Classifiers: Naïve Bayes, evaluation

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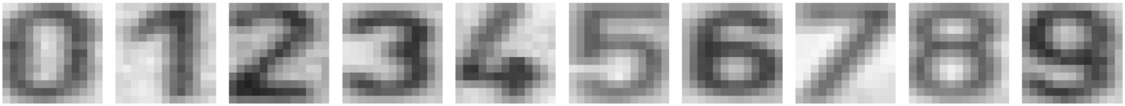
Notes

Example: Digit recognition/classification



- Input: 8-bit image 13×13, pixel intensities $0 - 255$. (0 means black, 255 means white)
- Output: Digit $0 - 9$. Decision about the class, classification.
- Features: Pixel intensities . . .

Notes

Digit recognition is a very classical example of classification problem. It has been used for decades, and it is used till today, see e.g. MNIST demo at PyTorch

Classification as a special case of statistical decision theory

- Attribute vector $\vec{x} = (x_1, x_2, \dots)$: pixels 1, 2, ...
- **State set $\mathcal{S}$ = decision set $\mathcal{D} = \{0, 1, \dots 9\}$.**
- State = actual class, Decision = recognized class
- Loss function:

$$l(s,d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg\min_d \sum_s \underbrace{l(s,d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg\min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg\min_d [1 - P(d|\vec{x})] = \arg\max_d P(d|\vec{x})$$

Notes

We are using different word – *classification* instead of *decision* but the reasoning and methods can be well applied in both. In classification problem we usually treat all mistakes – wrong classificaions – equally painful, contrary to decision problem – remember "What to cook for dinner" problem?

Bayes classification in practice

- Usually we are not given $P(s|\vec{x})$
- It has to be estimated from already classified examples – training data
- For discrete $\vec{x}$, training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \ldots (\vec{x}_l, s_l)$
  - so-called i.i.d (independent, identically distributed) multiset
  - every $(\vec{x}_i, s)$ is drawn independently from $P(\vec{x}, s)$
- Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) = \frac{P(\vec{x}, s)}{P(\vec{x})} \approx \frac{\#\text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\#\text{ examples where } \vec{x}_i = \vec{x}}$$

- Hard in practice:
  - To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of $\vec{x}$.
    - e.g. with the number of pixels in images
    - curse of dimensionality
    - denominator often 0

Notes

Why hard? Way too many various $\vec{x}$. Think about simple binary 10×10 image - $\vec{x}$ contains $0, 1$, position matters.
What is the total number of unique images? Think binary, 1×8 binary image?
What is the difference between set and multiset?

Reminder about math notation. In literature, vectors are mostly denoted by bold lower case $\mathbf{x}$. In lectures, we use $\vec{x}$ to match notation used on blackboard. It is difficult to write bold with a chalk.

Naïve Bayes classification

- For efficient classification we must thus rely on additional assumptions.
- In the exceptional case of statistical independence between components of $\vec{x}$ for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

- Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots =$$

- No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s.
- No need to estimate $P(\vec{x})$. (Why?)
- $P(s)$ may be provided apriori.
- naïve = when used despite statistical dependence

Notes

Why naïve at all? Consider $N-$ dimensional space, $8-bit$ values. Instead of problem 8^N we have $8 \times N$ problem. Think about statistical independence. Example1: person's weight and height. Are they independent? Example2: pixel values in images.

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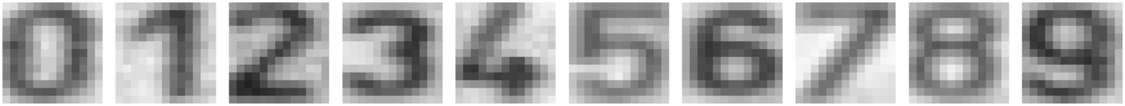

- Input: 8-bit image 13×13, pixel intensities $0 - 255$. (0 means black, 255 means white)
- Output: Digit $0 - 9$. Decision about the class, classification.
- Features: Pixel intensities ...

Collect data , ...

- $P(\vec{x})$. What is the dimension of $\vec{x}$? How many possible images?
- Learn $P(\vec{x}|s)$ per each class (digit).
- Classify $s^* = \operatorname{argmax}_s P(s|\vec{x})$.

Notes

We can create many more features than just pixel intensities. But first things first.
We are assuming all errors are equally important - minimizing the number of wrong decisions.
Dimension of $\vec{x}$ is $13 \times 13 = 169$. There are 255^{169} possible images.

