

# Sequential decisions under uncertainty

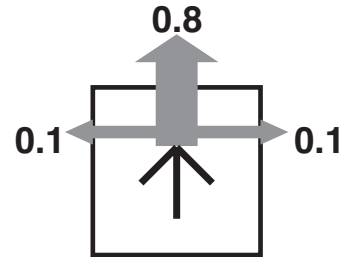
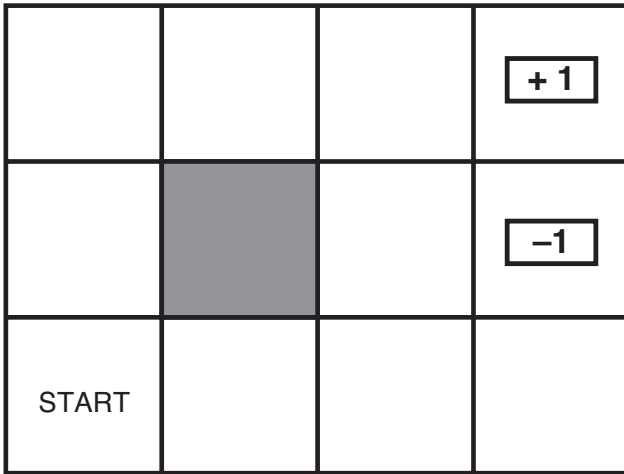
## Markov Decision Processes (MDP)

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Faculty of Electrical Engineering, Czech Technical University in Prague

April 1, 2020

# Unreliable actions in observable grid world



States  $s \in \mathcal{S}$ , actions  $a \in \mathcal{A}$

(Transition) Model  $T(s, a, s') \equiv p(s'|s, a)$  = probability that  $a$  in  $s$  leads to  $s'$

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## Notes

Beginning of semester – search – *deterministic* and (fully) *observable* environment

Now:

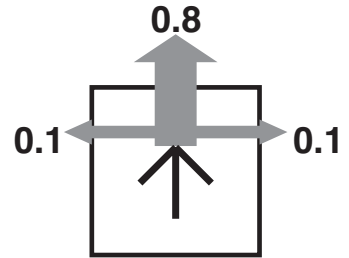
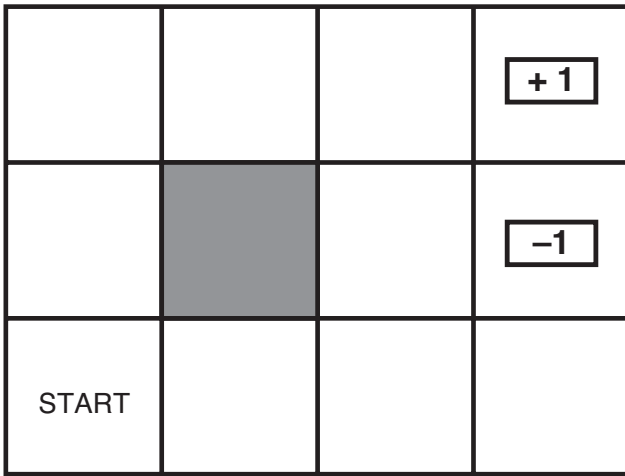
- Observable – we keep for now – agent knows where it is.
- Deterministic – We introduce “imperfect” agent that does not always obey the command – *stochastic action outcomes*.

There is a treasure (desired goal/end state) but there is also some danger (unwanted goal/end state).

The danger state: think about a mountainous area with safer but longer and shorter but more dangerous paths – a dangerous node may represent a chasm.

Notation note: calligraphic letters like  $\mathcal{S}, \mathcal{A}$  will denote the set(s) of all states/actions.

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# Unreliable actions



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## Notes

Actions: go over a glacier bridge or around?

# Plan? Policy

► In deterministic world: **Plan** – sequence of actions from **Start** to **Goal**.

