Optical ray: Spatial line that projects to a single image point.

1. consider the following line

   $\mathbf{d}$ unit line direction vector, $\|\mathbf{d}\| = 1$, $\lambda \in \mathbb{R}$, Cartesian representation

$$\mathbf{X}(\lambda) = \mathbf{C} + \lambda\,\mathbf{d}$$

2. the projection of the (finite) point $X(\lambda)$ is

$$\underline{\mathbf{m}} \simeq \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{X}(\lambda) \\ 1 \end{bmatrix} = \mathbf{Q}(\mathbf{C} + \lambda\mathbf{d}) + \mathbf{q} = \lambda\,\mathbf{Q}\,\mathbf{d} =$$

$$= \lambda \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{d} \\ 0 \end{bmatrix}$$

   ...which is also the image of a point at infinity in $\mathbb{P}^3$

- optical ray line corresponding to image point m is the set

$$\mathbf{X}(\lambda) = \mathbf{C} + \mu\,\mathbf{Q}^{-1}\underline{\mathbf{m}}, \qquad \mu \in \mathbb{R}$$

- optical ray direction may be represented by a point at infinity $(\mathbf{d}, 0)$ in $\mathbb{P}^3$
- optical ray is expressed in world coordinate frame
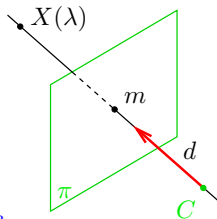
Optical axis: Optical ray that is perpendicular to image plane π

1. points on a line parallel to π project to line at infinity in π:

$$\begin{bmatrix} u \\ v \\ 0 \end{bmatrix} \simeq \mathbf{P}\underline{\mathbf{X}} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} \begin{bmatrix} \mathbf{X} \\ 1 \end{bmatrix}$$

2. therefore the set of points X is parallel to π iff

$$\mathbf{q}_3^\top \mathbf{X} + q_{34} = 0$$

3. this is a plane with $\pm\mathbf{q}_3$ as the normal vector
4. optical axis direction: substitution $\mathbf{P} \mapsto \lambda\mathbf{P}$ must not change the direction
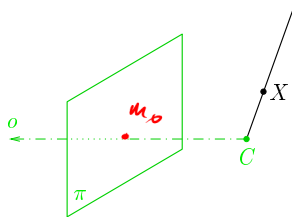5. we select (assuming $\det(\mathbf{R}) > 0$)

$$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$$

$$\frac{\mathbf{o}}{\|\mathbf{o}\|}$$

if $\mathbf{P} \mapsto \lambda\mathbf{P}$ then $\det(\mathbf{Q}) \mapsto \lambda^3 \det(\mathbf{Q})$ and $\mathbf{q}_3 \mapsto \lambda\,\mathbf{q}_3$ \hfill [H&Z, p. 161]

- the axis is expressed in world coordinate frame

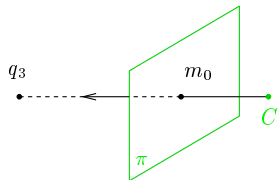Principal point: The intersection of image plane and the optical axis

1. as we saw, $\mathbf{q}_3$ is the directional vector of optical axis
2. we take point at infinity on the optical axis that must project to the principal point m_0

3. then

$$\underline{\mathbf{m}}_0 \simeq \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} \begin{bmatrix} \mathbf{q}_3 \\ 0 \end{bmatrix} = \mathbf{Q}\,\mathbf{q}_3$$
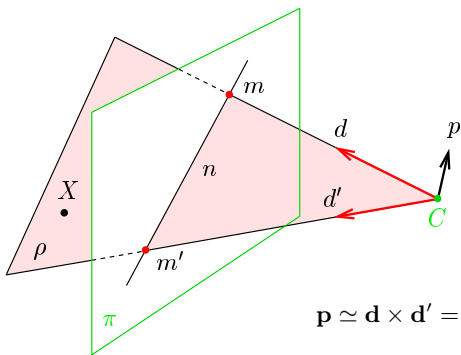
principal point:　$\boxed{\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3}$

- principal point is also the center of radial distortion

A spatial plane with normal p containing the projection center C and a given image line n.

