

Probabilistic classification

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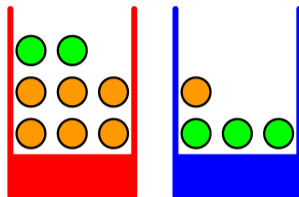
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(Re-)introduction uncertainty/probability

- ▶ Markov Decision Processes - uncertainty about outcome of **actions**
- ▶ Now: uncertainty may be also associated with **states**
 - ▶ Different states may have different **prior probabilities**
 - ▶ The states $s \in S$ may not be directly observable
 - ▶ They need to be inferred from **features** $x \in X$
- ▶ This is addressed by the rules of probability (*such as Bayes theorem*) and leads on to
 - ▶ Bayesian classification
 - ▶ Bayesian decision making

Probability example: Picking fruits

- ▶ red box: 2 apples, 6 oranges
- ▶ blue box: 3 apples, 1 orange

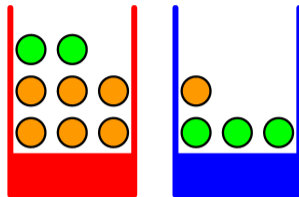


- ▶ Scenario: Pick a box (say red box in 40% cases), then pick a fruit at random
- ▶ (Frequent) questions:
 - ▶ What is the overall probability that the selection procedure will pick an apple?
 - ▶ Given that we have chosen an orange, what is the probability that the box we chose was the blue one?

Example from Chapter 1.2 [1]

Picking fruits. What is the probability that ... ?

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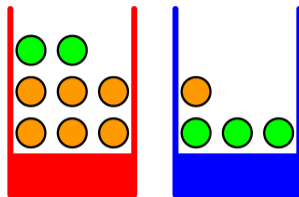


Scenario: Pick a box (say red box in 40% cases), then pick a fruit at random. What is the overall probability that the selection procedure will pick an apple?

- A: $11/20$
- B: $6/8$
- C: $1/2$
- D: Different value.

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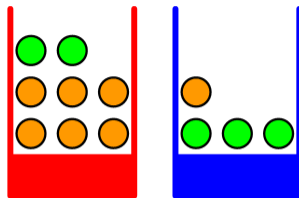
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- A: $1/4$
- B: $3/5$
- C: $1/3$
- D: Different value.

Rules of probability and notation I

- ▶ random variables X, Y
- ▶ x_i where $i = 1, \dots, M$ – values taken by variable X
- ▶ y_j where $j = 1, \dots, L$ – values taken by variable Y
- ▶ $P(X = x_i, Y = y_j)$ – probability that X takes the value x_i and Y takes y_j – joint probability
- ▶ $P(X = x_i)$ – probability that X takes the value x_i
- ▶ Sum rule of probability :
 - ▶ $P(X = x_i) = \sum_{j=1}^L P(X = x_i, Y = y_j)$
 - ▶ $P(X = x_i)$ is sometimes called marginal probability – obtained by marginalizing / summing out the other variables
 - ▶ general rule, compact notation: $P(X) = \sum_Y P(X, Y)$

Rules of probability and notation II

- ▶ **Conditional probability** : $P(Y = y_j | X = x_i)$
- ▶ **Product rule of probability** :
 - ▶ $P(X = x_i, Y = y_i) = P(Y = y_j | X = x_i)P(X = x_i)$
 - ▶ general rule, compact notation: $P(X, Y) = P(Y|X)P(X)$
- ▶ **Bayes theorem** :
 - ▶ from $P(X, Y) = P(Y, X)$ and product rule

$$P(Y|X) = \frac{P(X|Y)P(Y)}{P(X)}$$

$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}}$$

- ▶ **Independence** : $P(X, Y) = P(X)P(Y)$

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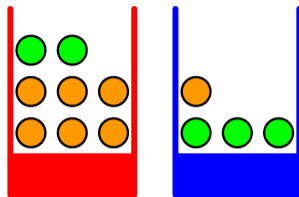
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Boxes and Fruits: posterior? likelihood? prior? evidence?

