Constraint Satisfaction Problems (CSP)

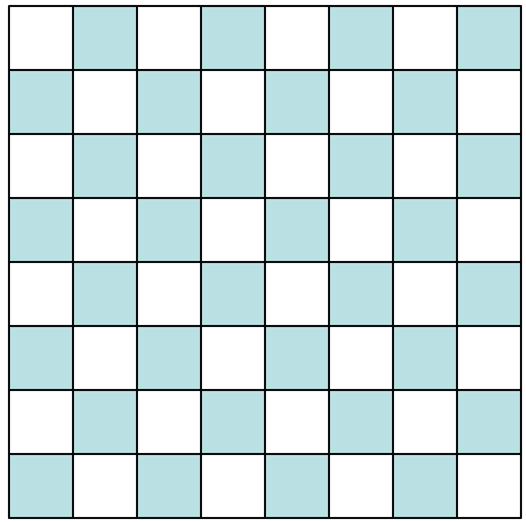
(Where we postpone making difficult decisions until they become easy to make)

R&N: Chap. 5

What we will try to do ...

- Search techniques make choices in an often arbitrary order. Often little information is available to make each of them
- In many problems, the same states can be reached independent of the order in which choices are made ("commutative" actions)
- Can we solve such problems more efficiently by picking the order appropriately? Can we even avoid making any choice?

Constraint Propagation



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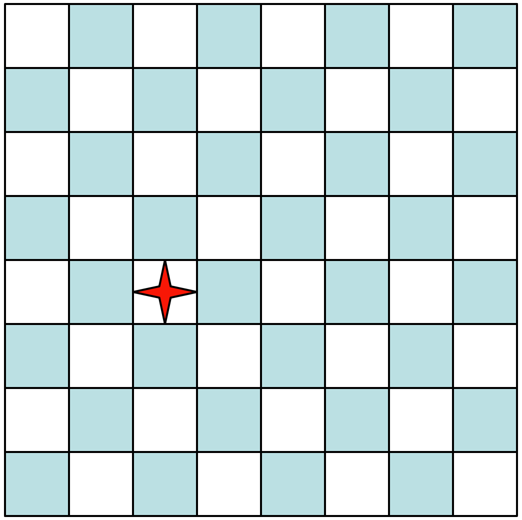

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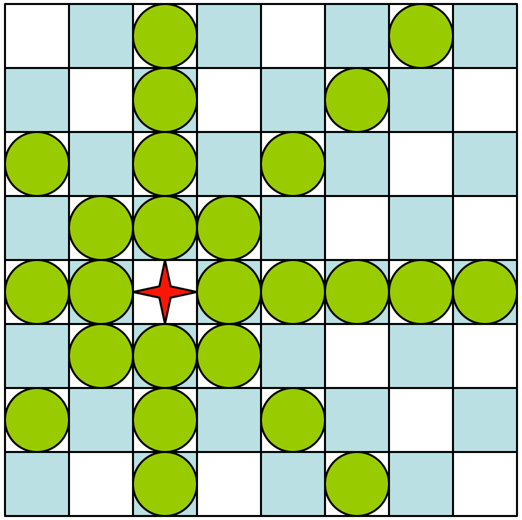
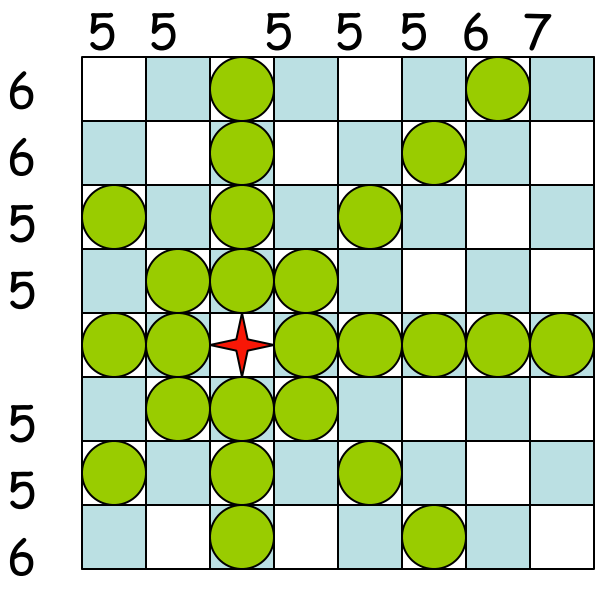


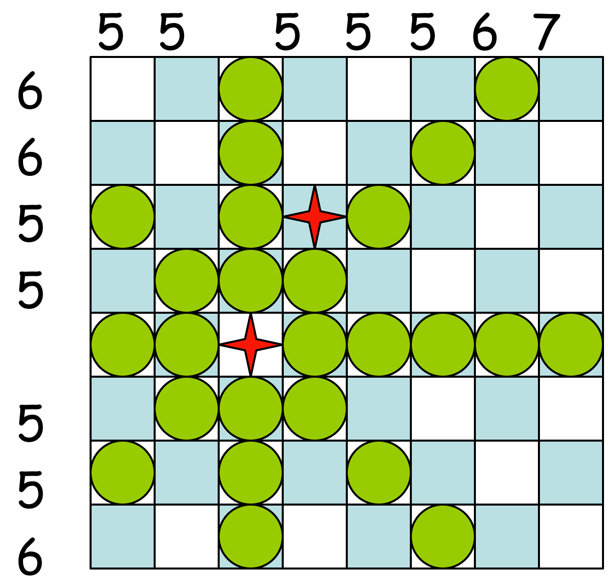
- Place a queen in a square
- Remove the attacked squares from future consideration

Constraint Propagation



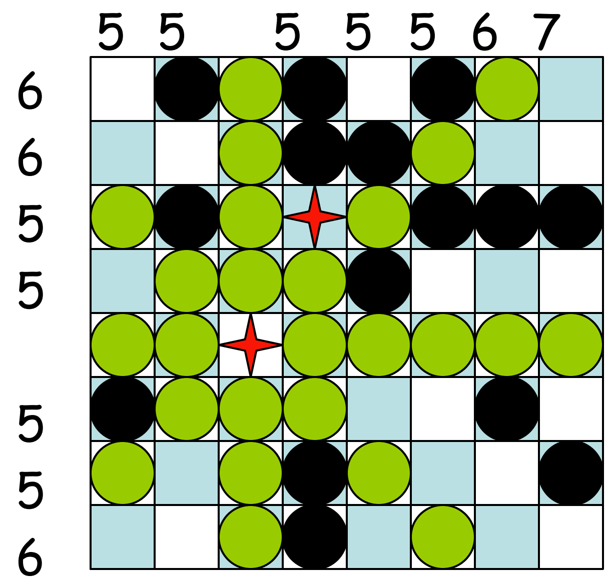
- Count the number of non-attacked squares in every row and column

Constraint Propagation



- Count the number of non-attacked squares in every row and column
- Place a queen in a row or column with minimum number

Constraint Propagation



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- Place a queen in a row or column with minimum number
- Remove the attacked squares from future consideration

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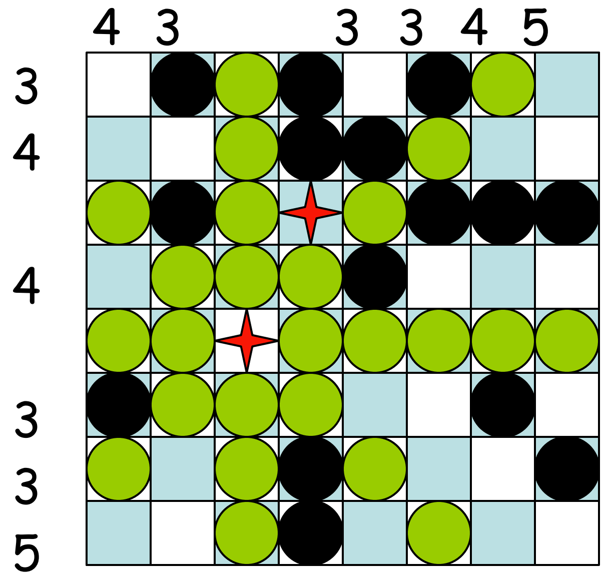
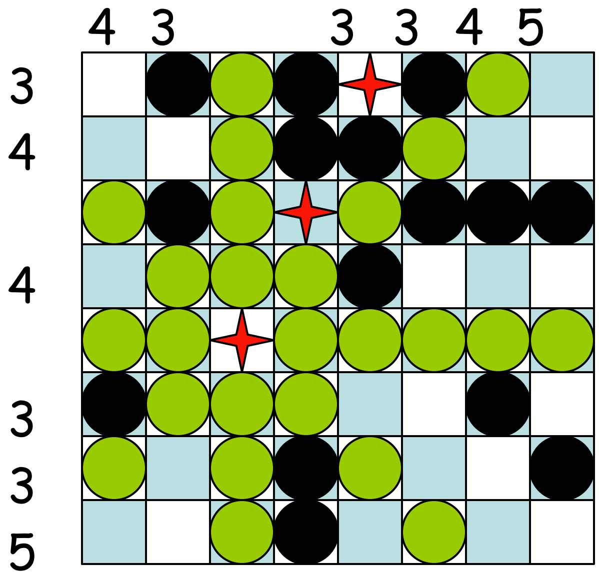


- Repeat

Constraint Propagation



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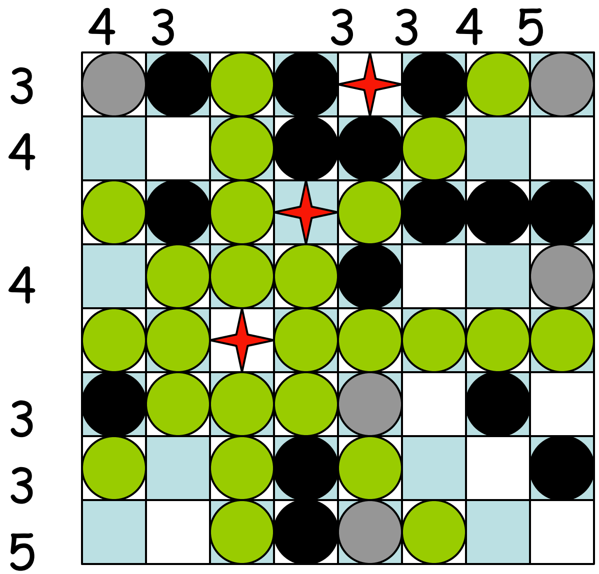

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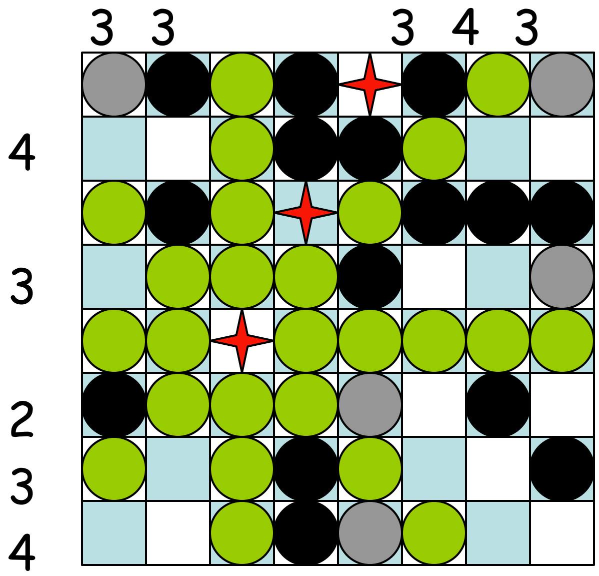

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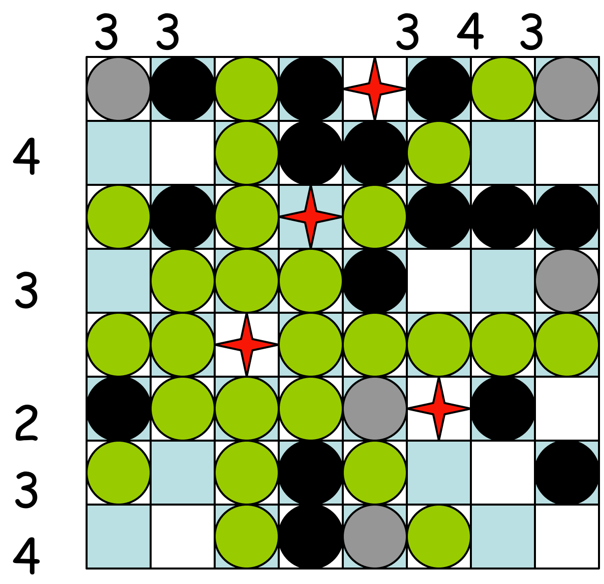

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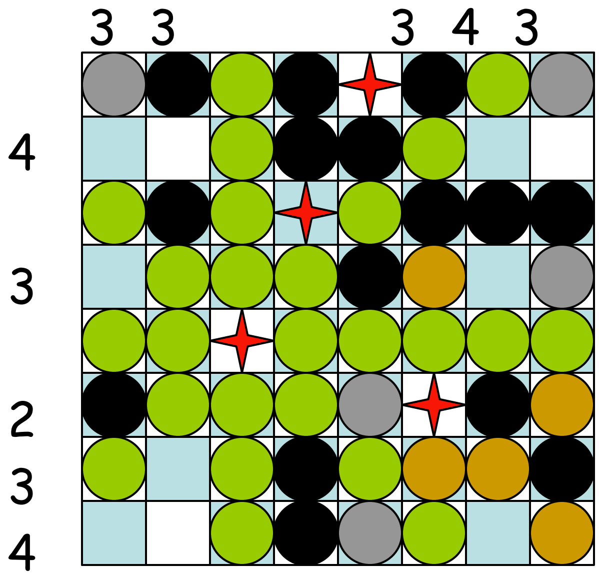

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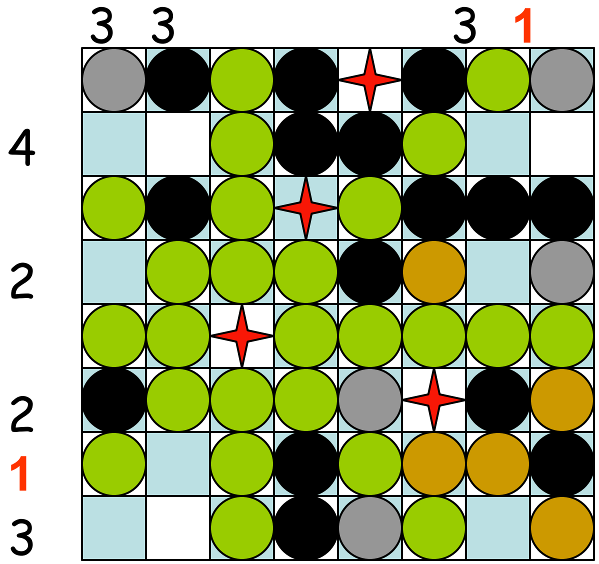

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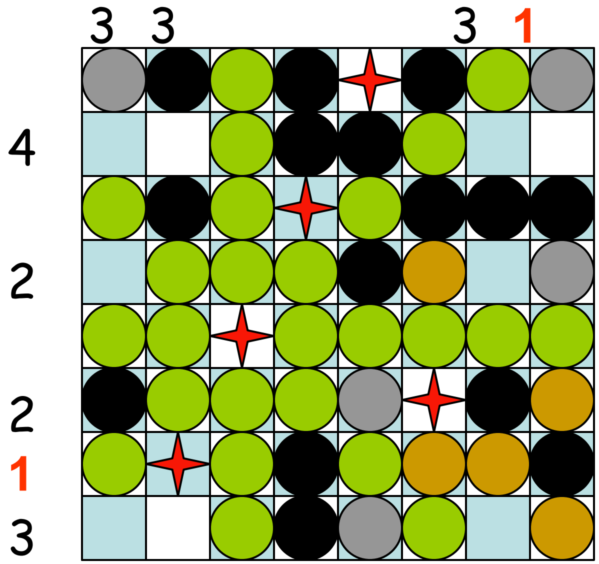

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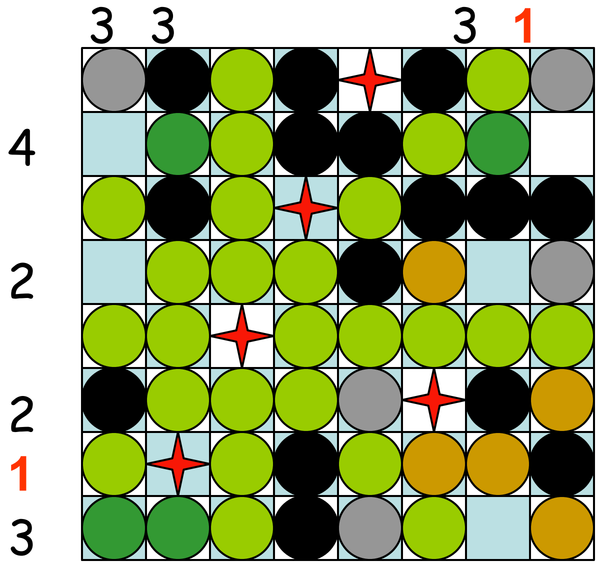

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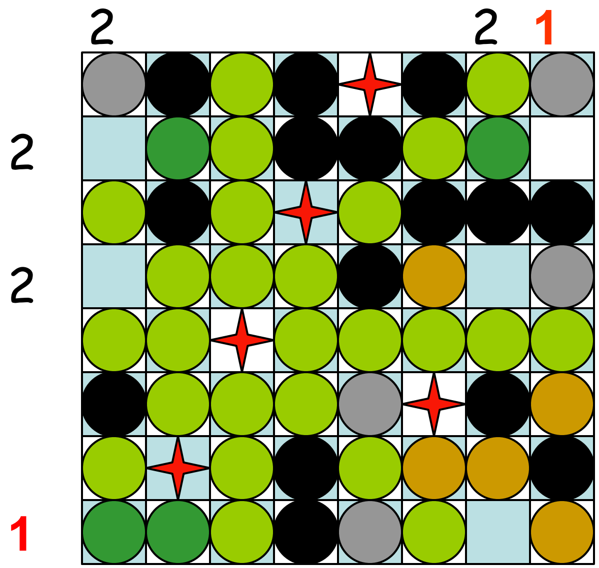

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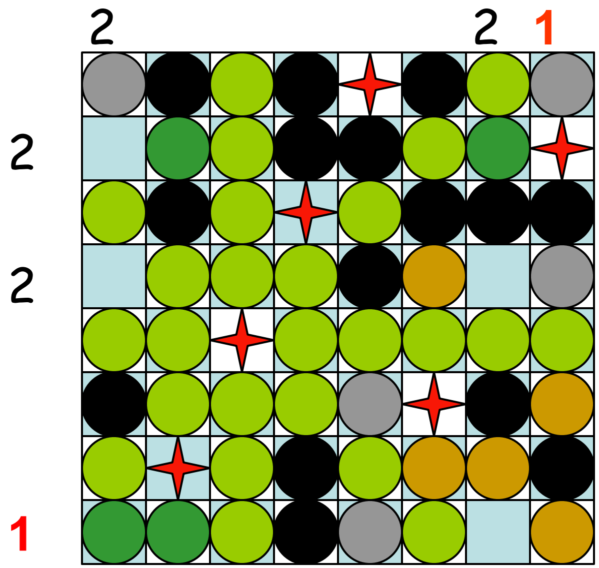

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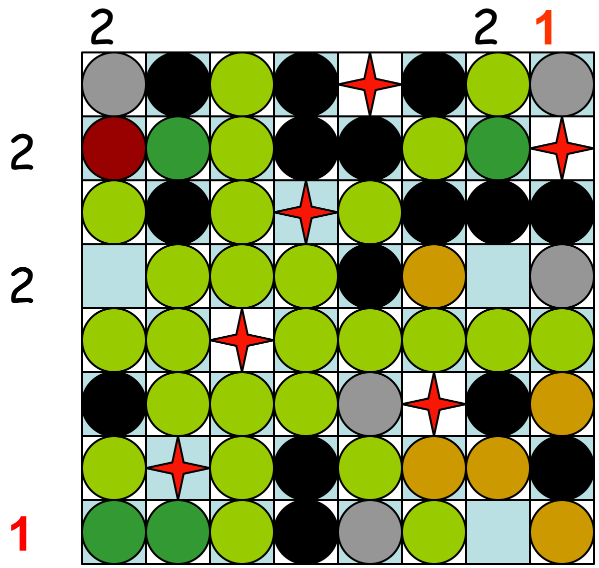

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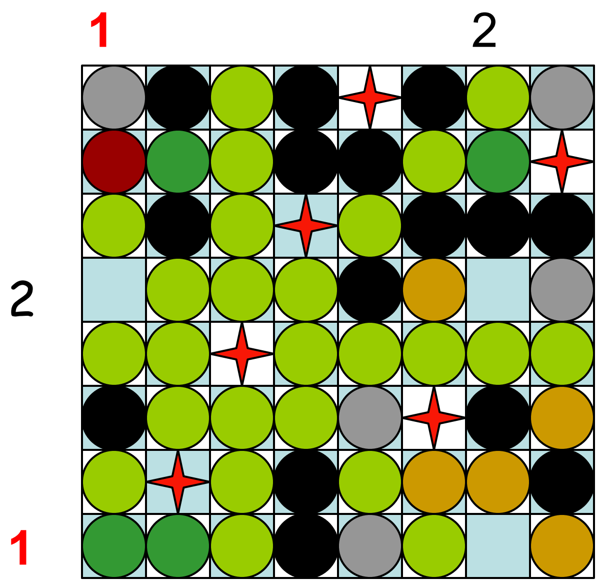

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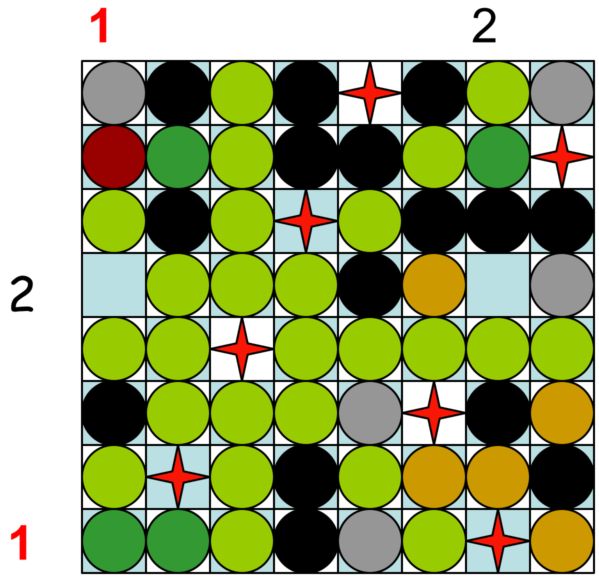

