

Classifiers: Naïve Bayes, evaluation

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Example: Digit recognition/classification



- ▶ **Input:** 8-bit image 13×13 , pixel intensities 0 – 255. (0 means black, 255 means white)
- ▶ **Output:** Digit 0 – 9. Decision about the class, classification.
- ▶ **Features:** Pixel intensities ...

Classification as a special case of statistical decision theory

- ▶ Attribute vector $\vec{x} = (x_1, x_2, \dots)$: pixels 1, 2, ...
- ▶ **State set \mathcal{S} = decision set $\mathcal{D} = \{0, 1, \dots, 9\}$.**
- ▶ State = actual class, Decision = recognized class
- ▶ Loss function:

$$l(s, d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg \min_d \sum_s \underbrace{l(s, d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg \min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg \min_d [1 - P(d|\vec{x})] = \arg \max_d P(d|\vec{x})$$

Bayes classification in practice

- ▶ Usually we are not given $P(s|\vec{x})$
- ▶ It has to be estimated from already classified examples – training data
- ▶ For discrete \vec{x} , training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \dots, (\vec{x}_l, s_l)$
 - ▶ so-called i.i.d (independent, identically distributed) multiset
 - ▶ every (\vec{x}_i, s) is drawn independently from $P(\vec{x}, s)$
- ▶ Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) = \frac{P(\vec{x}, s)}{P(\vec{x})} \approx \frac{\# \text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\# \text{ examples where } \vec{x}_i = \vec{x}}$$

- ▶ Hard in practice:
 - ▶ To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of \vec{x} .
 - ▶ e.g. with the number of pixels in images
 - ▶ curse of dimensionality
 - ▶ denominator often 0

Naïve Bayes classification

- ▶ For efficient classification we must thus rely on additional assumptions.
- ▶ In the exceptional case of **statistical independence** between components of \vec{x} for each class s it holds

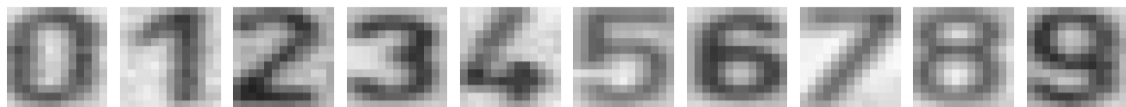
$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

- ▶ Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots =$$

- ▶ No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s .
- ▶ No need to estimate $P(\vec{x})$. (Why?)
- ▶ $P(s)$ may be provided apriori.
- ▶ **naïve** = when used despite statistical dependence

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Collect data , ...

- ▶ $P(\vec{x})$. What is the dimension of \vec{x} ? How many possible images?
- ▶ Learn $P(\vec{x}|s)$ per each class (digit).
- ▶ Classify $s^* = \operatorname{argmax}_s P(s|\vec{x})$.

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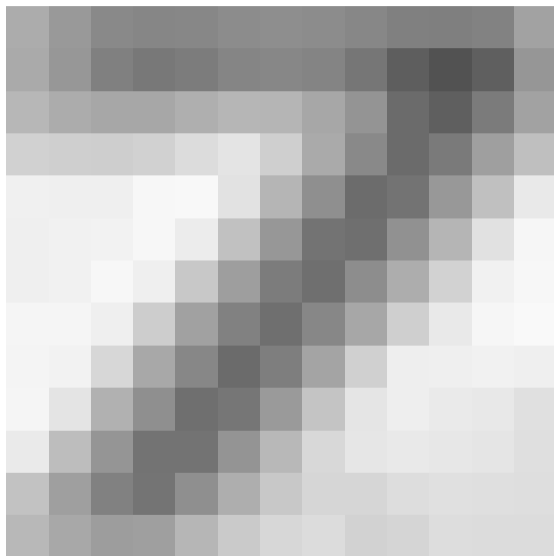
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From images to \vec{x}