- MDPs, we need a *policy*  $\pi : \mathcal{S} \rightarrow \mathcal{A}$ .
- An action for each possible state. *Why each?*
- What is the *best* policy?

0.00	0.00	0.00	0.00
0.00		0.00	0.00
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Unlike in deterministic environment (also search problems), with stochastic action outcomes, we can end up in any state. Thus, in any state, the robot/agent has to know what to do.

What is the best policy? We will come to that in a minute, ...

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# Rewards

-0.04	-0.04	-0.04	1.00
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**Reward** : Robot/Agent takes an action  $a$  and it is **immediately** rewarded.

**Reward function**  $r(s)$  (or  $r(s, a)$ ,  $r(s, a, s')$ )

$$= \begin{cases} -0.04 & \text{(small penalty) for nonterminal states} \\ \pm 1 & \text{for terminal states} \end{cases}$$

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## Notes

What do the rewards express? Reward to an agent to be/dwell in that state? Obviously we want the robot to go to the goal and do not stay too long in the maze. The negative reward of  $-0.04$  gives the agent an incentive to reach the goal state quickly, so our environment is a *stochastic generalization of the search problems*.

**Thinking about Reward:** Robot/Agent takes an action  $a$  and it is immediately rewarded for this. The reward may depend on

- current state  $s$ ,
- the action taken  $a$
- the next state  $s'$  - result of the action.

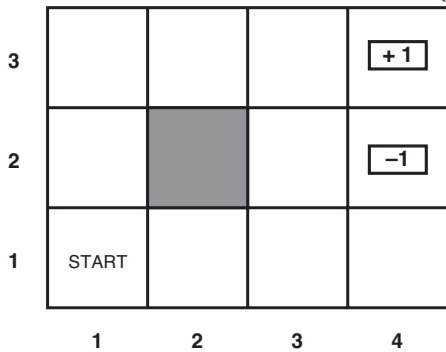
**Rewards for terminal states** can be understood in a way: there is only one action:  $a = \text{exit}$ . We will come to this soon.

The **reward function** is a property of (is related to) the problem.

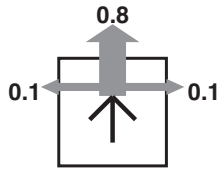
**Notation remark:** lowercase letters will be used for functions like  $p, r, v, f, \dots$



# Markov Decision Processes (MDPs)



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(b)

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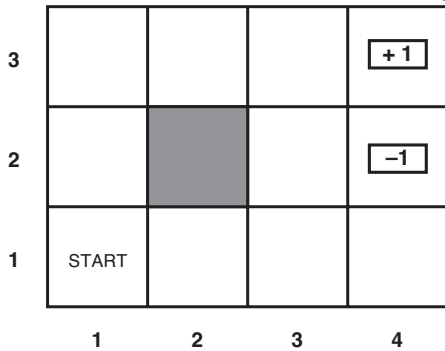
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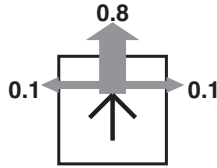
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Actions: UP, LEFT, RIGHT or N,W,E

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# Markovian property

- ▶ Given the present state, the future and the past are independent.
- ▶ MDP: Markov means action depends only on the current state.
- ▶ In search: successor function (transition model) depends on the current state only.

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## Notes

- Properties are somewhat obvious, reasonable.
- However, you may break it if wrongly formalized.
- Always check, before you go (do the calculations)

# Optimal(?) policies

On-line demos.

▶  $r(s) = \{-0.04, 1, -1\}$

▶  $r(s) = \{-2, 1, -1\}$

▶  $r(s) = \{-0.01, 1, -1\}$

How to measure quality of a policy?

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We run `mdp_agents.py` changing reward functions.