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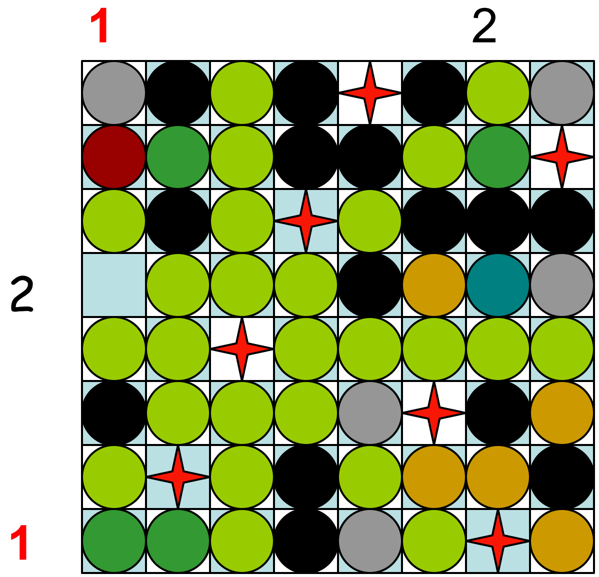

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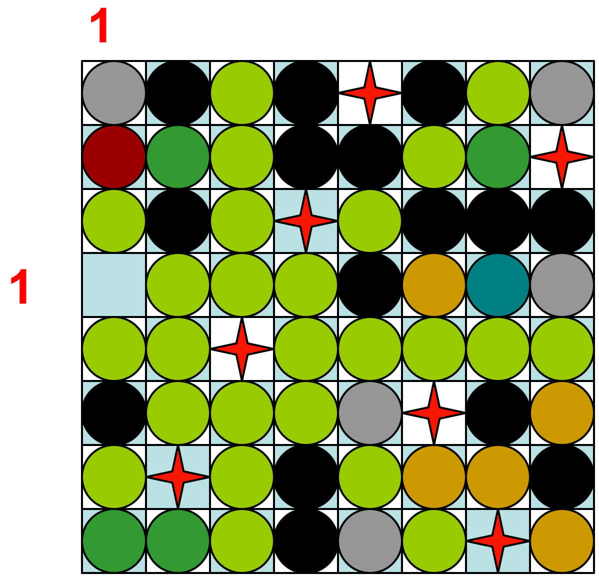

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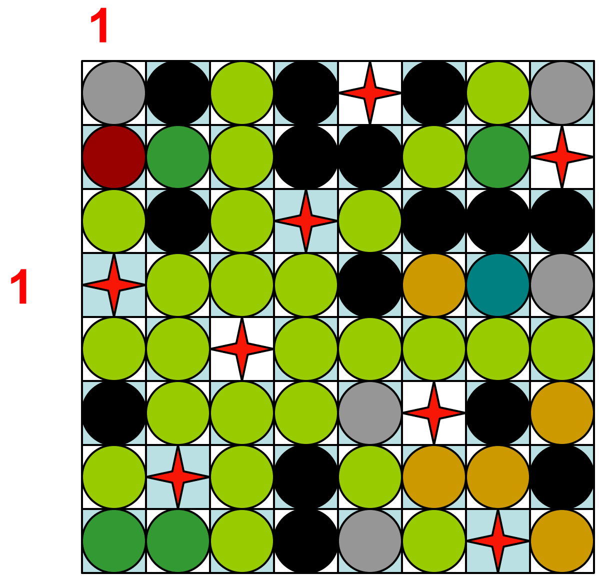

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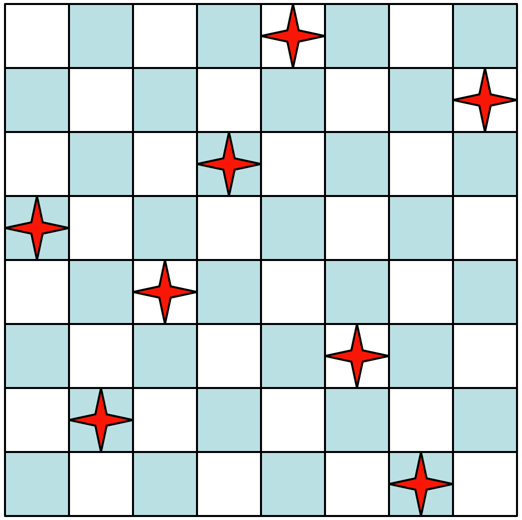


What do we need?

- More than just a successor function and a goal test

- We also need:
  - A means to propagate the constraints imposed by one queen's position on the positions of the other queens
  - An early failure test

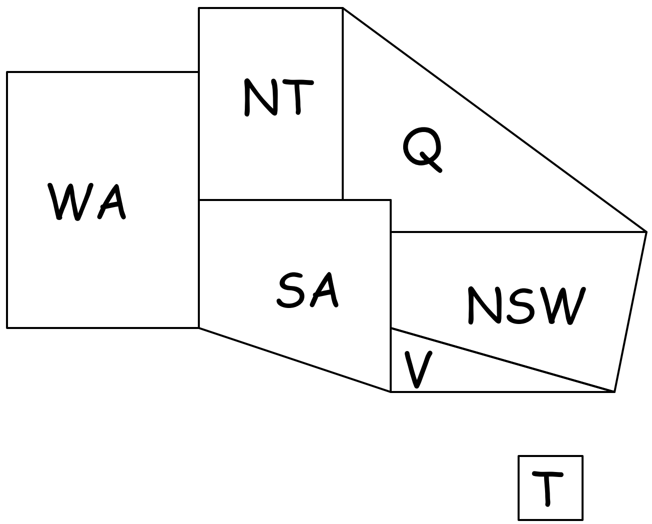
→ Explicit representation of constraints

→ Constraint propagation algorithms

Constraint Satisfaction Problem (CSP)

- Set of variables $\{X_1, X_2, \ldots, X_n\}$
- Each variable X_i has a domain D_i of possible values. Usually, D_i is finite
- Set of constraints $\{C_1, C_2, \ldots, C_p\}$
- Each constraint relates a subset of variables by specifying the valid combinations of their values
- Goal: Assign a value to every variable such that all constraints are satisfied

Map Coloring



- 7 variables {WA,NT,SA,Q,NSW,V,T}
- Each variable has the same domain:
  {red, green, blue}
- No two adjacent variables have the same value:
  WA≠NT, WA≠SA, NT≠SA, NT≠Q, SA≠Q,
  SA≠NSW, SA≠V, Q≠NSW, NSW≠V

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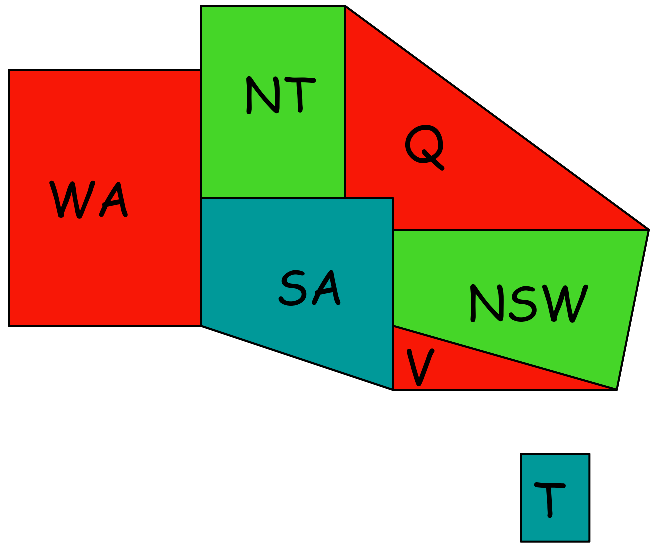

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8-Queen Problem

- 8 variables X_i, $i = 1$ to 8
- The domain of each variable is: {1,2,…,8}
- Constraints are of the forms:
  - $X_i = k \rightarrow X_j \neq k$ for all $j = 1$ to 8, $j \neq i$
  - Similar constraints for diagonals

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  - Similar constraints for diagonals

All constraints are binary

Street Puzzle

1 2 3 4 5

N_i = {English, Spaniard, Japanese, Italian, Norwegian}
C_i = {Red, Green, White, Yellow, Blue}
D_i = {Tea, Coffee, Milk, Fruit-juice, Water}
J_i = {Painter, Sculptor, Diplomat, Violinist, Doctor}
A_i = {Dog, Snails, Fox, Horse, Zebra}

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The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor’s
The Horse is next to the Diplomat’s

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Who owns the Zebra?
Who drinks Water?

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$\forall i,j \in [1,5],\ i \neq j,\ N_i \neq N_j$

$\forall i,j \in [1,5],\ i \neq j,\ C_i \neq C_j$

...

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The Spaniard has a Dog
The Japanese is a Painter → $(N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$
The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house →
$$\begin{cases} (C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green}) \\ (C_5 \neq \text{White}) \end{cases}$$
$(C_1 \neq \text{Green})$
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
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The Spaniard has a Dog
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The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house → $(C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green})$, $(C_5 \neq \text{White})$, $(C_1 \neq \text{Green})$
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
The Horse is next to the Diplomat's → left as an exercise

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The Spaniard has a Dog
The Japanese is a Painter → $(N_i = \text{Japanese}) \Leftrightarrow (J_i = \text{Painter})$
The Italian drinks Tea
The Norwegian lives in the first house on the left → $(N_1 = \text{Norwegian})$
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
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The owner of the middle house drinks Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice
The Fox is in the house next to the Doctor's
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The Green house is on the right of the White house →
- $(C_i = \text{White}) \Leftrightarrow (C_{i+1} = \text{Green})$
- $(C_5 \neq \text{White})$
- $(C_1 \neq \text{Green})$

unary constraints

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The Englishman lives in the Red house
The Spaniard has a Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left → N_1 = Norwegian
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk → D_3 = Milk
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The Englishman lives in the Red house → $C_1 \neq$ Red
The Spaniard has a Dog → $A_1 \neq$ Dog
The Japanese is a Painter
The Italian drinks Tea
The Norwegian lives in the first house on the left → N_1 = Norwegian
The owner of the Green house drinks Coffee
The Green house is on the right of the White house
The Sculptor breeds Snails
The Diplomat lives in the Yellow house
The owner of the middle house drinks Milk → D_3 = Milk
The Norwegian lives next door to the Blue house
The Violinist drinks Fruit juice → $J_3 \neq$ Violinist
The Fox is in the house next to the Doctor's
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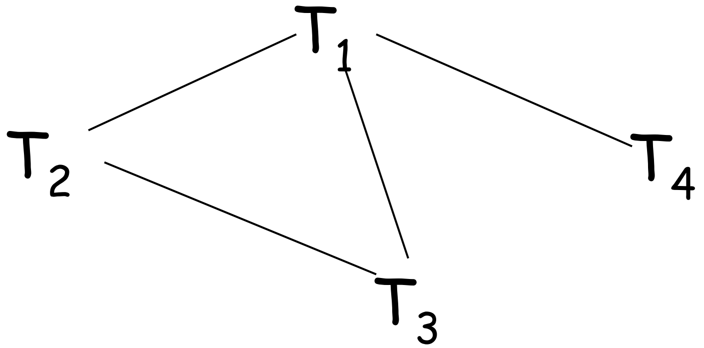
Task Scheduling



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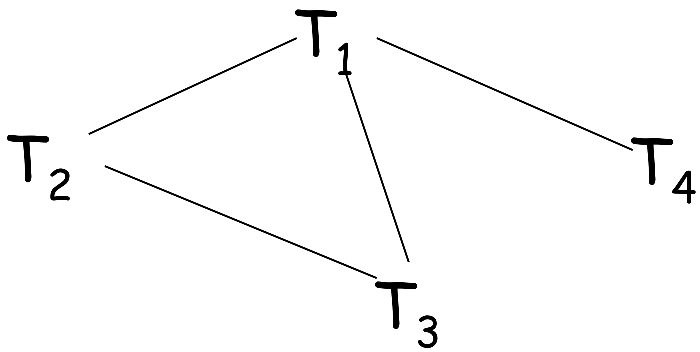

Four tasks T_1, T_2, T_3, and T_4 are related by time constraints:

- T_1 must be done during T_3
- T_2 must be achieved before T_1 starts
- T_2 must overlap with T_3
- T_4 must start after T_1 is complete

- Are the constraints compatible?
- What are the possible time relations between two tasks?
- What if the tasks use resources in limited supply?

How to formulate this problem as a CSP?

3-SAT

- n Boolean variables $u_1, \ldots, u_n$

- p constraints of the form

  $$u_i^* \vee u_j^* \vee u_k^* = 1$$

  where u* stands for either u or ¬u

- Known to be NP-complete

Finite vs. Infinite CSP

- Finite CSP: each variable has a finite domain of values
- Infinite CSP: some or all variables have an infinite domain

  E.g., linear programming problems over the reals:

  for i = 1, 2, ..., p : $a_{i,1}x_1+a_{i,2}x_2+\ldots+a_{i,n}x_n = a_{i,0}$

  for j = 1, 2, ..., q : $b_{j,1}x_1+b_{j,2}x_2+\ldots+b_{j,n}x_n \leq b_{j,0}$
- We will only consider finite CSP

CSP as a Search Problem

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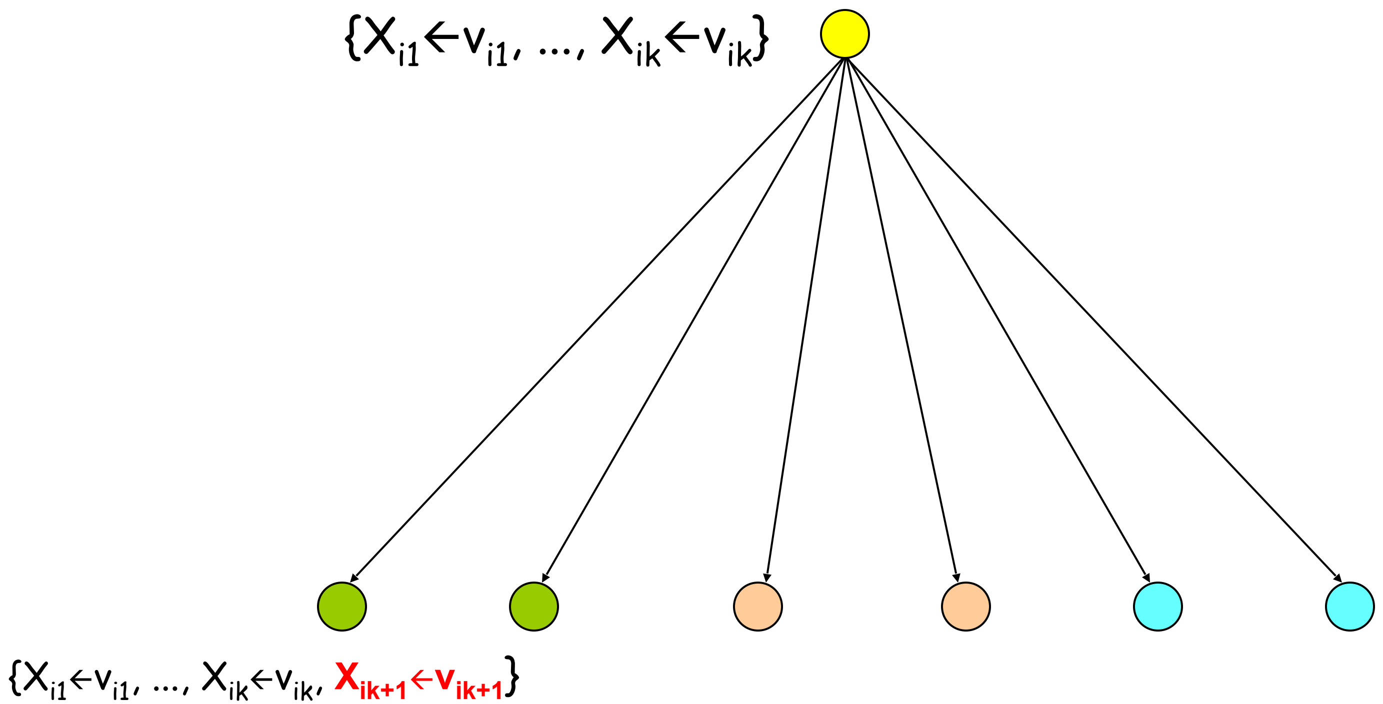
- n variables $X_1, \ldots, X_n$

CSP as a Search Problem

- n variables $X_1, \ldots, X_n$
- Valid assignment: $\{X_{i1} \leftarrow v_{i1}, \ldots, X_{ik} \leftarrow v_{ik}\}$, $0 \leq k \leq n$, such that the values $v_{i1}, \ldots, v_{ik}$ satisfy all constraints relating the variables $X_{i1}, \ldots, X_{ik}$
- Complete assignment: one where k = n
  [if all variable domains have size d, there are $O(d^n)$ complete assignments]
- States: valid assignments
- Initial state: empty assignment {}, i.e. k = 0
- Successor of a state:
  $\{X_{i1}\leftarrow v_{i1}, \ldots, X_{ik}\leftarrow v_{ik}\} \rightarrow \{X_{i1}\leftarrow v_{i1}, \ldots, X_{ik}\leftarrow v_{ik}, \mathbf{X_{ik+1}\leftarrow v_{ik+1}}\}$
- Goal test: k = n



$r = n-k$ variables with s values $\rightarrow$ $r \times s$ branching factor

A Key property of CSP: Commutativity

The order in which variables are assigned values has no impact on the reachable complete valid assignments

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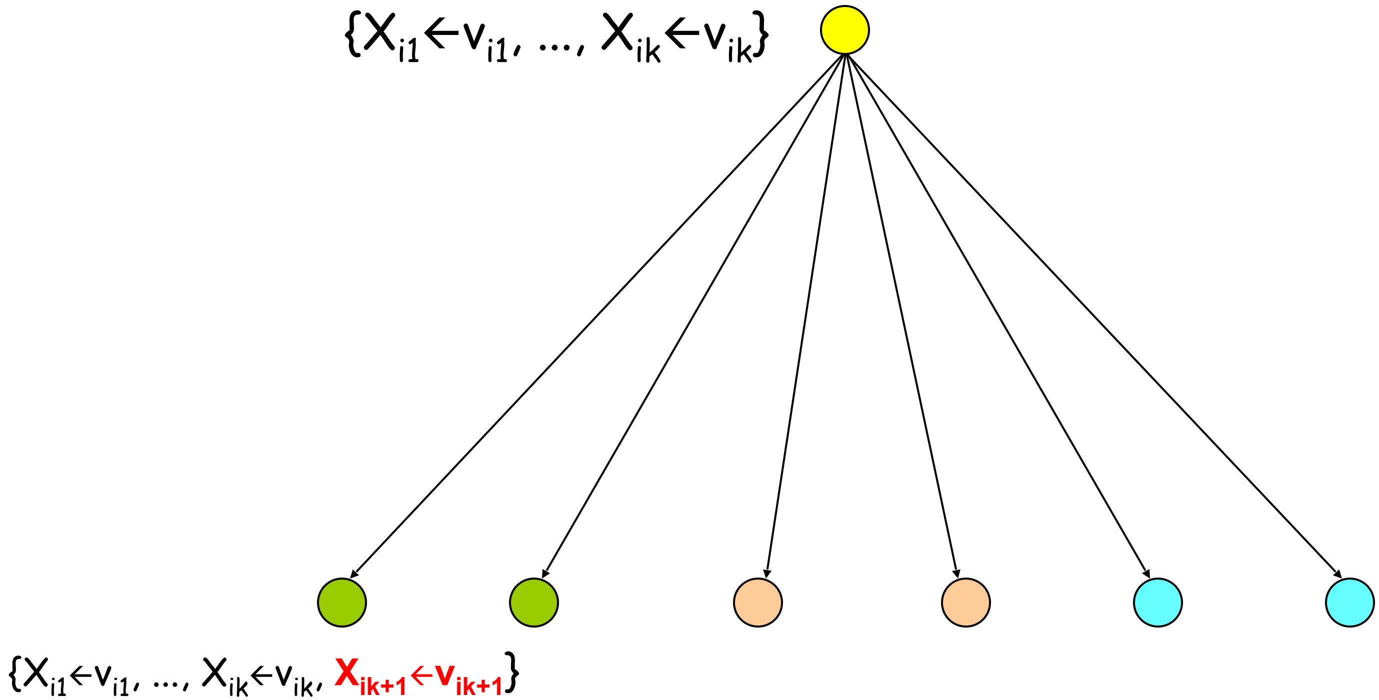
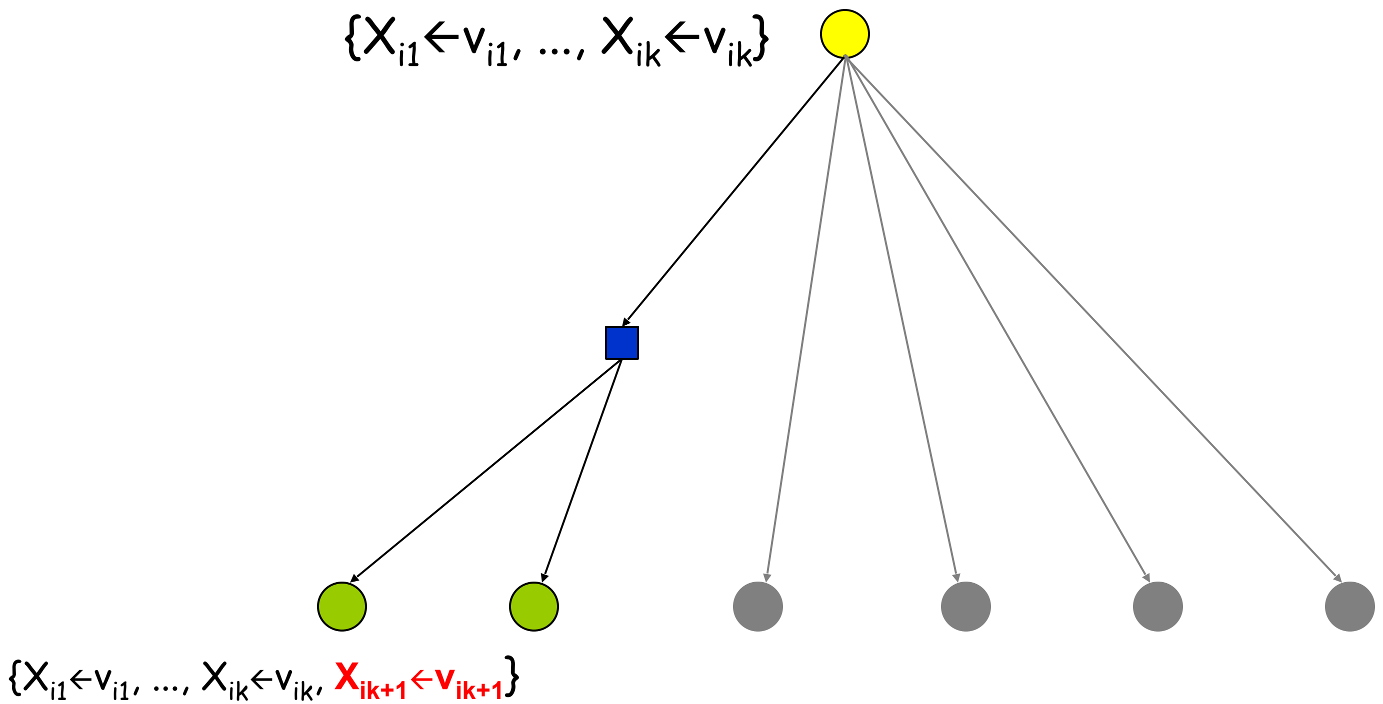
Hence:

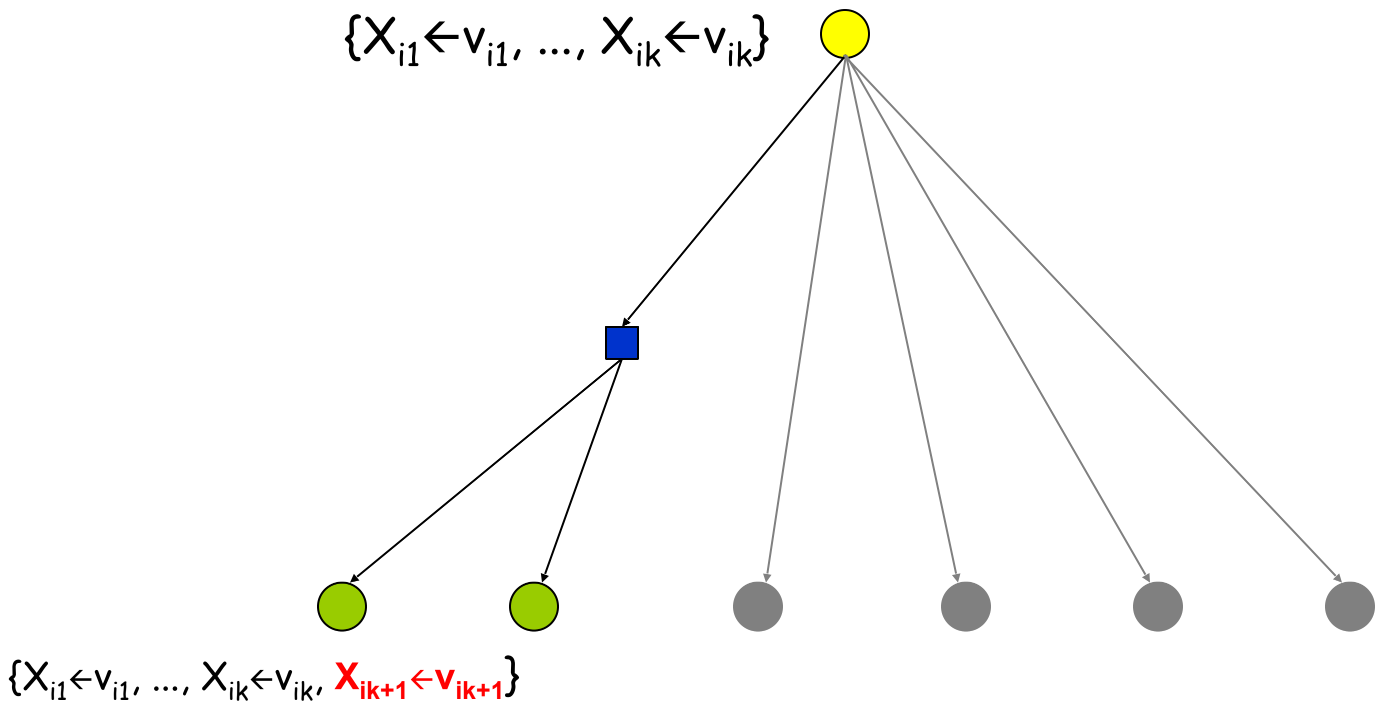
1) One can expand a node N by first selecting **one** variable X not in the assignment *A* associated with N and then assigning every value v in the domain of X
[→ big reduction in branching factor]



r = n-k variables with s values → r×s branching factor



r = n–k variables with s values → **s** branching factor



$r = n - k$ variables with s values → **s** branching factor

The depth of the solutions in the search tree is un-changed (n)

- 4 variables $X_1, \ldots, X_4$
- Let the valid assignment of N be:

  $A = \{X_1 \leftarrow v_1, X_3 \leftarrow v_3\}$
- For example pick variable X_4
- Let the domain of X_4 be $\{v_{4,1}, v_{4,2}, v_{4,3}\}$
- The successors of *A* are all the valid assignments among:

  $\{X_1 \leftarrow v_1, X_3 \leftarrow v_3 , X_4 \leftarrow v_{4,1}\}$

  $\{X_1 \leftarrow v_1, X_3 \leftarrow v_3 , X_4 \leftarrow v_{4,2}\}$

  $\{X_1 \leftarrow v_1, X_3 \leftarrow v_3 , X_4 \leftarrow v_{4,2}\}$

A Key property of CSP: Commutativity

The order in which variables are assigned values has no impact on the reachable complete valid assignments

Hence:

1) One can expand a node N by first selecting **one** variable X not in the assignment *A* associated with N and then assigning every value v in the domain of X
[→ big reduction in branching factor]

2) One need not store the path to a node

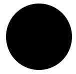
→ Backtracking search algorithm

Backtracking Search

Essentially a simplified depth-first algorithm using recursion

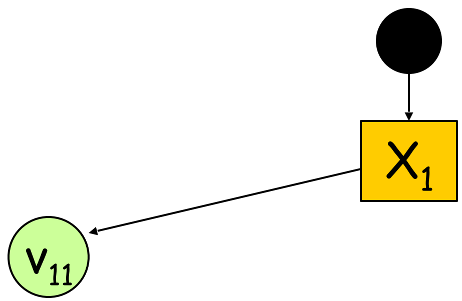
Backtracking Search
(3 variables)

Backtracking Search
(3 variables)



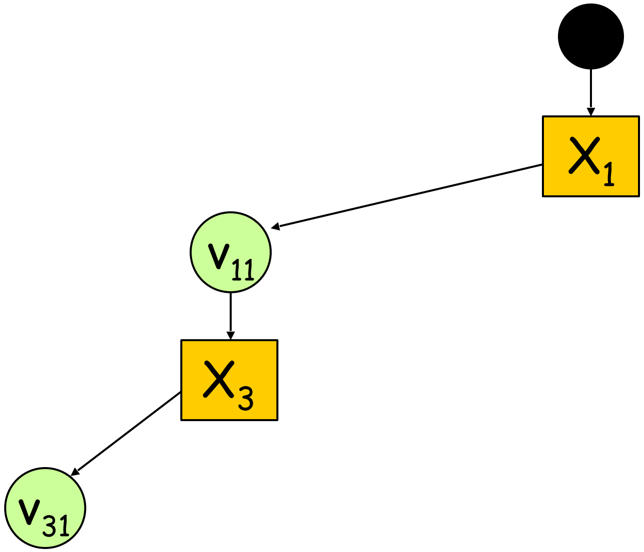
Assignment = {}

Backtracking Search
(3 variables)



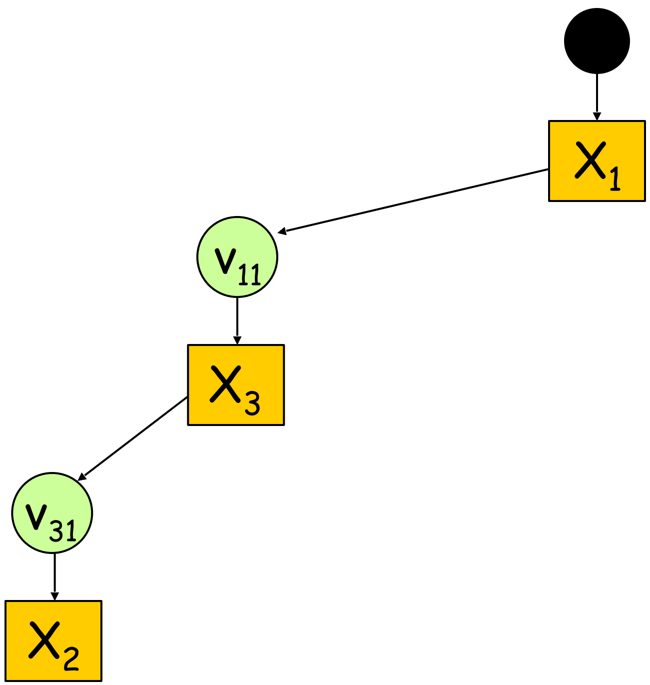
Assignment = $\{(X_1, v_{11})\}$

Backtracking Search
(3 variables)



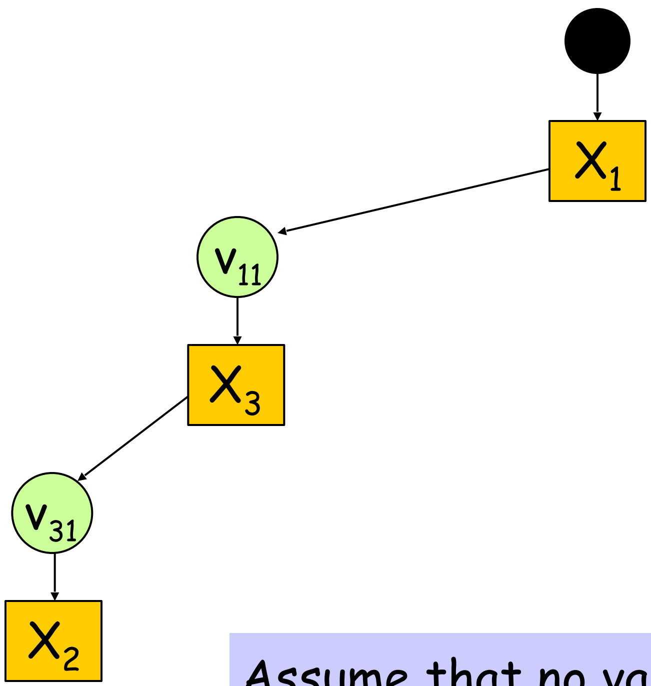
Assignment = $\{(X_1,v_{11}), (X_3,v_{31})\}$

Backtracking Search
(3 variables)



Assignment = $\{(X_1, v_{11}), (X_3, v_{31})\}$

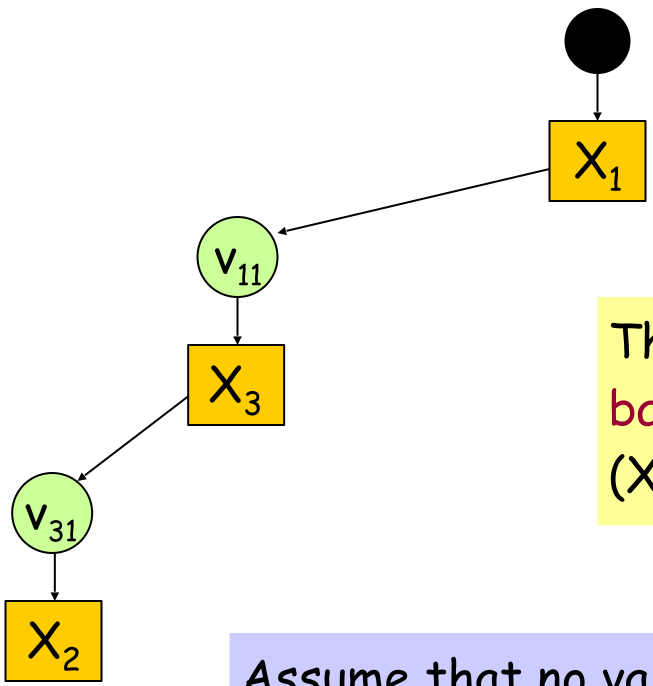
Backtracking Search
(3 variables)



Assume that no value of X_2 leads to a valid assignment

Assignment = $\{(X_1,v_{11}), (X_3,v_{31})\}$

Backtracking Search
(3 variables)

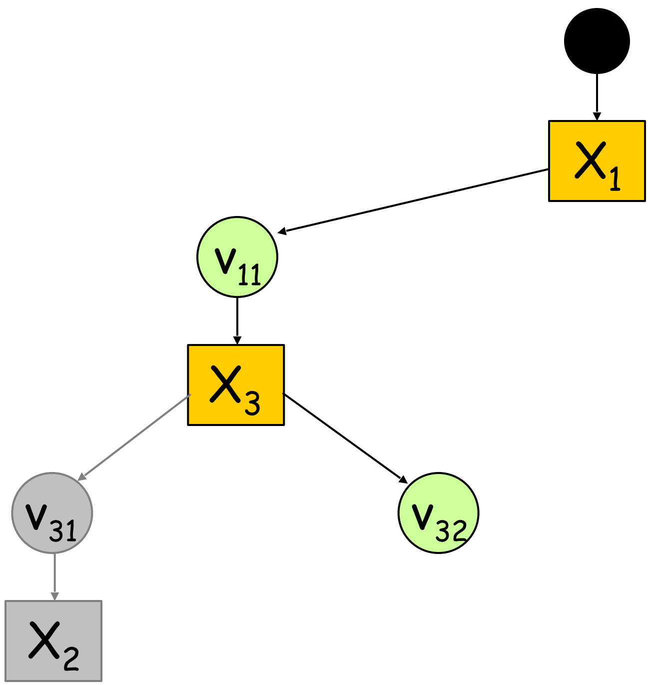


Then, the search algorithm backtracks to the previous variable (X_3) and tries another value

Assume that no value of X_2 leads to a valid assignment

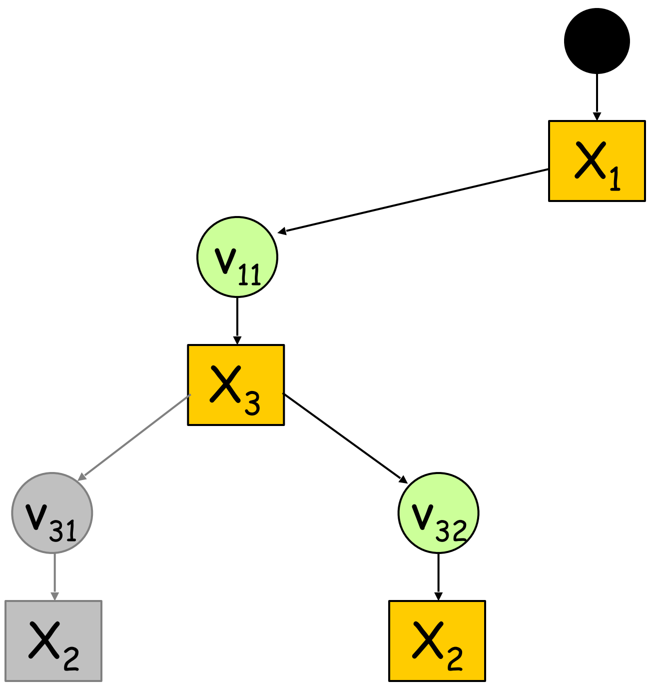
Assignment = $\{(X_1,v_{11}), (X_3,v_{31})\}$

Backtracking Search
(3 variables)



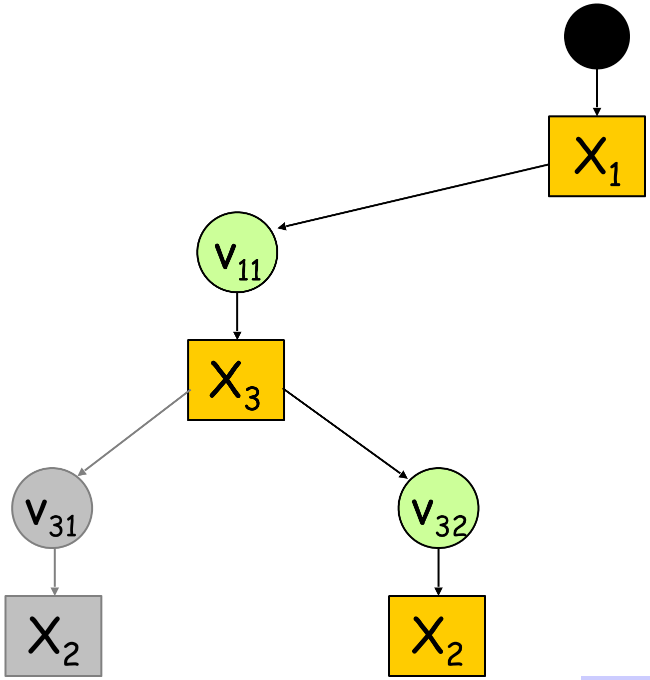
Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

Backtracking Search
(3 variables)



Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

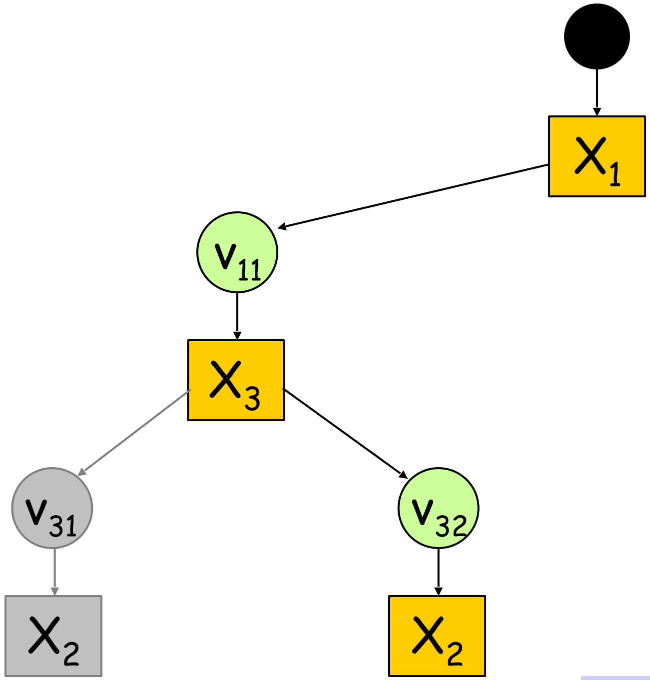
Backtracking Search
(3 variables)



Assume again that no value of X_2 leads to a valid assignment

Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

Backtracking Search
(3 variables)

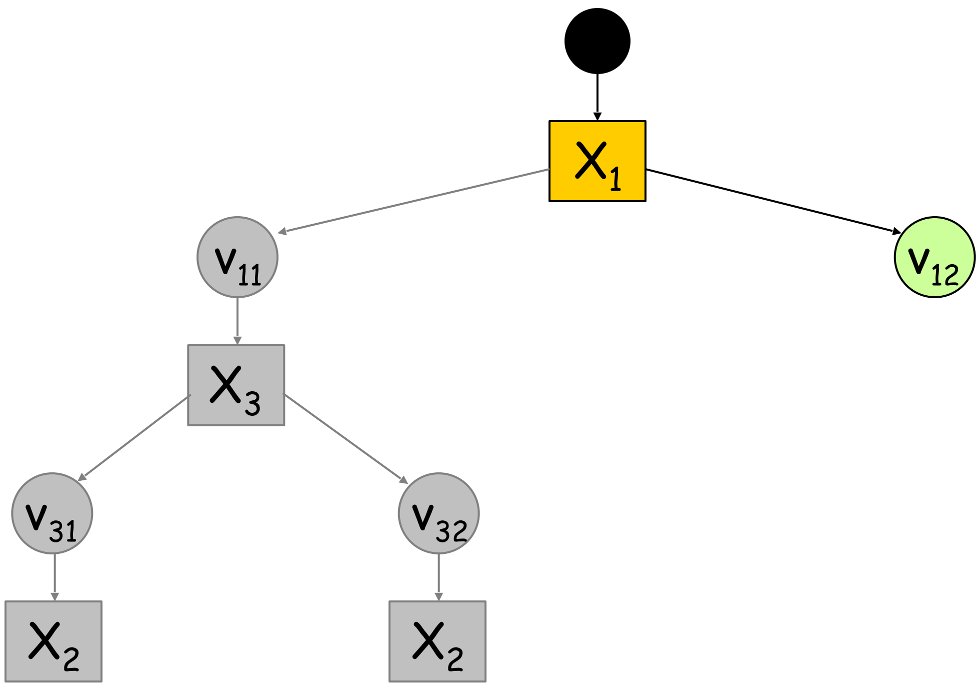


The search algorithm backtracks to the previous variable (X_3) and tries another value. But assume that X_3 has only two possible values. The algorithm backtracks to X_1