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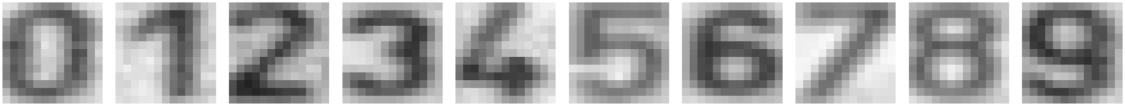

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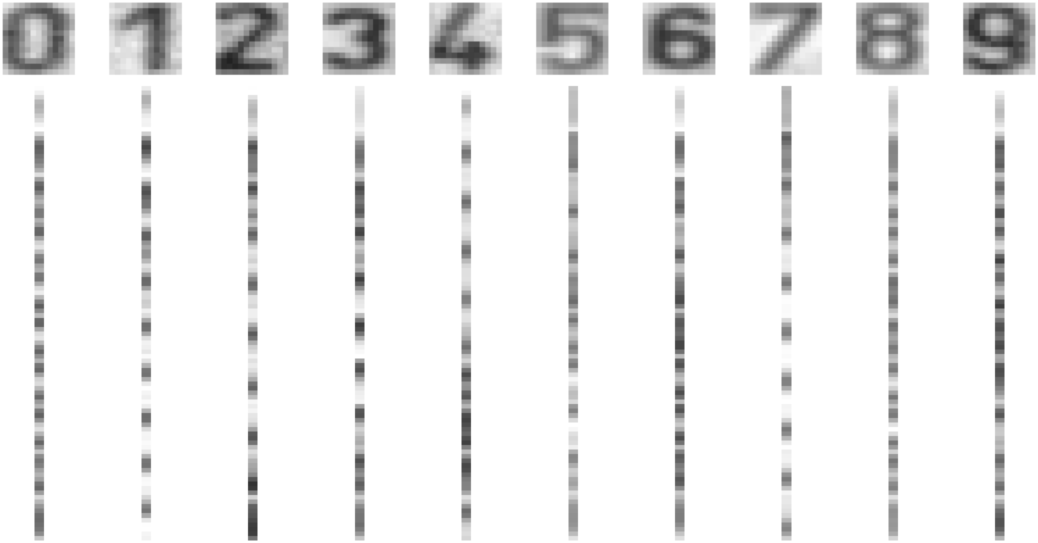
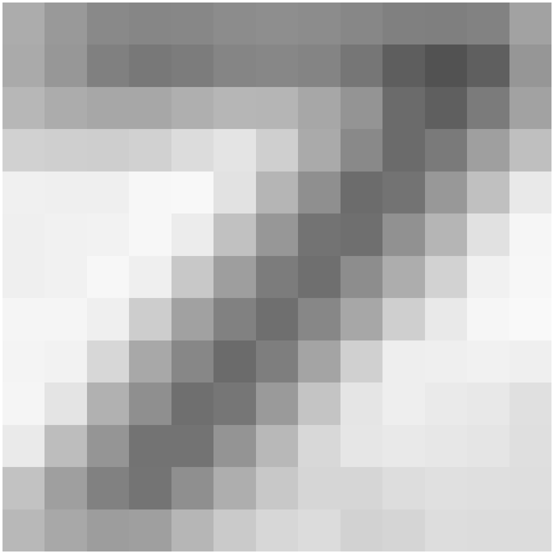
From images to $\vec{x}$



Notes

Conditional probabilities, likelihoods



- Apriori digit probabilities $P(s_k)$
- Likelihoods for pixels. $P(x_{r,c} = I_i | s_k)$

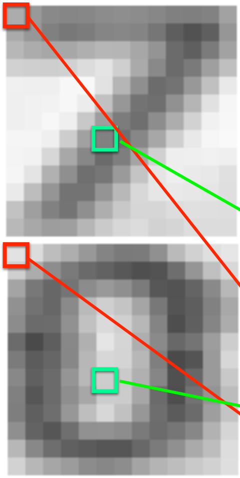
Notes

A lexical, especially for Czech speakers. *probability* as well as *likelihood* can be translated as *pravděpodobnost*. I suggest the following mental model than can work for our purposes.