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b

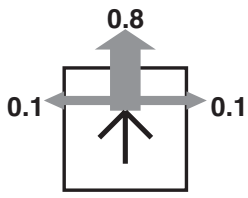
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c

- A: A-a, B-b, C-c
- B: A-b, B-a, C-c
- C: A-b, B-c, C-a
- D: A-c, B-a, C-b



Notes

- $r(s) = \{-0.04, 1, -1\}$
- $r(s) = \{-2, 1, -1\}$  - environment very hostile - heading for nearest exit even if it's with negative reward
- $r(s) = \{-0.01, 1, -1\}$  - environment very mildly unpleasant - conservative policy (banging head against the wall to avoid negative terminal state at all cost)
- C: option the right assignment



# Utilities of sequences

- ▶ State reward at time/step  $t$ ,  $R_t$ .
- ▶ State at time  $t$ ,  $S_t$ . State sequence  $[S_0, S_1, S_2, \dots, ]$

Typically, consider stationary preferences on reward sequences:

$$[R, R_1, R_2, R_3, \dots] \succ [R, R'_1, R'_2, R'_3, \dots] \Leftrightarrow [R_1, R_2, R_3, \dots] \succ [R'_1, R'_2, R'_3, \dots]$$

If stationary preferences :

Utility ( $h$ -history)

$$U_h([S_0, S_1, S_2, \dots, ]) = R_1 + R_2 + R_3 + \dots$$

If the horizon is finite - limited number of steps - preferences are nonstationary (depends on how many steps left).

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## Notes

We consider discrete time  $t$ .  $S_t, R_t$  notation emphasises the time sequence - not a sequence of particular states. The reward is for an action (transition)

Finite vs non-finite horizon. Think about the simple  $3 \times 4$  grid from the last slides and having limited budget of 3,4,5 steps.

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# Returns and Episodes

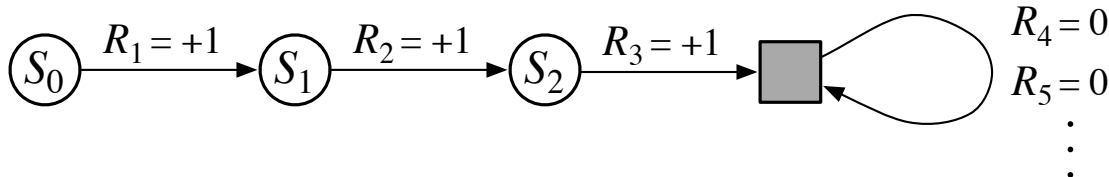
- ▶ Executing policy - sequence of states and **rewards**.
- ▶ **Episode** starts at  $t$ , ends at  $T$  (ending in a terminal state).
- ▶ **Return** (Utility) of the episode (policy execution)

$$G_t = R_{t+1} + R_{t+2} + R_{t+3} + \dots + R_T$$

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## Notes



Solid square – *absorbing state* – end of an episode.

(transitions only to itself and generates only rewards of zero)

Allows to unify two formulations of return ( $G_t$ ) as a finite and infinite sum of rewards.

# Comparing policies; Finite vs infinite horizon

Problem: Infinite lifetime  $\Rightarrow$  additive utilities are infinite.

- ▶ Finite horizon: termination at a fixed time  $\Rightarrow$  nonstationary policy,  $\pi(s)$  depends on the time left.
- ▶ Discounted return,  $\gamma < 1, R_t \leq R_{\max}$

$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{\max}}{1-\gamma}$$

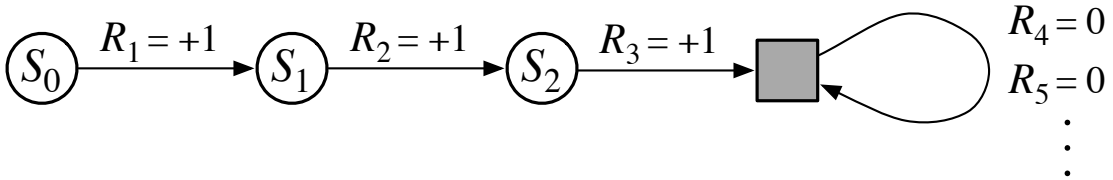
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Returns are successive steps related to each other