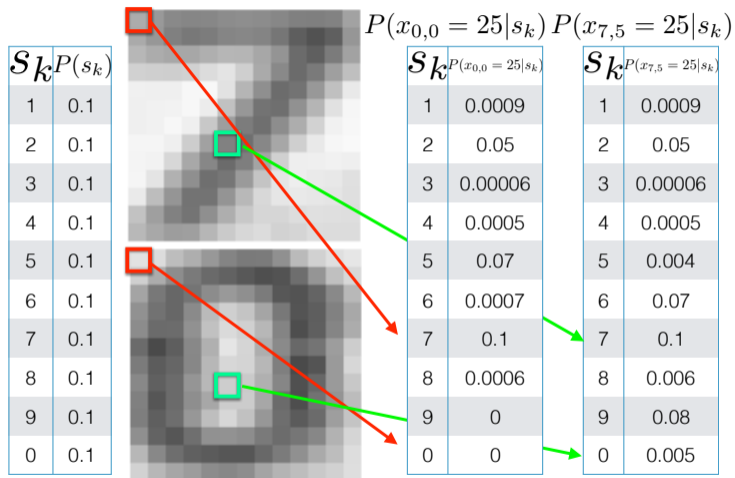


Conditional probabilities, likelihoods

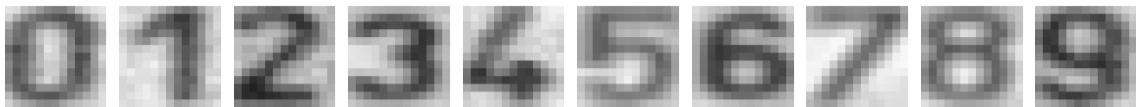


- ▶ Apriori digit probabilities $P(s_k)$
- ▶ Likelihoods for pixels. $P(x_{r,c} = l_i | s_k)$

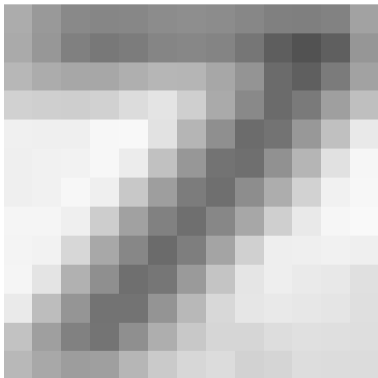
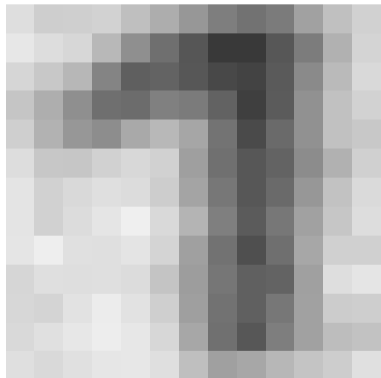
Conditional likelihoods



Unseen events



Images 13×13 , intensities 0 – 255, 100 exemplars per each class.



Laplace smoothing (“additive smoothing”)

$$P(x) = \frac{\text{count}(x)}{\text{total samples}}$$

Problem: $\text{count}(x) = 0$

Pretend you see the (any) sample one more time.

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{\sum_x [c(x) + 1]}$$

$$P_{\text{LAP}}(x) = \frac{c(x) + 1}{N + |X|}$$

where N is the number of (total) observations; $|X|$ is the number of possible values X can take (cardinality).

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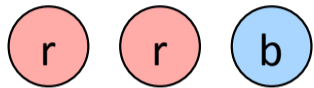
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$$P_{\text{LAP}}(X) = ?$$

Observation:



What is $P_{\text{LAP}}(X = \text{red})$ and $P_{\text{LAP}}(X = \text{blue})$?

A: $P_{\text{LAP}}(X = \text{red}) = 7/10$, $P_{\text{LAP}}(X = \text{blue}) = 3/10$

B: $P_{\text{LAP}}(X = \text{red}) = 2/3$, $P_{\text{LAP}}(X = \text{blue}) = 1/3$

C: $P_{\text{LAP}}(X = \text{red}) = 3/5$, $P_{\text{LAP}}(X = \text{blue}) = 2/5$

D: None of the above.

Laplace smoothing - as a hyperparameter k

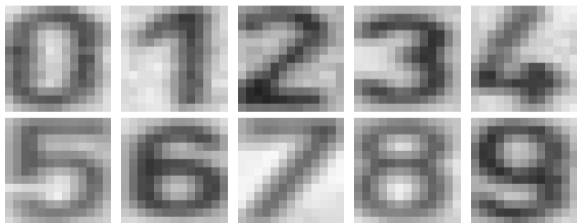
Pretend you see every sample k extra times:

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For conditional, smooth each condition independently

$$P_{\text{LAP}}(x|s) = \frac{c(x, s) + k}{c(s) + k|X|}$$



What is $|X|$ equal to?

