Combinatorial Optimization

Zdeněk Hanzálek
hanzalek@fel.cvut.cz

CTU FEE Department of Control Engineering

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Scheduling

Zdeněk Hanzálek, Přemysl Šůcha
hanzalek@fel.cvut.cz

CTU FEE Department of Control Engineering

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Scheduling - Basic Terminology

- **set of** n **tasks** $\mathcal{T} = \{T_1, T_2, \dots, T_n\}$
- set of m types of resources (processors, machines, employees,...) with capacities R_k, $\mathcal{P} = \left\{P_1^1, \dots, P_1^{R_1}, P_2^1, \dots, P_2^{R_2}, \dots\dots, P_m^1, \dots, P_m^{R_m}\right\}$
- **Scheduling is an assignment of a task to a resources in time**
- Each task must be **completed**
  - this differs from planning which decides which task will be scheduled and processed
- Set of tasks is known when executing the scheduling algorithm (this is called **off-line scheduling**)
  - this differs from on-line scheduling - OS scheduler, for example, schedules new tasks using some policy (e.g. priority levels)
- A result is a schedule which determines which task is run on which resource and when. Often depicted as a **Gantt chart**.

General and Specific Constraints

General constraints:

- Each task is to be processed **by at most one resource** at a time (task is sequential)
- Each resource is capable of processing **at most one task** at a time

Specific constraints:

- Task T_i has to be processed during time interval $\left\langle r_i, \widetilde{d}_i \right\rangle$
- When the precedence constraint is defined between T_i and T_j, i.e. $T_i \prec T_j$, then the processing of task T_j can't start before task T_i was completed
- If scheduling is non-preemptive, a task cannot be stopped and completed later
- If scheduling is preemptive, the number of preemptions must be finite

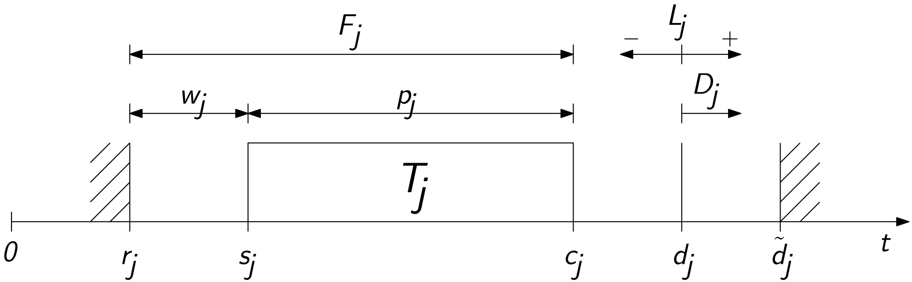
Task Parameters and Variables

Parameters

- **release time** r_j
- **processing time** p_j
- **due date** d_j, time in which task T_j should be completed
- **deadline** $\tilde{d}_j$, time in which task T_j has to be completed

Variables

- **start time** s_j
- **completion time** C_j
- waiting time $w_j = s_j - r_j$
- flow time $F_j = C_j - r_j$
- **lateness** $L_j = C_j - d_j$
- tardiness $D_j = \max\{C_j - d_j, 0\}$



Graham's Notation $\alpha\,|\beta|\,\gamma$

Classify scheduling problems by
resources | **tasks** | **criterion**

Example: $P2\,|\text{pmtn}|\,C_{max}$ represents scheduling on two parallel identical resources, and preemption is allowed. The optimization criterion is the completion time of the last task.

α - **resources**

- **Parallel resources** - a task can run on any resource (only one type of resource exists with capacity R, i.e. $\mathcal{P} = \left\{P^1, \ldots, P^R\right\}$).
- **Dedicated resources** - a task can run only on one resource (m resource types with unit capacity, i.e. $\mathcal{P} = \{P_1, P_2 \ldots, P_m\}$).
- **Project Scheduling** - m resource types, each with capacity R_k, i.e. $\mathcal{P} = \left\{P_1^1, \ldots, P_1^{R_1}, P_2^1, \ldots, P_2^{R_2}, \ldots\ldots, P_m^1, \ldots, P_m^{R_m}\right\}$.

Resources Characteristics α_1, α_2

α_1	$=$	1	single resource
		P	**parallel identical** resources
		Q	**parallel uniform** resources, computation time is inversely related to resource speed
		R	**parallel unrelated** resources, computation times are given as a matrix (resources x tasks)
		O	dedicated resources - **open-shop** - tasks are independent
		F	dedicated resources - **flow-shop** - tasks are grouped in the sequences (jobs) in the same order, each job visits each machine once
		J	dedicated resources - **job-shop** - order of tasks in jobs is arbitrary, resource can be used several times in a job
		PS	**Project Scheduling** - most general (several resource types with capacities, general precedence constraints)
α_2	$=$	$\emptyset$	arbitrary **number** of resources
		2	2 resources (or other specified number)
		m, R	m resource types with capacities R (Project Scheduling)

Task Characteristics $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6, \beta_7, \beta_8$

β_1	$=$	pmtn	**preemption is allowed**
		$\emptyset$	preemption is not allowed
β_2	$=$	prec	**precedence** constraints
		in-tree,out-tree	tree constraints
		chain	chain constraints
		tmpn	**temporal** constraints (for Project Sched.)
		$\emptyset$	independent tasks
β_3	$=$	r_j	release time
β_4	$=$	$p_j = k$	uniform processing time
		$p_L \leq p_j \leq p_U$	restricted processing time
		$\emptyset$	**arbitrary processing time**
β_5	$=$	$\widetilde{d}_j, d_j$	deadline, due-date
β_6	$=$	$n_j \leq k$	maximal number of tasks in a job
β_7	$=$	no-wait	buffers of zero capacity
β_8	$=$	set-up	time for resource **reconfiguration**

Optimality Criterion γ

γ	$=$	C_{max}	minimize schedule length $C_{max} = \max\{C_j\}$ (makespan, i.e. completion time of the last task)
		$\sum C_j$	minimize the sum of completion times
		$\sum w_j C_j$	minimize weighted completion time
		L_{max}	minimize max. lateness $L_{max} = \max\{C_j - d_j\}$
		$\emptyset$	decision problem
		$\ldots$	

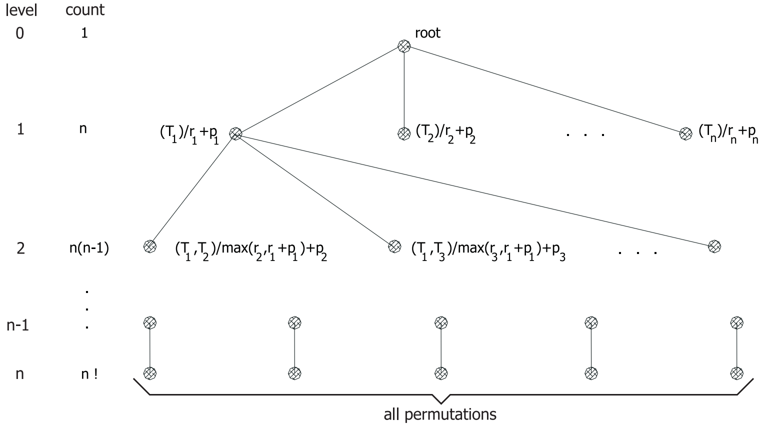
An Example: $P \,||\, C_{max}$ means:
Scheduling on an arbitrary number of parallel identical resources, no preemption, independent tasks (no precedence), tasks arrive to the system at time 0, processing times are arbitrary, objective is to minimize the schedule length.