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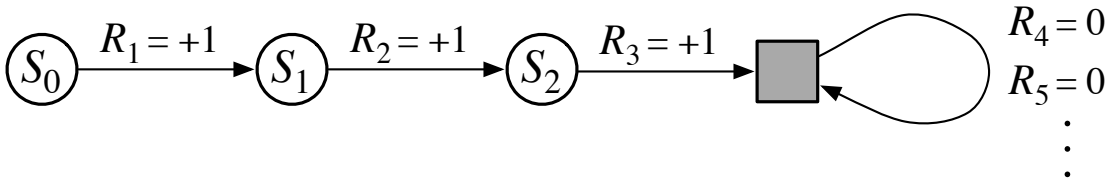
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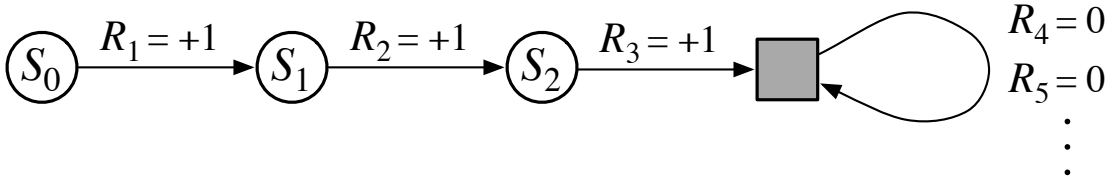
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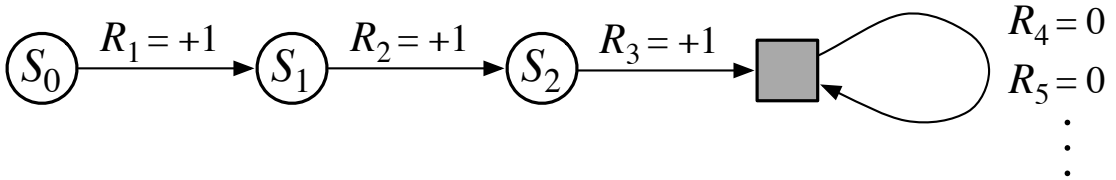
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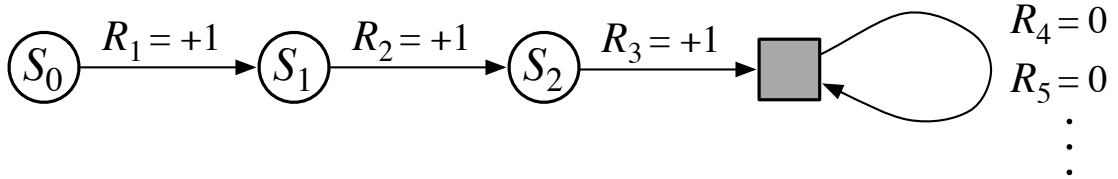
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$$G_t = R_{t+1} + \gamma R_{t+2} + \gamma^2 R_{t+3} + \dots = \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \leq \frac{R_{\max}}{1-\gamma}$$

- ▶ Absorbing (terminal) state.

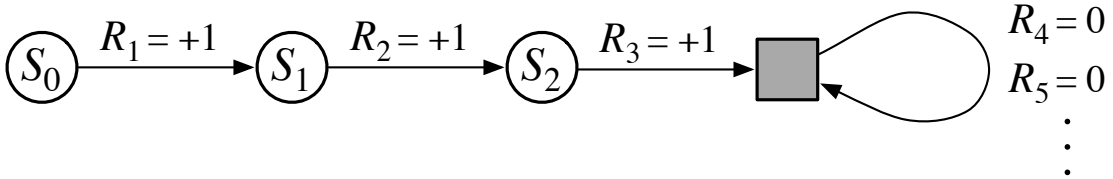
Returns are successive steps related to each other

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## Notes

Discounting is quite natural choice. Think about your preferences/rewards. Go to pub with friends tonight, studying (for the far future reward of getting A in the course)?