Assume again that no value of X_2 leads to a valid assignment

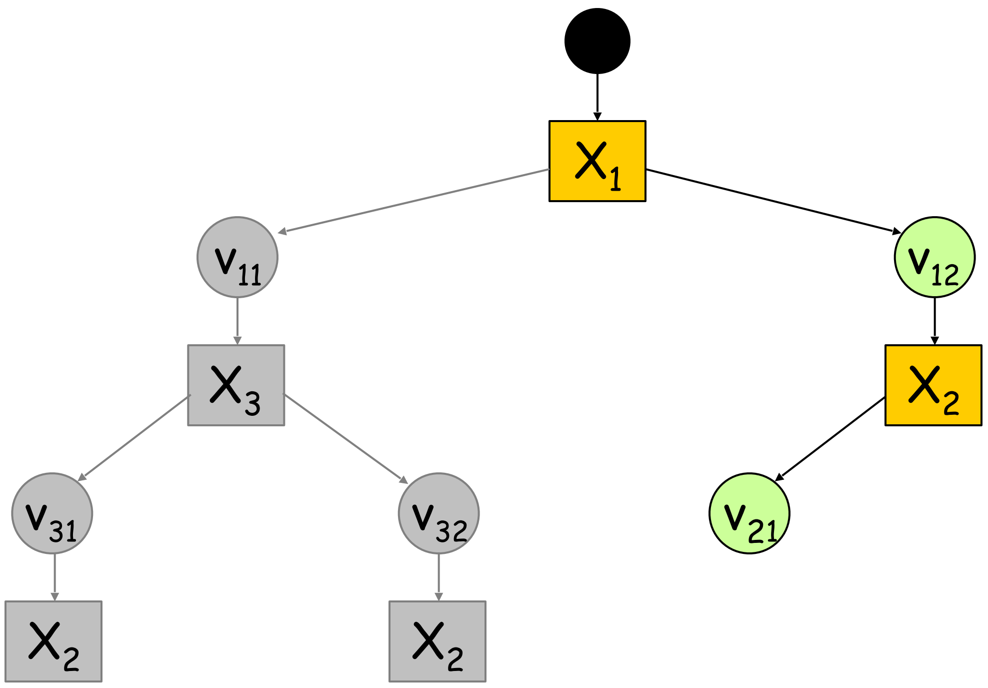
Assignment = $\{(X_1,v_{11}), (X_3,v_{32})\}$

Backtracking Search
(3 variables)



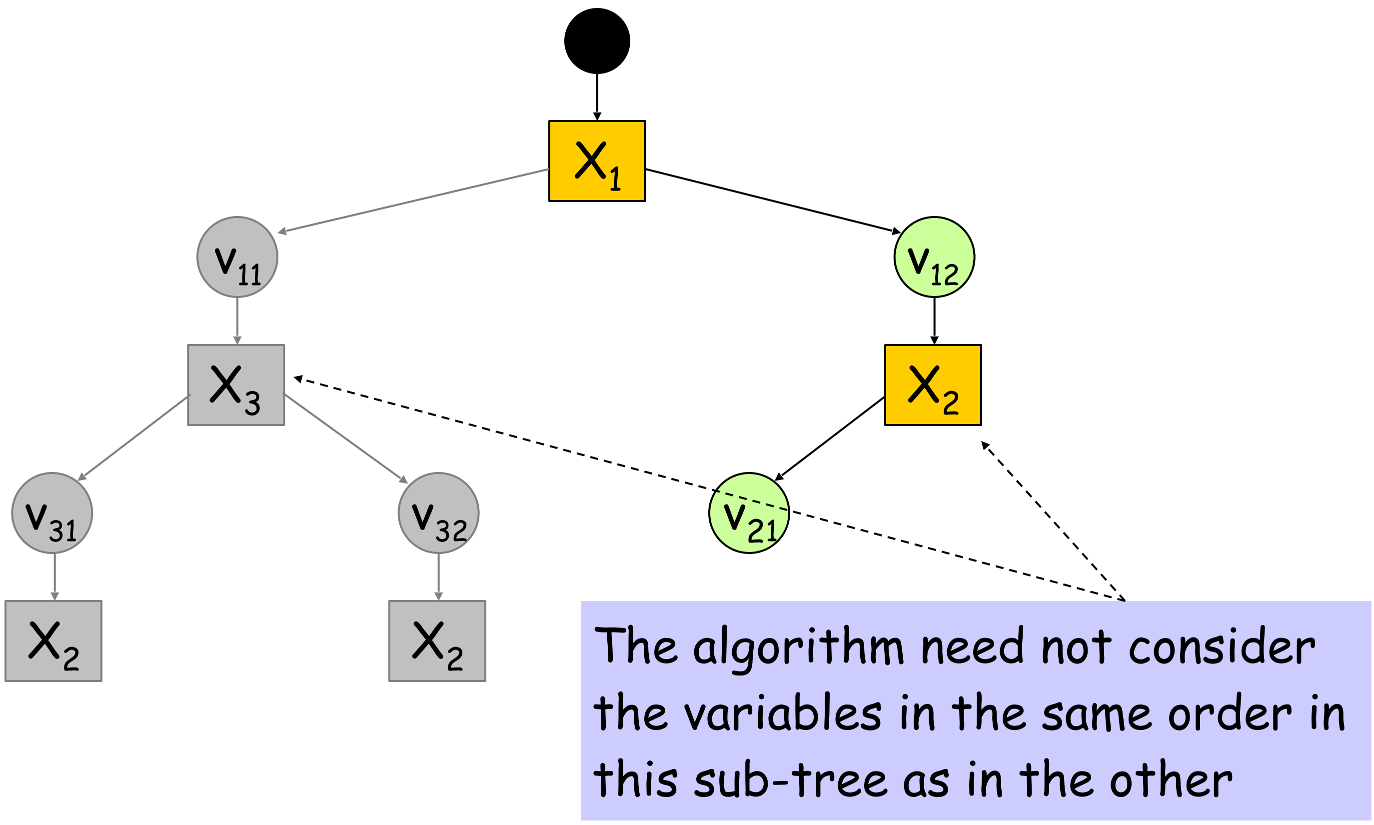
Assignment = $\{(X_1, v_{12})\}$

Backtracking Search
(3 variables)



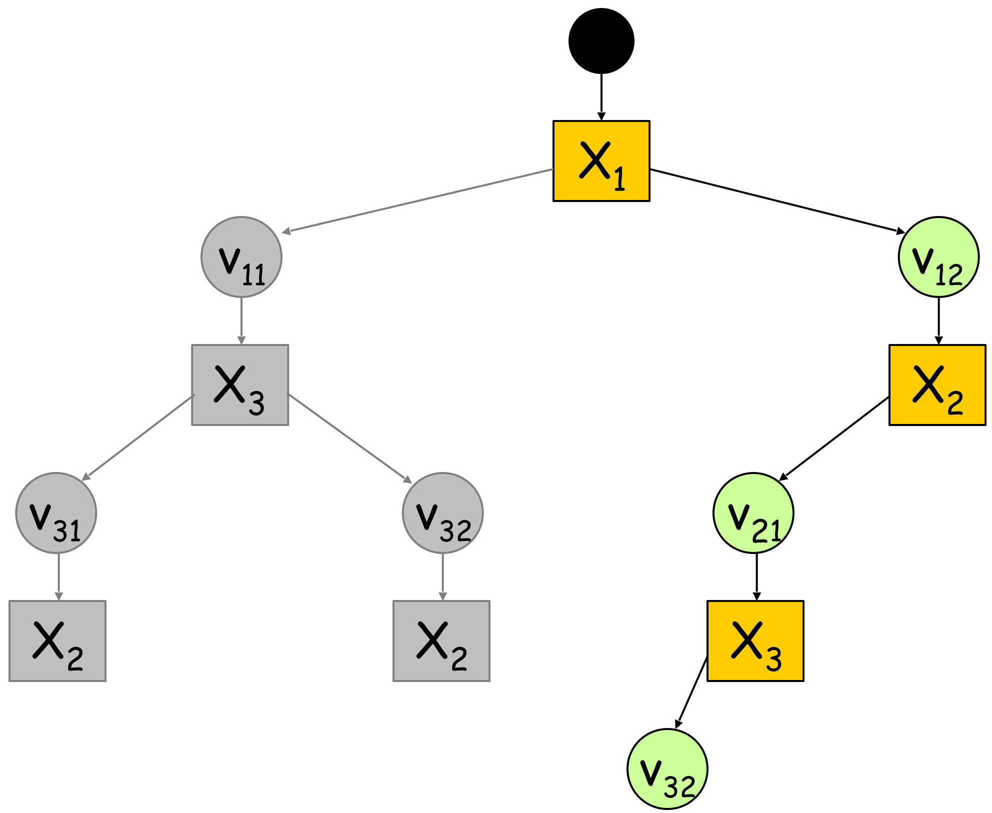
Assignment = $\{(X_1, v_{12}), (X_2, v_{21})\}$

Backtracking Search
(3 variables)



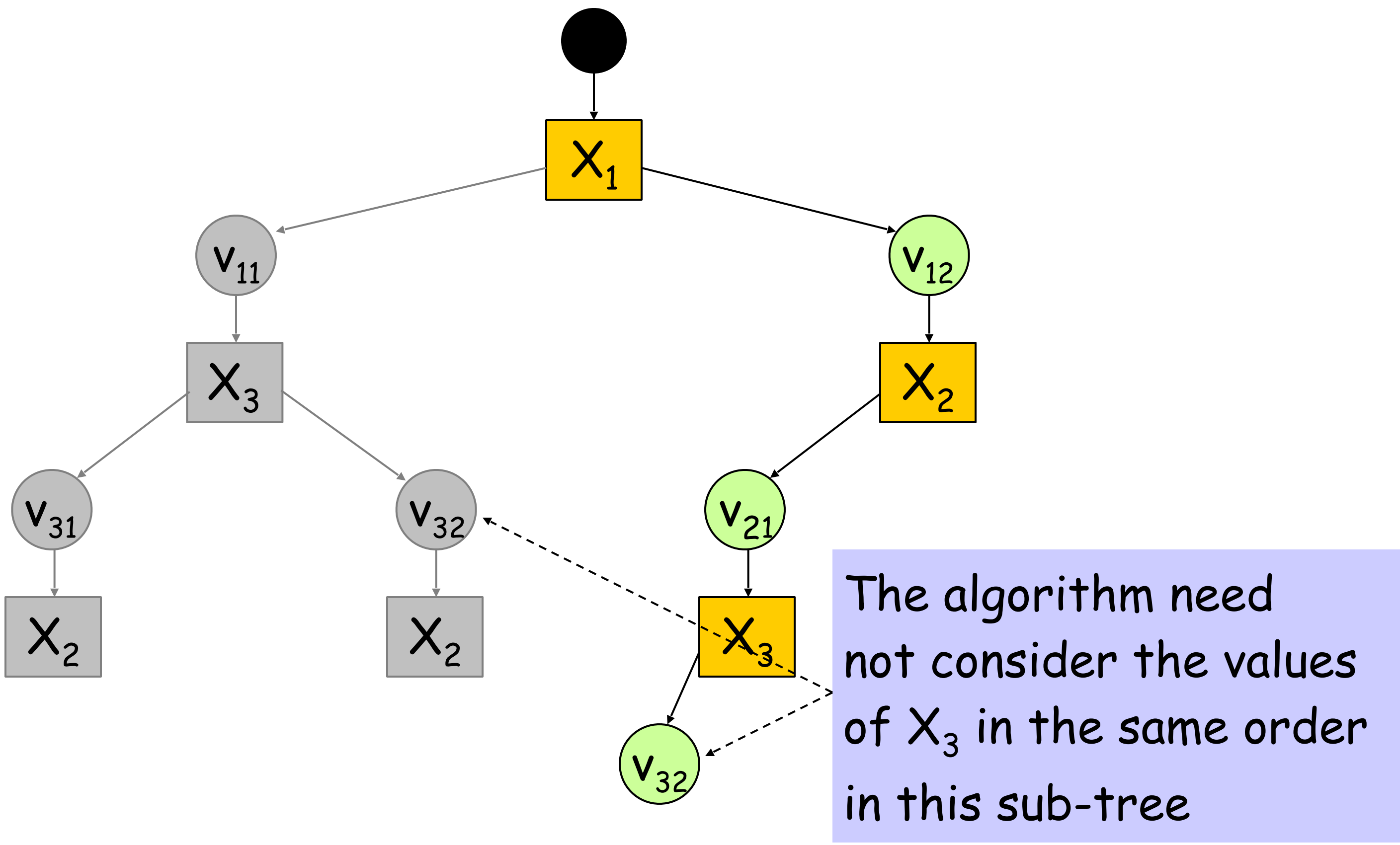
Assignment = $\{(X_1, v_{12}), (X_2, v_{21})\}$

Backtracking Search
(3 variables)



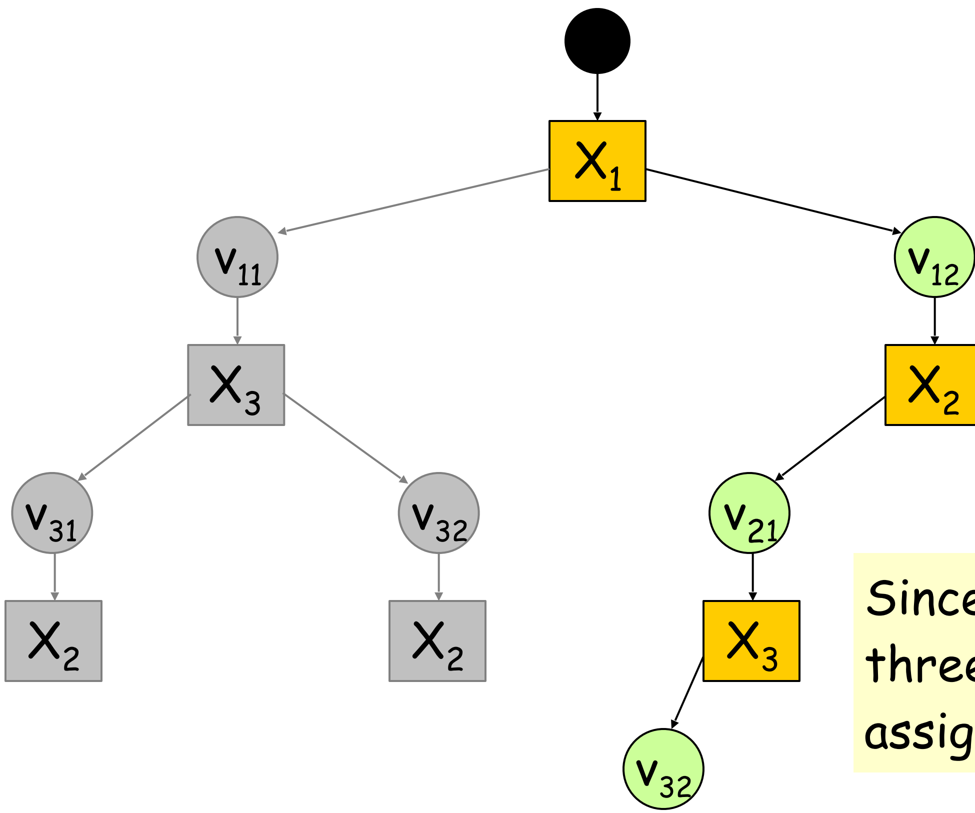
Assignment = $\{(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})\}$

Backtracking Search
(3 variables)



Assignment = $\{(X_1, v_{12}), (X_2, v_{21}), (X_3, v_{32})\}$

Backtracking Search
(3 variables)



Since there are only three variables, the assignment is complete

Assignment = $\{(X_1,v_{12}), (X_2,v_{21}), (X_3,v_{32})\}$

Backtracking Algorithm

CSP-BACKTRACKING(A)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
    1. Add (X←v) to A
    2. If A is valid then
        1. result ← CSP-BACKTRACKING(A)
        - » If result ≠ failure then return result
    3. Remove (X←v) from A

\- Return failure

Call CSP-BACKTRACKING({})

Backtracking Algorithm

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- Return failure

Call CSP-BACKTRACKING({})

[This recursive algorithm keeps too much data in memory.
An iterative version could save memory (left as an exercise)]

Critical Questions for the Efficiency of CSP-Backtracking

CSP-BACKTRACKING(A)

1. If assignment A is complete then return A
2. X ← **select** a variable not in A
3. D ← **select** an ordering on the domain of X
4. For each value v in D do
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Call CSP-BACKTRACKING({})

Critical Questions for the Efficiency of CSP-Backtracking

1) Which variable X should be assigned a value next?

2) In which order should X's values be assigned?

Critical Questions for the Efficiency of CSP-Backtracking

1) Which variable X should be assigned a value next?
   The current assignment may not lead to any solution, but the algorithm does not know it yet. Selecting the right variable X may help discover the contradiction more quickly

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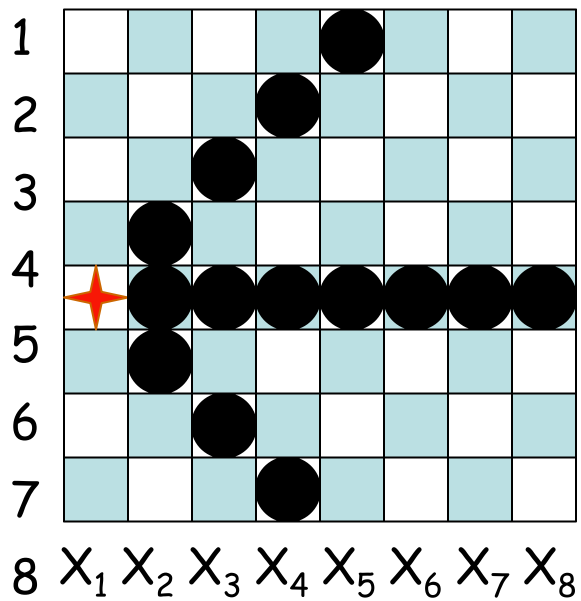
2) In which order should X's values be assigned?
   The current assignment may be part of a solution. Selecting the right value to assign to X may help discover this solution more quickly

More on these questions very soon ...

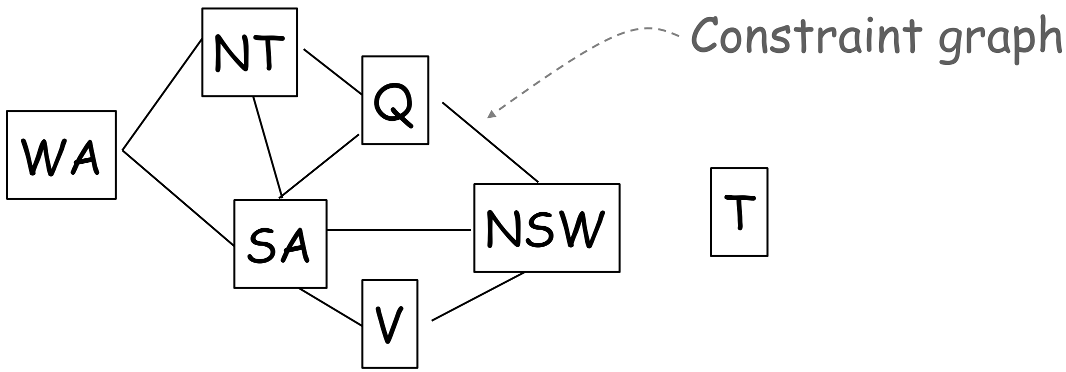
Forward Checking

A simple constraint-propagation technique:



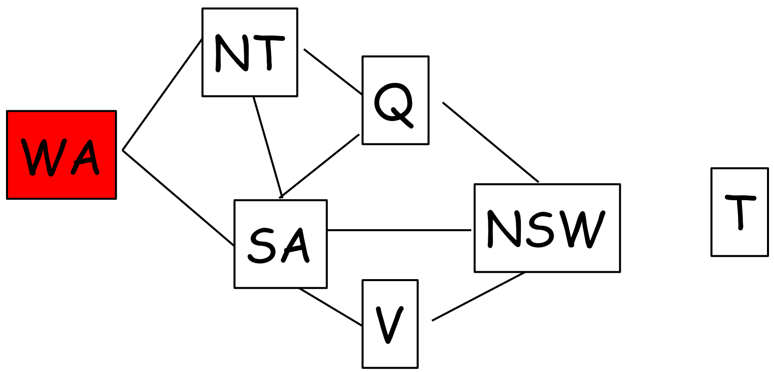
Assigning the value 5 to X1 leads to removing values from the domains of X2, X3, ..., X8

Forward Checking in Map Coloring



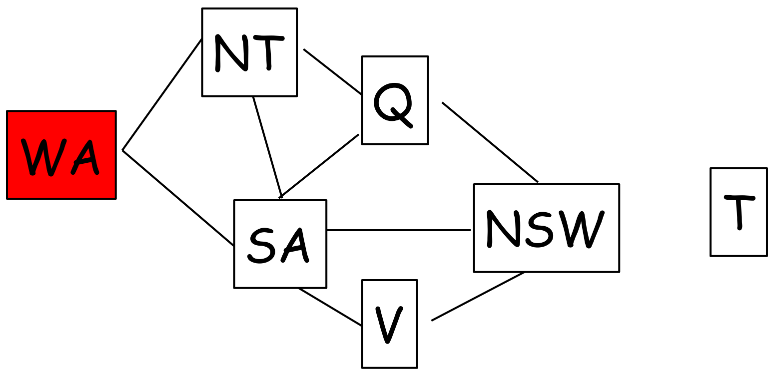
WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	RGB	RGB	RGB	RGB	RGB	RGB

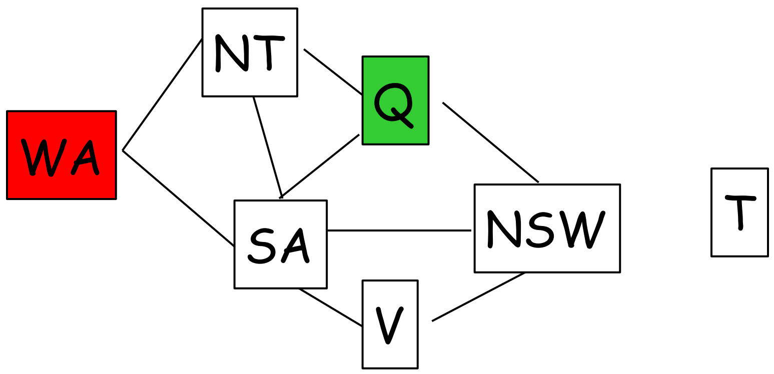
Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB

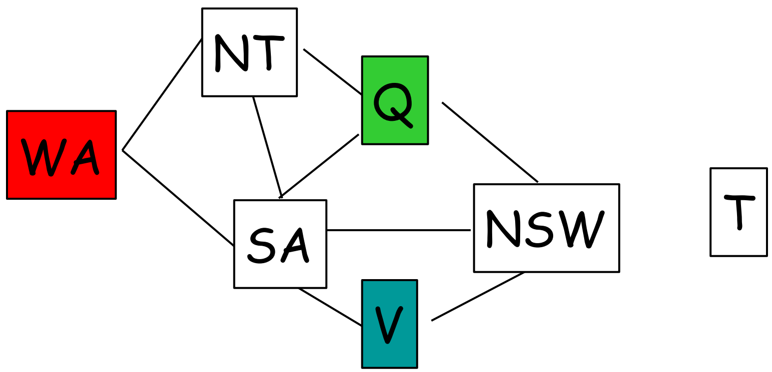
Forward checking removes the value Red of NT and of SA

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB

Forward Checking in Map Coloring



WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking in Map Coloring

Empty set: the current assignment
{(WA ← R), (Q ← G), (V ← B)}
does not lead to a solution

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	RGB	RGB	RGB	GB	RGB
R	B	G	RB	RGB	B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking (General Form)

Whenever a pair (X←v) is added to assignment *A* do:

- For each variable Y not in *A* do:
  - For every constraint *C* relating Y to the variables in *A* do:
    - Remove all values from Y's domain that do not satisfy *C*

Modified Backtracking Algorithm

CSP-BACKTRACKING(*A*, var-domains)

1. If assignment *A* is complete then return *A*
2. X ← select a variable not in *A*
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to *A*
   b. var-domains ← forward checking(var-domains, X, v, *A*)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(*A*, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from *A*
5. Return failure

Modified Backtracking Algorithm

CSP-BACKTRACKING(*A*, var-domains)

1. If assignment *A* is complete then return *A*
2. X ← select a variable not in *A*
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to *A* → No need any more to verify that *A* is valid
   b. var-domains ← forward checking(var-domains, X, v, *A*)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(*A*, var-domains)
      (ii) If result ≠ failure then return result
   d. Remove (X←v) from *A*
5. Return failure

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Need to pass down the updated variable domains

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

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   1. Remove (X←v) from A
5. Return failure

1) Which variable X_i should be assigned a value next?