Scheduling on One Resource
Minimizing Makespan (i.e. schedule length C_{max})

- $1 \left| prec \right| C_{max}$ - easy
  - the tasks are processed in an arbitrary order that satisfies the precedence relation, $C_{max} = \sum_{j=1}^{n} p_j$
- $1 \left|\right| C_{max}$ - easy
- $1 \left| r_j \right| C_{max}$ - easy
  - the tasks are processed in a non-descending order of r_j (T_j with the lowest r_j first)
- $1 \left| \widetilde{d}_j \right| C_{max}$ - easy
  - the tasks are processed in a non-descending order of $\widetilde{d}_j$
  - can be solved by EDF - Earliest Deadline First
  - the feasible schedule doesn't have to exist
- $1 \left| r_j, \widetilde{d}_j \right| C_{max}$ - NP-hard
  - NP-hardness proved by the polynomial transformation from 3-Partition problem
  - for $p_j = 1$ there exists a polynomial algorithm

Bratley's Algorithm for $1 \left| r_j, \widetilde{d}_j \right| C_{max}$

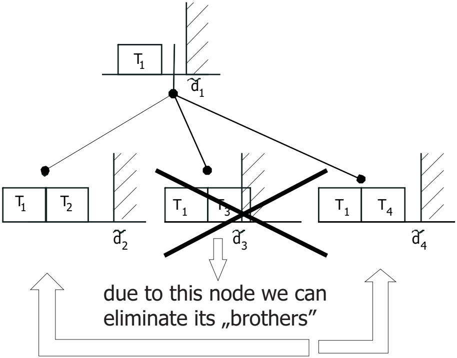
A **branch and bound** (B&B) algorithm.
Branching - without bounding it is an **enumerative method** that creates a solution tree (some of the nodes are infeasible). Every node is labeled by: (the order of tasks)/(completion time of the last task).



Reduction of the Tree - Bounding

(i) **eliminate the node** exceeding the deadline (and all its "brothers")

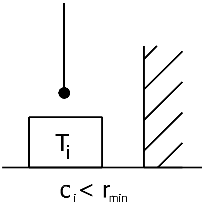
- If there is a node which exceeds any deadline, all its descendants should be eliminated
- Critical task (here T_3) will have to be scheduled anyway - therefore, all of its "brothers" should be eliminated as well

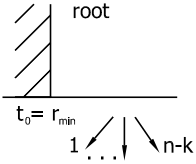


Tree Size Reduction - Decomposition

(ii) **problem decomposition** due to idle waiting - e.g. when the employee waits for the material, his work was optimal

- Consider node v on level k. If C_i of the last scheduled task is less than or equal to r_i of all unscheduled tasks, there is no need for backtrack above v
- v becomes a new root and there are $n - k$ levels ($n - k$ unscheduled tasks) to be scheduled





Optimality Test - Termination of Bratley's Algorithm

Definition: BRTP - Block with Release Time Property

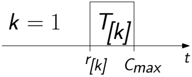
BRTP is a set of k tasks that satisfy:

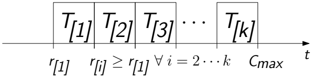
- first task $T_{[1]}$ starts at it's release time
- all tasks till the end of the schedule run without "idle waiting"
- $r_{[1]} \leq r_{[i]}$ for all $i = 2 \dots k$

Note: "till the end of the schedule" implies there is at most one BRTP

Lemma: sufficient condition of optimality

If BRTP exists, the schedule is optimal (the search is finished).





Proof:

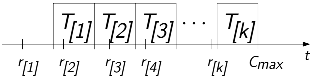
- this schedule is optimal since the last task $T_{[k]}$ can not be completed earlier
- order of prec. tasks is not important - see (ii)
- no task from BRTP can be done before $r_{[1]}$
- there is no task after C_{max}

BRTP is Sufficient and Necessary Condition of Optimality

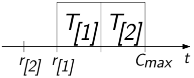
Proposition - necessary condition of optimality

If the schedule for $1 \left| r_j, \widetilde{d}_j \right| C_{max}$ is optimal, it contains BRTP.

Proof by contradiction: we show that the schedule without BRTP is not optimal. There are two cases of the schedule without BRTP:



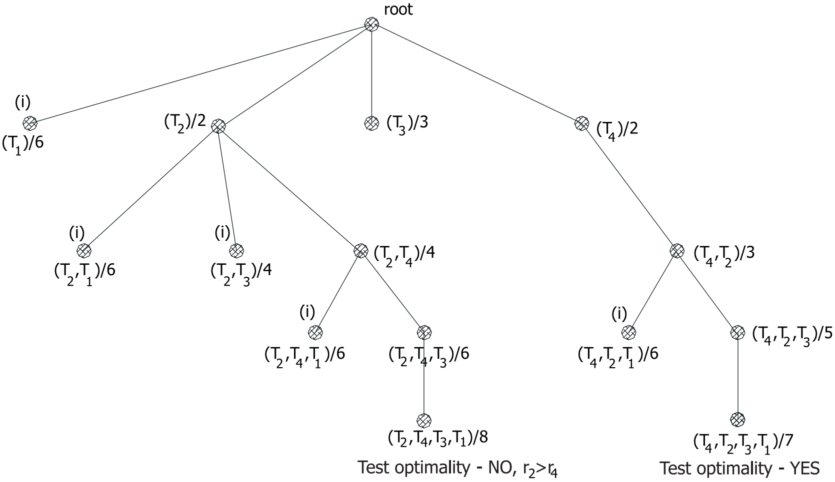
1. the last block does not start at $r_{[1]}$:
   - $r_{[1]} < s_{[1]}$
   - $s_{[1]} < r_{[i]} < s_{[i]}\ \forall i = 2 \dots k$ (note that if $i = 2 \dots k$ such that $r_{[i]} = s_{[i]}$ exists, than BRTP exists from $T_{[i]}$)
   - block can be moved left while maintaining actual order