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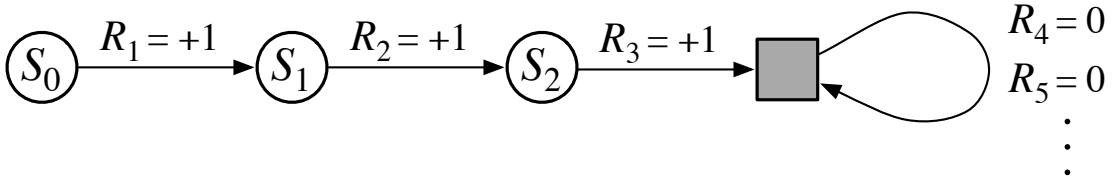
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# MDPs recap

## Markov decision processes (MDPs):

- ▶ Set of states  $\mathcal{S}$
- ▶ Set of actions  $\mathcal{A}$
- ▶ Transitions  $p(s'|s, a)$  or  $T(s, a, s')$
- ▶ Reward function  $r(s, a, s')$ ; and discount  $\gamma$

## MDP quantities:

- ▶ (deterministic) Policy  $\pi(s)$  – choice of action for each state
- ▶ Return (Utility) of an episode (sequence) – sum of (discounted) rewards.

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# Value functions

- ▶ Executing policy  $\pi \rightarrow$  sequence of states (and rewards).
- ▶ Utility of a state sequence.
  - ▶ But actions are unreliable - environment is stochastic.
  - ▶ Expected return of a policy  $\pi$ .

Starting at time  $t$ , i.e.  $S_t$ ,

$$U^\pi(S_t) = E^\pi \left[ \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \right]$$

## Value function

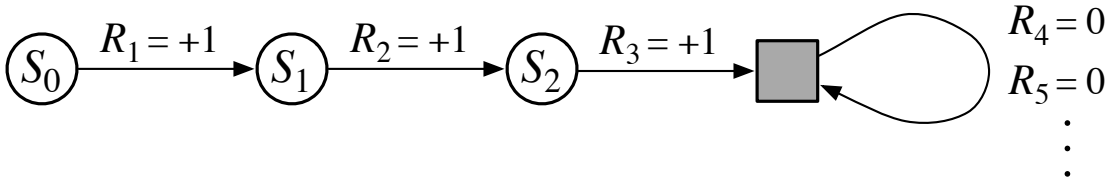
$$v^\pi(s) = E^\pi [G_t | S_t = s] = E^\pi \left[ \sum_{k=0}^{\infty} \gamma^k R_{t+k+1} \mid S_t = s \right]$$

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Contrast *return* of a particular episode vs. *value* – expected utility of a state sequence in general – *expected return*  
Expected value can be also computed by running (executing) the policy many times and then computing average  
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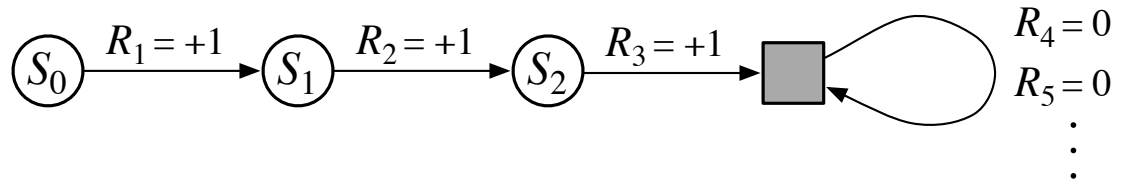
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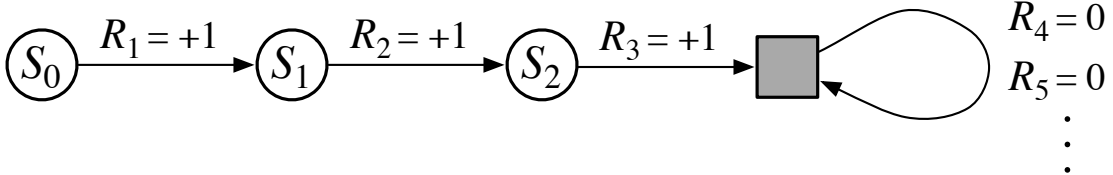
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# Optimal policy $\pi^*$ , and optimal value $v^*(s)$