optical ray given by m $\mathbf{d} \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}$

optical ray given by m' $\mathbf{d}' \simeq \mathbf{Q}^{-1}\underline{\mathbf{m}}'$

$$P = [Q \;\; q]$$

$$\mathbf{p} \simeq \mathbf{d} \times \mathbf{d}' = (\mathbf{Q}^{-1}\underline{\mathbf{m}}) \times (\mathbf{Q}^{-1}\underline{\mathbf{m}}') \cong \mathbf{Q}^{\top}(\underline{\mathbf{m}} \times \underline{\mathbf{m}}') = \mathbf{Q}^{\top}\underline{\mathbf{n}}$$

• note the way $\mathbf{Q}$ factors out!

hence, $0 = \mathbf{p}^{\top}(\mathbf{X} - \mathbf{C}) = \underline{\mathbf{n}}^{\top}\underbrace{\mathbf{Q}(\mathbf{X} - \mathbf{C})}_{\to 30} \cong \underline{\mathbf{n}}^{\top}\mathbf{P}\underline{\mathbf{X}} = (\mathbf{P}^{\top}\underline{\mathbf{n}})^{\top}\underline{\mathbf{X}}$ for every X in plane ρ

optical plane is given by n: $\boxed{\underline{\rho} \simeq \mathbf{P}^{\top}\underline{\mathbf{n}}}$ $\rho_1\, x + \rho_2\, y + \rho_3\, z + \rho_4 = 0$

Cross-Check: Optical Ray as Optical Plane Intersection
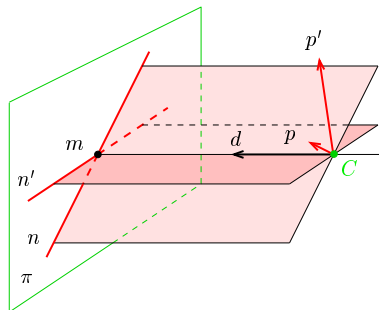
optical plane normal given by n $\mathbf{p} = \mathbf{Q}^{\top}\underline{\mathbf{n}}$

optical plane normal given by n' $\mathbf{p}' = \mathbf{Q}^{\top}\underline{\mathbf{n}}'$

$$\mathbf{d} = \mathbf{p} \times \mathbf{p}' = (\mathbf{Q}^{\top}\underline{\mathbf{n}}) \times (\mathbf{Q}^{\top}\underline{\mathbf{n}}') = \mathbf{Q}^{-1}(\underline{\mathbf{n}} \times \underline{\mathbf{n}}') = \mathbf{Q}^{-1}\underline{\mathbf{m}}$$

General (finite) camera

$$\mathbf{P} = \begin{bmatrix} \mathbf{Q} & \mathbf{q} \end{bmatrix} = \begin{bmatrix} \mathbf{q}_1^\top & q_{14} \\ \mathbf{q}_2^\top & q_{24} \\ \mathbf{q}_3^\top & q_{34} \end{bmatrix} = \mathbf{K} \begin{bmatrix} \mathbf{R} & \mathbf{t} \end{bmatrix} = \mathbf{K}\,\mathbf{R} \begin{bmatrix} \mathbf{I} & -\mathbf{C} \end{bmatrix}$$

$\underline{\mathbf{C}} \simeq \mathrm{rnull}(\mathbf{P}), \quad \mathbf{C} = -\mathbf{Q}^{-1}\mathbf{q}$ — projection center (world coords.) →34

$\mathbf{d} = \mathbf{Q}^{-1}\,\underline{\mathbf{m}}$ — optical ray direction (world coords.) →35

$\mathbf{o} = \det(\mathbf{Q})\,\mathbf{q}_3$ — outward optical axis (world coords.) →36

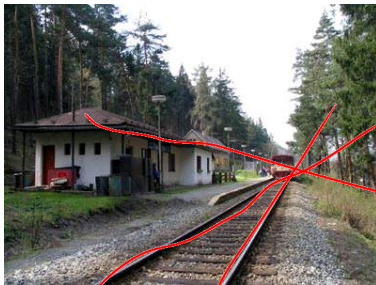$\underline{\mathbf{m}}_0 \simeq \mathbf{Q}\,\mathbf{q}_3$ — principal point (in image plane) →37

$\underline{\boldsymbol{\rho}} = \mathbf{P}^\top\,\underline{\mathbf{n}}$ — optical plane (world coords.) →38

$\mathbf{K} = \begin{bmatrix} f & -f\cot\theta & u_0 \\ 0 & f/(a\sin\theta) & v_0 \\ 0 & 0 & 1 \end{bmatrix}$ — camera (calibration) matrix (f, u_0, v_0 in pixels) →31

$\mathbf{R}$ — camera rotation matrix (cam coords.) →30

$\mathbf{t}$ — camera translation vector (cam coords.) →30

What Can We Do with An 'Uncalibrated' Perspective Camera?