→ Most-constrained-variable heuristic

→ Most-constraining-variable heuristic

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   → Most-constrained-variable heuristic
   → Most-constraining-variable heuristic

2) In which order should its values be assigned?
   → Least-constraining-value heuristic

These heuristics can be quite confusing

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These heuristics can be quite confusing

Keep in mind that **all** variables must eventually get a value, while only **one** value from a domain must be assigned to each variable

Most-Constrained-Variable Heuristic

Most-Constrained-Variable Heuristic

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Most-Constrained-Variable Heuristic

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Most-Constrained-Variable Heuristic

1) Which variable X_i should be assigned a value next?

   Select the variable with the smallest remaining domain

Most-Constrained-Variable Heuristic

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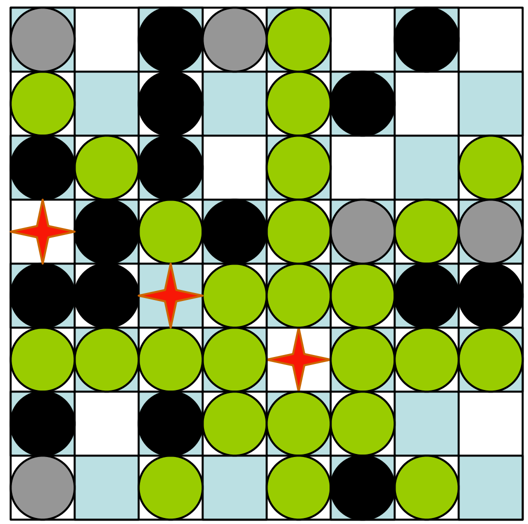
Most-Constrained-Variable Heuristic

1) Which variable X_i should be assigned a value next?

Select the variable with the smallest remaining domain

[Rationale: Minimize the branching factor]

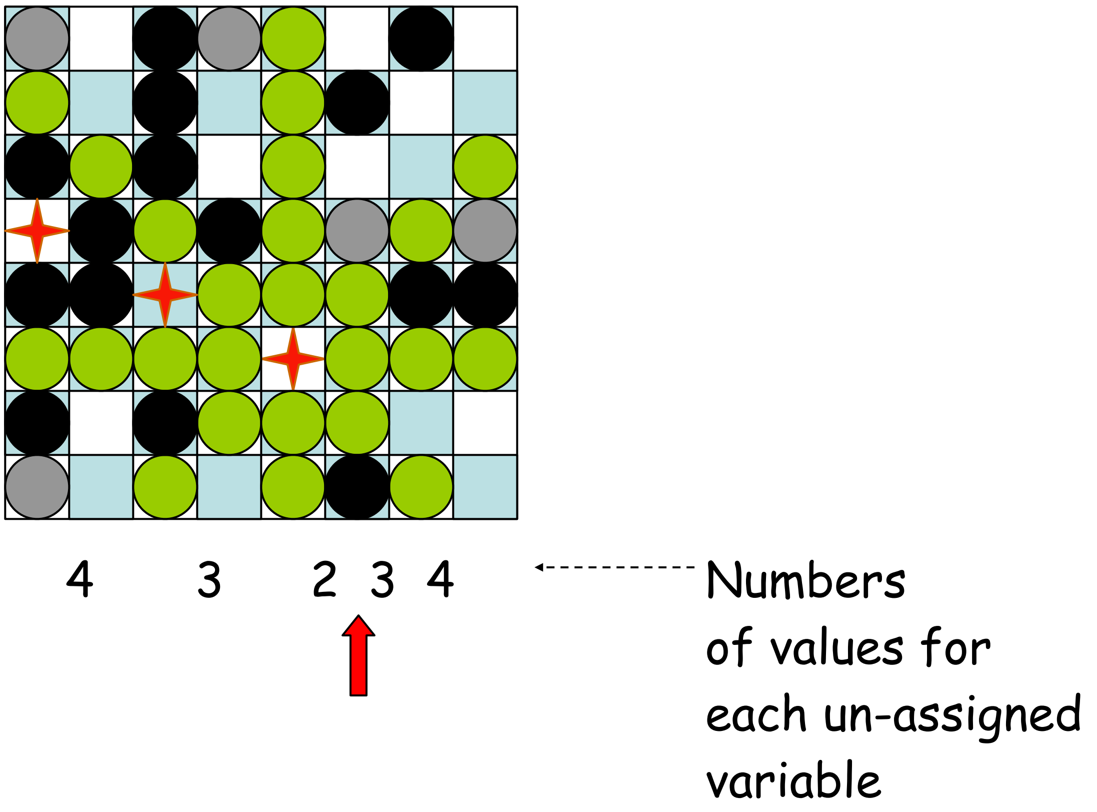
8-Queens



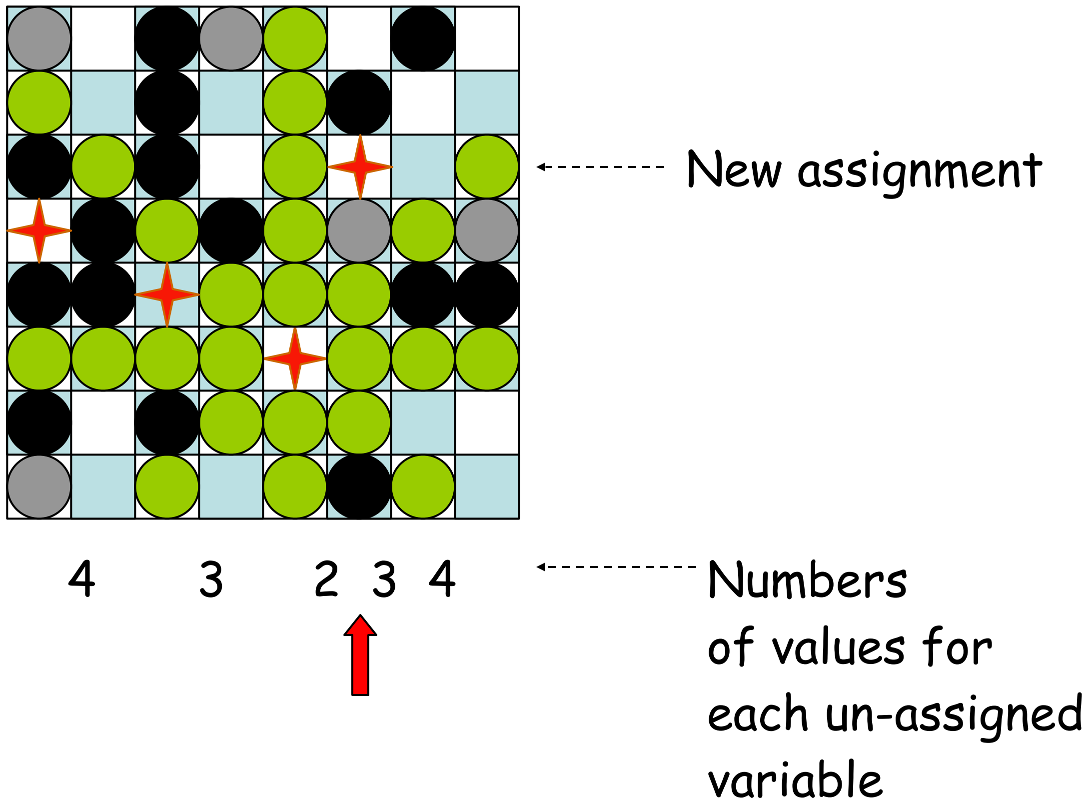
4 3 2 3 4

Numbers of values for each un-assigned variable

8-Queens

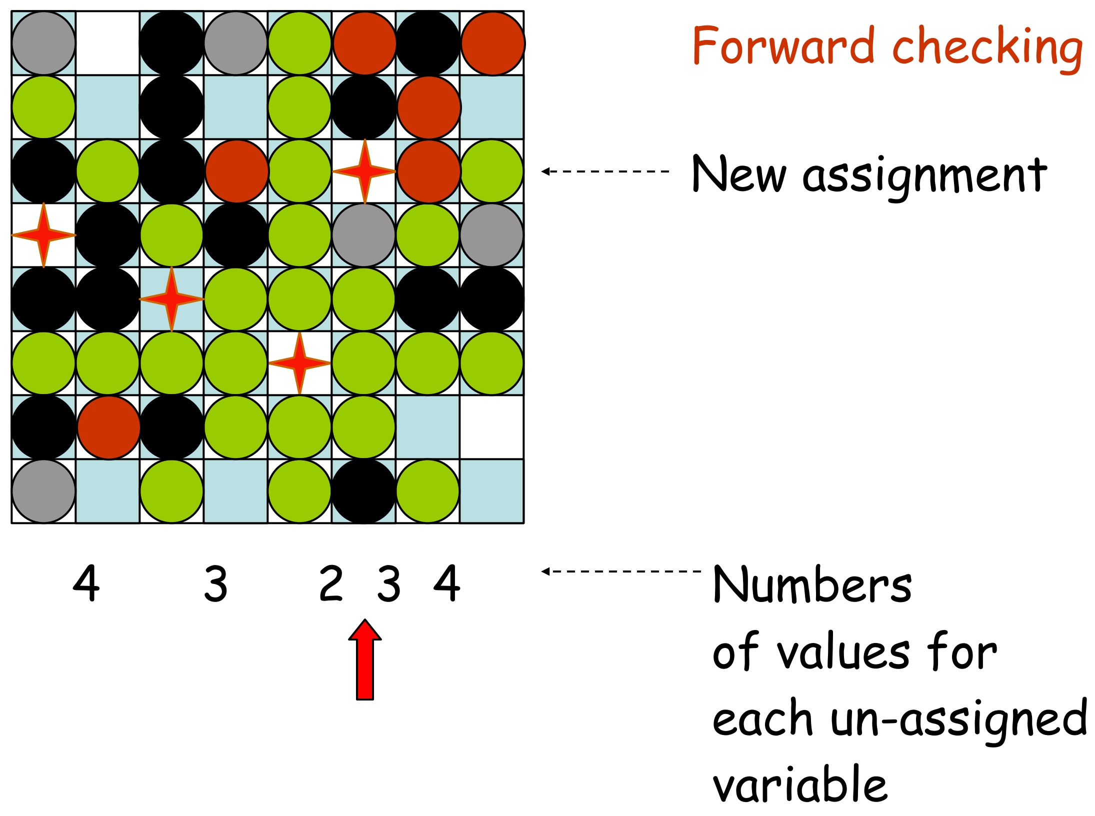


8-Queens



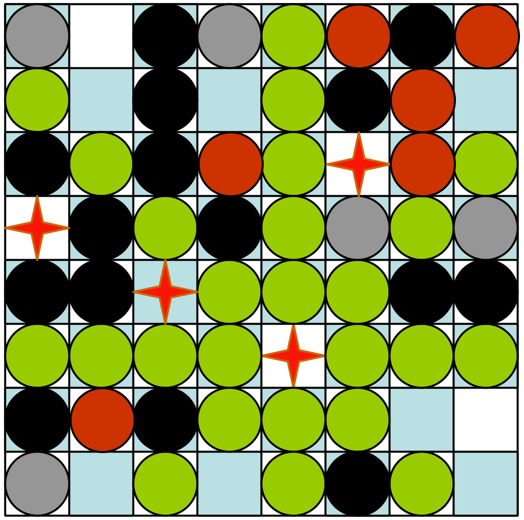
New assignment
4 3 2 3 4
Numbers
of values for
each un-assigned
variable

8-Queens



Forward checking
New assignment
4 3 2 3 4
Numbers of values for each un-assigned variable

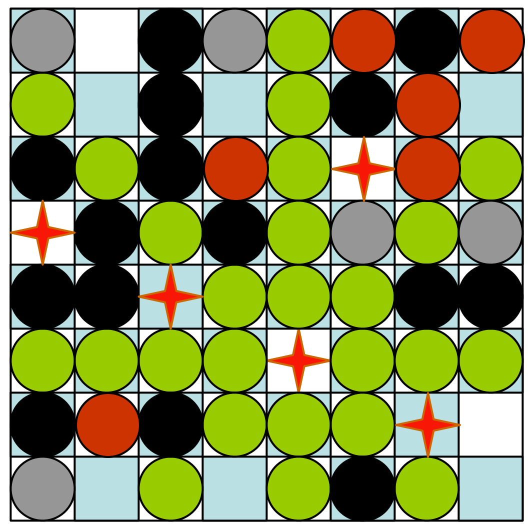
8-Queens



3 2 1 3

New numbers of values for each un-assigned variable

8-Queens

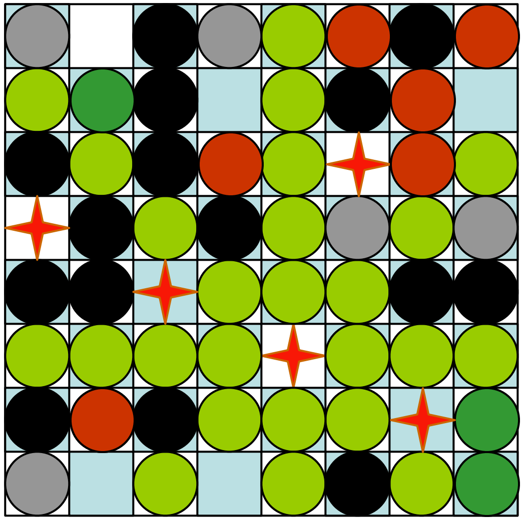


New assignment

3 2 1 3

New numbers of values for each un-assigned variable

8-Queens



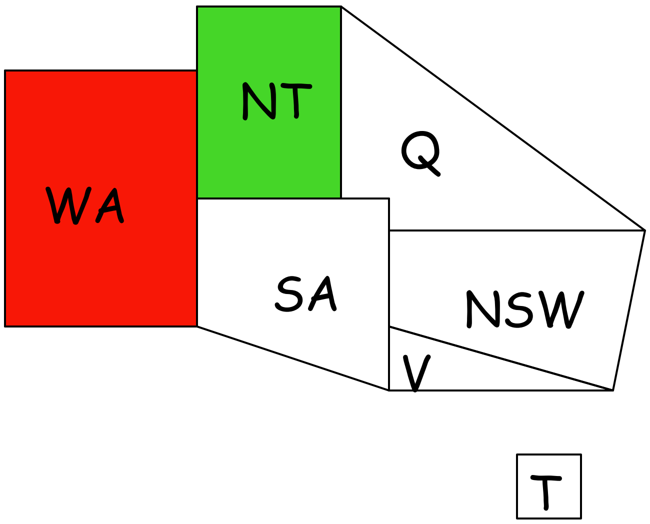
Forward checking

New assignment

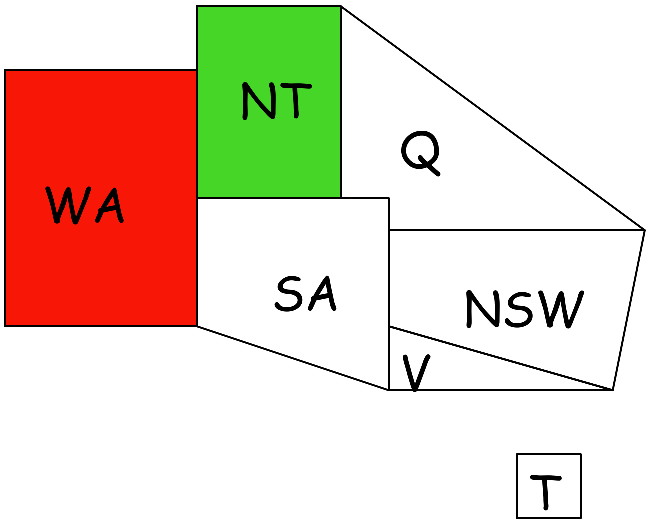
3 2 1 3

New numbers of values for each un-assigned variable

Map Coloring

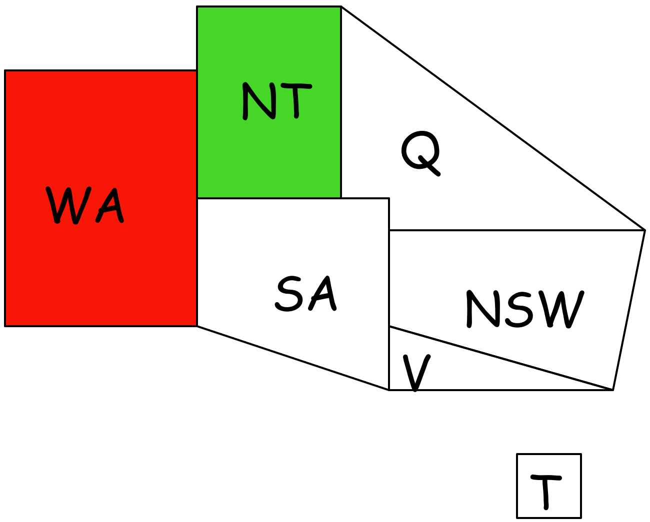


Map Coloring



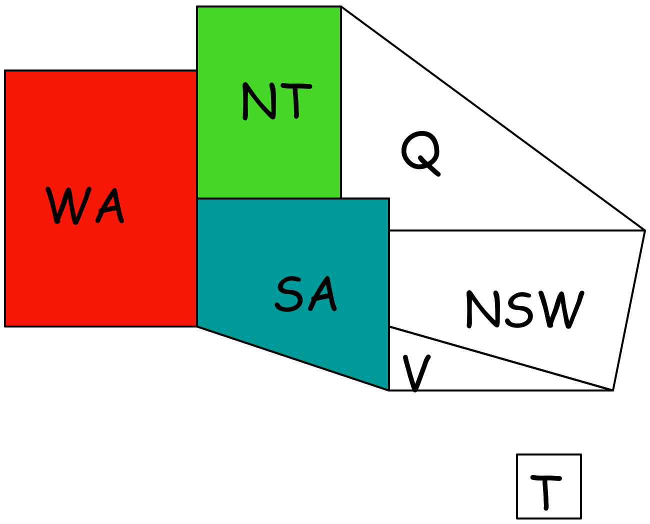
- SA's remaining domain has size 1 (value Blue remaining)

Map Coloring



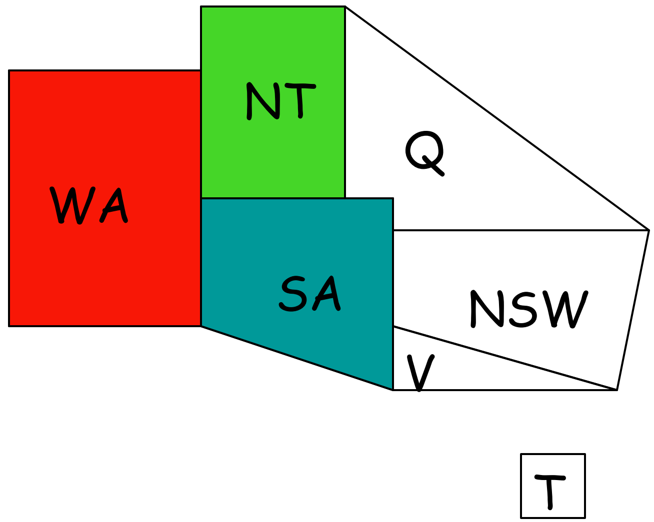
- SA's remaining domain has size 1 (value Blue remaining)
- Q's remaining domain has size 2

Map Coloring



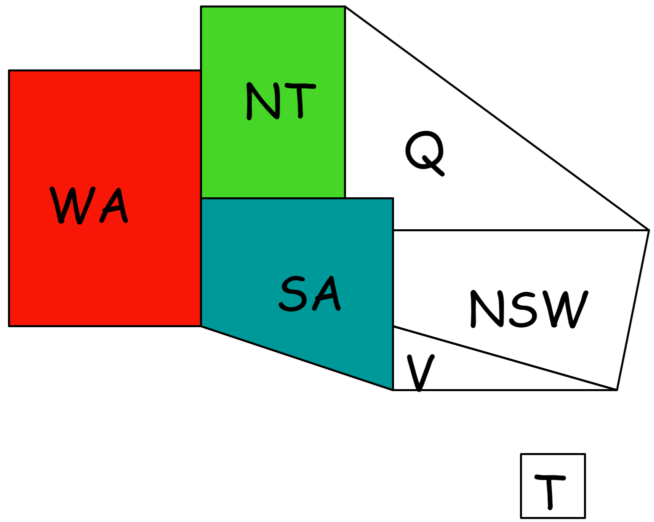
- SA's remaining domain has size 1 (value Blue remaining)
- Q's remaining domain has size 2

Map Coloring



- SA's remaining domain has size 1 (value Blue remaining)
- Q's remaining domain has size 2
- NSW's, V's, and T's remaining domains have size 3

Map Coloring



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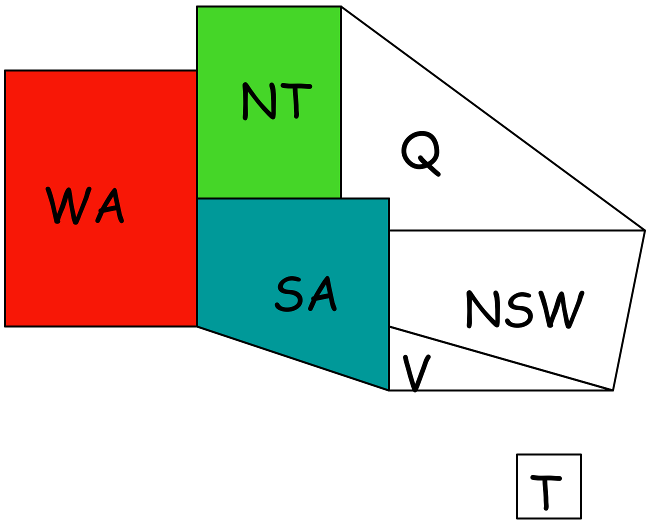
- NSW's, V's, and T's remaining domains have size 3

Map Coloring



- SA's remaining domain has size 1 (value Blue remaining)
- Q's remaining domain has size 2
- NSW's, V's, and T's remaining domains have size 3

→ Select SA

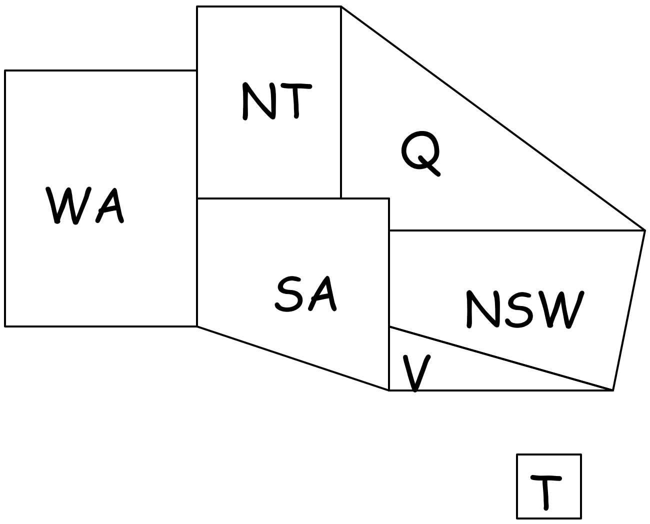
Most-Constraining-Variable Heuristic

1) Which variable X_i should be assigned a value next?