$v^*(s)$  = expected (discounted) sum of rewards (until termination) assuming *optimal* actions.

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## Notes

Showing cases for

- $r(s) = \{-0.04, 1, -1\}$ ,  $\gamma = 0.999999$ ,  $\epsilon = 0.03$
- $r(s) = \{-0.01, 1, -1\}$ ,  $\gamma = 0.999999$ ,  $\epsilon = 0.03$

What is the difference in the optimal policy? Try to explain why it happened.

We still do not know *how* to compute the optimality, ... right?

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	0	1	2	3	
0	0.88	0.92	0.96	1.00	0
1	0.84		0.92	-1.00	1
2	0.80	0.84	0.88	0.84	2
	0	1	2	3	

	0	1	2	3	
0	>	>	>	None	0
1	$\wedge$		$\wedge$	None	1
2	$\wedge$	>	$\wedge$	<	2
	0	1	2	3	

$$r(s) = \{-0.04, 1, -1\}, \gamma = 0.999999, \epsilon = 0.03, \text{ Robot } \textit{deterministic}$$

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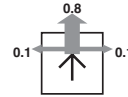
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2	0.92	0.91	0.90	0.80
	0	1	2	3

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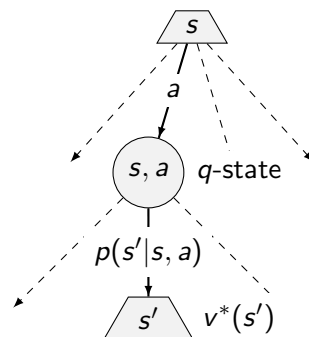
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The value of a  $q$ -state  $(s, a)$ :

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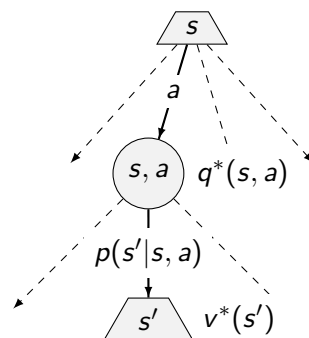
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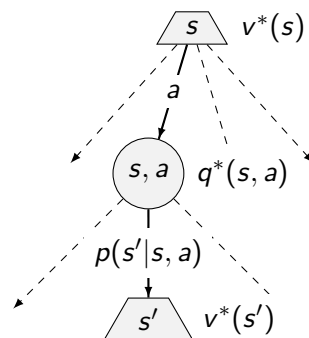
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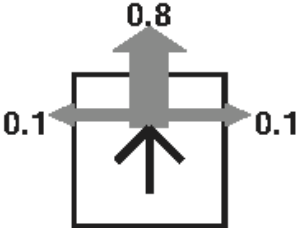
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# Bellman (optimality) equation

$$v^*(s) = \max_{a \in A(s)} \sum_{s'} p(s'|a, s) [r(s, a, s') + \gamma v^*(s')]$$