How far is the engine?

distance between sleepers (ties) 0.806m but we cannot count them, the image resolution is too low
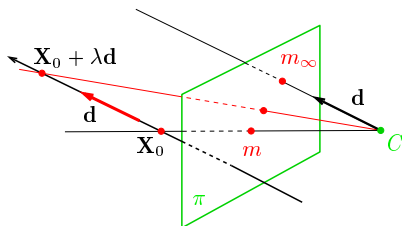
We will review some life-saving theory. . .
. . . and build a bit of geometric intuition. . .

In fact

- 'uncalibrated' = the image contains a calibrating object that suffices for the task at hand

Vanishing point: the limit of the projection of a point that moves along a space line infinitely in one direction.   the image of the point at infinity on the line

$$\underline{\mathbf{m}}_\infty \simeq \lim_{\lambda \to \pm\infty} \mathbf{P} \begin{bmatrix} \mathbf{X}_0 + \lambda\mathbf{d} \\ 1 \end{bmatrix} = \cdots \simeq \mathbf{Q}\,\mathbf{d}$$

⊛ P1; 1pt: Prove (use Cartesian coordinates and L'Hôpital's rule)

- the V.P. of a spatial line with directional vector $\mathbf{d}$ is $\underline{\mathbf{m}}_\infty \simeq \mathbf{Q}\,\mathbf{d}$
- V.P. is independent on line position $\mathbf{X}_0$, it depends on its directional vector only
- all parallel lines share the same V.P., including the optical ray defined by m_∞
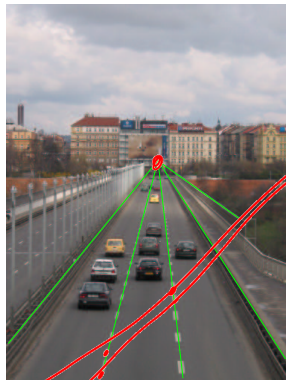
Some Vanishing Point "Applications"

where is the sun?

what is the wind direction?
(must have video)
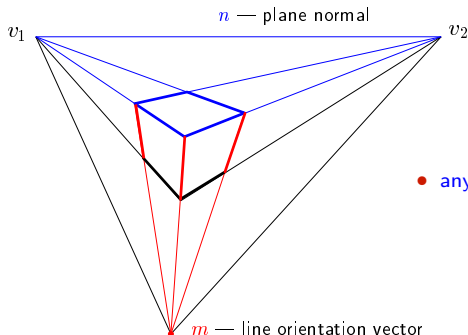
fly above the lane,
at constant altitude!

Vanishing line: The set of vanishing points of all lines in a plane

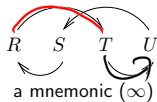the image of the line at infinity in the plane
and in all parallel planes

- V.L. n corresponds to spatial plane of normal vector $\mathbf{p} = \mathbf{Q}^\top \underline{\mathbf{n}}$

  because this is the normal vector of a parallel optical plane (!) →38

- a spatial plane of normal vector $\mathbf{p}$ has a V.L. represented by $\underline{\mathbf{n}} = \mathbf{Q}^{-\top}\mathbf{p}$.

Four distinct finite collinear spatial points R, S, T, U define cross-ratio

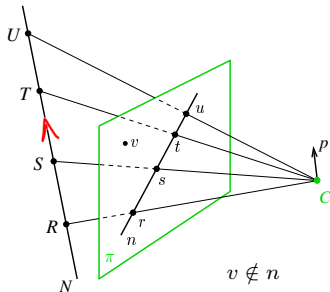$$[RSTU] = \frac{|\overrightarrow{RT}|\,|\overrightarrow{US}|}{|\overrightarrow{SR}|\,|\overrightarrow{TU}|}$$

a mnemonic (∞)

$|\overrightarrow{RT}|$ – signed distance from R to T in the arrow direction

6 cross-ratios from four points:

$$[SRUT] = [RSTU], \quad [RSUT] = \frac{1}{[RSTU]}, \quad [RTSU] = 1 - [RSTU], \quad \cdots$$