$$\text{posterior} = \frac{\text{likelihood} \times \text{prior}}{\text{evidence}}$$



▶ posterior
after observation

▶ $P(B)$

▶ likelihood
of an observation

▶ $P(F)$

▶ prior
before observation

▶ $P(F | B)$

▶ evidence
total observations

▶ $P(B | F)$

Decision example: Insure or not? (from late 1980s) [4]

A doctor calls: “Your HIV test is positive, 999/1000 you will die in 10 years. I’m sorry . . .”.
Insurance company does not want to insure a married couple.

- ▶ Was the doctor right?
- ▶ Was the insurance company rational?

What the doctor (and the company) knew:

- ▶ HIV test falsely positive only in 1 case out of 1000.
- ▶ Heterosexual male, has family, no drugs, no risk behavior.

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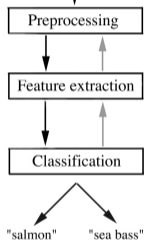
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Decision: guilty or not? (people of CA vs Collins, 1968) [4]

- ▶ Robbery, LA 1964, fuzzy evidence of the offenders:
 - ▶ female, around 65 kg
 - ▶ wearing something dark
 - ▶ hairs of light color, between light and dark blond, in a ponytail
- ▶ At the same time, additional evidence close to the crime scene:
 - ▶ loud scream, yelling, looking at the this direction
 - ...
 - ▶ a woman sitting into a yellow car
 - ▶ car starts immediately and passes close to the additional witness
 - ▶ a black man with beard and moustache was driving
- ▶ No more evidence
- ▶ Testimony of both the victim and the witness not unambiguous (didn't recognize suspects)
- ▶ Still, the suspects were sentenced to jail.

Classification example: What's the fish?



- ▶ Factory for fish processing
- ▶ 2 classes $s_{1,2}$:
 - ▶ salmon
 - ▶ sea bass
- ▶ Features \vec{x} : length, width, lightness etc. from a camera

Fish – classification using probability

$$\textit{posterior} = \frac{\textit{likelihood} \times \textit{prior}}{\textit{evidence}}$$

- ▶ Notation for classification problem
 - ▶ Classes $s_j \in \mathcal{S}$ (e.g., salmon, sea bass)
 - ▶ Features $x_i \in \mathcal{X}$ or feature vectors (\vec{x}_i) (also called attributes)

- ▶ Optimal classification of \vec{x} :

$$\delta^*(\vec{x}) = \arg \max_j P(s_j | \vec{x})$$

- ▶ We thus choose the most probable class for a given feature vector.
- ▶ Both likelihood and prior are taken into account – recall Bayes rule:

$$P(s_j | \vec{x}) = \frac{P(\vec{x} | s_j) P(s_j)}{P(\vec{x})}$$

- ▶ Can we do (classify) better?

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Bayes classification in practice

- ▶ Usually we are not given $P(s|\vec{x})$
 - ▶ It has to be estimated from already classified examples – training data
 - ▶ For discrete \vec{x} , training examples $(\vec{x}_1, s_1), (\vec{x}_2, s_2), \dots, (\vec{x}_I, s_I)$
 - ▶ so-called i.i.d (independent, identically distributed) multiset
 - ▶ every (\vec{x}_i, s) is drawn independently from $P(\vec{x}, s)$
 - ▶ Without knowing anything about the distribution, a non-parametric estimate:

$$P(s|\vec{x}) \approx \frac{\# \text{ examples where } \vec{x}_i = \vec{x} \text{ and } s_i = s}{\# \text{ examples where } \vec{x}_i = \vec{x}}$$

- ▶ Hard in practice:
 - ▶ To reliably estimate $P(s|\vec{x})$, the number of examples grows exponentially with the number of elements of \vec{x} .
 - ▶ e.g. with the number of pixels in images
 - ▶ curse of dimensionality
 - ▶ denominator often 0