2. some task can be placed before $T_{[1]}$, i.e. there is $i = 2 \dots k$ such that $r_{[i]} < s_{[1]}$ exists
   - schedule can be improved while moving $T_{[2]}$ before $T_{[1]}$

Bratley's Algorithm - Example

r =[4,1,1,0], p =[2,1,2,2], $\tilde{d}$ =[8,5,6,4]



Scheduling on One Resource
Minimizing $\sum w_j C_j$

- $1 || \sum C_j$ - easy
  - SPT rule (Shortest Processing Time first) - schedule the tasks in a non-decreasing order of p_j
- $1 || \sum w_j C_j$ - easy
  - Weighted SPT - schedule the tasks in a non-decreasing order of $\frac{p_j}{w_j}$
- $1 | r_j | \sum C_j$ - NP-hard
- $1 | \mathrm{pmtn}, r_j | \sum C_j$ - can be solved by modified SPT
- $1 | \mathrm{pmtn}, r_j | \sum w_j C_j$ - NP-hard
- $1 \left| \widetilde{d}_j \right| \sum C_j$ - can be solved by modified SPT
- $1 \left| \widetilde{d}_j \right| \sum w_j C_j$ - NP-hard
- $1 | \mathrm{prec} | \sum C_j$ - NP-hard

Branch and Bound with LP for $1 \left| prec \right| \sum w_j C_j$

First, we formulate the problem as a ILP:

- we use **variable** $x_{ij} \in \{0, 1\}$ such that $x_{ij} = 1$ iff T_i precedes T_j or $i = j$
- we encode **precedence relations** into $e_{ij} \in \{0, 1\}$ such that $e_{ij} = 1$ iff there is a directed edge from T_i to T_j in the precedence graph G or $i = j$
- **criterion** - completion time of task T_j consists of p_j and the processing time of its predecessors:

$$\begin{aligned} C_j &= \textstyle\sum_{i=1}^{n} p_i \cdot x_{ij} \\ w_j \cdot C_j &= \textstyle\sum_{i=1}^{n} p_i \cdot x_{ij} \cdot w_j \\ J = \textstyle\sum_{j=1}^{n} w_j \cdot C_j &= \textstyle\sum_{j=1}^{n} \sum_{i=1}^{n} p_i \cdot x_{ij} \cdot w_j \end{aligned}$$

  from all feasible schedules x we look for the one that minimizes $J(x)$, i.e. $\min_x J(x)$

ILP formulation for $1\,|\text{prec}|\sum w_j C_j$

$$\begin{array}{lll}
\min & \sum_{j=1}^{n} \sum_{i=1}^{n} p_i \cdot x_{ij} \cdot w_j & \\
\text{subject to:} & & \\
x_{i,j} \geq e_{i,j} & i,j \in 1..n & \text{if } T_i \text{ precedes } T_j \text{ in } G, \\
 & & \text{then it precedes } T_j \\
 & & \text{in the schedule} \\
x_{i,j} + x_{j,i} = 1 & i,j \in 1..n, i \neq j & \text{either } T_i \text{ precedes } T_j, \\
 & & \text{or vice versa} \\
1 \leq x_{i,j} + x_{j,k} + x_{k,i} \leq 2 & i,j,k \in 1..n, & \text{no cycle exists in the} \\
 & i \neq j \neq k & \text{digraph of } x \\
x_{i,i} = 1 & i \in 1..n &
\end{array}$$

parameters: $p_{i \in 1..n} \in \mathbb{R}_0^+ \; e_{i \in 1..n, j \in 1..n} \in \{0,1\}$

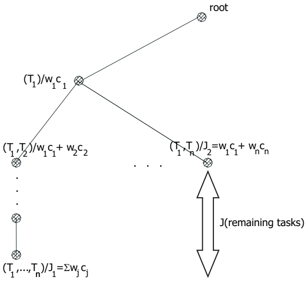
variables: $x_{i \in 1..n, j \in 1..n} \in \{0,1\}$

Branch and Bound with LP Bounding

We relax on the integrality of variable x:

- $0 \leq x_{ij} \leq 1$ and $x_{i \in 1..n, j \in 1..n} \in \mathbb{R}$
- This does not give us the right solution, however we can use the J^{LP}(remaining tasks) value of this LP formulation as a lower bound on the "amount of remaining work"

The Branch and Bound algorithm creates a similar tree as Bradley's algorithm.



- Let J_1 be the value of the best solution known up to now
- We discard the partial solution of value J_2 not only when $J_2 \geq J_1$, but also when $J_2 + J^{LP}(\text{remaining tasks}) \geq J_1$.
  Since the solution space of ILP is a subspace of LP we know:
  $J(\text{remaining tasks}) \geq J^{LP}(\text{remaining tasks})$.

Scheduling on Parallel Identical Resources Minimizing C_{max}

- $P2 \,||\, C_{max}$ - NP-hard
  - schedule n non-preemptive tasks on two parallel identical resources minimizing makespan, i.e. the completion time of the last task
  - the problem is NP-hard because the **2 partition problem** (see ILP lecture) can be reduced to $P2 \,||\, C_{max}$ while comparing the optimal C_{max} with the threshold of $0.5 * \sum_{i \in 1..n} p_i$.
- $P \,|pmtn|\, C_{max}$ - easy
  - can be solved by the **McNaughton** algorithm in $O(n)$
- $P \left| pmtn, r_j, \widetilde{d}_j \right| C_{max}$ - easy
  - can be formulated as a **maximum flow problem** (see the lecture on Flows)
- $P \,|prec|\, C_{max}$ - NP-hard
  - **LS - approximation algorithm** with factor $r_{LS} = 2 - \frac{1}{R}$, where R is the number of parallel identical resources
- $P \,||\, C_{max}$ - NP-hard
  - **LPT - approximation algorithm** with factor $r_{LPT} = \frac{4}{3} - \frac{1}{3R}$
  - **dynamic programming** - Rothkopf's pseudopolynomial algorithm
- $P \,|pmtn, prec|\, C_{max}$ - NP-hard
  - Muntz&Coffman's **level algorithm** with factor $r_{MC} = 2 - \frac{2}{R}$

McNaughton's Algorithm for $P\,|\text{pmtn}|\,C_{max}$

Input: R, number of parallel identical resources, n, number of preemptive tasks and computation times $[p_1, p_2, ..., p_n]$.

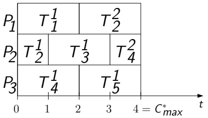
Output: n-element vectors s^1, s^2, z^1, z^2 where s_i^1 (resp. s_i^2) is start time of the first (resp. second) part of task T_i and z_i^1 (resp. z_i^2) is the resource ID on which the first (resp. second) part of task T_i will be executed.