A: 10

B: 2

C: 256

D: None of the above

Laplace smoothing - as a hyperparameter k

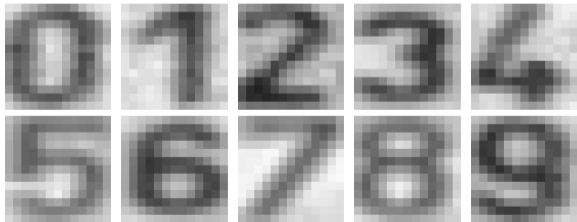
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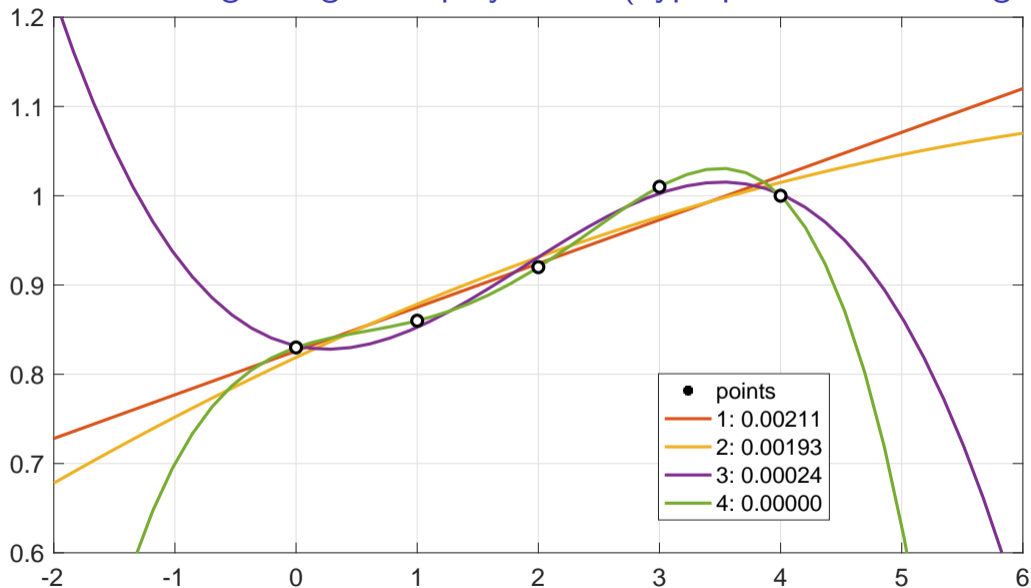
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What is $|X|$ equal to?

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- D: None of the above

What is the right degree of polynomial (hyperparameter of a regressor)



Generalization and overfitting

- ▶ **Data: training, validating, testing** . Wanted classifier performs well on what data?
- ▶ Overfitting: too close to training, poor on testing.

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Training and testing

Data labeled instances.

- ▶ Training set
- ▶ Held-out (validation) set
- ▶ Testing set.

Features : Attribute-value pairs.

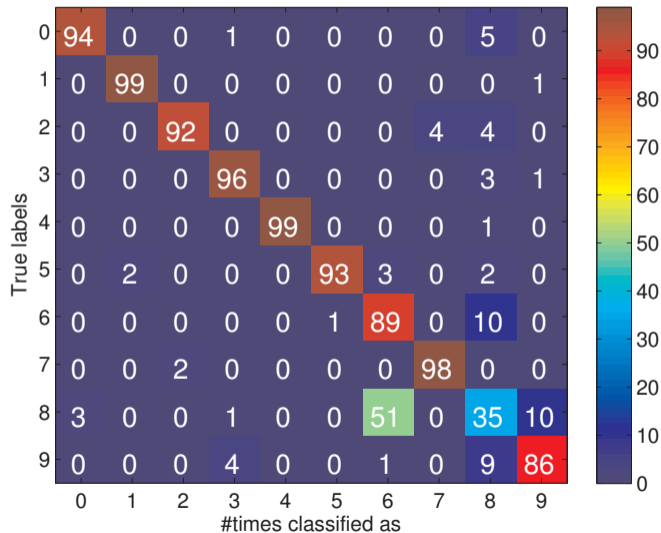
Learning cycle:

- ▶ **Learn** parameters (e.g. probabilities) on training set.
- ▶ **Tune** hyperparameters on held-out (validation) set.
- ▶ **Evaluate** performance on testing set.



How to evaluate a classifier? Confusion table

Matching table for test set



Precision and Recall, and ...

Consider digit **detection** (is there a digit?) or SPAM/HAM classification.

Recall :

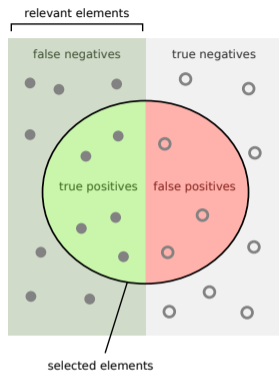
- ▶ How many relevant items are selected?
- ▶ Are we missing some items?
- ▶ Also called: **True positive rate** (TPR), sensitivity, hit rate ...

