Method 1: Geometric Error Optimization

- we need to encode the constraints $\hat{\underline{\mathbf{y}}}_i \, \mathbf{F} \, \hat{\underline{\mathbf{x}}}_i = 0$, $\operatorname{rank} \mathbf{F} = 2$
- idea: reconstruct 3D point via equivalent projection matrices and use reprojection error
- equivalent projection matrices are see [H&Z,Sec. 9.5] for complete characterization

$$\mathbf{P}_1 = \begin{bmatrix} \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{P}_2 = \begin{bmatrix} [\underline{\mathbf{e}}_2]_\times \mathbf{F} + \underline{\mathbf{e}}_2 \underline{\mathbf{e}}_1^\top & \underline{\mathbf{e}}_2 \end{bmatrix}$$

⊛ H3; 2pt: Verify that $\mathbf{F}$ is a f.m. of $\mathbf{P}_1$, $\mathbf{P}_2$, for instance that $\mathbf{F} \simeq \mathbf{Q}_2^{-\top} \mathbf{Q}_1^\top [\mathbf{e}_1]_\times$

1. compute $\mathbf{F}^{(0)}$ by the 7-point algorithm → Slide 81; construct camera $\mathbf{P}_2^{(0)}$ from $\mathbf{F}^{(0)}$
2. triangulate 3D points $\hat{X}_i^{(0)}$ from correspondences (x_i, y_i) for all $i = 1, \ldots, k$ → Slide 85
3. express the energy function as reprojection error

$$W_i(x_i, y_i \mid \hat{X}_i, \mathbf{P}_2) = \|\mathbf{x}_i - \hat{\mathbf{x}}_i\|^2 + \|\mathbf{y}_i - \hat{\mathbf{y}}_i\|^2 \quad \text{where} \quad \hat{\underline{\mathbf{x}}}_i \simeq \mathbf{P}_1 \hat{\underline{\mathbf{X}}}_i, \; \hat{\underline{\mathbf{y}}}_i \simeq \mathbf{P}_2 \hat{\underline{\mathbf{X}}}_i$$

4. starting from $\mathbf{P}_2^{(0)}$, $\hat{X}^{(0)}$ minimize

$$(\hat{X}^*, \mathbf{P}_2^*) = \arg\min_{\mathbf{P}_2, \hat{X}} \sum_{i=1}^{k} W_i(x_i, y_i \mid \hat{X}_i, \mathbf{P}_2)$$

5. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2^*$

$P_2 \simeq \lambda P_2$ $\quad \|P_2\|^2 = 1 \quad P_{34} = 1$

$D = \operatorname{diag}(1, \alpha, 0)$

- $3k + 12$ parameters to be found: latent: $\hat{\mathbf{X}}_i$, for all i (correspondences!), non-latent: $\mathbf{P}_2$
- minimal representation: $3k + 7$ parameters, $\mathbf{P}_2 = \mathbf{P}_2(\mathbf{F})$ → Slide 138 $\quad UDV^T = F$
- there are pitfalls; this is essentially bundle adjustment; we will return to this later Slide 131

▶Method 2: First-Order Error Approximation

An elegant method for solving problems like (14):

- we will get rid of the latent parameters [H&Z, p. 287], [Sampson 1982]
- we will recycle the algebraic error $\boldsymbol{\varepsilon} = \underline{\mathbf{y}}^\top \mathbf{F}\, \underline{\mathbf{x}}$ from Slide 81

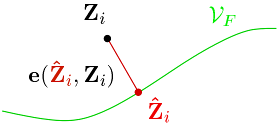
Observations:

- correspondences $\hat{x}_i \leftrightarrow \hat{y}_i$ satisfy $\underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i = 0$, $\quad \underline{\hat{\mathbf{x}}}_i = (\hat{u}^1, \hat{v}^1, 1)$, $\underline{\hat{\mathbf{y}}}_i = (\hat{u}^2, \hat{v}^2, 1)$
- this is a manifold $\mathcal{V}_F \in \mathbb{R}^4$: a set of points $\hat{\mathbf{Z}} = (\hat{u}^1,\, \hat{v}^1,\, \hat{u}^2,\, \hat{v}^2)$ consistent with $\mathbf{F}$
- let $\hat{\mathbf{Z}}_i$ be the closest point on $\mathcal{V}_F$ to measurement $\mathbf{Z}_i$, then (see (13))