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Naïve Bayes classification

- ▶ For efficient classification we must thus rely on additional assumptions.
- ▶ In the exceptional case of **statistical independence** between \vec{x} components for each class s it holds

$$P(\vec{x}|s) = P(x[1]|s) \cdot P(x[2]|s) \cdot \dots$$

- ▶ Use simple Bayes law and maximize:

$$P(s|\vec{x}) = \frac{P(\vec{x}|s)P(s)}{P(\vec{x})} = \frac{P(s)}{P(\vec{x})} P(x[1]|s) \cdot P(x[2]|s) \cdot \dots =$$

- ▶ No combinatorial curse in estimating $P(s)$ and $P(x[i]|s)$ separately for each i and s .
- ▶ No need to estimate $P(\vec{x})$. (Why?)
- ▶ $P(s)$ may be provided apriori.
- ▶ **naïve** = when used despite statistical dependence

Decision making under uncertainty

- ▶ An important feature of intelligent systems
 - ▶ make the best possible decision
 - ▶ in uncertain conditions.
- ▶ Example: Take a tram OR subway from *A* to *B*?
 - ▶ Tram: timetables imply a quicker route, but adherence uncertain.
 - ▶ Subway: longer route, but adherence almost certain.
- ▶ Example: where to route a letter with this ZIP?
 - ▶ 15700? 15706? 15200? 15206?
- ▶ What is the optimal decision ?
- ▶ Both examples fall into the same framework.

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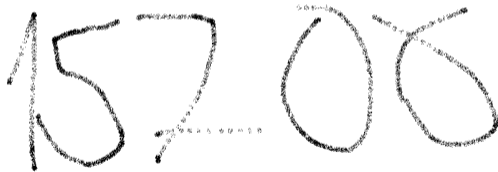
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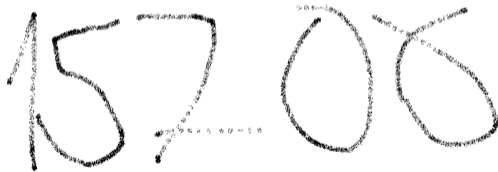
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A handwritten ZIP code '15700' is shown. The digits are somewhat blurry and the ink is dark. The '1' is a simple vertical stroke. The '5' has a curved top. The '7' has a horizontal top bar and a vertical stem. The '0' is a simple oval. The second '0' is also a simple oval.

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A handwritten ZIP code '15700' is shown. The digits are somewhat blurry and the '0's are written with a loop, making them difficult to read. The first digit is '1', the second is '5', the third is '7', and the last two are '0's.

- ▶ 15700? 15706? 15200? 15206?
- ▶ What is the **optimal decision** ?
- ▶ Both examples fall into the same framework.

Example: What to cook for a dinner [3]

- ▶ *Wife coming back from work. Husband: what to cook for dinner?*
- ▶ 3 dishes (decisions) in his repertoire:
 - ▶ *nothing ... don't bother cooking* \Rightarrow no work but makes wife upset
 - ▶ *pizza ... microwave a frozen pizza* \Rightarrow not much work but won't impress
 - ▶ *g.T.c. ... general Tso's chicken* \Rightarrow will make her day, but very laborious.
- ▶ Hassle incurred by the individual options depends wife's feeling
- ▶ For each of the 9 possible situation (3 possible decisions \times 3 possible states) the hassle is quantified by a loss function $l(d, s)$:

$l(s, d)$	$d = \textit{nothing}$	$d = \textit{pizza}$	$d = \textit{g.T.c.}$
$s = \textit{good}$	0	2	4
$s = \textit{average}$	5	3	5
$s = \textit{bad}$	10	9	6

Wife's state of mind is an uncertain state.

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Wife's state of mind is an **uncertain state**.

Example (cont'd), State uncertain, ...

