▶The Representation Theorem for Essential Matrices

Theorem

Let $\mathbf{E}$ *be a* 3×3 *matrix with SVD* $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$*. Then* $\mathbf{E}$ *is an essential matrix iff* $\mathbf{D} \simeq \mathrm{diag}(1, 1, 0)$*.*

Proof.

1. Part I: General properties of antisymmetric 3×3 matrices

2. Part II (direct):

   If $\mathbf{E}$ is essential then the it has two equal singular values and the third is zero.

3. Part III (converse):

   Let $\mathbf{A} = \hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top$ s.t. $\mathbf{D} = \mathrm{diag}(1, 1, 0)$ then $\mathbf{A} = [\hat{\mathbf{u}}_3]_\times \mathbf{R}$, where $\mathbf{R}$ is orthogonal, $\hat{\mathbf{u}}_3$ is the 3rd column of $\hat{\mathbf{U}}$, and $\mathbf{R} = \hat{\mathbf{U}}\mathbf{W}\hat{\mathbf{V}}^\top$, where $\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ and $|\alpha| = 1$.

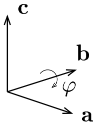
□

Proof, Part I: More Properties of Antisymmetric 3×3 Matrices

Given vector $\mathbf{b}$, let there be matrices $\mathbf{D}$, $\mathbf{W}$, $\mathbf{V}$

$$\mathbf{D} = \|\mathbf{b}\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \mathbf{V} = \left[\mathbf{a}, \ \mathbf{c}, \ \frac{\mathbf{b}}{\|\mathbf{b}\|}\right] \tag{11}$$

such that



1. $|\alpha| = 1$
2. $\|\mathbf{a}\| = \|\mathbf{c}\| = 1$
3. $\mathbf{a}$, $\mathbf{c}$, $\mathbf{b}$ mutually orthogonal: $\mathbf{V}^\top \mathbf{V} = \mathbf{I}$
4. $\det \mathbf{V} = 1$

note that

- $\mathbf{W}^\top \mathbf{W} = \mathbf{I}$; $\mathbf{W}$ is a rotation by 90°
- if $\alpha \mapsto -\alpha$ then $\mathbf{W} \mapsto \mathbf{W}^\top$
- $\mathbf{a}$, $\mathbf{c}$ are determined up to a rotation φ about $\mathbf{b}$, $\hat{\mathbf{V}} = \mathbf{T}_\varphi \mathbf{V}$, $\mathbf{T}_\varphi \mathbf{b} = \mathbf{b}$

Theorem (A)

Let $\mathbf{V}$, $\mathbf{D}$, $\mathbf{W}$, $\mathbf{T}_\varphi$ *be defined as above. Then* $\hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top$ *is an SVD of* $[\mathbf{b}]_\times$ *iff* $\hat{\mathbf{U}} = \mathbf{T}_\varphi \mathbf{V} \mathbf{W}^\top$, $\hat{\mathbf{V}} = \mathbf{T}_\varphi \mathbf{V}$ *for some* φ.

It follows $\hat{\mathbf{U}} = \hat{\mathbf{V}}\mathbf{W}^\top$ for any φ and $\hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top = \hat{\mathbf{V}}\mathbf{W}^\top\mathbf{D}\hat{\mathbf{V}}^\top = \hat{\mathbf{U}}\mathbf{D}\mathbf{W}^\top\hat{\mathbf{U}}$

Proof of Theorem A.

1. Converse ($\hat{\mathbf{U}}$, $\hat{\mathbf{V}}$, $\mathbf{D}$, $\mathbf{V}$, $\mathbf{W}$, $\mathbf{T}_\varphi$ as defined $\Rightarrow$ $\hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top$ is an SVD of $[\mathbf{b}]_\times$):
   a. $\hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top = \underbrace{\mathbf{T}_\varphi \mathbf{V}\mathbf{W}^\top}_{\hat{\mathbf{U}}} \mathbf{D} \underbrace{\mathbf{V}^\top \mathbf{T}_\varphi^\top}_{\hat{\mathbf{V}}^\top}$ is indeed an SVD of some matrix for any φ.
   b. what matrix?

$$\begin{aligned}\mathbf{T}_\varphi \mathbf{V}\mathbf{W}^\top \mathbf{D}\mathbf{V}^\top \mathbf{T}_\varphi^\top &= \mathbf{T}_\varphi \|\mathbf{b}\| \left(\mathbf{c}\mathbf{a}^\top - \mathbf{a}\mathbf{c}^\top\right)\mathbf{T}_\varphi^\top = \|\mathbf{b}\|\, \mathbf{T}_\varphi [\mathbf{a}\times\mathbf{c}]_\times \mathbf{T}_\varphi^\top = \\ &= \mathbf{T}_\varphi [\mathbf{b}]_\times \mathbf{T}_\varphi^\top = [\mathbf{T}_\varphi \mathbf{b}]_\times = [\mathbf{b}]_\times\end{aligned} \tag{12}$$

   hence it is an SVD of $[\mathbf{b}]_\times$ but also of $[\mathbf{T}_\varphi \mathbf{b}]_\times$ for any φ

2. Direct: For every φ we go backward in (12) and obtain an SVD.

□

We are proving (from Slide 78):

Part II

If $\mathbf{E}$ is essential then the it has two equal singular values and the third is zero.