- Probability is related to the future events (unknown outcome). E.g. what is the probability of selecting blue box? What is the probability that a random zup number begins with 7?
- Likelihood refers to past events (known outcome). In my data, how many images of 7 have dark pixel in top right corner? We can think about relative frequency (relativní četnost).

Conditional likelihoods

s_k	$P(s_k)$
1	0.1
2	0.1
3	0.1
4	0.1
5	0.1
6	0.1
7	0.1
8	0.1
9	0.1
0	0.1



$P(x_{0,0} = 25|s_k)$

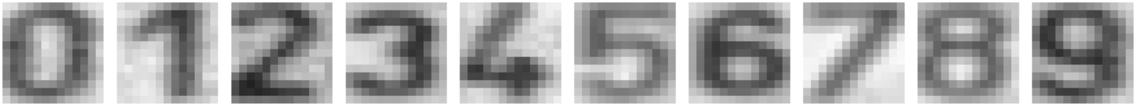
s_k	$P(x_{0,0} = 25\|s_k)$
1	0.0009
2	0.05
3	0.00006
4	0.0005
5	0.07
6	0.0007
7	0.1
8	0.0006
9	0
0	0

$P(x_{7,5} = 25|s_k)$

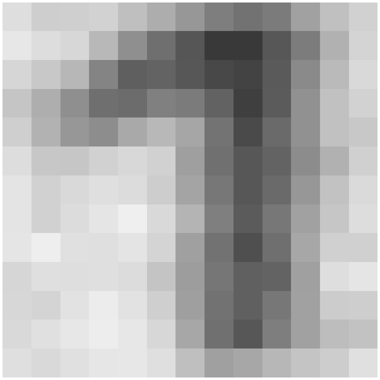
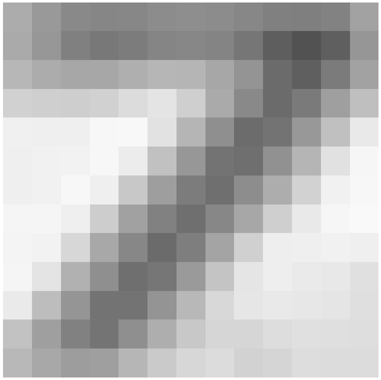
s_k	$P(x_{7,5} = 25\|s_k)$
1	0.0009
2	0.05
3	0.00006
4	0.0005
5	0.004
6	0.07
7	0.1
8	0.006
9	0.08
0	0.005

Notes

Unseen events



Images 13×13, intensities $0 - 255$, 100 exemplars per each class.

Notes

Think about the problem of classifying numerals. Some $P(x_{r,c} = I \mid s) = 0$. What about an example:

$$
\begin{aligned}
P(x_{0,0} = 100 \mid s = 7) &= 0.05 \\
P(x_{0,0} = 101 \mid s = 7) &= 0 \\
P(x_{0,0} = 102 \mid s = 7) &= 0.06
\end{aligned}
$$

A new (not in training) query image with $x_{0,0} = 101$. How would you classify?

Laplace smoothing ("additive smoothing")

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem: $\text{count}(x) = 0$

Pretend you see the (any) sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{N + |X|}$$

where N is the number of (total) observations; $|X|$ is the number of possible values X can take (cardinality).

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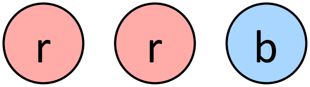
$P_{\text{LAP}}(x) = ?$

Observation:



What is $P_{\text{LAP}}(X = \text{red})$ and $P_{\text{LAP}}(X = \text{blue})$?

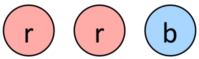
A: $P_{\text{LAP}}(X = \text{red}) = 7/10$, $P_{\text{LAP}}(X = \text{blue}) = 3/10$

B: $P_{\text{LAP}}(X = \text{red}) = 2/3$, $P_{\text{LAP}}(X = \text{blue}) = 1/3$

C: $P_{\text{LAP}}(X = \text{red}) = 3/5$, $P_{\text{LAP}}(X = \text{blue}) = 2/5$

D: None of the above.

Notes



$P_{ML}(X) =$

$P_{LAP}(X) =$

originally:

- $P(red) = 2/3$
- $P(blue) = 1/3$

after Laplace smoothing – adding one red ball and blue ball to the actual observations:

- $P_{LAP}(red) = (2+1)/(2+1+1+1) = 3/5$
- $P_{LAP}(blue) = (1+1)/(2+1+1+1) = 2/5$

this slide: courtesy of P. Abeel, http://ai.berkeley.edu. 21st lecture of CS 188.