Precision

- ▶ How many selected items are relevant?
- ▶ Also called: Positive predictive value

False positive rate (FPR)

- ▶ Probability of false alarm



How many selected items are relevant?



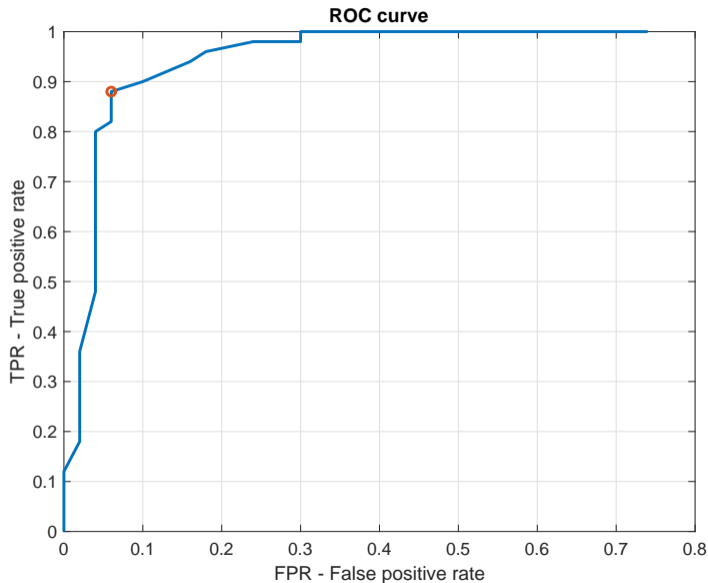
$$\text{Precision} = \frac{\text{true positives}}{\text{true positives} + \text{false positives}}$$

How many relevant items are selected?



$$\text{Recall} = \frac{\text{true positives}}{\text{true positives} + \text{false negatives}}$$

ROC – Receiver operating characteristics curve



$$\text{TPR} = \frac{\text{TP}}{P} = \frac{\text{TP}}{\text{TP} + \text{FN}}$$

$$\text{FPR} = \frac{\text{FP}}{N} = \frac{\text{FP}}{\text{FP} + \text{TN}}$$

Discriminant functions $f(\vec{x}, s)$

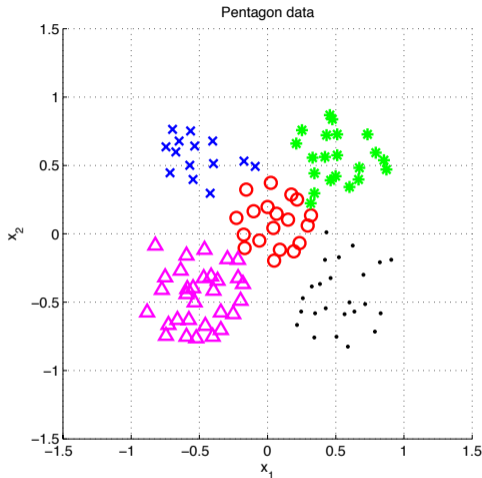
$$s^* = \operatorname{argmax}_{s \in \mathcal{S}} f(\vec{x}, s)$$

Conditional likelihoods: $\mathcal{N}(\vec{x} | \vec{\mu}_s, \Sigma_s)$

$$\frac{1}{2\pi |\Sigma_s|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\right\}$$

Bayes:

$$s^* = \operatorname{argmax}_{s \in \mathcal{S}} P(s | \vec{x}) = \frac{P(\vec{x} | s)P(s)}{P(\vec{x})}$$



$$f(\vec{x}, s) = P(s) \frac{1}{2\pi |\Sigma_s|^{1/2}} \exp\left\{-\frac{1}{2}(\vec{x} - \vec{\mu}_s)^\top \Sigma_s^{-1}(\vec{x} - \vec{\mu}_s)\right\}$$

Towards linear classifier, geometrical thoughts ...

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Product of many small numbers . . .

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$P(\vec{x})$ not needed,

$$\log(P(x[1]|s)P(x[2]|s) \cdot \dots) = \log(P(x[1]|s)) + \log(P(x[2]|s)) + \dots$$

Product of many small numbers ...

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References I

Further reading: Chapter 13 and 14 of [4]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. This lecture has been also inspired by the 21st lecture of CS 188 at <http://ai.berkeley.edu> (e.g., Laplace smoothing). Many Matlab figures created with the help of [3].

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[2] Richard O. Duda, Peter E. Hart, and David G. Stork.

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