$$\|\mathbf{Z}_i - \hat{\mathbf{Z}}_i\|^2 = (u_i^1 - \hat{u}_i^1)^2 + (v_i^1 - \hat{v}_i^1)^2 + (u_i^2 - \hat{u}_i^2)^2 + (v_i^2 - \hat{v}_i^2)^2 =$$

$$= V_i(x_i, y_i \mid \hat{x}_i, \hat{y}_i, \mathbf{F}) \stackrel{\text{def}}{=} \|\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)\|^2$$

which is what we needed in (14)



$\mathbf{Z}_i = (u^1, v^1, u^2, v^2)$ – measurement

algebraic error: $\boxed{\boldsymbol{\varepsilon}(\hat{\mathbf{Z}}_i) \stackrel{\text{def}}{=} \underline{\hat{\mathbf{y}}}_i^\top \mathbf{F}\, \underline{\hat{\mathbf{x}}}_i}$ $(= \mathbf{0})$

Sampson's idea: Linearize $\boldsymbol{\varepsilon}(\hat{\mathbf{Z}}_i)$ (with hat!) at $\mathbf{Z}_i$ (no hat!) and estimate $\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$ with it

▶Sampson's Idea

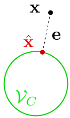
Linearize $\boldsymbol{\varepsilon}(\hat{\mathbf{Z}}_i)$ at $\mathbf{Z}_i$ per correspondence and estimate $\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$ with it

have: $\boldsymbol{\varepsilon}(\mathbf{Z}_i)$, want: $\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)$

$$\boldsymbol{\varepsilon}(\hat{\mathbf{Z}}_i) \approx \boldsymbol{\varepsilon}(\mathbf{Z}_i) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} \stackrel{\text{def}}{=} \boxed{\boldsymbol{\varepsilon}(\mathbf{Z}_i) + \mathbf{J}(\mathbf{Z}_i)\, \mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i) \stackrel{!}{=} 0}$$

Illustration on circle fitting

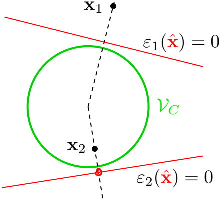
We are estimating distance from point $\mathbf{x}$ to circle $\mathcal{V}_C$ of radius r in canonical position. The circle is $\boldsymbol{\varepsilon}(\mathbf{x}) = \|\mathbf{x}\|^2 - r^2 = 0$. Then $\|\mathbf{x}\|^2 = \mathbf{x}^\top \mathbf{x}$

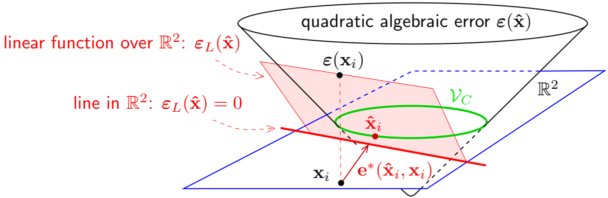


$$\boldsymbol{\varepsilon}(\hat{\mathbf{x}}) \approx \boldsymbol{\varepsilon}(\mathbf{x}) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}(\mathbf{x})}{\partial \mathbf{x}}}_{\mathbf{J}(\mathbf{x}) = 2\mathbf{x}^\top} \underbrace{(\hat{\mathbf{x}} - \mathbf{x})}_{\mathbf{e}(\hat{\mathbf{x}}, \mathbf{x})} = \cdots = 2\,\mathbf{x}^\top \hat{\mathbf{x}} - (r^2 + \|\mathbf{x}\|^2) \stackrel{\text{def}}{=} \boldsymbol{\varepsilon}_L(\hat{\mathbf{x}})$$

$= 0$

and $\boldsymbol{\varepsilon}_L(\hat{\mathbf{x}}) = 0$ is a <u>line</u> with normal $\frac{\mathbf{x}}{\|\mathbf{x}\|}$ and intercept $\frac{r^2 + \|\mathbf{x}\|^2}{2\|\mathbf{x}\|}$ not tangent to $\mathcal{V}_C$, outside!