Laplace smoothing - as a hyperparameter k

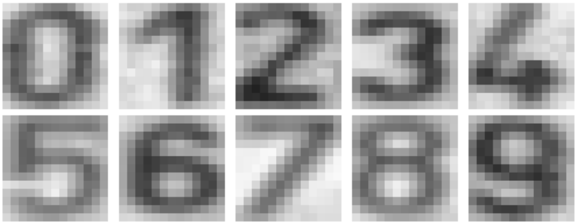
Pretend you see every sample k extra times:

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{\sum_x [c(x) + k]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + k}{N + k|X|}$$

For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x, s) + k}{c(s) + k|X|}$$



What is $|X|$ equal to?

A: 10

B: 2

C: 256

D: None of the above

Notes

Hyperparameter would be tuned along with your classifier
For $k = 100$ and blue and red, you would get:

- $P_{LAP}(red) = (2 + 100)/(3 + 100 * 2) = 102/203$
- $P_{LAP}(blue) = (1 + 100)/(3 + 100 * 2) = 101/203$

In this case, smoothing ("prior") would dominate over the observations - shifting estimate from empirical to uniform.
In the digit recognition from pixels example: 256 intensity values; $13 \times 13 = 169$ pixels: Applying Laplace smoothing with $k = 1$ to $P(x)$ (prior probability of a particular pixel will take an intensity value i): $P(x_{r,c} = i) = (c(x) + 1)/(N + 256)$

Conditional: relevant for the Naïve Bayes case.

Laplace smoothing - as a hyperparameter k

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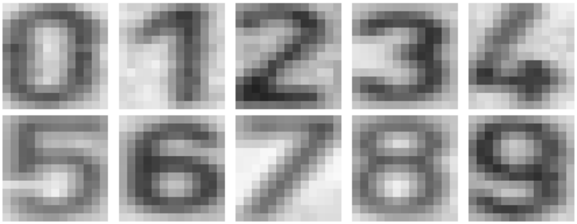

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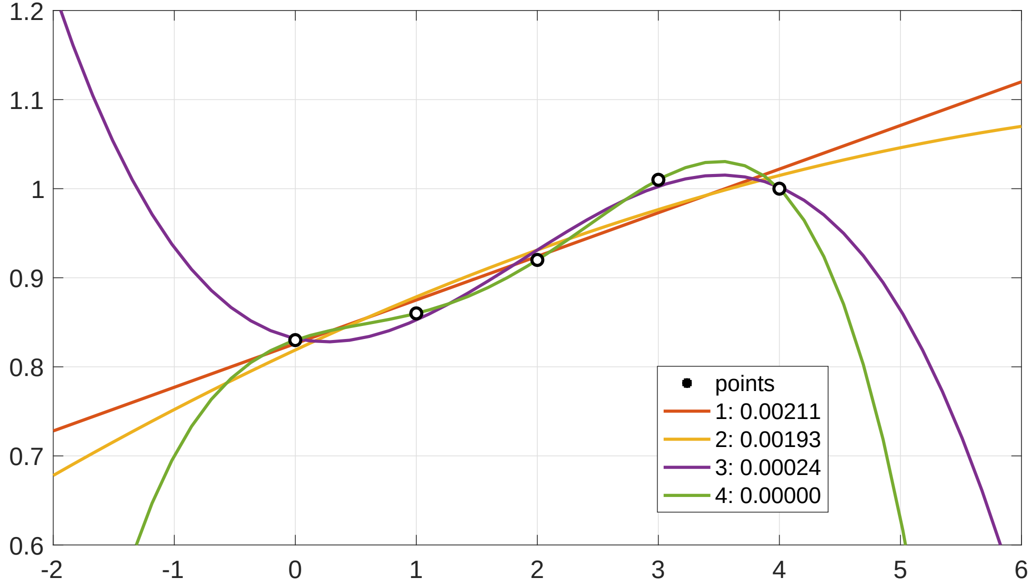
What is the right degree of polynomial (hyperparameter of a regressor)



Notes

see the `overfit.m` demo

Generalization and overfiting

- Data: training, validating, testing . Wanted classifier performs well on what data?
- Overfitting: too close to training, poor on testing.

Notes

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Notes

Training and testing

Data labeled instances.