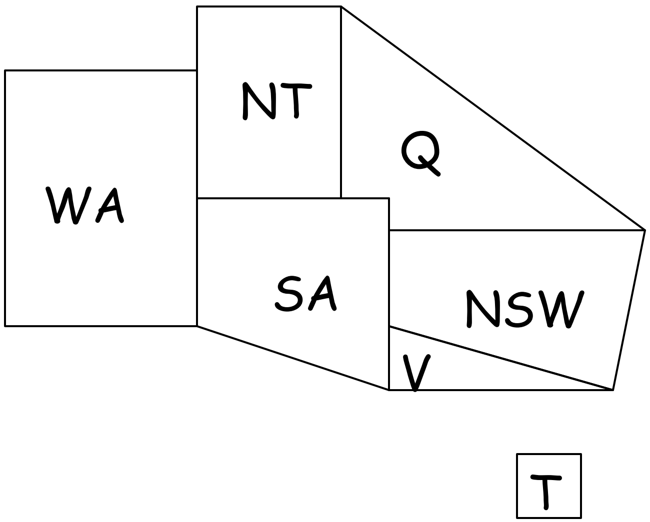
Among the variables with the smallest remaining domains (ties with respect to the most-constrained-variable heuristic), select the one that appears in the largest number of constraints on variables not in the current assignment

[Rationale: Increase future elimination of values, to reduce future branching factors]

Map Coloring

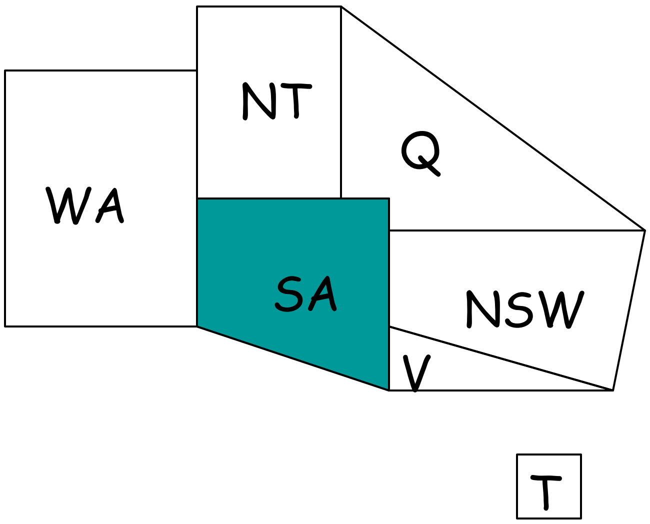


Map Coloring



- Before any value has been assigned, all variables have a domain of size 3, but SA is involved in more constraints (5) than any other variable

Map Coloring



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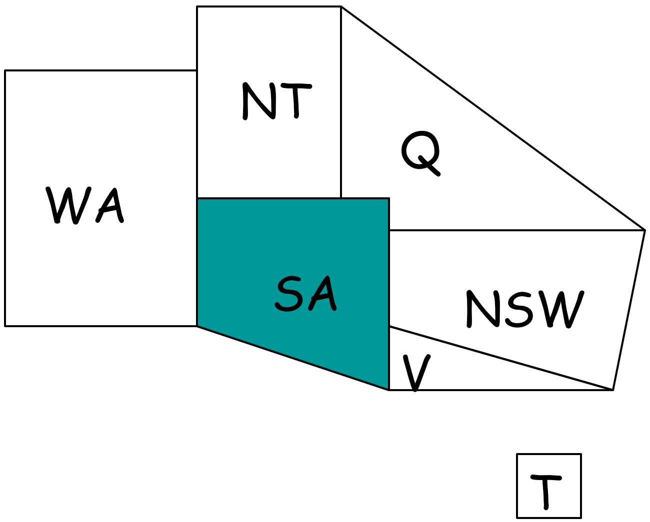

- Before any value has been assigned, all variables have a domain of size 3, but SA is involved in more constraints (5) than any other variable

→ Select SA and assign a value to it (e.g., Blue)

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
3. D ← select an ordering on the domain of X
4. For each value v in D do
   a. Add (X←v) to A
   b. var-domains ← forward checking(var-domains, X, v, A)
   c. If no variable has an empty domain then
      (i) result ← CSP-BACKTRACKING(A, var-domains)
      (ii) If result ≠ failure then return result
   1. Remove (X←v) from A
5. Return failure

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5. Return failure

1) Most-constrained-variable heuristic
2) Most-constraining-variable heuristic

1) Select the variable with the smallest remaining domain
2) Select the variable that appears in the largest number of constraints on variables not in the current assignment

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5. Return failure

1) Most-constrained-variable heuristic
2) Most-constraining-variable heuristic

3) Least-constraining-value heuristic

1) Select the variable with the smallest remaining domain
2) Select the variable that appears in the largest number of constraints on variables not in the current assignment

Least-Constraining-Value Heuristic

Least-Constraining-Value Heuristic

2) In which order should X's values be assigned?

Least-Constraining-Value Heuristic

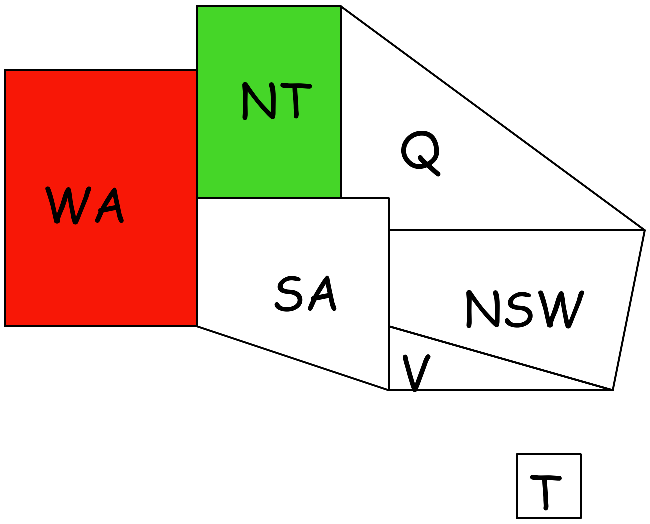
2) In which order should X's values be assigned?

Select the value of X that removes the smallest number of values from the domains of those variables which are not in the current assignment

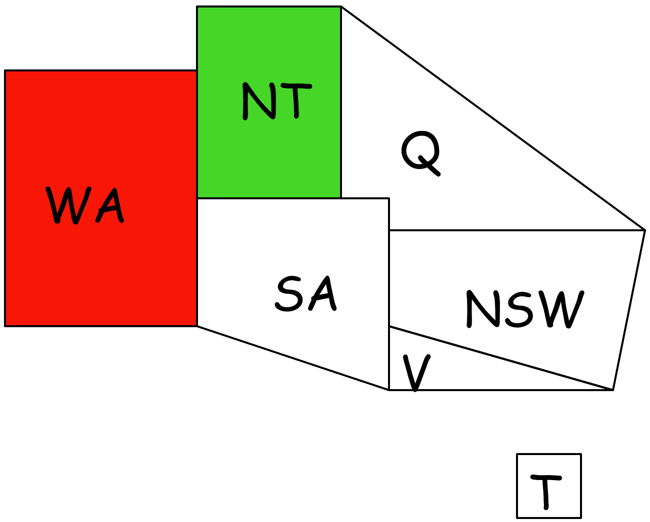
[Rationale: Since only one value will eventually be assigned to X, pick the least-constraining value first, since it is the most likely not to lead to an invalid assignment]

[Note: Using this heuristic requires performing a forward-checking step for every value, not just for the selected value]

Map Coloring

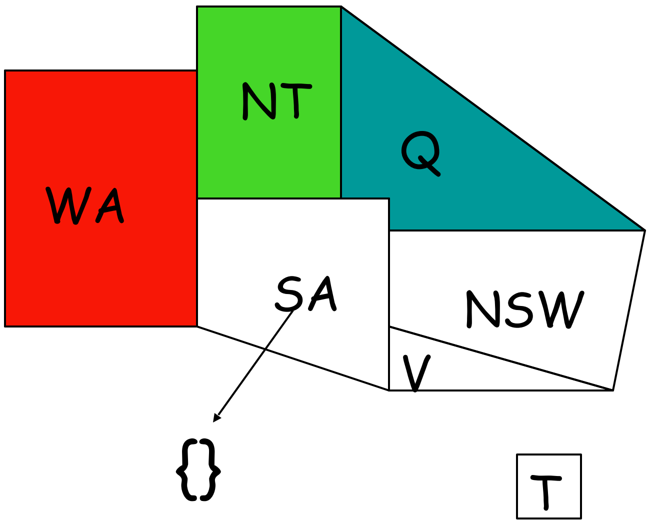


Map Coloring



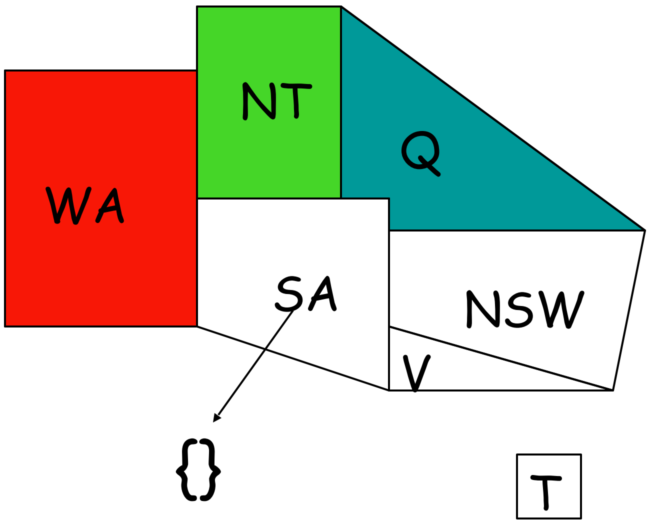
- Q's domain has two remaining values: Blue and Red

Map Coloring



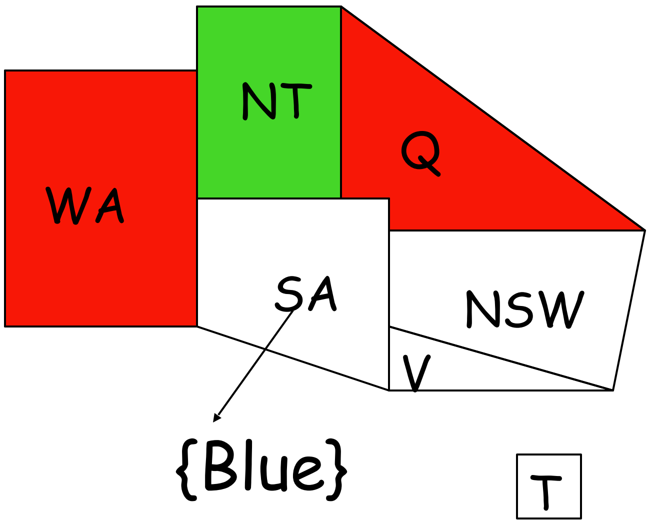
- Q's domain has two remaining values: Blue and Red

Map Coloring



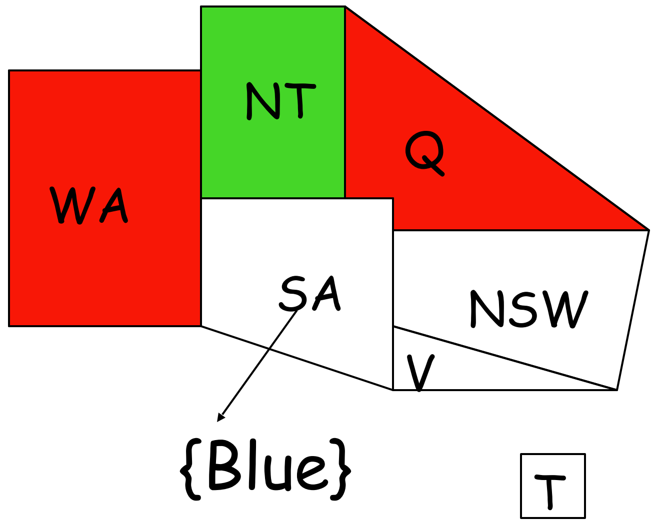
- Q's domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 value for SA, while assigning Red would leave 1 value

Map Coloring



- Q's domain has two remaining values: Blue and Red

Map Coloring



- Q's domain has two remaining values: Blue and Red
- Assigning Blue to Q would leave 0 value for SA, while assigning Red would leave 1 value

→ So, assign Red to Q

Modified Backtracking Algorithm

CSP-BACKTRACKING(A, var-domains)

1. If assignment A is complete then return A
2. X ← select a variable not in A
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4. For each value v in D do
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   c. If no variable has an empty domain then
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      (ii) If result ≠ failure then return result
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5. Return failure

1) Most-constrained-variable heuristic
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Constraint Propagation

(Where a better exploitation of the constraints further reduces the need to make decisions)

Constraint Propagation …

… is the process of determining how the constraints and the possible values of one variable affect the possible values of other variables

It is an important form of "least-commitment" reasoning

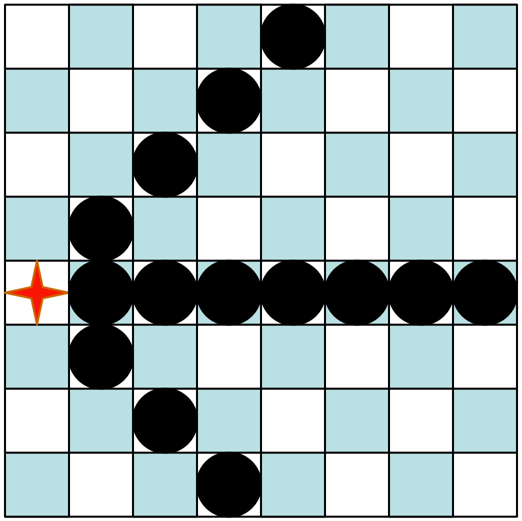
Forward checking is only on simple form of constraint propagation

When a pair (X→v) is added to assignment A do:

  For each variable Y not in A do:

    For every constraint C relating Y to variables in A do:

      Remove all values from Y's domain that do not satisfy C



- n = number of variables
- d = size of initial domains
- s = maximum number of constraints involving a given variable ($s \leq n-1$)
- Forward checking takes O(nsd) time

Forward Checking in Map Coloring

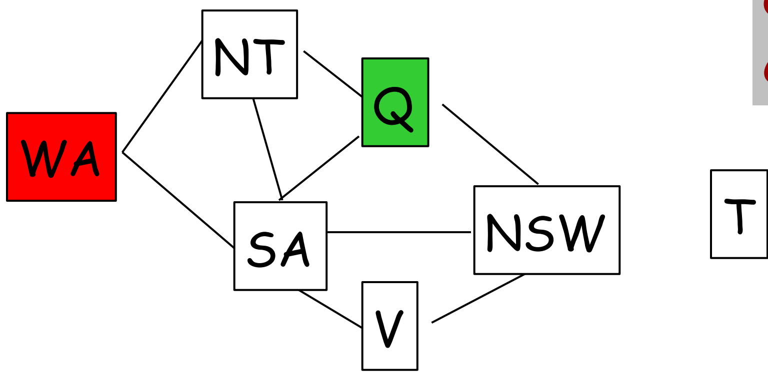
Empty set: the current assignment

$\{(WA \leftarrow R), (Q \leftarrow G), (V \leftarrow B)\}$

does not lead to a solution

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

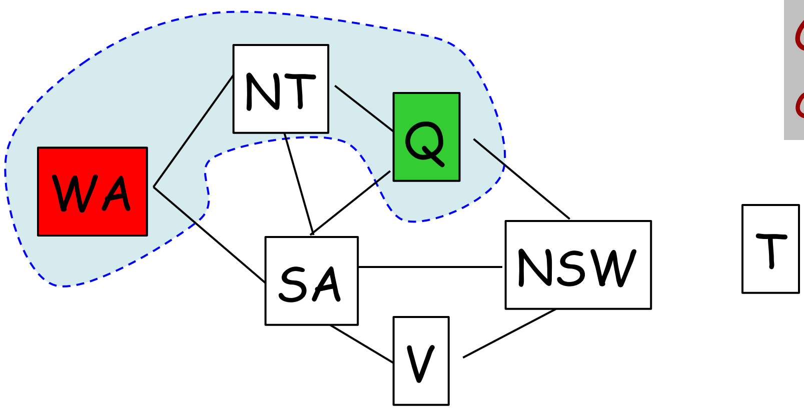
Forward Checking in Map Coloring



Contradiction that forward checking did not detect

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

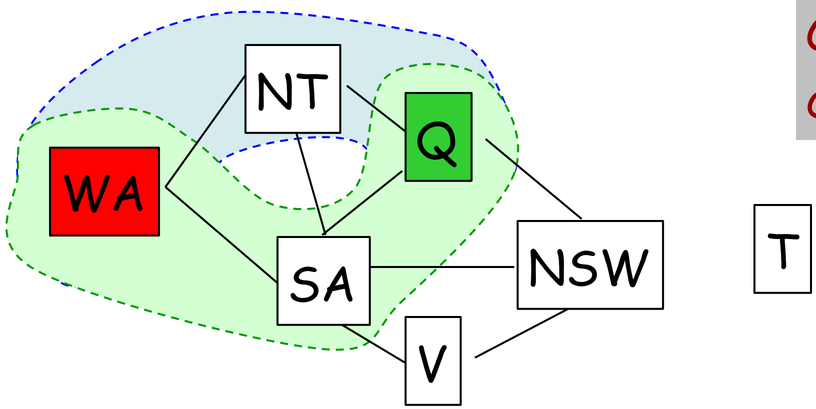
Forward Checking in Map Coloring



Contradiction that forward checking did not detect

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

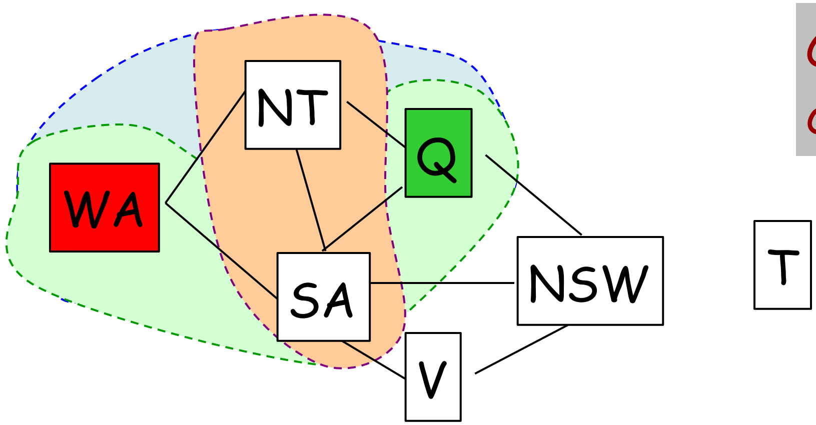
Forward Checking in Map Coloring



Contradiction that forward checking did not detect

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

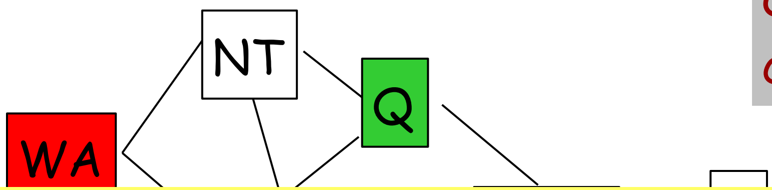
Forward Checking in Map Coloring



Contradiction that forward checking did not detect

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	~~R~~GB	RGB	RGB	RGB	~~R~~GB	RGB
R	~~G~~B	G	R~~G~~B	RGB	~~G~~B	RGB
R	B	G	R~~B~~	B	~~B~~	RGB

Forward Checking in Map Coloring



Contradiction that forward checking did not detect

Detecting this contradiction requires a more powerful constraint propagation technique