►Sampson Error Approximation

In general, the Taylor expansion is

$$\boldsymbol{\varepsilon}(\mathbf{Z}_i) + \underbrace{\frac{\partial \boldsymbol{\varepsilon}(\mathbf{Z}_i)}{\partial \mathbf{Z}_i}}_{\mathbf{J}_i(\mathbf{Z}_i)} \underbrace{(\hat{\mathbf{Z}}_i - \mathbf{Z}_i)}_{\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)} = \underbrace{\boldsymbol{\varepsilon}(\mathbf{Z}_i)}_{\boldsymbol{\varepsilon}_i \in \mathbb{R}^n} + \underbrace{\mathbf{J}(\mathbf{Z}_i)}_{\mathbf{J}_i \in \mathbb{R}^{n,d}} \underbrace{\mathbf{e}(\hat{\mathbf{Z}}_i, \mathbf{Z}_i)}_{\mathbf{e}_i \in \mathbb{R}^d} \stackrel{!}{=} 0$$

to find $\hat{\mathbf{Z}}_i$ closest to $\mathbf{Z}_i$, we estimate $\mathbf{e}_i$ from $\boldsymbol{\varepsilon}_i$ by minimizing per correspondence $\mathbf{X}_i$

$$\mathbf{e}_i^* = \arg\min_{\mathbf{e}_i} \|\mathbf{e}_i\|^2 \quad \text{subject to} \quad \boldsymbol{\varepsilon}_i + \mathbf{J}_i\, \mathbf{e}_i = 0$$

which gives a closed-form solution ⊛ P1; 1pt: derive $\mathbf{e}_i^*$

$$\mathbf{e}_i^* = -\mathbf{J}_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \boldsymbol{\varepsilon}_i$$
$$\|\mathbf{e}_i^*\|^2 = \boldsymbol{\varepsilon}_i^\top (\mathbf{J}_i \mathbf{J}_i^\top)^{-1} \boldsymbol{\varepsilon}_i$$

- note that $\mathbf{J}_i$ is not invertible!
- we often do not need $\hat{\mathbf{Z}}_i$, just the squared distance $\|\mathbf{e}_i\|^2$ exception: triangulation → Slide 100
- the unknown parameters $\mathbf{F}$ are inside: $\mathbf{e}_i = \mathbf{e}_i(\mathbf{F})$, $\boldsymbol{\varepsilon}_i = \boldsymbol{\varepsilon}_i(\mathbf{F})$, $\mathbf{J}_i = \mathbf{J}_i(\mathbf{F})$

▶Sampson Error: Result for Fundamental Matrix Estimation

The fundamental matrix estimation problem becomes

$$\mathbf{F}^* = \arg\min_{\mathbf{F},\,\mathrm{rank}\,\mathbf{F}=2} \sum_{i=1}^{k} e_i^2(\mathbf{F})$$

Let $\mathbf{F} = \begin{bmatrix}\mathbf{F}_1 & \mathbf{F}_2 & \mathbf{F}_3\end{bmatrix}$ (per columns) $= \begin{bmatrix}(\mathbf{F}^1)^\top \\ (\mathbf{F}^2)^\top \\ (\mathbf{F}^3)^\top\end{bmatrix}$ (per rows), $\mathbf{S} = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$, then

Sampson

$\varepsilon_i = \underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i$	$\varepsilon_i \in \mathbb{R}$	scalar algebraic error from Slide 81
$\mathbf{J}_i = \left[\frac{\partial \varepsilon_i}{\partial u_i^1}, \frac{\partial \varepsilon_i}{\partial v_i^1}, \frac{\partial \varepsilon_i}{\partial u_i^2}, \frac{\partial \varepsilon_i}{\partial v_i^2}\right]$	$\mathbf{J}_i \in \mathbb{R}^{1,4}$	derivatives over point coords.
$e_i^2(\mathbf{F}) = \frac{\varepsilon_i^2}{\lVert \mathbf{J}_i \rVert^2}$	$e_i \in \mathbb{R}$	Sampson error

$$\mathbf{J}_i = \left[(\mathbf{F}_1)^\top \underline{\mathbf{y}}_i,\ (\mathbf{F}_2)^\top \underline{\mathbf{y}}_i,\ (\mathbf{F}^1)^\top \underline{\mathbf{x}}_i,\ (\mathbf{F}^2)^\top \underline{\mathbf{x}}_i\right] \qquad e_i^2(\mathbf{F}) = \frac{(\underline{\mathbf{y}}_i^\top \mathbf{F} \underline{\mathbf{x}}_i)^2}{\lVert \mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i \rVert^2 + \lVert \mathbf{S}\mathbf{F}^\top \underline{\mathbf{y}}_i \rVert^2}$$