- ▶ Husband's experiment. He tells her he accidentally overtaped their wedding video and observes her reaction
- ▶ Anticipates 4 possible reactions:
 - ▶ *mild* ... all right, we keep our memories.
 - ▶ *irritated* ... how many times do I have to tell you...
 - ▶ *upset* ... Why did I marry this guy?
 - ▶ *alarming* ... silence
- ▶ The reaction is a measurable attribute ("feature") of the mind state.
- ▶ From experience, the husband knows how individual reactions are probable in each state of mind; this is captured by the joint distribution $P(x, s)$.

$P(x, s)$	$x = mild$	$x = irritated$	$x = upset$	$x = alarming$
$s = good$	0.35	0.28	0.07	0.00
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Decision strategy

- ▶ **Decision strategy** : a rule selecting a decision for *any given value* of the measured attribute(s).
- ▶ i.e. function $d = \delta(x)$.
- ▶ Example of husband's possible strategies:

$\delta(x)$	$x = \text{mild}$	$x = \text{irritated}$	$x = \text{upset}$	$x = \text{alarming}$
$\delta_1(x) =$	<i>nothing</i>	<i>nothing</i>	<i>pizza</i>	<i>g.T.c.</i>
$\delta_2(x) =$	<i>nothing</i>	<i>pizza</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
$\delta_3(x) =$	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
$\delta_4(x) =$	<i>nothing</i>	<i>nothing</i>	<i>nothing</i>	<i>nothing</i>

- ▶ How many strategies?
- ▶ How to define which strategy is best? How to sort them by quality?
- ▶ Define the *risk* of a strategy as a mean (expected) loss value .

$$r(\delta) = \sum_x \sum_s l(s, \delta(x)) P(x, s)$$

Decision strategy

- ▶ **Decision strategy** : a rule selecting a decision for *any given value* of the measured attribute(s).
- ▶ i.e. function $d = \delta(x)$.
- ▶ Example of husband's possible strategies:

$\delta(x)$	$x = \text{mild}$	$x = \text{irritated}$	$x = \text{upset}$	$x = \text{alarming}$
$\delta_1(x) =$	<i>nothing</i>	<i>nothing</i>	<i>pizza</i>	<i>g.T.c.</i>
$\delta_2(x) =$	<i>nothing</i>	<i>pizza</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
$\delta_3(x) =$	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
$\delta_4(x) =$	<i>nothing</i>	<i>nothing</i>	<i>nothing</i>	<i>nothing</i>

- ▶ How many strategies?
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Calculating $r(\delta) = \sum_x \sum_s l(s, \delta(x))P(x, s)$

$l(s, d)$	$d = \textit{nothing}$	$d = \textit{pizza}$	$d = \textit{g.T.c.}$
$s = \textit{good}$	0	2	4
$s = \textit{average}$	5	3	5
$s = \textit{bad}$	10	9	6

$P(x, s)$	$x = \textit{mild}$	$x = \textit{irritated}$	$x = \textit{upset}$	$x = \textit{alarming}$
$s = \textit{good}$	0.35	0.28	0.07	0.00
$s = \textit{average}$	0.04	0.10	0.04	0.02
$s = \textit{bad}$	0.00	0.02	0.05	0.03

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$\delta_3(x) =$	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
\vdots	\vdots	\vdots	\vdots	\vdots

Do we need to evaluate all possible strategies? $P(x, s) = P(s|x)P(x)$

Calculating $r(\delta) = \sum_x \sum_s I(s, \delta(x))P(x, s)$

$I(s, d)$	$d = \textit{nothing}$	$d = \textit{pizza}$	$d = \textit{g.T.c.}$	
$s = \textit{good}$	0	2	4	
$s = \textit{average}$	5	3	5	
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$\delta_3(x) =$	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
\vdots	\vdots	\vdots	\vdots	\vdots

Do we need to evaluate all possible strategies? $P(x, s) = P(s|x)P(x)$

Calculating $r(\delta) = \sum_x \sum_s l(s, \delta(x))P(x, s)$

$l(s, d)$	$d = \textit{nothing}$	$d = \textit{pizza}$	$d = \textit{g.T.c.}$	
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$\delta_3(x) =$	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>	<i>g.T.c.</i>
\vdots	\vdots	\vdots	\vdots	\vdots