- The $\mathbf{E}$ is essential, hence $\mathbf{E} \simeq [\mathbf{t}]_\times \mathbf{R}$
- Let $\hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top$ be the SVD of $[\mathbf{t}]_\times$. Then, by Theorem A, $\underbrace{\hat{\mathbf{U}}}_{\text{orthogonal}}\ \mathbf{D}\ \underbrace{\hat{\mathbf{V}}^\top\mathbf{R}}_{\text{orthogonal}}$ is an SVD of $\mathbf{E}$ with singular values $\mathbf{D} = \|\mathbf{t}\| \operatorname{diag}(1,1,0)$.

Part III

Let $\mathbf{A} = \hat{\mathbf{U}}\mathbf{D}\hat{\mathbf{V}}^\top$ s.t. $\mathbf{D} = \operatorname{diag}(1,1,0)$ then $\mathbf{A} = [\hat{\mathbf{u}}_3]_\times \mathbf{R}$, where $\mathbf{R}$ is orthogonal.

$$\hat{\mathbf{U}}\mathbf{D}\underbrace{\hat{\mathbf{V}}^\top}_{\text{choice: } \mathbf{W}\hat{\mathbf{U}}^\top\mathbf{R}} = \hat{\mathbf{U}}\mathbf{D}\mathbf{W}\hat{\mathbf{U}}^\top\mathbf{R} = [\hat{\mathbf{u}}_3]_\times \mathbf{R}$$

$$\mathbf{D}\mathbf{W} = \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ hence } \hat{\mathbf{U}}\mathbf{D}\mathbf{W}\hat{\mathbf{U}}^\top = \underbrace{\hat{\mathbf{u}}_1\hat{\mathbf{u}}_2^\top - \hat{\mathbf{u}}_2\hat{\mathbf{u}}_1^\top}_{\text{antisymmetric with null space } \mathbf{u}_3} = [\hat{\mathbf{u}}_3]_\times$$

where we have defined $\hat{\mathbf{V}}$ s.t. $\mathbf{R} = \hat{\mathbf{U}}\mathbf{W}\hat{\mathbf{V}}^\top$

▶Essential Matrix Decomposition

Essential matrix captures relative camera position [Longuet-Higgins 1981]

$$\mathbf{E} = [-\mathbf{t}_{21}]_\times \mathbf{R}_{21} = [\mathbf{R}_2 \mathbf{b}]_\times \mathbf{R}_{21} = \mathbf{R}_{21} [\mathbf{R}_1 \mathbf{b}]_\times$$

1. $\operatorname{rank} \mathbf{E} = 2$ since $\operatorname{rank} [\mathbf{t}_{21}]_\times = 2$
2. Let $\mathbf{E} = \mathbf{U}\mathbf{D}\mathbf{V}^\top$ be the SVD of $\mathbf{E}$ s.t. $\mathbf{D} = \operatorname{diag}(1,1,0)$. Then [H&Z, sec. 9.6]
   a. in case $\det \mathbf{U} < 0$ transform it to $-\mathbf{U}$, do the same for $\mathbf{V}$
   b. compute

$$\mathbf{R}_{21} = \mathbf{U} \begin{bmatrix} 0 & \alpha & 0 \\ -\alpha & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{V}^\top, \quad \mathbf{t}_{21} = -\mathbf{U} \begin{bmatrix} 0 \\ 0 \\ \beta \end{bmatrix}, \qquad |\alpha| = 1, \quad \beta \neq 0 \tag{13}$$