- Sampson correction 'normalizes' the algebraic error
- automatically copes with multiplicative factors $\mathbf{F} \mapsto \lambda\mathbf{F}$
- actual optimization not yet covered → Slide 103

▸Back to Triangulation: The Golden Standard Method

We are given $\mathbf{P}_1$, $\mathbf{P}_2$ and a single correspondence $x \leftrightarrow y$ and we look for 3D point $\mathbf{X}$ projecting to x and y. → Slide 85

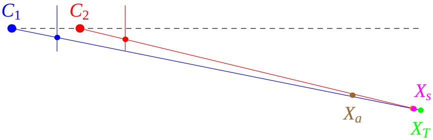
Idea:

1. compute $\mathbf{F}$ from $\mathbf{P}_1$, $\mathbf{P}_2$, e.g. $\mathbf{F} = (\mathbf{Q}_1\mathbf{Q}_2^{-1})^\top[\mathbf{q}_1 - (\mathbf{Q}_1\mathbf{Q}_2^{-1})\mathbf{q}_2]_\times$
2. correct measurement by the linear estimate of the correction vector → Slide 98

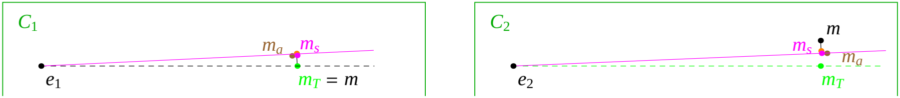
$$\begin{bmatrix}\hat{u}^1\\ \hat{v}^1\\ \hat{u}^2\\ \hat{v}^2\end{bmatrix} \approx \begin{bmatrix}u^1\\ v^1\\ u^2\\ v^2\end{bmatrix} - \frac{\varepsilon}{\|\mathbf{J}\|^2}\mathbf{J}^\top = \begin{bmatrix}u^1\\ v^1\\ u^2\\ v^2\end{bmatrix} - \frac{\underline{\mathbf{y}}^\top\mathbf{F}\underline{\mathbf{x}}}{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}\|^2 + \|\mathbf{S}\mathbf{F}^\top\underline{\mathbf{y}}\|^2}\begin{bmatrix}(\mathbf{F}_1)^\top\mathbf{y}\\ (\mathbf{F}_2)^\top\mathbf{y}\\ (\mathbf{F}^1)^\top\mathbf{x}\\ (\mathbf{F}^2)^\top\mathbf{x}\end{bmatrix}$$

3. use the SVD algorithm with numerical conditioning → Slide 86

Ex (cont'd from Slide 89):



X_T – noiseless ground truth position
● – reprojection error minimizer
X_s – Sampson-corrected algebraic error minimizer
X_a – algebraic error minimizer
m – measurement (m_T with noise in v^2)



Levenberg-Marquardt (LM) Iterative Estimation

Consider error function $\mathbf{e}_i(\boldsymbol{\theta}) = f(\mathbf{x}_i, \mathbf{y}_i, \boldsymbol{\theta}) \in \mathbb{R}^m$, with $\mathbf{x}_i, \mathbf{y}_i$ given, $\boldsymbol{\theta} \in \mathbb{R}^q$ unknown

$\theta = \mathbf{F}$, $q = 9$, $m = 1$ for f.m. estimation

Our goal: $\boldsymbol{\theta}^* = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{k} \|\mathbf{e}_i(\boldsymbol{\theta})\|^2$