$s_i^1 = s_i^2 = z_i^1 = z_i^2 := 0$ for all $i \in 1 \ldots n$;
$t := 0; v := 1; i := 1;$
$C^*_{max} = \max\left\{\max_{i=1\ldots n}\{p_i\}, \frac{1}{R}\sum_1^n p_i\right\};$
while $i \leq n$ **do**
  if $t + p_i \leq C^*_{max}$ **then**
    $s_i^1 := t; z_i^1 := v;\ t := t + p_i; i := i + 1;$
  else
    $s_i^2 := t; z_i^2 := v;\ p_i := p_i - (C^*_{max} - t); t := 0; v := v + 1;$
  end
end

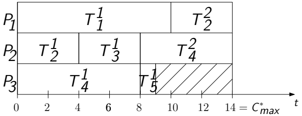
The term $C^*_{max} = \max\left\{\max_{i=1\dots n}\{p_i\}, \frac{1}{R}\sum_1^n p_i\right\}$ should be interpreted as follows:

- component $\max_{i=1\dots n}\{p_i\}$ represents the sequential nature of each task - it's parts can be assigned to different resources, but these parts can not be run simultaneously. Note that each task can be divided into two parts at most.
- component $\frac{1}{R}\sum_1^n p_i$ represents a situation when all resources work without idle waiting

Example 1:
$p = [2, 3, 2, 3, 2], R = 3$
compute $C^*_{max} = \max\left\{3, \frac{12}{3}\right\} = 4$



Example 2:
$p = [10, 8, 4, 14, 1], R = 3$
compute $C^*_{max} = \max\left\{14, \frac{37}{3}\right\} = 14$



List Scheduling - Approximation Alg. for $P\,|\text{prec}|\,C_{max}$

Input: R, number of parallel identical resources, n, number of non-preemptive tasks and computation times $[p_1, p_2, ..., p_n]$. G, digraph of precedence constraints.

Output: n-element vectors s and z where s_i is the start time of T_i and z_i is the resource ID.

$t_v := 0$ for all $v \in 1 \dots R$; `// availability of resource`
$s_i = z_i := 0$ for all $i \in 1 \dots n$;
Sort tasks in list L;
for $count := 1$ **to** n **do** `// for all tasks`
  $k = arg\min_{v=1 \dots R} \{t_v\}$; `// choose res. with the lowest` t_v
  Remove the first free task T_i from L ;
  $s_i = \max\{t_k, \max_{j \in Pred(T_i)}\{s_j + p_j\}\}; z_i = k$; `// assign` T_i `to` P_k
  $t_k = s_i + p_i$; `// update availability time of` P_k
end

Task T_i is **free** if its predecessors have been completed. $Pred(T_i)$ is a set of the task IDs that are **predecessors** of T_i. Complexity is $O(n)$.

List Scheduling - Approximation algorithm for $P\,|\text{prec}|\,C_{max}$

List Scheduling (LS) is a general heuristic useful in many problems.

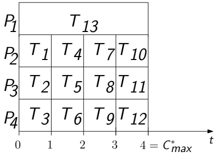
- We have a list (n-tuple) of tasks and when some resource is free, we assign the first free task from the list to this resource.
- The accuracy of LS depends on the criterion and sorting procedure.

Approximation factor of LS algorithm [Graham 1966]

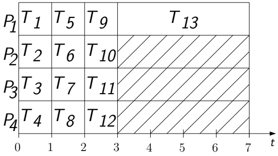
For $P\,|\text{prec}|\,C_{max}$ (and also for $P\,||\,C_{max}$) and arbitrary (unsorted) list L, List Scheduling is an approximation algorithm with factor $r_{LS} = 2 - \frac{1}{R}$

An example illustrating the case when the factor is attained:

$n = (R-1)\cdot R + 1,$
$p = [1, 1, \ldots, 1, R],$
$\prec$ empty.
Illustration for $R = 4$
$r_{LS} = 2 - \frac{1}{4} = \frac{7}{4}$



$L = [T_n, T_1, \ldots, T_{n-1}]$



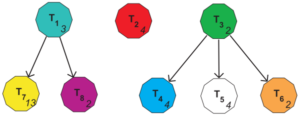
$L' = [T_1, T_2, \ldots, T_n]$

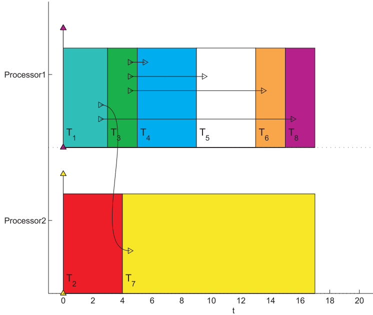
Anomalies of List Scheduling Algorithm

The LS algorithm depends not only on the **order of tasks in L**, but it exhibits **anomalies** (C_{max} surprisingly increases when relaxing some constraints/parameters) caused by:

1. **the decrease of processing time p_i**
2. **the removal of some precedence constraints**
3. **the increase of the number of resources R**

Example illustrating different anomalies for
$R = 2,\ n = 8,\ p = [3, 4, 2, 4, 4, 2, 13, 2]$

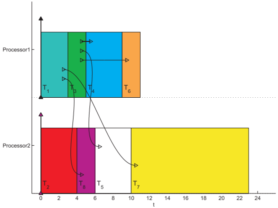




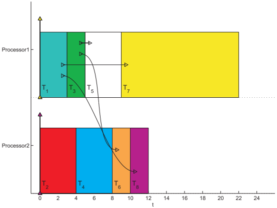
Using list $L = [T_1, T_2, T_3, T_4, T_5, T_6, T_7, T_8]$,
LS finds solution with $C^*_{max} = 17$.

List Scheduling Anomalies - Prolongation of C_{max}

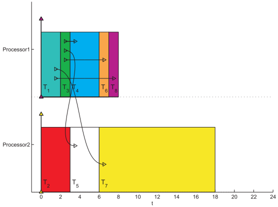
Exchange position of T_7 and T_8
$L = [T_1, T_2, T_3, T_4, T_5, T_6, T_8, T_7]$.



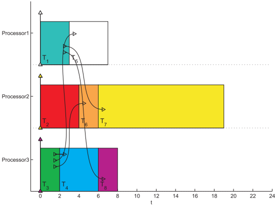
Remove prec. constraint $T_3 \prec T_4$.



Decrease p_i of all tasks by one.



Add resource ($R = 3$).



LPT (Longest Processing Time First) - Approximation Algorithm for $P \,||\, C_{max}$

The approximation factor of the LS algorithm can be decreased using the **Longest Processing Time first** (LPT) strategy

- During initialization of LS, we sort list L in a non-increasing order of p_i

Approximation factor of LPT algorithm [Graham 1966]

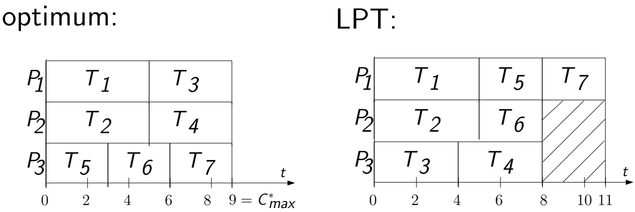
LPT algorithm for $P||C_{max}$ is an approximation algorithm with factor $r_{LPT} = \frac{4}{3} - \frac{1}{3R}$

Time complexity of LPT algorithm is $O(n \cdot log(n))$ due to the sorting.