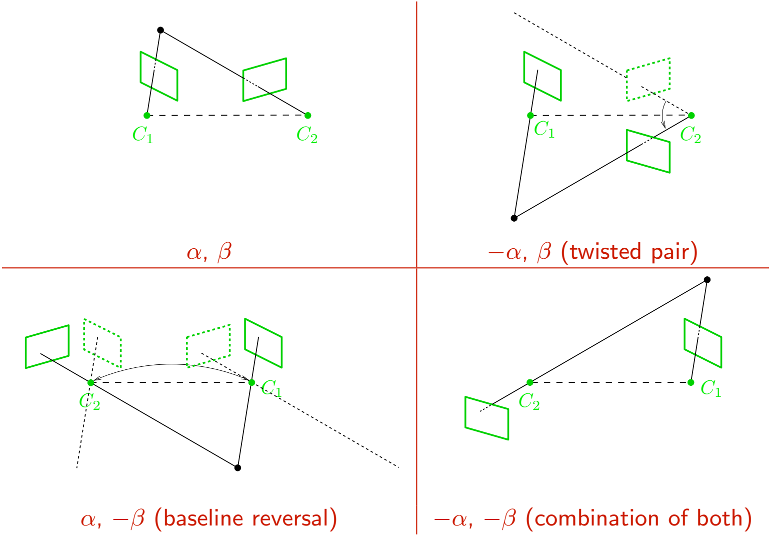
Notes

- the result for $\mathbf{R}_{21}$ is unique up to $\alpha = \pm 1$ despite non-uniqueness of SVD
- change of sign in $\mathbf{W}$ rotates the solution by 180° about $\mathbf{t}$
  $\mathbf{R}_1 = \mathbf{U}\mathbf{W}\mathbf{V}^\top$, $\mathbf{R}_2 = \mathbf{U}\mathbf{W}^\top\mathbf{V}^\top \Rightarrow \mathbf{T} = \mathbf{R}_2\mathbf{R}_1^\top = \cdots = \mathbf{U}\operatorname{diag}(-1,-1,1)\mathbf{U}^\top$ which is a rotation by 180° about $\mathbf{u}_3 = \mathbf{t}_{21}$:

$$\mathbf{U}\operatorname{diag}(-1,-1,1)\mathbf{U}^\top \mathbf{u}_3 = \mathbf{U} \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \mathbf{u}_3$$

- $\mathbf{t}_{21}$ recoverable up to scale β and direction $\operatorname{sign} \beta$
- 4 solution sets for 4 sign combinations of α, β see next for geometric interpretation

►Four Solutions to Essential Matrix Decomposition



α, β

$-\alpha$, β (twisted pair)

α, $-\beta$ (baseline reversal)

$-\alpha$, $-\beta$ (combination of both)

- chirality constraint: all 3D points are in front of both cameras
- this singles-out the upper left case [H&Z, Sec. 9.6.3]

▶7-Point Algorithm for Estimating Fundamental Matrix

Problem: Given a set $\{(x_i, y_i)\}_{i=1}^{k}$ of $k = 7$ correspondences, estimate f. m. $\mathbf{F}$.

$$\underline{\mathbf{y}}_i^\top \mathbf{F}\, \underline{\mathbf{x}}_i = 0, \quad i = 1, \ldots, k, \qquad \text{known:} \quad \underline{\mathbf{x}}_i = (x_{i1}, x_{i2}, 1), \quad \underline{\mathbf{y}}_i = (y_{i1}, y_{i2}, 1)$$

terminology: correspondence = truth, later: match = algorithm's result; hypothesised corresp.

Solution:

$$\mathbf{D} = \begin{bmatrix} x_{11}y_{11} & x_{11}y_{12} & x_{11} & x_{12}y_{11} & x_{12}y_{12} & x_{12} & y_{11} & y_{12} & 1 \\ x_{21}y_{21} & x_{21}y_{22} & x_{21} & x_{22}y_{21} & x_{22}y_{22} & x_{22} & y_{21} & y_{22} & 1 \\ \vdots & & & & & & & & \vdots \\ x_{k1}y_{k1} & x_{k1}y_{k2} & x_{k1} & x_{k2}y_{k1} & x_{k2}y_{k2} & x_{k2} & y_{k1} & y_{k2} & 1 \end{bmatrix}, \qquad \mathbf{D} \in \mathbb{R}^{k,9}$$

$$\mathbf{D}\mathbf{f} = \mathbf{0}, \qquad \mathbf{f} = \begin{bmatrix} f_{11} & f_{21} & f_{31} & \ldots & f_{33} \end{bmatrix}^\top, \qquad \mathbf{f} \in \mathbb{R}^9,$$

- for $k = 7$ we have a rank-deficient system, the null-space of $\mathbf{D}$ is 2-dimensional
- but we know that $\det \mathbf{F} = 0$
- 7-point algorithm:
  1. find a basis of the null space of $\mathbf{D}$: $\mathbf{F}_1$, $\mathbf{F}_2$ — by SVD or QR factorization
  2. get up to 3 real solutions for α from

     $$\det(\alpha \mathbf{F}_1 + (1-\alpha)\mathbf{F}_2) = 0 \qquad \text{cubic equation in } \alpha$$