Idea 1 (Gauss-Newton approximation): proceed iteratively for $s = 0, 1, 2, \ldots$

$$\boldsymbol{\theta}^{s+1} := \boldsymbol{\theta}^s + \mathbf{d}_s\,, \qquad \text{where} \qquad \mathbf{d}_s = \arg\min_{\mathbf{d}} \sum_{i=1}^{k} \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d})\|^2 \qquad (15)$$

$$\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}) \approx \|\mathbf{e}_i(\boldsymbol{\theta}^s) + \mathbf{L}_i\,\mathbf{d},\|^2$$

$$(\mathbf{L}_i)_{jl} = \frac{\partial\big(\mathbf{e}_i(\boldsymbol{\theta})\big)_j}{\partial(\boldsymbol{\theta})_l}, \qquad \mathbf{L}_i \in \mathbb{R}^{m,q}$$

typically a long matrix

Then the solution to Problem (15) is a set of normal eqs

$$-\underbrace{\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s)}_{\mathbf{e}\in\mathbb{R}^{q,1}} = \underbrace{\left(\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{L}_i\right)}_{\mathbf{L}\in\mathbb{R}^{q,q}} \mathbf{d}_s, \qquad (16)$$

$e = L d_s$

8 $k < q = 9$

$d_s = e \backslash L$

$d_s = \mathrm{inv}(L)\, e$

- $\mathbf{d}_s$ can be solved for by Gaussian elimination using Choleski decomposition of $\mathbf{L}$
  $\mathbf{L}$ symmetric $\Rightarrow$ use Choleski, almost $2\times$ faster than Gauss-Seidel, see bundle adjustment slide 134
- such updates do not lead to stable convergence $\longrightarrow$ ideas of Levenberg and Marquardt

LM (cont'd)

Idea 2 (Levenberg): replace $\sum_i \mathbf{L}_i^\top \mathbf{L}_i$ with $\sum_i \mathbf{L}_i^\top \mathbf{L}_i + \lambda \mathbf{I}$ for some damping factor $\lambda \geq 0$

Idea 3 (Marquardt): replace $\lambda \mathbf{I}$ with $\lambda \sum_i \operatorname{diag}(\mathbf{L}_i^\top \mathbf{L}_i)$ to adapt to local curvature:

$$-\sum_{i=1}^{k} \mathbf{L}_i^\top \mathbf{e}_i(\boldsymbol{\theta}^s) = \left(\sum_{i=1}^{k} (\mathbf{L}_i^\top \mathbf{L}_i + \lambda \operatorname{diag} \mathbf{L}_i^\top \mathbf{L}_i) \right) \mathbf{d}_s$$

diag $\ell \mapsto (1+\lambda)\,\ell$

non-diag $\ell \mapsto \ell$

Idea 4 (Marquardt): adaptive λ small $\lambda \to$ Gauss-Newton, large $\lambda \to$ gradient descend

1. choose $\lambda \approx 10^{-3}$ and compute $\mathbf{d}_s$
2. if $\sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s + \mathbf{d}_s)\|^2 < \sum_i \|\mathbf{e}_i(\boldsymbol{\theta}^s)\|^2$ then accept $\mathbf{d}_s$ and set $\lambda := \lambda/10$, $s := s + 1$
3. otherwise set $\lambda := 10\lambda$ and recompute $\mathbf{d}_s$

- sometimes different constants are needed for the 10 and 10^{-3}
- note that $\mathbf{L}_i \in \mathbb{R}^{m,q}$ (long matrix) but each contribution $\mathbf{L}_i^\top \mathbf{L}_i$ is a square singular $q \times q$ matrix (always singular for $k < q$)
- error can be made robust to outliers, see the trick on Slide 106
- we have approximated the least squares Hessian by ignoring second derivatives of the error function (Gauss-Newton approximation) See [Triggs et al. 1999, Sec. 4.3]
- λ helps avoid the consequences of gauge freedom → Slide 136

LM with Sampson Error for Fundamental Matrix Estimation

Sampson (derived by linearization over point coordinates u^1, v^1, u^2, v^2)

$$e_i^2(\mathbf{F}) = \frac{\varepsilon_i^2}{\|\mathbf{J}_i\|^2} = \frac{(\underline{\mathbf{y}}_i^\top \mathbf{F}\underline{\mathbf{x}}_i)^2}{\|\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i\|^2 + \|\mathbf{S}\mathbf{F}^\top\underline{\mathbf{y}}_i\|^2} \qquad \mathbf{S} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