WA	NT	Q	NSW	V	SA	T
RGB	RGB	RGB	RGB	RGB	RGB	RGB
R	RGB	RGB	RGB	RGB	RGB	RGB
R	GB	G	RGB	RGB	GB	RGB
R	B	G	RB	B	B	RGB

Constraint Propagation for Binary Constraints

REMOVE-VALUES(X,Y) removes every value of Y that is incompatible with the values of X

REMOVE-VALUES(X,Y)

1. removed ← false
2. For every value v in the domain of Y do
   - If there is no value u in the domain of X such that the constraint on (X,Y) is satisfied then
     1. Remove v from Y's domain
     2. removed ← true
3. Return removed

Constraint Propagation for Binary Constraints

AC3

1. Initialize queue Q with all variables (not yet instantiated)
2. While Q ≠ ∅ do
   a. X ← Remove(Q)
   - For every (not yet instantiated) variable Y related to X by a (binary) constraint do
     1. If REMOVE-VALUES(X,Y) then
        a. If Y's domain = ∅ then exit
        1. Insert(Y,Q)

Complexity Analysis of AC3

- n = number of variables
- d = size of initial domains
- s = maximum number of constraints involving a given variable ($s \leq n-1$)
- Each variable is inserted in Q up to d times
- REMOVE-VALUES takes $O(d^2)$ time
- AC3 takes $O(n \times d \times s \times d^2) = O(n \times s \times d^3)$ time
- Usually more expensive than forward checking

AC3

1. Initialize queue Q with all variables (not yet instantiated)
2. While Q ≠ ∅ do
   a. X ← Remove(Q)
   - For every (not yet instantiated) variable Y related to X by a (binary) constraint do
     1. If REMOVE-VALUES(X,Y) then
        a. If Y's domain = ∅ then exit
        1. Insert(Y,Q)

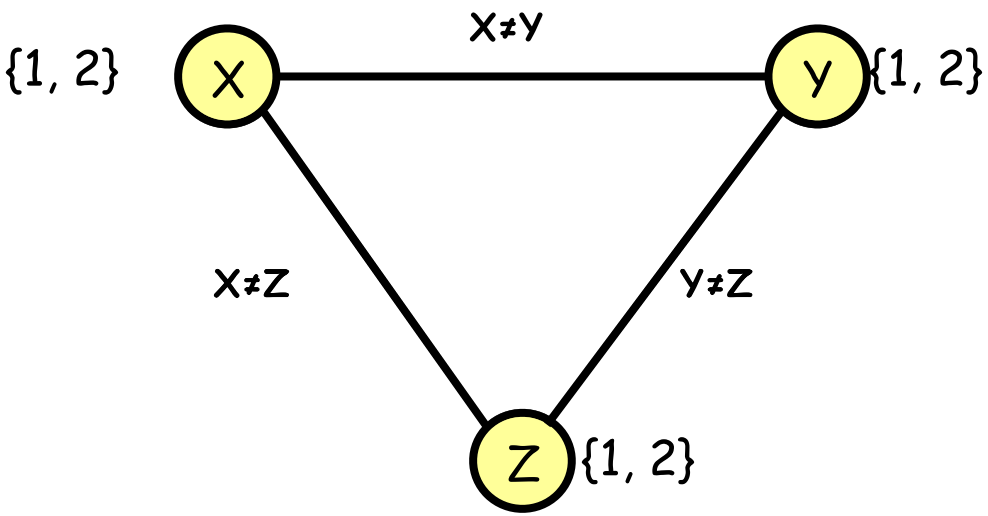
REMOVE-VALUES(X,Y)

1. removed ← false
2. For every value v in the domain of Y do
   - If there is no value u in the domain of X such that the constraint on (x,y) is satisfied then
     a. Remove v from Y's domain
     b. removed ← true
3. Return removed

Is AC3 all that we need?

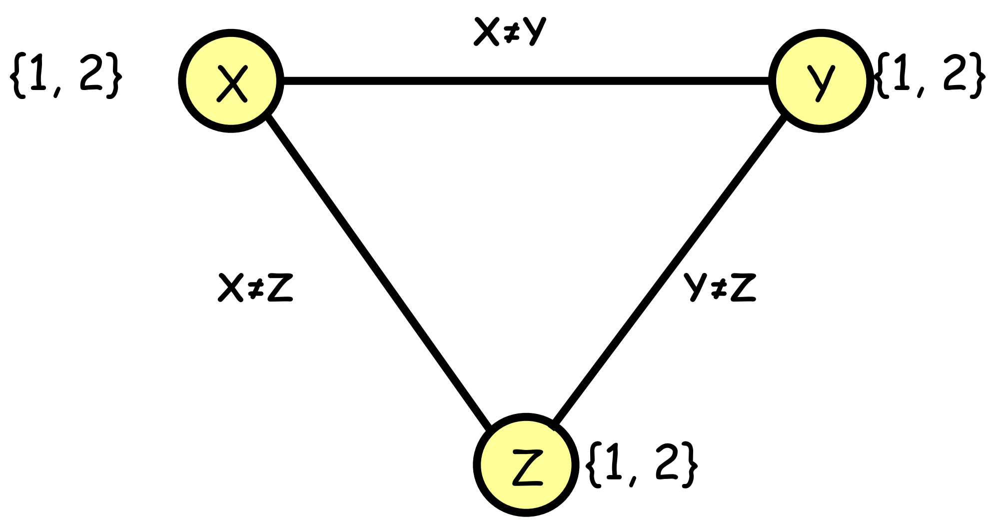
Is AC3 all that we need?

- No !!



Is AC3 all that we need?

- No !!
- AC3 can't detect all contradictions among binary constraints



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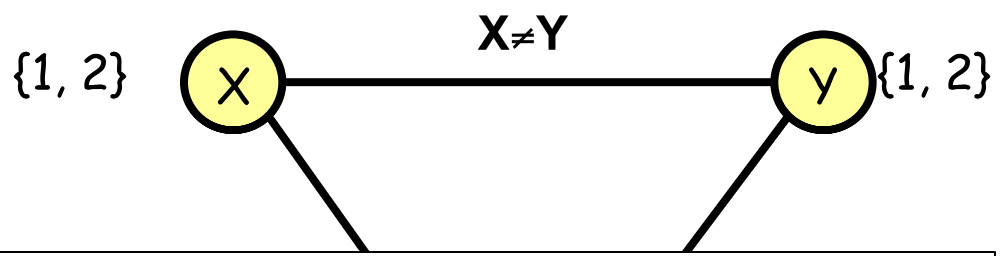

REMOVE-VALUES(X,Y)

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     2. removed ← true
- Return removed

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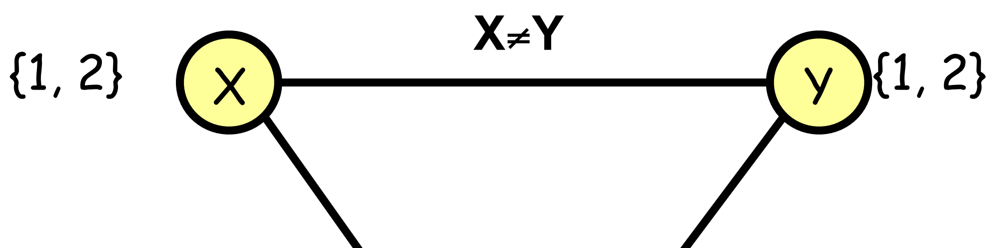

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1. removed ← false
2. For every value v in the domain of Y do
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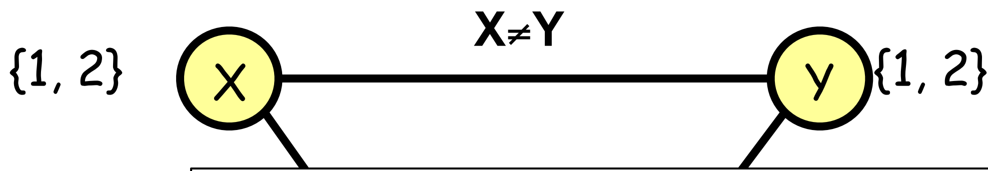


REMOVE-VALUES(X,Y

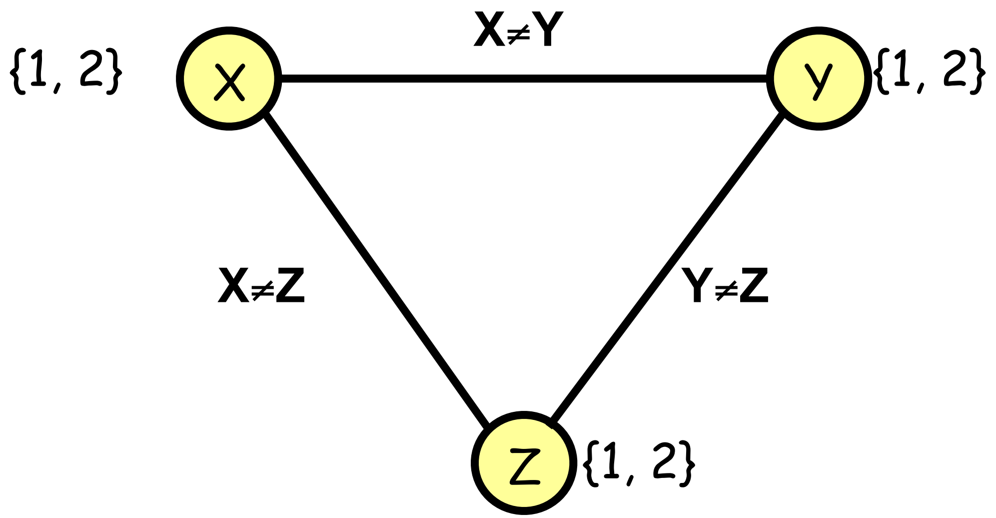
1. removed ← false
2. For every value v
   - If there is no
     that the const
     1. Remove v
     2. removed
- Return removed

REMOVE-VALUES(X,Y,Z)

1. removed ← false
2. For every value w in the domain of Z do
   - If there is no pair (u,v) of values in the domains of X and Y verifying the constraint on (X,Y) such that the constraints on (X,Z) and (Y,Z) are satisfied then
     - Remove w from Z's domain
     1. removed ← true

Is AC3 all that we need?

- No !!
- AC3 can't detect all contradictions among binary constraints



- Not all constraints are binary

Tradeoff

Generalizing the constraint propagation algorithm increases its time complexity

> Tradeoff between time spent in backtracking search and time spent in constraint propagation

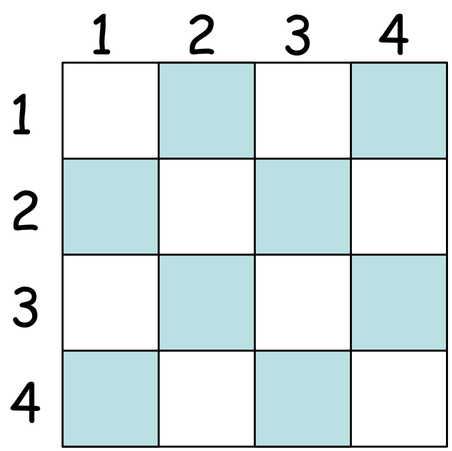
A good tradeoff when all or most constraints are binary is often to combine backtracking with forward checking and/or AC3 (with REMOVE-VALUES for two variables)

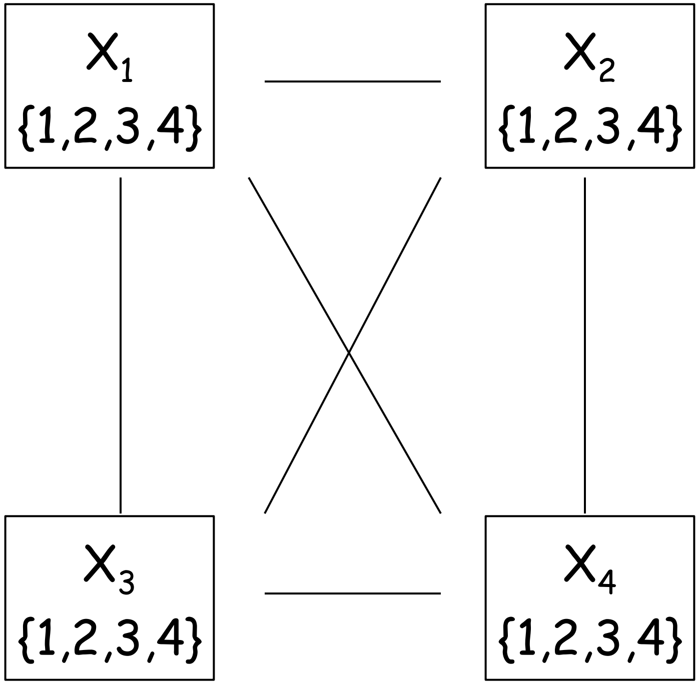
Modified Backtracking Algorithm with AC3

CSP-BACKTRACKING(A, var-domains)

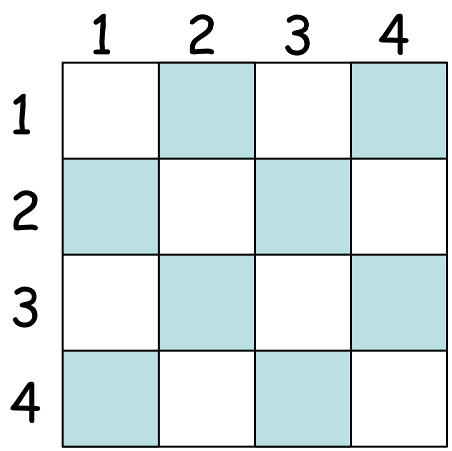
1. If assignment A is complete then return A
2. Run AC3 and update var-domains accordingly
3. If a variable has an empty domain then return failure
4. X ← select a variable not in A
5. D ← select an ordering on the domain of X
6. For each value v in D do
    a. Add (X←v) to A
    b. var-domains ← forward checking(var-domains, X, v, A)
    c. If no variable has an empty domain then
       (i) result ← CSP-BACKTRACKING(A, var-domains)
       (ii) If result ≠ failure then return result
    - Remove (X←v) from A
7. Return failure

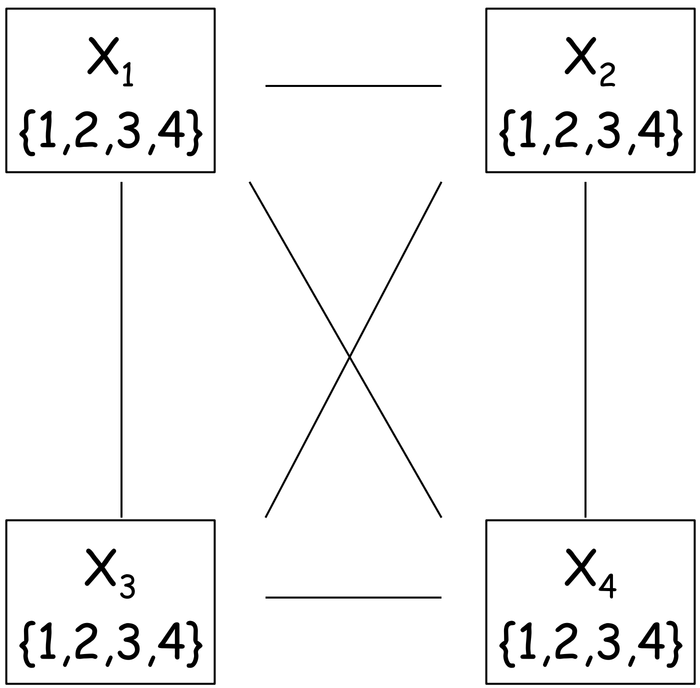
A Complete Example:4-Queens Problem





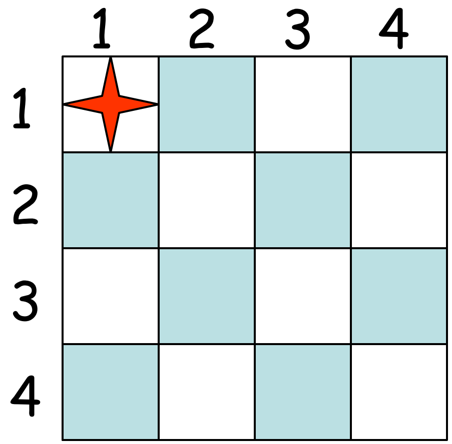
A Complete Example:4-Queens Problem

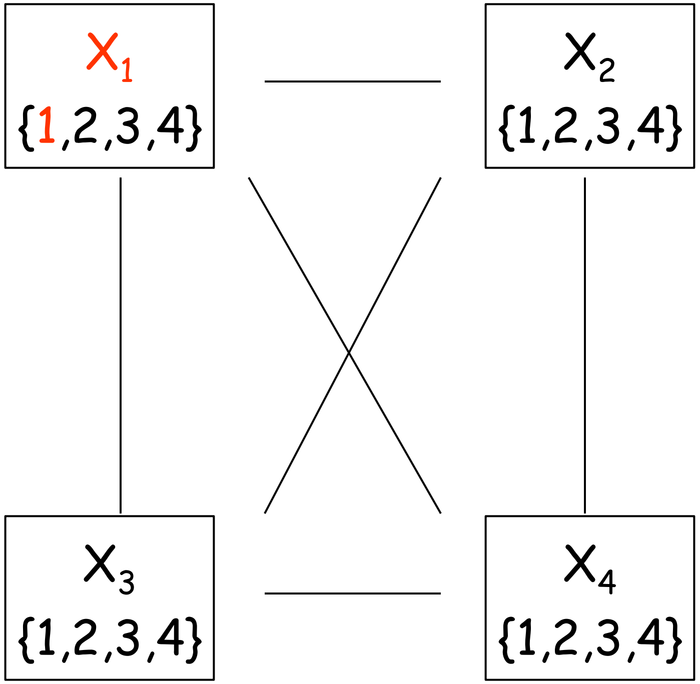




1) The modified backtracking algorithm starts by calling *AC3*, which removes no value

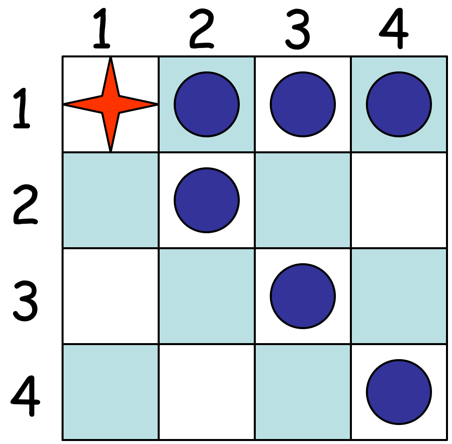
4-Queens Problem

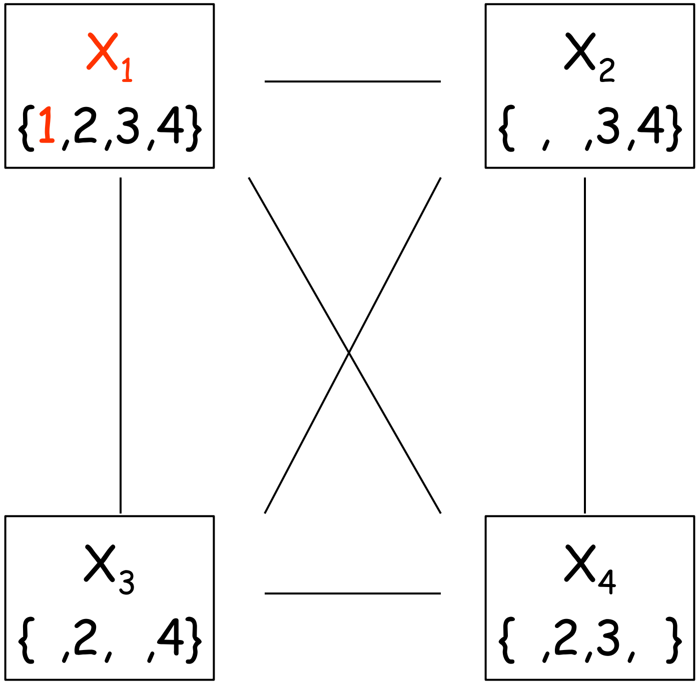




2) The backtracking algorithm then selects a variable and a value for this variable. No heuristic helps in this selection. X_1 and the value 1 are arbitrarily selected

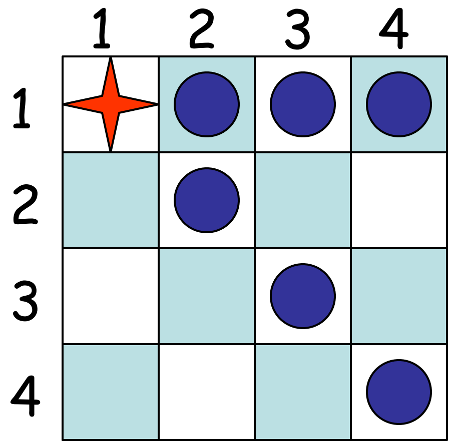
4-Queens Problem

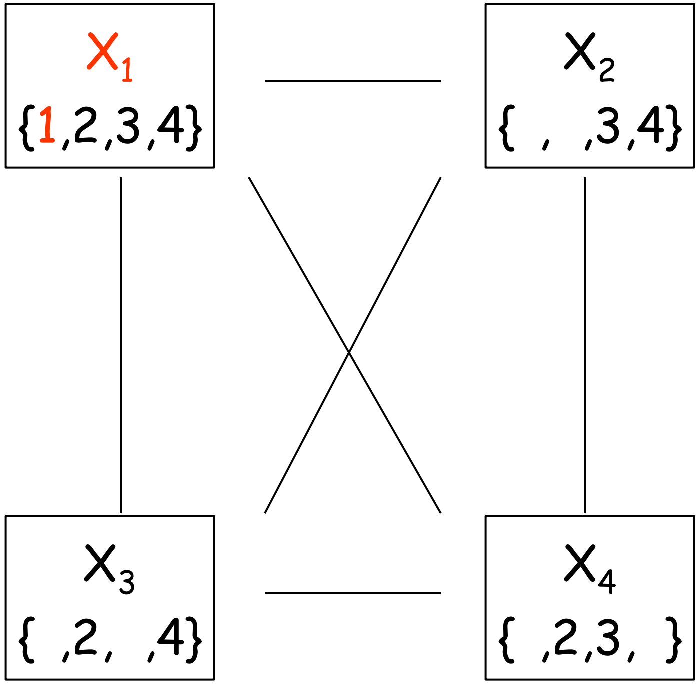




3) The algorithm performs forward checking, which eliminates 2 values in each other variable's domain

