

3D Computer Vision

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Open Informatics Master's Course

Perspective Camera

- 2.1 Basic Entities: Points, Lines
- 2.2 Homography: Mapping Acting on Points and Lines
- 2.3 Canonical Perspective Camera
- 2.4 Changing the Outer and Inner Reference Frames
- 2.5 Projection Matrix Decomposition
- 2.6 Anatomy of Linear Perspective Camera
- 2.7 Vanishing Points and Lines

covered by

[H&Z] Secs: 2.1, 2.2, 3.1, 6.1, 6.2, 8.6, 2.5, Example: 2.19

► Basic Geometric Entities, their Representation, and Notation

- entities have names and representations
- names and their components:

entity	in 2-space	in 3-space
point	$m = (u, v)$	$X = (x, y, z)$
line	n	O
plane		π, φ

- associated vector representations

$$\mathbf{m} = \begin{bmatrix} u \\ v \end{bmatrix} = [u, v]^{\top}, \quad \mathbf{X} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad \mathbf{n}$$

will also be written in an 'in-line' form as $\mathbf{m} = (u, v)$, $\mathbf{X} = (x, y, z)$, etc.

- vectors are always meant to be columns $\mathbf{x} \in \mathbb{R}^{n,1}$
- associated homogeneous representations

$$\underline{\mathbf{m}} = [m_1, m_2, m_3]^{\top}, \quad \underline{\mathbf{X}} = [x_1, x_2, x_3, x_4]^{\top}, \quad \underline{\mathbf{n}}$$

'in-line' forms: $\underline{\mathbf{m}} = (m_1, m_2, m_3)$, $\underline{\mathbf{X}} = (x_1, x_2, x_3, x_4)$, etc.

- matrices are $\mathbf{Q} \in \mathbb{R}^{m,n}$, linear map of a $\mathbb{R}^{n,1}$ vector is $\mathbf{y} = \mathbf{Q}\mathbf{x}$
- j -th element of vector \mathbf{m}_i is $(\mathbf{m}_i)_j$; element i, j of matrix \mathbf{P} is \mathbf{P}_{ij}

► Image Line (in 2D)

a finite line in the 2D (u, v) plane

$$a u + b v + c = 0$$

corresponds to a (homogeneous) vector

$$\underline{\mathbf{n}} \simeq (a, b, c) \in \mathbb{R}^3$$

and there is an equivalence class for $\lambda \in \mathbb{R}, \lambda \neq 0$ $(\lambda a, \lambda b, \lambda c) \simeq (a, b, c)$

'Finite' lines

- standard representative for finite $\underline{\mathbf{n}} = (n_1, n_2, n_3)$ is $\lambda \underline{\mathbf{n}}$, where $\lambda = \frac{1}{\sqrt{n_1^2 + n_2^2}}$
assuming $n_1^2 + n_2^2 \neq 0$; $\mathbf{1}$ is the unit, usually $\mathbf{1} = 1$

'Infinite' line

- we augment the set of lines for a special entity called the **line at infinity** (ideal line)

$$\underline{\mathbf{n}}_\infty \simeq (0, 0, 1) \quad (\text{standard representative})$$

- the set of equivalence classes of vectors in $\mathbb{R}^3 \setminus (0, 0, 0)$ forms the projective space \mathbb{P}^2
a set of rays $\rightarrow 21$
- line at infinity is a proper member of \mathbb{P}^2
- I may sometimes wrongly use $=$ instead of \simeq , if you are in doubt, ask me

► Image Point

Finite point $\mathbf{m} = (u, v)$ is incident on a finite line $\mathbf{n} = (a, b, c)$ iff iff = works either way!

$$a u + b v + c = 0$$

can be rewritten as (with scalar product): $(u, v, \mathbf{1}) \cdot (a, b, c) = \mathbf{m}^\top \mathbf{n} = 0$

'Finite' points

- a finite point is also represented by a homogeneous vector $\mathbf{m} \simeq (u, v, \mathbf{1})$
- the equivalence class for $\lambda \in \mathbb{R}, \lambda \neq 0$ is $(m_1, m_2, m_3) = \lambda \mathbf{m} \simeq \mathbf{m}$
- the standard representative for finite point \mathbf{m} is $\lambda \mathbf{m}$, where $\lambda = \frac{1}{m_3}$ assuming $m_3 \neq 0$
- when $\mathbf{1} = 1$ then units are pixels and $\lambda \mathbf{m} = (u, v, 1)$
- when $\mathbf{1} = f$ then all elements have a similar magnitude, $f \sim$ image diagonal

use $\mathbf{1} = 1$ unless you know what you are doing;

all entities participating in a formula must be expressed in the same units

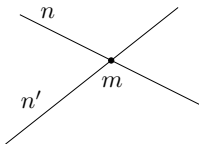
'Infinite' points

- we augment for **points at infinity** (ideal points) $\mathbf{m}_\infty \simeq (m_1, m_2, 0)$
proper members of \mathbb{P}^2
- all such points lie on the line at infinity (ideal line) $\mathbf{n}_\infty \simeq (0, 0, 1)$, i.e. $\mathbf{m}_\infty^\top \mathbf{n}_\infty = 0$

