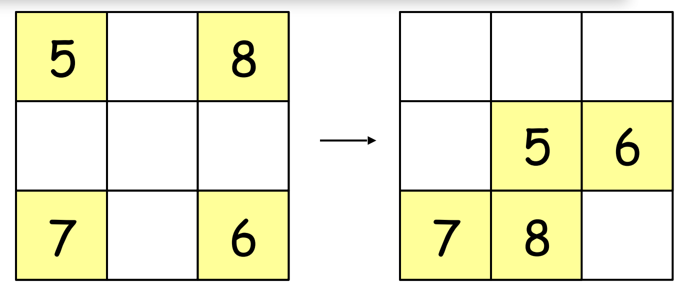


d = length of the

→ Several order-of-magnitude speedups for the 15- and 24-puzzle (see R&N)

ignoring the other tiles





On Completeness and Optimality

- A* with a consistent heuristic function has nice properties: completeness, optimality, no need to revisit states
- Theoretical completeness does not mean "practical" completeness if you must wait too long to get a solution (remember the time limit issue)
- So, if one can't design an accurate consistent heuristic, it may be better to settle for a non-admissible heuristic that "works well in practice", even through completeness and optimality are no longer guaranteed