- Training set
- Held-out (validation) set
- Testing set.

Features : Attribute-value pairs.

Learning cycle:

- Learn parameters (e.g. probabilities) on training set.
- Tune hyperparameters on held-out (validation) set.
- Evaluate performance on testing set.

Notes

Training set - biggest part.

How to evaluate a classifier? Confusion table

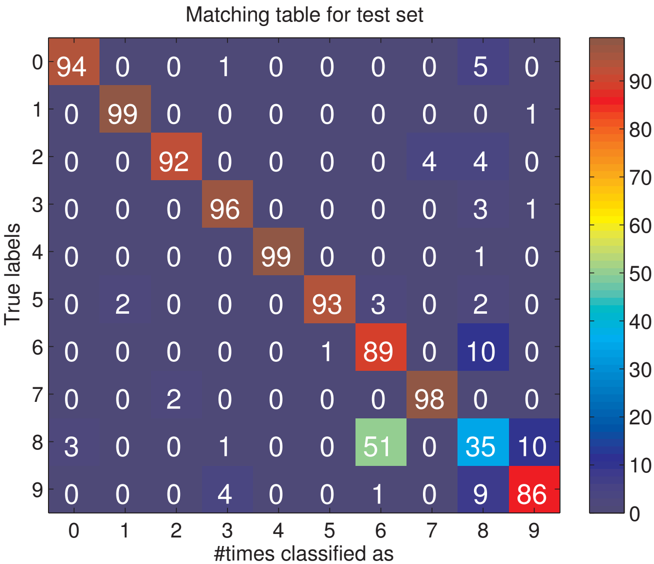


Figure from [5]

Notes

A result for a one particular classifer and its setting (parameters), one particular testing set

Precision and Recall, and ...

Consider digit detection (is there a digit?) or SPAM/HAM classification.

Recall :

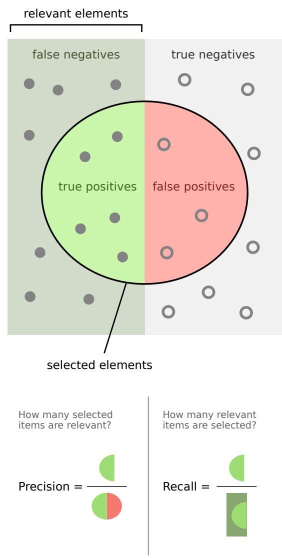
- How many relevant items are selected?
- Are we missing some items?
- Also called: True positive rate (TPR), sensitivity, hit rate ...

Precision

- How many selected items are relevant?
- Also called: Positive predictive value

False positive rate (FPR)

- Probability of false alarm



Notes

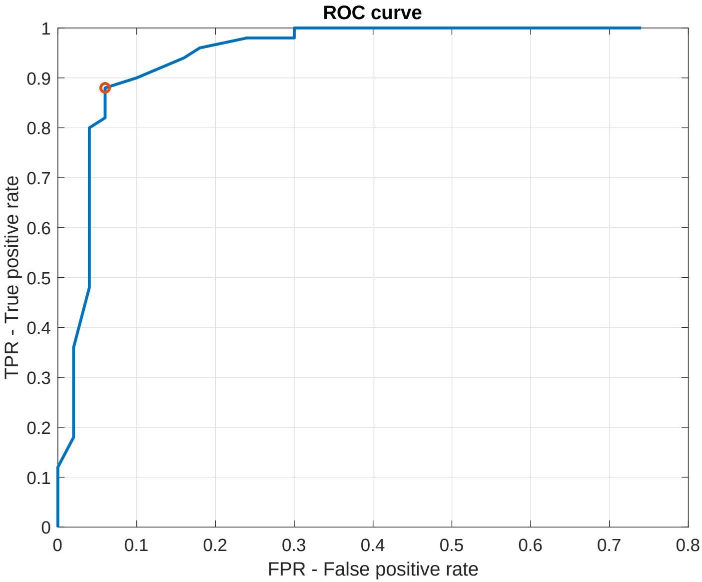
$$\mathrm{TPR} = \frac{\mathrm{TP}}{P} = \frac{\mathrm{TP}}{\mathrm{TP} + \mathrm{FN}}$$

$$\mathrm{Precision} = \frac{\mathrm{TP}}{\mathrm{TP} + \mathrm{FP}}$$

$$\mathrm{FPR} = \frac{\mathrm{FP}}{N} = \frac{\mathrm{FP}}{\mathrm{FP} + \mathrm{TN}}$$

Think about TPR vs FPR graph, what is the best classifier?