LPT (Longest Processing Time First) - Approximation Algorithm for $P \,||\, C_{max}$

An example illustrating the case when the factor is attained:
$p = [2R-1, 2R-1, 2R-2, 2R-2, \ldots, R+1, R+1, R, R, R]$
$n = 2 \cdot R + 1, \prec$ empty,

Illustration for $R = 3$



$r_{LPT} = \frac{4}{3} - \frac{1}{9} = \frac{11}{9}$

Factor of LPT algorithm

If the number of tasks is big, the factor can get better depending on k - the number of tasks assigned to the resource which finishes last:
$r_{LPT} = 1 + \frac{1}{k} - \frac{1}{kR}$

Dynamic Programming for $P \,||\, C_{max}$ [Rothkopf]

Pseudopolynomial algorithm - the range of discreet values is limited by the upper bound. In some special cases there exists a polynomial algorithm for such a restricted problem.

- we add a binary variable $x_i(t_1, t_2, \ldots, t_R)$ where
  - $i = 1, 2 \ldots n$ is the task index
  - $v = 1, 2, \ldots R$ is the index of the resource
  - $t_v = 0, 1, 2, \ldots UB$ is the time variable associated to the resource v
  - UB is upper bound on C_{max}
- $x_i(t_1, t_2, \ldots, t_R) = 1$ iff tasks T_1, $T_2, \ldots, T_i$ can be assigned to the resource such that P_v is occupied during the time interval $\langle 0, t_v \rangle \, ; v = 1, 2, \ldots R$

Input: R, the number of parallel identical resources, n, the number of nonpreemptive tasks and their processing time $[p_1, p_2, ..., p_n]$.

Output: n-elements vectors s and z where s_i is the start time and z_i is the resource ID.

for $(t_1, t_2, \ldots, t_R) \in \{1, 2, \ldots UB\}^R$ **do** $x_0(t_1, t_2, \ldots, t_R) := 0$;

$x_0(0, 0, \ldots, 0) := 1$;

for $i := 1$ **to** n **do** `// for all tasks`

  for $(t_1, t_2, \ldots, t_R) \in \{0, 1, 2, \ldots UB\}^R$ **do** `// in the whole space`

    $x_i(t_1, t_2, \ldots, t_R) := \mathrm{OR}_{v=1}^{R} x_{i-1}(t_1, t_2, \ldots, t_v - p_i, \ldots t_R)$;

    `//` $x_i() = 1$ `iff there existed`

    `//` $x_{i-1}() = 1$ `‘‘smaller’’ by` p_i `in any direction`

  end

end

$C^*_{max} = \min_{x_n(t_1, t_2, \ldots, t_R)=1} \{\max_{v=1,2,\ldots R} \{t_v\}\}$;

Assign tasks $T_n, T_{n-1}, \ldots, T_1$ in the reverse direction;

Time complexity is $O(n \cdot UB^R)$. Example n=3, R=2, p=[2,1,2], C=5.

Muntz&Coffman's Level Algorithm for $P\,|\text{pmtn, prec}|\,C_{max}$

Principle:

- tasks are picked from **the list ordered by the level** of tasks
- **the level of task** T_j - sum of p_i (including p_j) along the **longest path** from T_j to a terminal task (a task with no successor)
- when more tasks of the same level are assigned to less resources, each task gets **part of the resource capacity** β
- the algorithm moves forward to time τ when **one of the tasks ends** or the task with a lower level would be processed by a bigger capacity β than the tasks with a higher level

For $P2\,|\text{pmtn, prec}|\,C_{max}$ and $P\,|\text{pmtn, forest}|\,C_{max}$, the algorithm is **exact**.
For $P\,|\text{pmtn, prec}|\,C_{max}$ **approximation** alg. with factor $r_{MC} = 2 - \frac{2}{R}$.
Time complexity is $O(n^2)$.

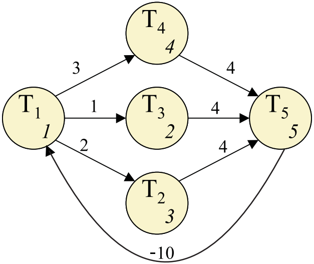
Input:	R, the number of parallel identical resources, n, the number of preemptive tasks and proc. times $[p_1, p_2, ..., p_n]$. Prec. graph G.
Output:	n-elements vectors s and z where s_i is the start time and z_i is the resource ID.

Muntz&Coffman's Level Algorithm for $P\,|\text{pmtn}, \text{prec}|\,C_{max}$

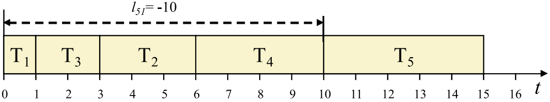
```
compute the level of all tasks ; t:=0; h:=R; // h represents free res
while unfinished tasks exists do
    construct Z;                  // subset T of free tasks in time t
    while h > 0 and |Z| > 0 do // free resources and free tasks
        construct S;  // subset Z of tasks of the highest level
        if |S| > h then                    // more tasks than resources
            assign part of capacity β := h/|S| to tasks in S; h := 0;
        else
            assign one resource to each task in S; β := 1; h := h − |S|;
        end
        Z := Z \ S;
    end
    compute τ;          // time when one of the tasks is finished
    decrease level of tasks by (τ − t) · β;    // finished part of task
    t := τ;h := R;
end
Use McNaughton's alg. to re-schedule parts with more tasks on less res.;
```

Project Scheduling with Temporal Constraints

- Set of non-preemptive tasks $\mathcal{T} = \{T_1, T_2, ..., T_n\}$ is represented by the nodes of the directed graph G.
- Processing time p_i is assigned to each task.



- The edges represent temporal constraints. Each edge from T_i to T_j has the length l_{ij}.
- Each temporal constraint is characterized by one inequality $s_i + l_{ij} \leq s_j$.