$v \notin n$

Obs: $[RSTU] = \dfrac{\begin{vmatrix} \underline{\mathbf{r}} & \underline{\mathbf{t}} & \underline{\mathbf{v}} \end{vmatrix}}{\begin{vmatrix} \underline{\mathbf{s}} & \underline{\mathbf{r}} & \underline{\mathbf{v}} \end{vmatrix}} \cdot \dfrac{\begin{vmatrix} \underline{\mathbf{u}} & \underline{\mathbf{s}} & \underline{\mathbf{v}} \end{vmatrix}}{\begin{vmatrix} \underline{\mathbf{t}} & \underline{\mathbf{u}} & \underline{\mathbf{v}} \end{vmatrix}}, \quad |\underline{\mathbf{r}} \ \underline{\mathbf{t}} \ \underline{\mathbf{v}}| = \det \begin{bmatrix} \underline{\mathbf{r}} & \underline{\mathbf{t}} & \underline{\mathbf{v}} \end{bmatrix} = (\underline{\mathbf{r}} \times \underline{\mathbf{t}})^{\top} \underline{\mathbf{v}}$ \quad (1)
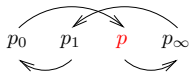
Corollaries:
- cross ratio is invariant under homographies $\underline{\mathbf{x}}' \simeq \mathbf{H}\underline{\mathbf{x}}$ plug $\mathbf{H}\underline{\mathbf{x}}$ in (1): $(\mathbf{H}^{-\top}(\underline{\mathbf{r}} \times \underline{\mathbf{t}}))^{\top} \mathbf{H}\underline{\mathbf{v}}$
- cross ratio is invariant under perspective projection: $[RSTU] = [r\,s\,t\,u]$

- 4 collinear points: any perspective camera will "see" the same cross-ratio of their images
- we measure the same cross-ratio in image as on the world line
- one of the points $R,\ S,\ T,\ U$ may be at infinity (we take the limit, in effect $\frac{\infty}{\infty} = 1$)

►1D Projective Coordinates

The 1-D projective coordinate of a point P is defined by the following cross-ratio:

$$[P] = [P_0\ P_1\ P\ P_\infty] = [p_0\ p_1\ p\ p_\infty] = \frac{|\overrightarrow{p_0\ p}|}{|\overrightarrow{p_1\ p_0}|}\ \frac{|\overrightarrow{p_\infty\ p_1}|}{|\overrightarrow{p\ p_\infty}|} = [p]$$
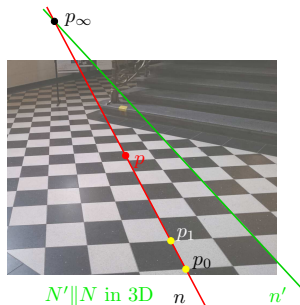

$p_0 \quad p_1 \quad p \quad p_\infty$

naming convention:

P_0 – the origin $\qquad\qquad [P_0] = 0$

P_1 – the unit point $\qquad\qquad [P_1] = 1$

P_∞ – the supporting point $\qquad [P_\infty] = \pm\infty$

$[P] = [p]$

$[P]$ is equal to Euclidean coordinate along N

$[p]$ is its measurement in the image plane
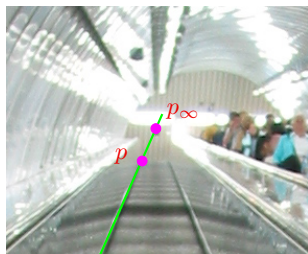
$N' \| N$ in 3D $\quad n \qquad n'$

Applications

- Given the image of a 3D line N, the origin, the unit point, and the vanishing point, then the Euclidean coordinate of any point $P \in N$ can be determined →47
- Finding v.p. of a line through a regular object →48
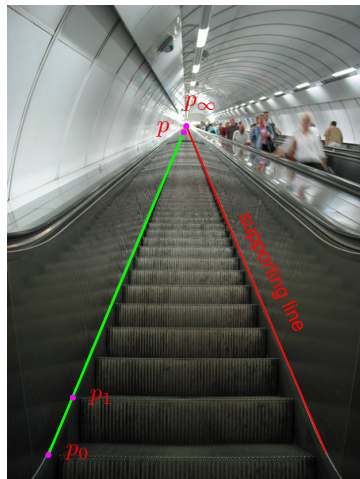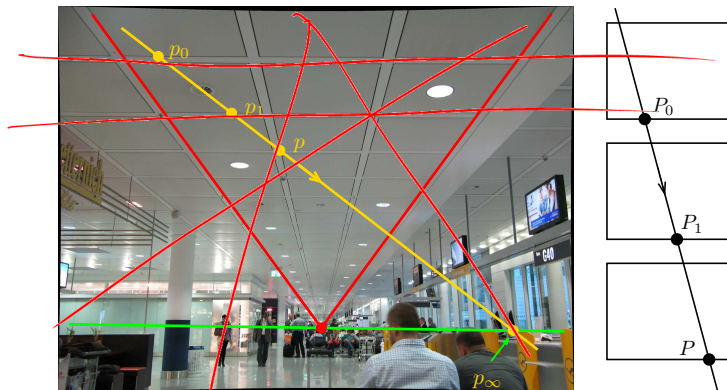
• Namesti Miru underground station in Prague