ROC – Receiver operating characteristics curve



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$$\text{FPR} = \frac{\text{FP}}{N} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

Notes

- How do you slide along the curve?
- What is the meaning of the diagonal?
- What would be the shape of the curve for the ideal/worst classifier?
- How would you compare various curve and select the best classifier?
- Think/read about other ways to evaluate/visualise classification results.

Discriminant functions $f(\vec{x}, s)$

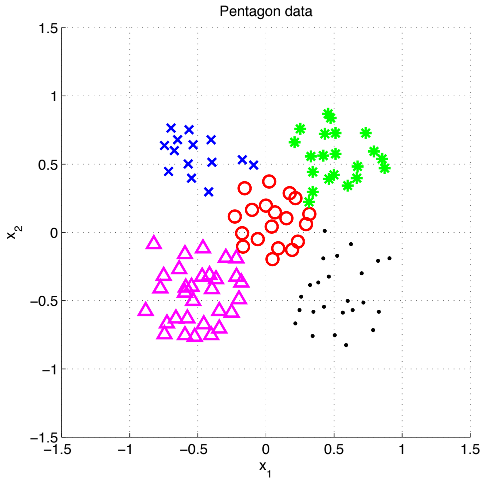
$$s^* = \underset{s \in \mathcal{S}}{\operatorname{argmax}} f(\vec{x}, s)$$

Conditional likelihoods: $\mathcal{N}(\vec{x}|\vec{\mu}_s, \Sigma_s)$

$$\frac{1}{2\pi|\Sigma_s|^{1/2}} \exp\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\}$$

Bayes:

$$s^* = \underset{s \in \mathcal{S}}{\operatorname{argmax}} P(s|\vec{x}) = \frac{P(\vec{x} \mid s)P(s)}{P(\vec{x})}$$



$$f(\vec{x}, s) = P(s)\frac{1}{2\pi|\Sigma_s|^{1/2}} \exp\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\}$$

Notes

Normal distribution for general dimensionality D:

$$\mathcal{N}(\vec{x}|\vec{\mu}, \Sigma) = \frac{1}{(2\pi)^{D/2}}\frac{1}{|\Sigma|^{1/2}} \exp\{-\frac{1}{2}(\vec{x} - \vec{\mu})^\top \Sigma^{-1}(\vec{x} - \vec{\mu})\}$$

How about learning $f(\vec{x}, s)$ directly without explicit modeling of underlying probabilities?
What about $f(\vec{x}, s) = \vec{w}_s^\top \vec{x} + w_{s0}$

Towards linear classifier, geometrical thoughts ...

$$f(\vec{x}, s) = P(s)\frac{1}{2\pi|\Sigma_s|^{1/2}}\exp\{-\frac{1}{2}(\vec{x}-\vec{\mu}_s)^\top\Sigma_s^{-1}(\vec{x}-\vec{\mu}_s)\}$$

Notes

Product of many small numbers . . .

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

$P(\vec{x})$ not needed, ...

$$\log(P(x[1]|s)P(x[2]|s)\cdots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \cdots$$

Notes

just try

- `prod(rand(1,100))` and `prod(rand(1,10000))` in Matlab.
- `prod(rand(1,100)) == 0` and `prod(rand(1,10000)) == 0` in Matlab.

or in python console:

- `>>> import numpy as np`
- `>>> np.prod(np.random.rand(100))==0`
- `>>> np.prod(np.random.rand(1000))==0`
-
```
>>> a = np.random.rand(1000)
>>> b = np.random.rand(1000)
>>> np.prod(a)>np.prod(b)
False
>>> np.prod(a)<np.prod(b)
False
>>> np.sum(np.log(a))>np.sum(np.log(b))
True
```

Hitting the limit of number representation.
What is the way out?
$P(\vec{x})$ not needed – does not depend on the class.
Laws of logarithms...

Product of many small numbers ...

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \ldots$$

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References I

Further reading: Chapter 13 and 14 of [4]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. This lecture has been also inspired by the 21st lecture of CS 188 at http://ai.berkeley.edu (e.g., Laplace smoothing). Many Matlab figures created with the help of [3].

[1] Christopher M. Bishop.
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Notes

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Notes