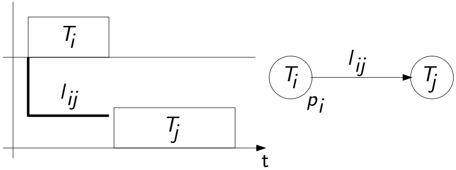


Temporal Constraints $s_i + l_{ij} \leq s_j$ with Positive l_{ij}

Temporal Constraints (also called a **generalized precedence constraint** or a **positive-negative time lag**)
- the start time of one task depends on the start time of another task

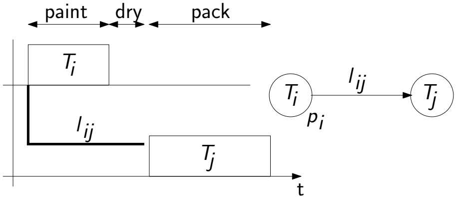
a) $l_{ij} = p_i$

- "normal" precedence relation
- the second task can start when the previous task is finished



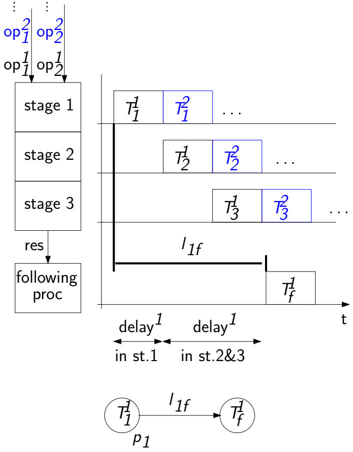
b) $l_{ij} > p_i$

- the second task can start some time after the completion of previous task
- b.1) example of a dry operation performed in sufficiently large space



b.2) another example with $l_{ij} > p_i$ - pipe-lined ALU

- We assume the processing time to be equal in all stages
- **Result is available** l_{1f} tics after stage 1 reads operands
- **Stage 1 reads new operands** each p_1 tics
- **Stages 2 and 3 are not modeled** since we have enough of these resources and they are synchronized with stage 1

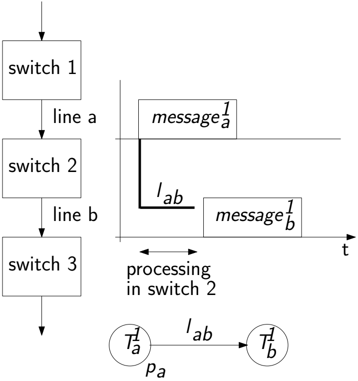


Temporal Constraints $s_i + l_{ij} \leq s_j$ with Positive l_{ij}

c) $0 < l_{ij} < p_i$

Partial results of the previous task may be used to start the execution of the following task.
E.g. the **cut-through** mechanism, where the switch starts transmission on the output port earlier than it receives the complete message on the input port.

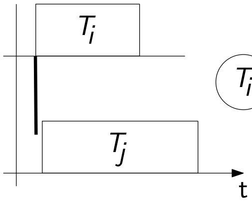
- time-triggered protocol
- resources are communication links
- l_{ab} represents the **processing** (of one bit) in the switch
- **different parts** of the same message are transmitted by several communication links at the same time

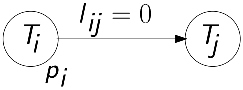


Temporal Constraints $s_i + l_{ij} \leq s_j$ with Zero or Negative l_{ij}

d) $l_{ij} = 0$

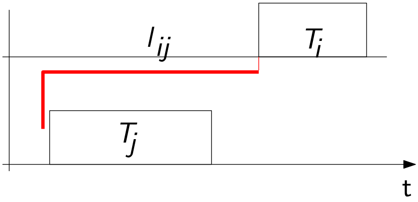
- Task T_i has to start earlier or at the same time as T_j

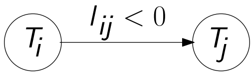




e) $l_{ij} < 0$

- Task T_i has to start earlier or **at most** $|l_{ij}|$ **later** than T_j
- It loses the sense of "normal " precedence relation, since T_i does not have to precede T_j
- It represents the **relative deadline** of T_i related to the start-time of T_j





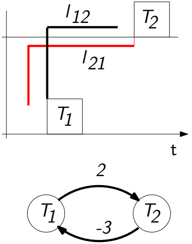
Cycles and Relative Time Windows

- Absence of a positive cycle in graph G
  - it is a necessary condition for schedulability
  - it is a necessary and sufficient condition for schedulability of the instance with unlimited resources capacity (the schedule is restricted only by the temporal constraints - can be computed easily by LP)
- For G we can create a complete digraph G' where weight l_{ij} is the length of the longest oriented path from T_i to T_j in G (if no oriented edge exists in G or G', the weight is $l_{ij} = -\infty$). In the following text, we think of l_{ij} as an edge in complete graph G' of the longest paths.
  - $s_j \geq \max_{\forall i \in 1 \ldots n} l_{ij}$, - start time of T_j is lower bounded by the longest path from arbitrary node.

Example - relative time window

If $l_{ij} \geq 0$ and $l_{ji} < 0$ exists, tasks T_i and T_j are constrained by the relative time window.

- the length of the negative cycle determines the "clearance" of the time window
- e.g. applying a catalyst to the chemical process



Project Scheduling
Minimizing C_{max}

- $PS1 \,|\text{temp}|\, C_{max}$ - NP-hard
  - Input: The number of non-preemptive tasks n and processing times $[p_1, p_2, ..., p_n]$. The temporal constraints defined by digraph G.
  - Output: n-element vector s, where s_i is the start time of T_i
  - We will show Time-indexed and Relative-order ILP formulations
- $PSm, 1 \,|\text{temp}|\, C_{max}$ - NP-hard
  - Input: The number of non-preemptive tasks n and processing times $[p_1, p_2, ..., p_n]$. The temporal constraints defined by digraph G. The number of dedicated resources m and the assignment of the tasks to the resources $[a_1, a_2, ..., a_n]$, where a_i is the index of the resource on which task T_i will be executed.
  - Output: n-element vector s, where s_i is the start time of T_i
  - We show the Relative-order ILP formulation

ILP formulation of $PS1\,|\text{temp}|\,C_{max}$

Task can be represented in two ways:

- **Time-indexed** - ILP model is based on variable x_{it}, which is equal to 1 iff $s_i = t$. Otherwise, it is equal to zero. Processing times are positive integers.
- **Relative-order** - ILP model is based on the relative order of tasks given by variable x_{ij}, which is equal to 1 iff task T_i precedes task T_j. Otherwise, it is equal to zero. The processing times are nonnegative real numbers.