4-Queens Problem



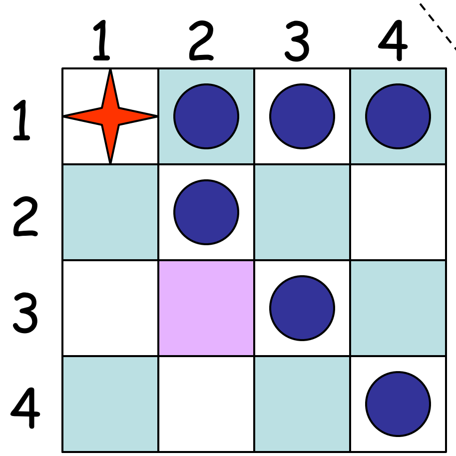


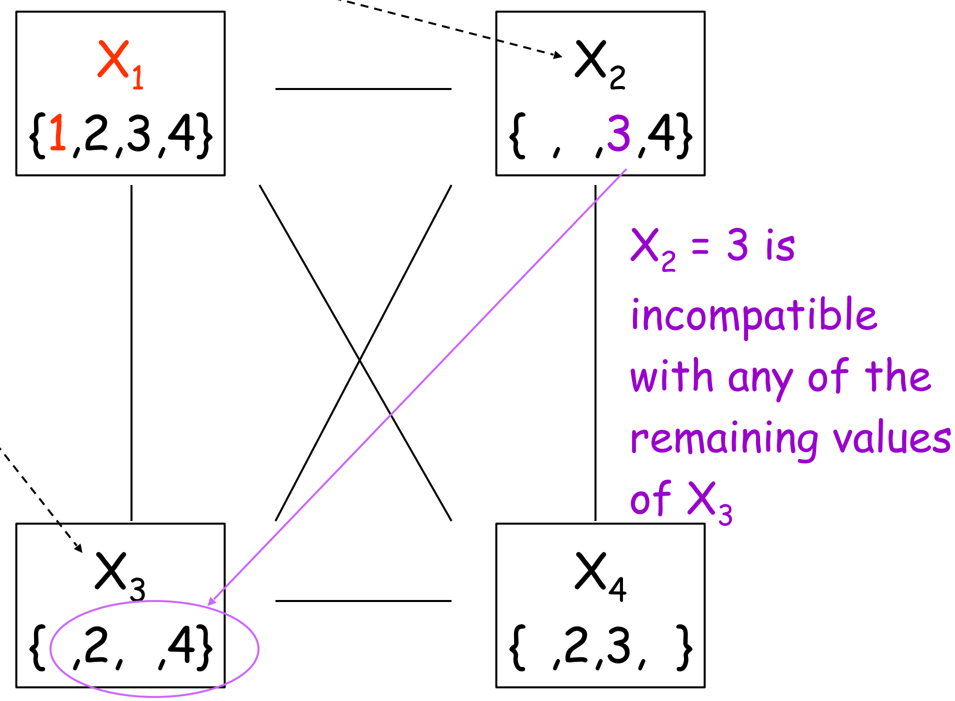
4) The algorithm calls AC3

4-Queens Problem

REMOVE-VALUES(X,Y)

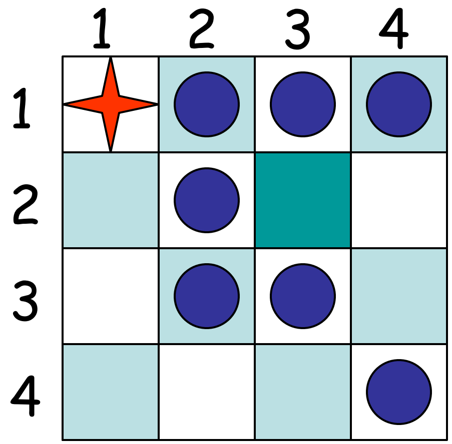
1. removed ← false
2. For every value v in the domain of Y do
   - If there is no value u in the domain of X such that the constraint on (x,y) is satisfied then
     a. Remove v from Y's domain
     b. removed ← true
3. Return removed

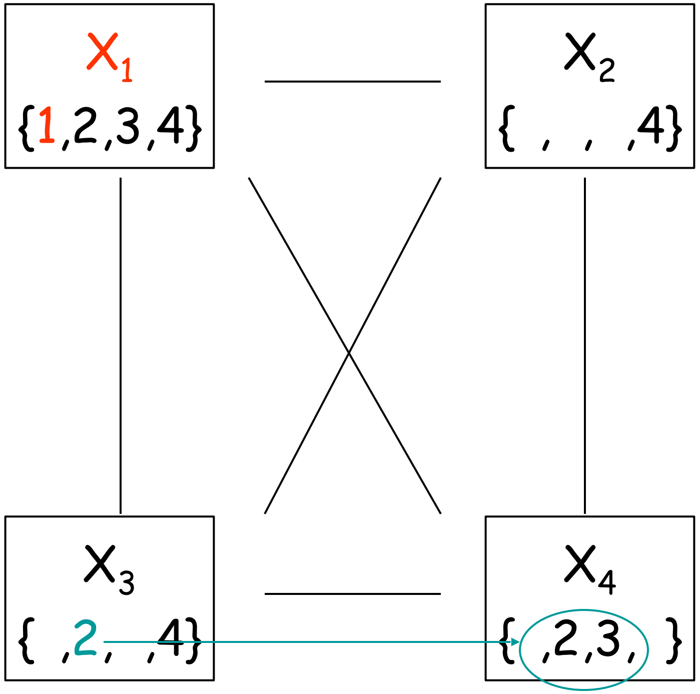




4) The algorithm calls AC3, which eliminates 3 from the domain of X_2

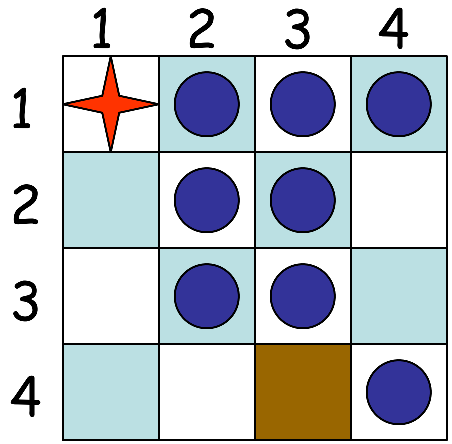
4-Queens Problem

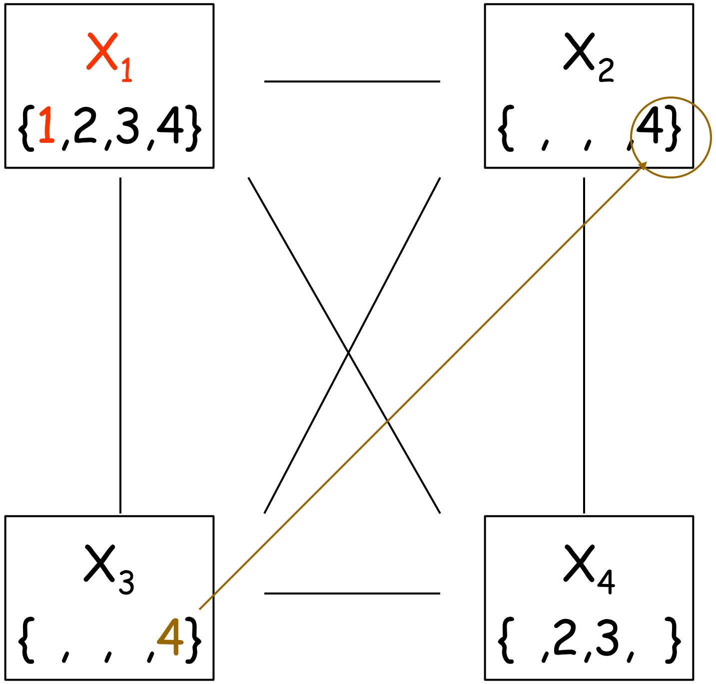




4) The algorithm calls *AC3*, which eliminates 3 from the domain of X_2, and 2 from the domain of X_3

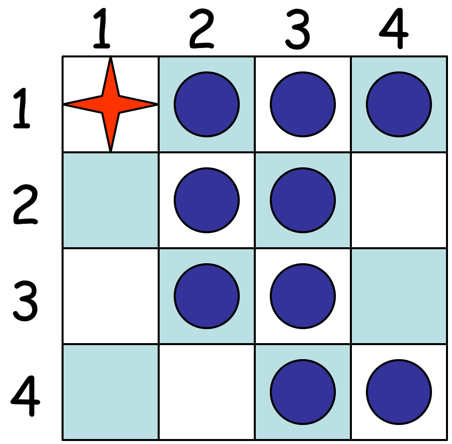
4-Queens Problem

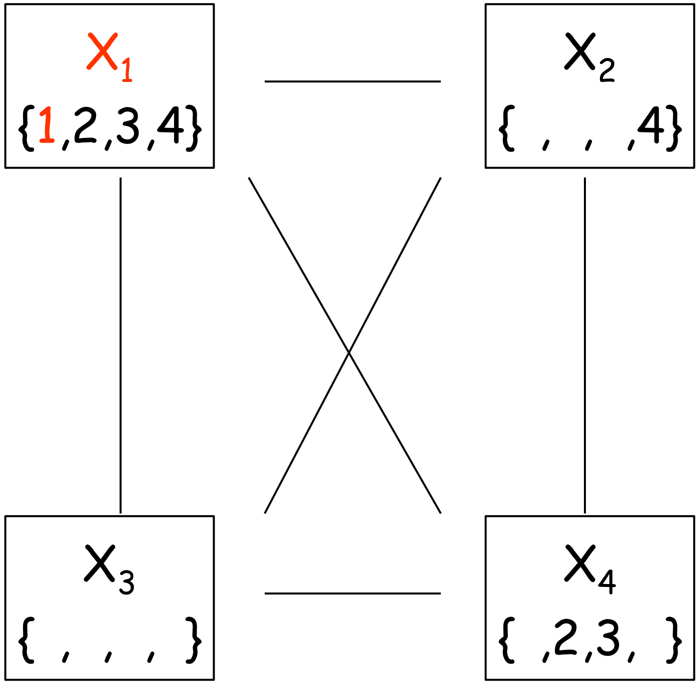




4) The algorithm calls AC3, which eliminates 3 from the domain of X_2, and 2 from the domain of X_3, and 4 from the domain of X_3

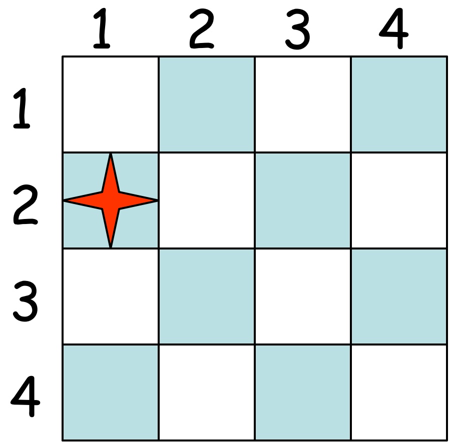
4-Queens Problem

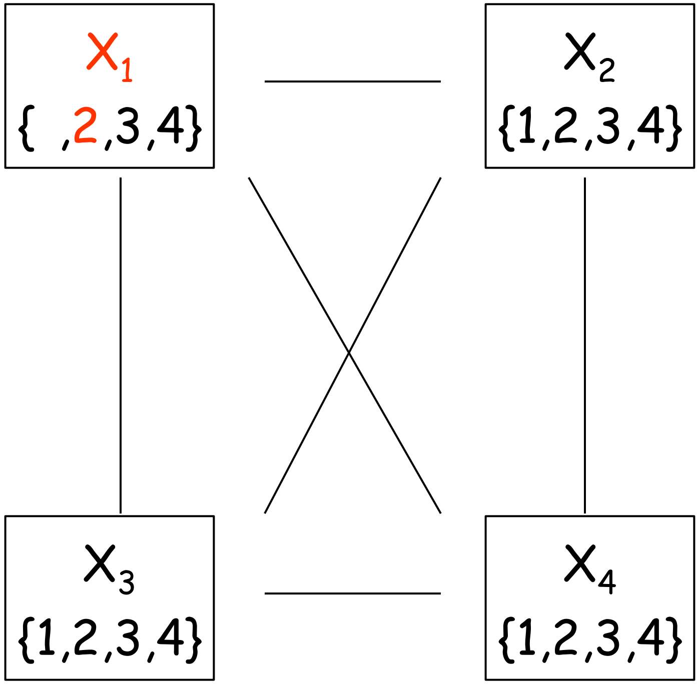




5) The domain of X_3 is **empty** → backtracking

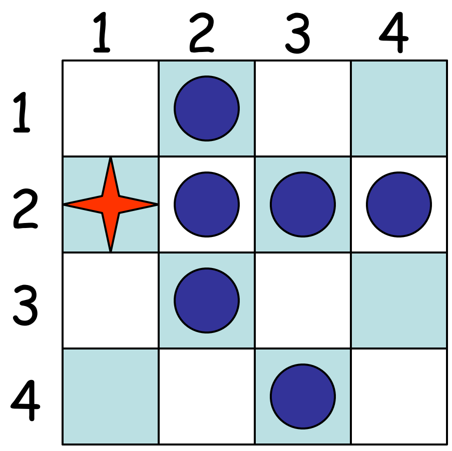
4-Queens Problem

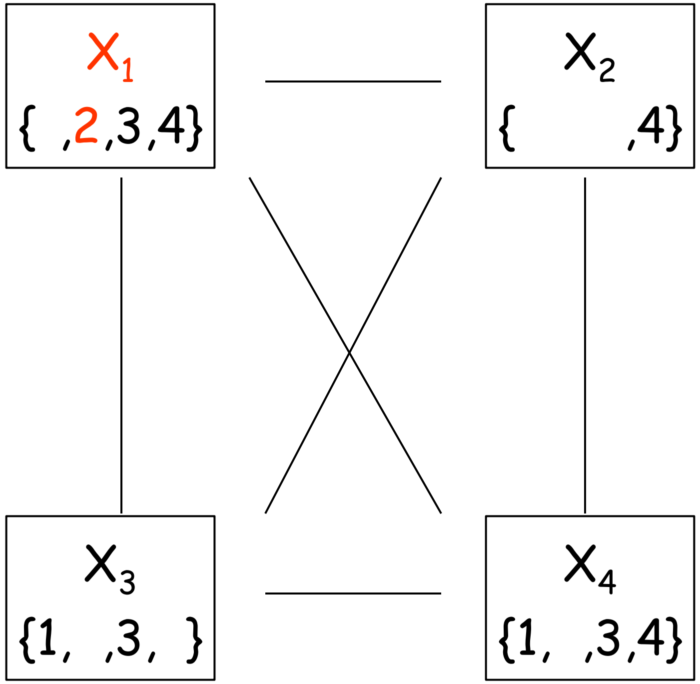




6) The algorithm removes 1 from X_1's domain and assign 2 to X_1

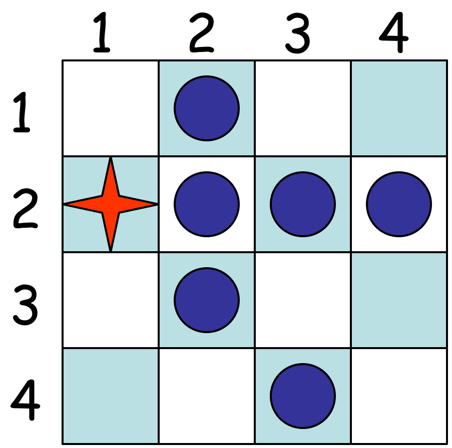
4-Queens Problem

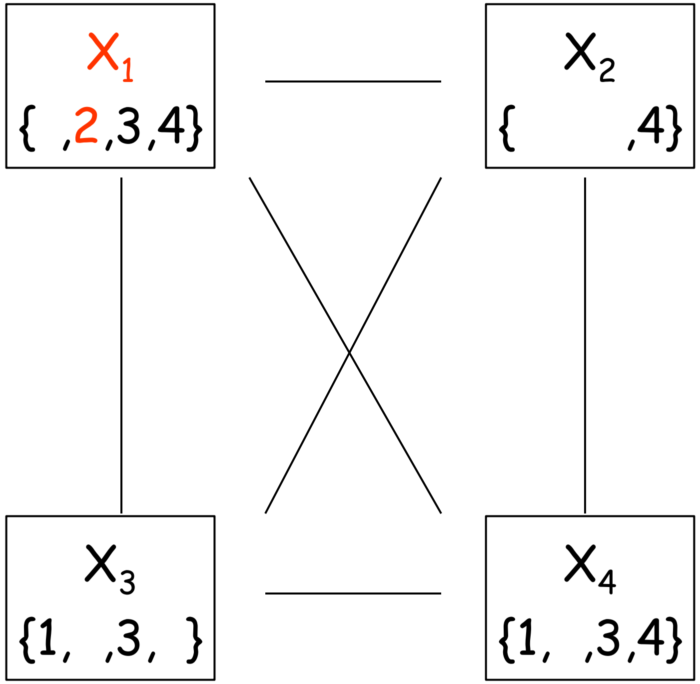




7) The algorithm performs forward checking

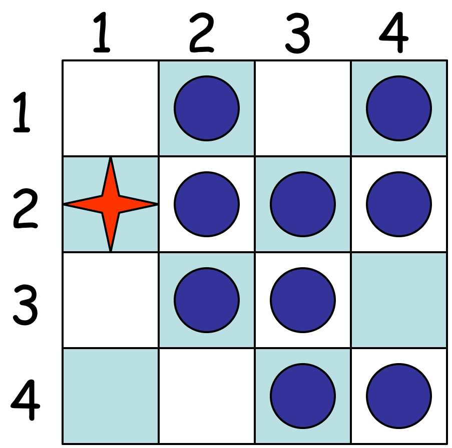
4-Queens Problem

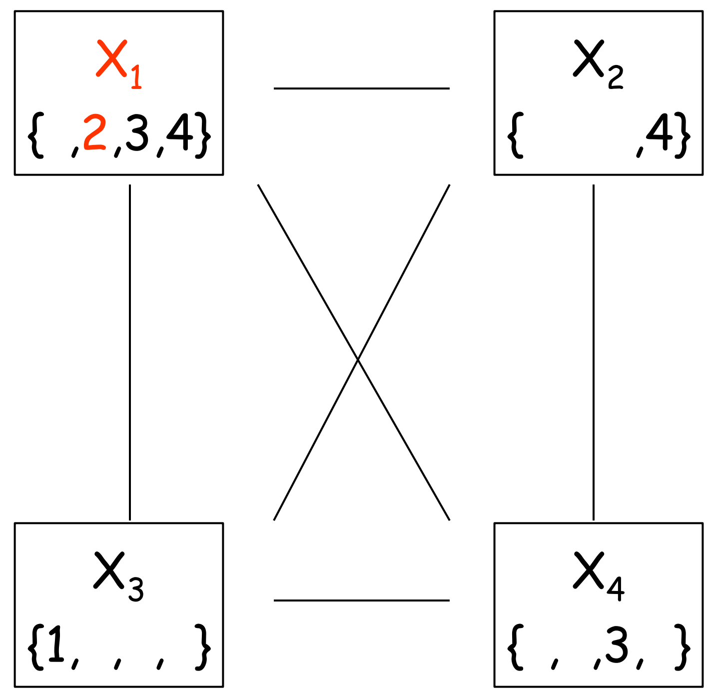




8) The algorithm calls AC3

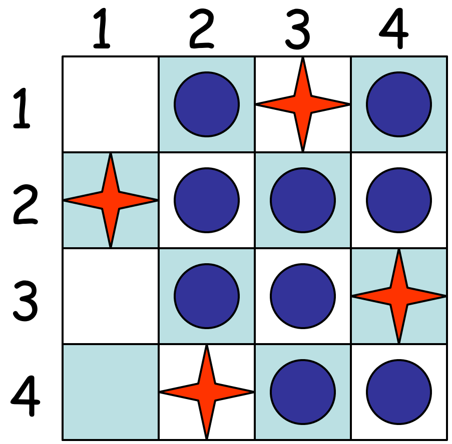
4-Queens Problem

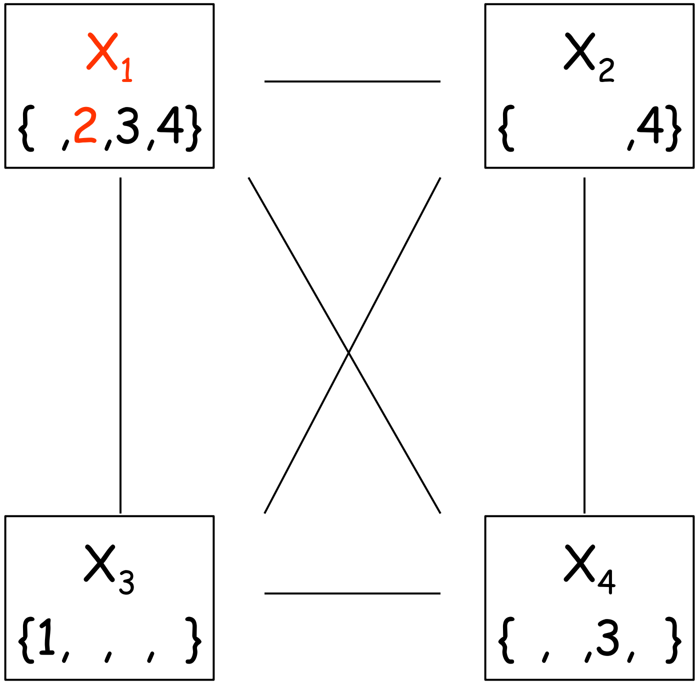




8) The algorithm calls AC3, which reduces the domains of X_3 and X_4 to a single value

4-Queens Problem





8) The algorithm calls *AC3*, which reduces the domains of X_3 and X_4 to a single value

Applications of CSP

- CSP techniques are widely used
- Applications include:
  - Crew assignments to flights
  - Management of transportation fleet
  - Flight/rail schedules
  - Job shop scheduling
  - Task scheduling in port operations
  - Design, including spatial layout design
  - Radiosurgical procedures