0			<b>+1</b>	
1			<b>-1</b>	
2	START			
	0	1	2	3

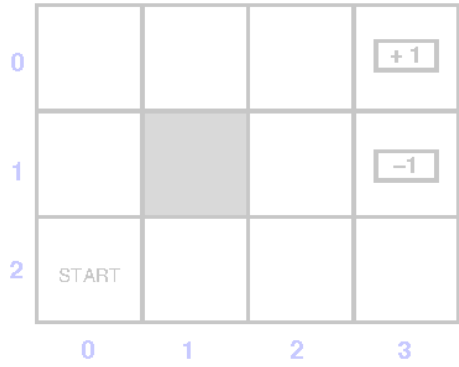


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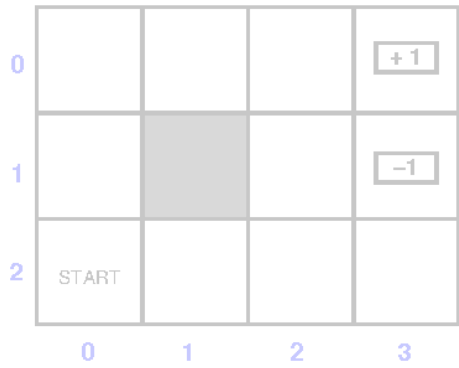
$v$  computation on the table - one row for each action. We got  $n$  equations for  $n$  unknown -  $n$  states. But max is a non-linear operator!



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---

**Notes**

Space for on-line drawing ...

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$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s, a) V_k(s')$$

- ▶ Repeat until convergence

The idea: Bellman update makes local consistency with the Bellmann equation. Everywhere locally consistent  $\Rightarrow$  globally optimal.

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## Notes

What is the complexity of each iteration?  $O(S^2A)$

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## Value iteration - Complexity of one estimation sweep

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s, a) V_k(s')$$

- ▶  $O(AS)$
- ▶  $O(S^2)$
- ▶  $O(AS^2)$
- ▶  $O(A^2S^2)$

---

### Notes

The sweep goes through all the states  $S$ , from each state we need evaluate all actions  $A$  and each action may, in principle, land in any other state  $S \rightarrow O(AS^2)$

# Convergence

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V_k(s')$$

$$\gamma < 1$$

$$-R_{\max} \leq R(s) \leq R_{\max}$$

Max norm:

$$\|V\| = \max_s |V(s)|$$

$$U([s_0, s_1, s_2, \dots, s_\infty]) = \sum_{t=0}^{\infty} \gamma^t R(s_t) \leq \frac{R_{\max}}{1-\gamma}$$

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## Notes

Keep in mind that  $V$  is a vector of all state values. If the problem has 12 states ( $3 \times 4$  grid) then it is a 12-dim vector.



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## Convergence cont'd

$V_{k+1} \leftarrow BV_k \dots B$  as the Bellman update  $V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} p(s'|s, a) V_k(s')$

$$\|BV_k - BV'_k\| \leq \gamma \|V_k - V'_k\|$$

$$\|BV_k - V_{\text{true}}\| \leq \gamma \|V_k - V_{\text{true}}\|$$

Rewards are bounded, at the beginning then Value error is

$$\|V_0 - V_{\text{true}}\| \leq \frac{2R_{\text{max}}}{1-\gamma}$$

We run  $N$  iterations and reduce the error by factor  $\gamma$  in each and want to stop the error is below  $\epsilon$ :

$$\gamma^N 2R_{\text{max}} / (1-\gamma) \leq \epsilon \text{ Taking logs, we find: } N \geq \frac{\log(2R_{\text{max}}/\epsilon(1-\gamma))}{\log(1/\gamma)}$$

To stop the iteration we want to find a bound relating the error to the size of *one* Bellman update for any given iteration.