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\vdots	\vdots	\vdots	\vdots	\vdots

Do we need to evaluate all possible strategies? $P(x, s) = P(s|x)P(x)$

Bayes optimal strategy

- ▶ The **Bayes optimal strategy** : one minimizing mean risk.

$$\delta^* = \arg \min_{\delta} r(\delta)$$

- ▶ From $P(x, s) = P(s|x)P(x)$ (Bayes rule), we have

$$\begin{aligned} r(\delta) &= \sum_x \sum_s l(s, \delta(x)) P(x, s) = \sum_s \sum_x l(s, \delta(x)) P(s|x) P(x) \\ &= \sum_x P(x) \underbrace{\sum_s l(s, \delta(x)) P(s|x)}_{\text{Conditional risk}} \end{aligned}$$

- ▶ The optimal strategy is obtained by minimizing the conditional risk *separately* for each x :

$$\delta^*(x) = \arg \min_d \sum_s l(s, d) P(s|x)$$

Optimal strategy: $\delta^*(x) = \arg \min_d \sum_s l(s, d)P(s|x)$

$l(s, d)$	$d = \text{nothing}$	$d = \text{pizza}$	$d = \text{g.T.c.}$
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$\delta(x)$	$x = \text{mild}$	$x = \text{irritated}$	$x = \text{upset}$	$x = \text{alarming}$
$\delta^*(x) =$??	??	??	??

Statistical decision making: wrapping up

▶ Given:

- ▶ A set of possible **states** : \mathcal{S}
- ▶ A set of possible **decisions** : \mathcal{D}
- ▶ A **loss function** $l : \mathcal{D} \times \mathcal{S} \rightarrow \mathfrak{R}$
- ▶ The range \mathcal{X} of the **attribute**
- ▶ Distribution $P(x, s)$, $x \in \mathcal{X}, s \in \mathcal{S}$.

▶ Define:

- ▶ **Strategy** : function $\delta : \mathcal{X} \rightarrow \mathcal{D}$
- ▶ **Risk of strategy** $\delta : r(\delta) = \sum_x \sum_s l(s, \delta(x))P(x, s)$

▶ Bayes problem:

- ▶ Goal: find the optimal strategy $\delta^* = \arg \min_{\delta \in \Delta} r(\delta)$
- ▶ Solution: $\delta^*(x) = \arg \min_d \sum_s l(s, d)P(s|x)$

A special case - Bayesian *classification*

- ▶ Bayesian classification is a special case of statistical decision theory:
 - ▶ Attribute vector $\vec{x} = (x_1, x_2, \dots)$: pixels 1, 2, ...
 - ▶ **State set \mathcal{S} = decision set $\mathcal{D} = \{0, 1, \dots, 9\}$.**
 - ▶ **State = actual class, Decision = recognized class**
 - ▶ Loss function:

$$l(s, d) = \begin{cases} 0, & d = s \\ 1, & d \neq s \end{cases}$$

$$\delta^*(\vec{x}) = \arg \min_d \sum_s \underbrace{l(s, d)}_{0 \text{ if } d=s} P(s|\vec{x}) = \arg \min_d \sum_{s \neq d} P(s|\vec{x})$$

Obviously $\sum_s P(s|\vec{x}) = 1$, then:

$$P(d|\vec{x}) + \sum_{s \neq d} P(s|\vec{x}) = 1$$

Inserting into above:

$$\delta^*(\vec{x}) = \arg \min_d [1 - P(d|\vec{x})] = \arg \max_d P(d|\vec{x})$$

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References I

Further reading: Chapter 13 and 14 of [6]. Books [1] and [2] are classical textbooks in the field of pattern recognition and machine learning. An interesting insights into how people think and interact with probabilities are presented in [4] (in Czech as [5])

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