Both models contain two types of constraints:

- precedence constraints
- resource constraints - prevent overlapping of tasks

Time-indexed Model for $PS1\,|temp|\,C_{max}$

$$\min C_{max}$$

$$\begin{array}{ll}
\sum_{t=0}^{UB-1}(t \cdot x_{it}) + l_{ij} \leq \sum_{t=0}^{UB-1}(t \cdot x_{jt}) & \forall l_{ij} \neq -\infty \text{ a } i \neq j \text{ (prec. const.)} \\
\sum_{i=1}^{n}\left(\sum_{k=\max(0,t-p_i+1)}^{t} x_{ik}\right) \leq 1 & \forall t \in \{0,\ldots UB-1\} \text{ (resource)} \\
\sum_{t=0}^{UB-1} x_{it} = 1 & \forall i \in \{1,\ldots n\} \text{ (}T_i\text{ is scheduled)} \\
\sum_{t=0}^{UB-1}(t \cdot x_{it}) + p_i \leq C_{max} & \forall i \in \{1,\ldots n\}
\end{array}$$

variables: $x_{it} \in \{0,1\},\ C_{max} \in \{0,\ldots UB\}$

UB - upper bound of C_{max} (e.g. $UB = \sum_{i=1}^{n} \max\left\{p_i, \max_{i,j\in\{1,\ldots,n\}} l_{ij}\right\}$).

Start time of T_i is $s_i = \sum_{t=0}^{UB-1}(t \cdot x_{it})$.

Model contains $n \cdot UB + 1$ variables and $|E| + UB + 2n$ constraints.

Constant $|E|$ represents the number of temporal constraints (edges in G).

Time-indexed Model for $PS1 \,|temp|\, C_{max}$

$\mathcal{T} = \{T_1, T_2, T_3\}$, $p = [1, 2, 1]$, $UB = 5$

T_1 is scheduled:

	0	1	2	3	4
T_1	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
T_2	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
T_3	x_{30}	x_{31}	x_{32}	x_{33}	x_{34}

$\Sigma = 1$

Resource constr. at time 2:

	0	1	2	3	4
T_1	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}
T_2	x_{20}	x_{21}	x_{22}	x_{23}	x_{24}
T_3	x_{30}	x_{31}	x_{32}	x_{33}	x_{34}

$\Sigma \leq 1$

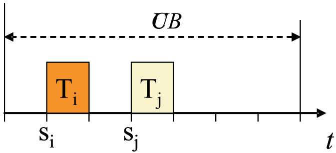
Relative-order Model for $PS1\,|\text{temp}|\,C_{max}$

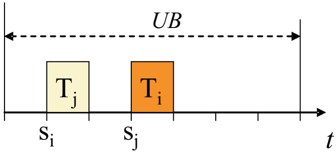
Resource constraint for couple of tasks:
$p_j \leq s_i - s_j + UB \cdot x_{ij} \leq UB - p_i$

The constraint uses "big M" (here UB - upper bound on C_{max}).

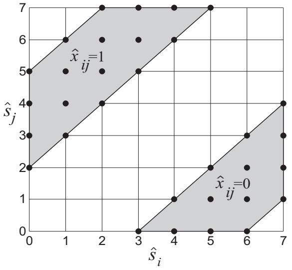
If $x_{ij} = 1$, T_i **precedes** task T_j and the constraint is formulated as
$s_i + p_i \leq s_j$.

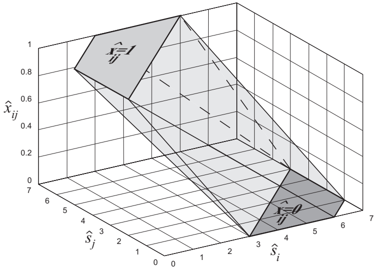
If $x_{ij} = 0$, T_i **follows** task T_j and the constraint is formulated as
$s_j + p_j \leq s_i$.





An example of a polytope which is determined by the resource constraint for a pair of tasks T_i and T_j with $p_i = 2$ and $p_j = 3$. There are no precedence constraints among the tasks and $UB = 8$.





Relative-order Model for $PS1\,|\text{temp}|\,C_{max}$

min C_{max}

$s_i + l_{ij} \leq s_j$ $\quad\forall l_{ij} \neq -\infty$ a $i \neq j$
(temporal constraint)

$p_j \leq s_i - s_j + UB \cdot x_{ij} \leq UB - p_i$ $\quad\forall i, j \in \{1, \dots, n\}$ a $i < j$
(resource constraint)

$s_i + p_i \leq C_{max}$ $\quad\forall i \in \{1, \dots, n\}$

variables: $x_{ij} \in \{0, 1\},\ C_{max} \in \langle 0, UB \rangle,\ s_i \in \langle 0, UB \rangle$

The model contains $n + \left(n^2 - n\right)/2 + 1$ variables and $|E| + \left(n^2 - n\right) + n$ constraints. $|E|$ is a number of temporal constraints (edges in G).

Comparison of the Two Models

Each model is suitable for different types of tasks:

Time-indexed model:

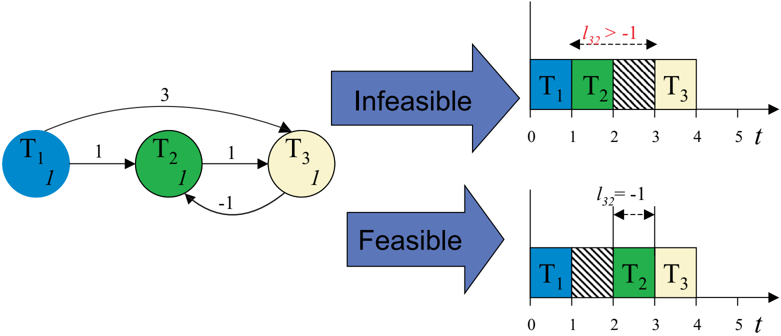
- (+) Can be easily extended for parallel identical processors.
- (+) ILP formulation does not need many constraints.
- (-) The size of the model grows with the size of UB.

Relative-order model:

- (+) The size of ILP model does not depend on UB.
- (-) Requires a big number of constraints.

Feasibility Test for Heuristic Algorithms

If the partial schedule (found for example by a greedy algorithm which inserts tasks in a topological order, or the partial result during the Branch and Bound algorithm) violates some time constraints, the order of tasks does not need to be infeasible.



When the optimal order of the tasks in the schedule is known (variables x_{ij} are constants), it is easy to find the start time of the tasks (for example by LP formulation involving time constraints only).

Relative-order Model for $PSm, 1 \,|\text{temp}|\, C_{max}$

Part of the input parameters are the number of resources m and **assignment of the tasks to the resources** $[a_1, ..., a_i, ..., a_n]$, where a_i is index of the resource on which task T_i will be running.

min C_{max}

$$
\begin{array}{ll}
s_i + l_{ij} \leq s_j & \forall l_{ij} \neq -\infty \text{ and } i \neq j \\
 & \text{(temporal constraints)} \\
p_j \leq s_i - s_j + UB \cdot x_{ij} \leq UB - p_i & \forall i,j \in \{1, \ldots, n\},\ i < j \text{ and } \underline{a_i = a_j} \\
 & \text{(independent on each resource)} \\
s_i + p_i \leq C_{max} & \forall i \in \{1, \ldots, n\}
\end{array}
$$

variables: $x_{ij} \in \{0, 1\}$, $C_{max} \in \langle 0, UB \rangle$, $s_i \in \langle 0, UB \rangle$

Model consists of less than $n + \left(n^2 - n\right)/2 + 1$ variables (exact number depends on the number of tasks scheduled on each resource).