LM (by linearization over parameters $\mathbf{F}$)

$$\mathbf{L}_i = \frac{\partial e_i(\mathbf{F})}{\partial \mathbf{F}} = \frac{1}{2\|\mathbf{J}_i\|} \left[\left(\underline{\mathbf{y}}_i - \frac{2e_i}{\|\mathbf{J}_i\|}\mathbf{S}\mathbf{F}\underline{\mathbf{x}}_i \right) \underline{\mathbf{x}}_i^\top + \underline{\mathbf{y}}_i \left(\underline{\mathbf{x}}_i - \frac{2e_i}{\|\mathbf{J}_i\|}\mathbf{S}\mathbf{F}^\top\underline{\mathbf{y}}_i \right)^\top \right]$$

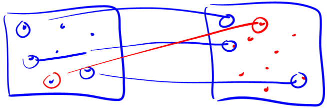
$F = UDV$

$D = (\sigma_1^2, \sigma_2^2, 0)$

- $\mathbf{L}_i$ is a 3×3 matrix, must be reshaped to dimension-9 vector
- $\underline{\mathbf{x}}_i$ and $\underline{\mathbf{y}}_i$ in Sampson error are normalized to unit homogeneous coordinate
- reinforce $\operatorname{rank} \mathbf{F} = 2$ after each LM update to stay in the fundamental matrix manifold and $\|\mathbf{F}\| = 1$ to avoid gauge freedom (by SVD, see Slide 104)
- LM linearization could be done by numerical differentiation (small dimension)

▶Local Optimization for Fundamental Matrix Estimation

Given a set $\{(x_i, y_i)\}_{i=1}^{k}$ of $k > 7$ inlier correspondences, compute an efficient estimate for fundamental matrix $\mathbf{F}$.



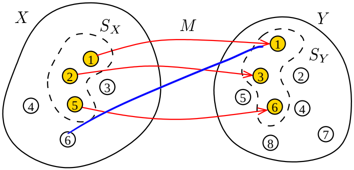
1. Find the conditioned (→ Slide 88) 7-point $\mathbf{F}_0$ (→ Slide 81) from a suitable 7-tuple
2. Improve the $\mathbf{F}_0^*$ using the LM optimization (→ Slides 101–102) and the Sampson error (→ Slide 103) on all inliers, reinforce rank-2, unit-norm $\mathbf{F}_k^*$ after each LM iteration using SVD

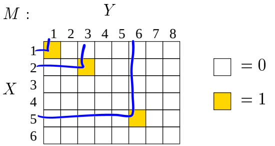
- if there are no wrong matches (outliers), this gives a local optimum
- contamination of (inlier) correspondences by outliers may wreak havoc with this algorithm
- the full problem involves finding the inliers!
- in addition, we need a mechanism for jumping out of local minima (and exploring the space of all fundamental matrices)

▶The Full Problem of Matching and Fundamental Matrix Estimation

Problem: Given two sets of image points $X = \{x_i\}_{i=1}^m$ and $Y = \{y_j\}_{j=1}^n$ and their descriptors D, find the most probable

1. inliers $S_X \subseteq X$, $S_Y \subseteq Y$
2. one-to-one perfect matching $M : S_X \to S_Y$ — perfect matching: 1-factor of the bipartite graph
3. fundamental matrix $\mathbf{F}$ such that $\operatorname{rank} \mathbf{F} = 2$
4. such that for each $x_i \in S_X$ and $y_j = M(x_i)$ it is probable that
   a. the image descriptor $D(x_i)$ is similar to $D(y_j)$, and
   b. the total geometric error $\sum_{ij} e_{ij}^2(\mathbf{F})$ is small — note a slight change in notation: e_{ij}
5. inlier-outlier and outlier-outlier matches are improbable