► Line Intersection and Point Join

The point of **intersection** m of image lines n and n' , $n \neq n'$ is

$$\underline{m} \simeq \underline{n} \times \underline{n}'$$

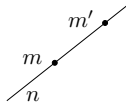


proof: If $\underline{m} = \underline{n} \times \underline{n}'$ is the intersection point, it must be incident on both lines. Indeed, using known equivalences from vector algebra

$$\underline{n}^\top \underbrace{(\underline{n} \times \underline{n}')}_{\underline{m}} \equiv \underline{n}'^\top \underbrace{(\underline{n} \times \underline{n}')}_{\underline{m}} \equiv 0$$

The **join** n of two image points m and m' , $m \neq m'$ is

$$\underline{n} \simeq \underline{m} \times \underline{m}'$$



Parallel lines intersect (somewhere) on the line at infinity $\underline{n}_\infty \simeq (0, 0, 1)$

$$a u + b v + c = 0,$$

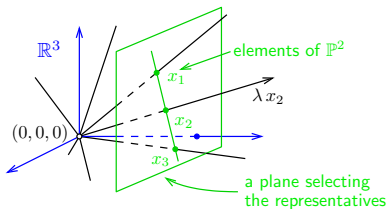
$$a u + b v + d = 0,$$

$$d \neq c$$

$$(a, b, c) \times (a, b, d) \simeq (b, -a, 0)$$

- all such intersections lie on \underline{n}_∞
- line at infinity represents a set of directions in the plane
- Matlab: `m = cross(n, n_prime);`

► Homography in \mathbb{P}^2



Projective plane \mathbb{P}^2 : Vector space of dimension 3 excluding the zero vector, $\mathbb{R}^3 \setminus (0, 0, 0)$, factorized to linear equivalence classes ('rays'), $\underline{x} \simeq \lambda \underline{x}$, $\lambda \neq 0$ including 'points at infinity'

Homography in \mathbb{P}^2 : Non-singular linear mapping in \mathbb{P}^2

$$\underline{x}' \simeq \mathbf{H} \underline{x}, \quad \mathbf{H} \in \mathbb{R}^{3,3} \text{ non-singular}$$

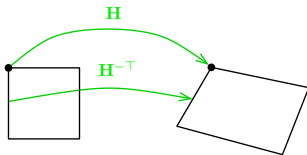
an analogic definition for \mathbb{P}^3



Defining properties

- collinear image points are mapped to collinear image points
lines of points are mapped to lines of points
- concurrent image lines are mapped to concurrent image lines
concurrent = intersecting at a point
- and point-line incidence is preserved
e.g. line intersection points mapped to line intersection points
- \mathbf{H} is a 3×3 non-singular matrix, $\lambda \mathbf{H} \simeq \mathbf{H}$ equivalence class, 8 degrees of freedom
- homogeneous matrix representant: $\det \mathbf{H} = 1$
- what we call homography here is often called 'projective collineation' in mathematics

► Mapping 2D Points and Lines by Homography



$$\underline{\mathbf{m}}' \simeq \mathbf{H} \underline{\mathbf{m}} \quad \text{image point}$$

$$\underline{\mathbf{n}}' \simeq \mathbf{H}^{-\top} \underline{\mathbf{n}} \quad \text{image line}$$

$$\mathbf{H}^{-\top} = (\mathbf{H}^{-1})^{\top} = (\mathbf{H}^{\top})^{-1}$$

- incidence is preserved: $(\underline{\mathbf{m}}')^{\top} \underline{\mathbf{n}}' \simeq \underline{\mathbf{m}}^{\top} \mathbf{H}^{\top} \mathbf{H}^{-\top} \underline{\mathbf{n}} = \underline{\mathbf{m}}^{\top} \underline{\mathbf{n}} = 0$

Mapping a finite 2D point $\mathbf{m} = (u, v)$ to $\underline{\mathbf{m}} = (u', v')$

- extend the Cartesian (pixel) coordinates to homogeneous coordinates, $\underline{\mathbf{m}} = (u, v, 1)$
- map by homography, $\underline{\mathbf{m}}' = \mathbf{H} \underline{\mathbf{m}}$
- if $m'_3 \neq 0$ convert the result $\underline{\mathbf{m}}' = (m'_1, m'_2, m'_3)$ back to Cartesian coordinates (pixels),

$$u' = \frac{m'_1}{m'_3} \mathbf{1}, \quad v' = \frac{m'_2}{m'_3} \mathbf{1}$$

- note that, typically, $m'_3 \neq 1$
- an infinite point $(u, v, 0)$ maps the same way

$m'_3 = 1$ when \mathbf{H} is affine

Some Homographic Tasters

Rectification of camera rotation: →59 (geometry), →126 (homography estimation)



$$\mathbf{H} \simeq \mathbf{K}\mathbf{R}^T\mathbf{K}^{-1}$$

maps from image plane to facade plane

Homographic Mouse for Visual Odometry: [Mallis 2007]



illustrations courtesy of AMSL Racing Team, Meiji University and LIBVISO: Library for VISual Odometry

$$\mathbf{H} \simeq \mathbf{K} \left(\mathbf{R} - \frac{\mathbf{t}\mathbf{n}^T}{d} \right) \mathbf{K}^{-1} \quad [\text{H\&Z, p. 327}]$$

Thank You