detail around the vanishing point

Result: $[P] = 214$ steps (correct answer is 216 steps)   4Mpx camera

in 3D: $|P_0P| = 2|P_0P_1|$ then   [H&Z, p. 218]

$$[P_0P_1PP_\infty] = \frac{|P_0P|}{|P_1P_0|} = 2 \quad \Rightarrow \quad x_\infty = \frac{x_0\,(2\,x - x_1) - x\,x_1}{x + x_0 - 2\,x_1}$$

- x – 1D coordinate along the yellow line, positive in the arrow direction
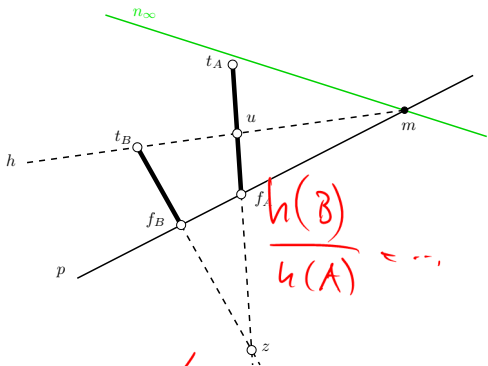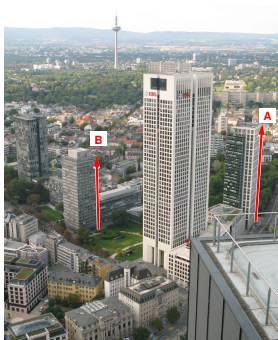- could be applied to counting steps ($\rightarrow$47) if there was no supporting line

⊛ P1; 1pt: How high is the camera above the floor?   sitting / standing

⊛ H2; 3pt: What is the ratio of heights of Building A to Building B?
- expected: conceptual solution; use notation from this figure
- deadline: LD+2 weeks

Hints

1. What are the interesting properties of line h connecting the top t_B of Building B with the point m at which the horizon intersects the line p joining the foots f_A, f_B of both buildings? [1 point]
2. How do we actually get the horizon n_∞? (we do not see it directly, there are some hills there...) [1 point]
3. Give the formula for measuring the length ratio. [formula = 1 point]
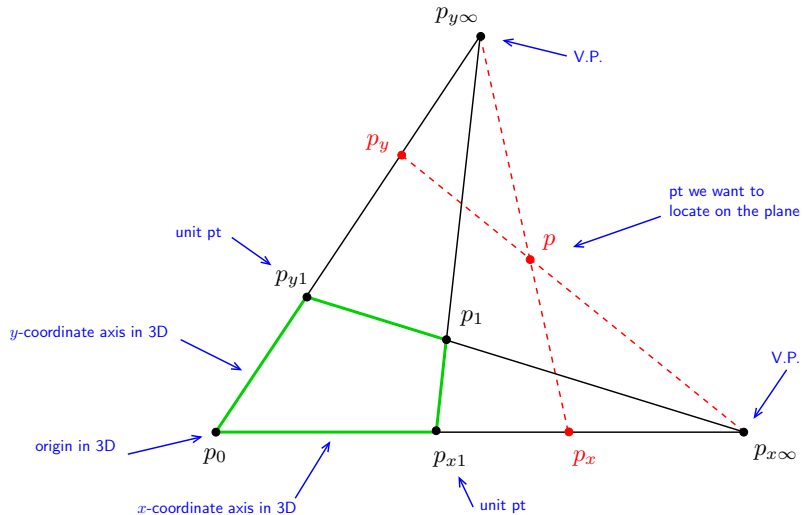
$$[P_x] = [P_0 \ P_{x1} \ P_x \ P_{x\infty}] \qquad\qquad [P_y] = [P_0 \ P_{y1} \ P_y \ P_{y\infty}]$$

San Giovanni in Laterano, Rome

- measuring distances on the floor in terms of tile units
- what are the dimensions of the seal? Is it circular (assuming square tiles)?
- needs no explicit camera calibration

because we can see the calibrating object (vanishing points)

Thank You