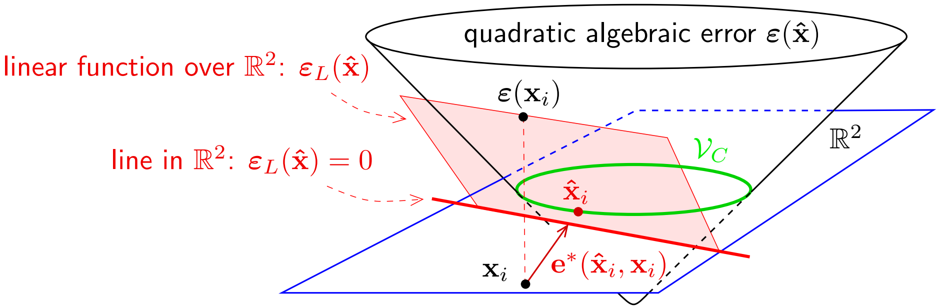


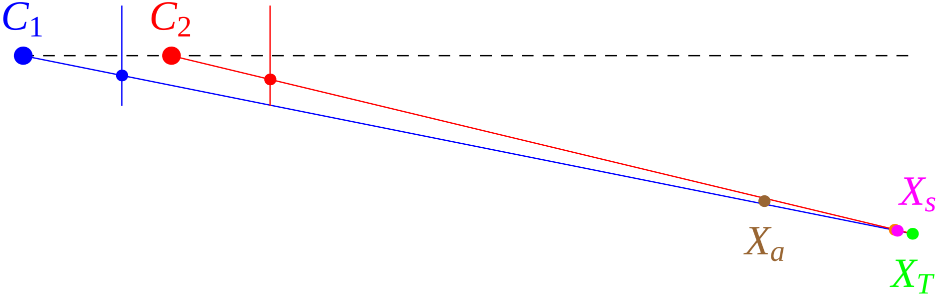


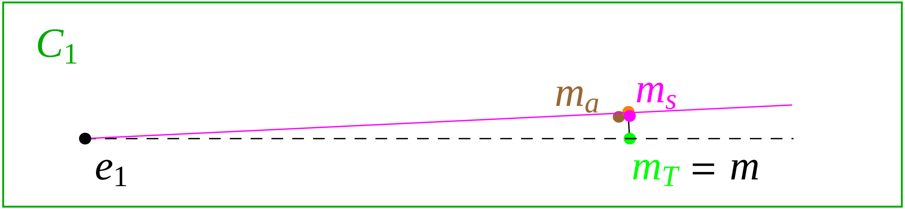
$$(M^*, \mathbf{F}^*) = \arg\max_{M,\mathbf{F}} p(M, \mathbf{F} \mid X, Y, D) \tag{17}$$

- probabilistic model: an efficient language for task formulation
- the (17) is a p.d.f. for all the involved variables (there is a constant number of variables!)
- binary matching table $M_{ij} \in \{0, 1\}$ of fixed size $m \times n$
  - each row/column contains at most one unity
  - zero rows/columns correspond to unmatched point x_i/y_j

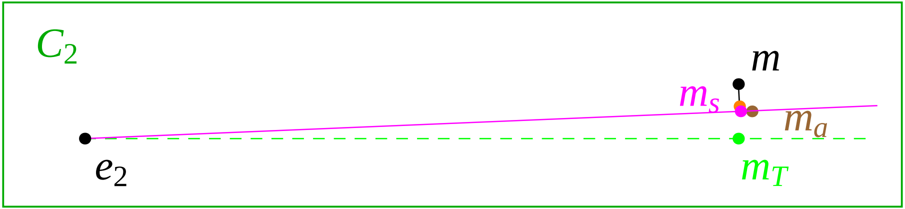
Thank You


quadratic algebraic error $\boldsymbol{\varepsilon}(\hat{\mathbf{x}})$
linear function over $\mathbb{R}^2$: $\boldsymbol{\varepsilon}_L(\hat{\mathbf{x}})$
$\boldsymbol{\varepsilon}(\mathbf{x}_i)$
line in $\mathbb{R}^2$: $\boldsymbol{\varepsilon}_L(\hat{\mathbf{x}}) = 0$
$\mathcal{V}_C$
$\mathbb{R}^2$
$\hat{\mathbf{x}}_i$
$\mathbf{x}_i$
$\mathbf{e}^*(\hat{\mathbf{x}}_i, \mathbf{x}_i)$


C_1
C_2
X_s
X_a
X_T



C_1
m_a
m_s
e_1
$m_T = m$


C_2
m
m_s
m_a
e_2
m_T