We stop if

$$\|V_{k+1} - V_k\| \leq \frac{\epsilon(1-\gamma)}{\gamma}$$

then also:  $\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$  Proof on the next slide

---

### Notes

Try to prove that:

$$\|\max f(a) - \max g(a)\| \leq \max \|f(a) - g(a)\|$$

## Convergence cont'd

$\|V_{k+1} - V_{\text{true}}\| \leq \epsilon$  is the same as  $\|V_{k+1} - V_{\infty}\| \leq \epsilon$

Assume  $\|V_{k+1} - V_k\| = \text{err}$

In each of the following iteration steps we reduce the error by the factor  $\gamma$  (because  $\|BV_k - V_{\text{true}}\| \leq \gamma\|V_k - V_{\text{true}}\|$ ). Till  $\infty$ , the total sum of reduced errors is:

$$\text{total} = \gamma \text{err} + \gamma^2 \text{err} + \gamma^3 \text{err} + \gamma^4 \text{err} + \dots = \frac{\gamma \text{err}}{(1 - \gamma)}$$

We want to have  $\text{total} < \epsilon$ .

$$\frac{\gamma \text{err}}{(1 - \gamma)} < \epsilon$$

From it follows that

$$\text{err} < \frac{\epsilon(1 - \gamma)}{\gamma}$$

Hence we can stop if  $\|V_{k+1} - V_k\| < \epsilon(1 - \gamma)/\gamma$

# Value iteration demo

$$V_{k+1}(s) \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V_k(s')$$

	0	1	2	3	
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## Notes

Run `mdp_agents.py` and try to compute next state value in advance. Remind the  $R(s) = -0.04$  and  $\gamma = 1$  in order to simplify computation. Then discuss the course of the Values.

# Value iteration algorithm

**function** VALUE-ITERATION( $\text{env}, \epsilon$ ) **returns:** state values  $V$

**input:** env - MDP problem,  $\epsilon$

$V' \leftarrow 0$  in all states

**repeat**

$V \leftarrow V'$

$\delta \leftarrow 0$

**for each state  $s$  in  $S$  do**

$V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V(s')$

**if**  $|V'[s] - V[s]| > \delta$  **then**  $\delta \leftarrow |V'[s] - V[s]|$

**end for**

**until**  $\delta < \epsilon(1 - \gamma)/\gamma$

**end function**

▷ iterate values until convergence

▷ keep the last known values

▷ reset the max difference

# Value iteration algorithm

**function** VALUE-ITERATION( $\text{env}, \epsilon$ ) **returns:** state values  $V$

**input:**  $\text{env}$  - MDP problem,  $\epsilon$

$V' \leftarrow 0$  in all states

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# Sync vs. async Value iteration

**function** VALUE-ITERATION( $\text{env}, \epsilon$ ) **returns:** state values  $V$

**input:**  $\text{env}$  - MDP problem,  $\epsilon$

$V' \leftarrow 0$  in all states

**repeat**

$V \leftarrow V'$

$\delta \leftarrow 0$

**for each** state  $s$  **in**  $S$  **do**

$V'[s] \leftarrow R(s) + \gamma \max_{a \in A(s)} \sum_{s'} P(s'|s, a) V(s')$

**if**  $|V'[s] - V[s]| > \delta$  **then**  $\delta \leftarrow |V'[s] - V[s]|$

**end for**

**until**  $\delta < \epsilon(1 - \gamma)/\gamma$

**end function**

▷ iterate values until convergence

▷ keep the last known values

▷ reset the max difference

## References

Some figures from [1] (chapter 17) but notation slightly changed in order to adapt notation from [2] (chapters 3, 4) which will help us in the Reinforcement Learning part of the course. Note that the book [2] is available on-line.

[1] Stuart Russell and Peter Norvig.

*Artificial Intelligence: A Modern Approach.*

Prentice Hall, 3rd edition, 2010.

<http://aima.cs.berkeley.edu/>.

[2] Richard S. Sutton and Andrew G. Barto.

*Reinforcement Learning; an Introduction.*

MIT Press, 2nd edition, 2018.

<http://www.incompleteideas.net/book/the-book-2nd.html>.