Modeling with Temporal Constraints

Using $PS1\left|\text{temp}\right|C_{max}$ we will model:

- $1\left|r_j, \widetilde{d}_j\right|C_{max}$
- scheduling on **dedicated resources** $PSm,1\left|\text{temp}\right|C_{max}$

Using $PSm,1\left|\text{temp}\right|C_{max}$ we will model:

- scheduling of **multiprocessor tasks** - task needs more than one resource type at a given moment
- scheduling with **setup times** - two subsequent tasks executed on one resource need to be separated by idle waiting, for example to change the tool.

Reduction from $1\left|r_j, \widetilde{d}_j\right| C_{max}$ to $PS1\,|\text{temp}|\,C_{max}$

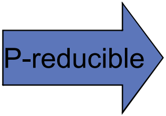
This polynomial reduction proves that $PS1\,|\text{temp}|\,C_{max}$ is NP-hard, since Bratley's problem is NP-hard.

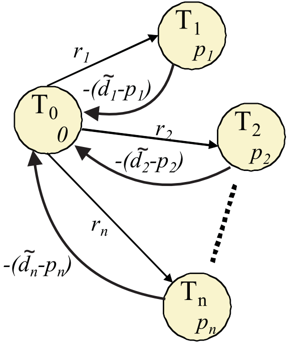
Instance $1\left|r_j, \widetilde{d}_j\right| C_{max}$

$r = [r_1, r_2, \ldots, r_n]$

$p = [p_1, p_2, \ldots, p_n]$

$\widetilde{d} = [\widetilde{d}_1, \widetilde{d}_2, \ldots, \widetilde{d}_n]$





Reduction from $PSm, 1 \,|\text{temp}|\, C_{max}$ to $PS1 \,|\text{temp}|\, C_{max}$

Reduction from $PSm, 1 \,|\text{temp}|\, C_{max}$ to $PS1 \,|\text{temp}|\, C_{max}$ is based on the projection of **each resource to the independent time window**. In other words, the schedule of tasks on P_j is projected into interval $\langle (j-1) \cdot UB, j \cdot UB \rangle$

Transformation consists of two steps:

- **Add dummy tasks** T_0 and T_{n+1} with $p_0 = p_{n+1} = 0$.
  - Task T_0, processed on P_1, precedes all tasks $T_i \in \mathcal{T}$, ie. $s_0 \leq s_i$.
  - Task T_{n+1}, processed on P_m, follows all task $T_i \in \mathcal{T}$, tj. $s_i + p_i \leq s_{n+1}$.
- **Transform the original temporal constraints** to $l'_{ij} = l_{ij} + (a_j - a_i) \cdot UB$.

Reduction from $PSm, 1 \, |\text{temp}| \, C_{max}$ to $PS1 \, |\text{temp}| \, C_{max}$

The new start time s'_i of each task on processor a_i is:
$s'_i = s_i + (a_i - 1) \cdot UB$.

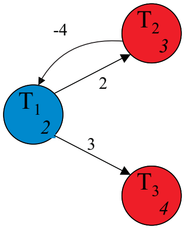
Temporal constraints $s_i + l_{ij} \leq s_j$ are transformed to:

$$s'_i - (a_i - 1) \cdot UB + l_{ij} \leq s'_j - (a_j - 1) \cdot UB$$
$$s'_i + l_{ij} + (a_j - a_i) \cdot UB \leq s'_j$$

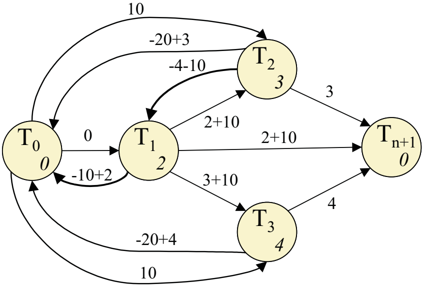
The transformed temporal constraint will look like $s'_i + l'_{ij} \leq s'_j$, where:

$$l'_{ij} = l_{ij} + (a_j - a_i) \cdot UB$$

Reduction from $PSm, 1\,|\text{temp}|\,C_{max}$ to $PS1\,|\text{temp}|\,C_{max}$



two dedicated resources



one resource

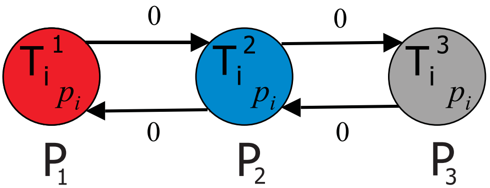
While minimizing the completion time of T_{n+1}, we push tasks T_1, T_2 and T_3 "to the left" due to the edges entering T_{n+1}

Multiprocessors Tasks

Transformation of multiprocessor tasks to $PSm, 1\,|\text{temp}|\,C_{max}$

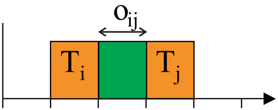
- create as many virtual tasks as there are processors needed to execute the physical tasks
- ensure that the virtual tasks of the given physical task start at the same time - this is done by two edges with weight $l_{ij} = l_{ji} = 0$. Consequently $s_i \leq s_j$ and $s_j \leq s_i$.

Example: Task T_i needs resources [P_1,P_2,P_3].



Changeover Time (i.e. Sequence Dependent Set-up Time)

The set-up time o_{ij} is a time needed to separate task T_i from T_j. It is used for example to change the tool in the machine.
Since the order of tasks is unknown in advance, we can not determine which set-up time will be used.



Reduction of the scheduling problem with the set-up time to $PSm, 1\,|\text{temp}|\,C_{max}$

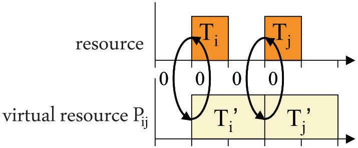
- for each pair of set-up constrained tasks add the virtual resource and a pair of extended virtual tasks
- ensure that the virtual task and the physical task start at the same time

Changeover Time (i.e. Sequence Dependent Set-up Time)

For each pair of tasks such that the set up time $o_{ij} > 0$ or $o_{ji} > 0$, the virtual resource P_{ij} and the corresponding virtual tasks T'_i and T'_j are added.

- Task T'_i has $p'_i = p_i + o_{ij}$ and task T'_j has $p'_j = p_j + o_{ji}$.
- Both tasks run on one virtual resource P_{ij}.
- Task T'_i (resp. T'_j) is synchronized with the original task by:

$$s_i \leq s'_i \quad s'_i \leq s_i \quad \text{resp.} \quad s_j \leq s'_j \quad s'_j \leq s_j$$



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