  3. get up to 3 fundamental matrices $\mathbf{F} = \alpha_i \mathbf{F}_1 + (1-\alpha_i)\mathbf{F}_2$

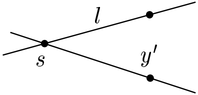
- the result may depend on image transformations
- normalization improves conditioning Slide 91
- this gives a good starting point for the full algorithm Slide 110

▸Degenerate Configurations for Fundamental Matrix Estimation

When is $\mathbf{F}$ not uniquely determined from any number of correspondences? [H&Z, Sec. 11.9]

1. camera centers coincide $C_1 = C_2$
   - epipolar geometry is not defined
   - images are related by homography $\mathbf{H}$
   - we do get an $\mathbf{F}$ from the 7-point algorithm but it is of the form of $\mathbf{F} = \mathbf{S}\mathbf{H}$, with $\mathbf{S}$ antisymmetric

$\underline{\mathbf{y}} \simeq \mathbf{H}\underline{\mathbf{x}}$



$\underline{\mathbf{l}} \simeq \underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}$ arbitrary s

$$y \in l:\, 0 = \underline{\mathbf{y}}^\top (\underline{\mathbf{s}} \times \mathbf{H}\underline{\mathbf{x}}) = \underline{\mathbf{y}}^\top \underbrace{[\underline{\mathbf{s}}]_\times \mathbf{H}}_{\mathbf{S}\mathbf{H}} \underline{\mathbf{x}}$$

2. <u>all 3D points lie in a plane</u>
   - images related by homography
   - again, $\mathbf{F}$ is not unique, $\mathbf{F} = \mathbf{S}\mathbf{H}$, where $\mathbf{S}$ is as above

note essential matrix estimation <u>can</u> deal with planes, Slide 87

3. both camera centers and all 3D points lie on a ruled quadric

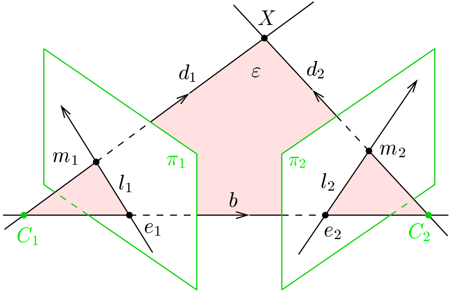
   hyperboloid of one sheet, cones, cylinders, two planes
   - there are 3 solutions for $\mathbf{F}$

notes

- a complete treatment with additional degenerate configurations in [H&Z, sec. 22.2]
- stronger epipolar constraint can reject some configurations
- we assume correct correspondences, dealing with mismatches need not be a part of the 7-point algorithm → Slide 112

A Note on Oriented Epipolar Constraint

- a tighter epipolar constraint preserves orientations
- requires all points and cameras be on the same side of the plane at infinity



$$\underline{\mathbf{e}}_2 \times \underline{\mathbf{m}}_2 \overset{+}{\sim} \mathbf{F}\, \underline{\mathbf{m}}_1$$

notation: $\underline{\mathbf{m}} \overset{+}{\sim} \underline{\mathbf{n}}$ means $\underline{\mathbf{m}} = \lambda \underline{\mathbf{n}}$, $\lambda > 0$

- note that the constraint is not invariant to the change of either sign of $\underline{\mathbf{m}}_i$
- all 7 correspondence in 7-point alg. must have the same sign — see later
- this may help reject some wrong matches, see Slide 112 — [Chum et al. 2004]
- an even more tight constraint: scene points in front of both cameras — expensive

this is called chirality constraint

►Five-Point Algorithm for Relative Camera Orientation

Problem: Given $\{\underline{\mathbf{m}}_i, \underline{\mathbf{m}}'_i\}_{i=1}^5$ corresponding image points and calibration matrix $\mathbf{K}$, recover the camera motion $\mathbf{R}$, $\mathbf{t}$.

Obs:

1. $\mathbf{R}$ – 3DOF, $\mathbf{t}$ – we can recover 2DOF only, in total 5 DOF $\rightarrow$ we need 3 constraints on $\mathbf{E}$
2. real $\mathbf{F} \in \mathbb{R}^{3,3}$ is a fundamental matrix iff $\det \mathbf{F} = 0$
3. fundamental matrix is essential iff its two non-zero eigenvalues are equal

This gives an equation system:

$$\underline{\mathbf{v}}_i^\top \mathbf{E}\, \underline{\mathbf{v}}'_i = 0 \qquad \text{5 linear constraints } (\underline{\mathbf{v}} = \mathbf{K}^{-1}\underline{\mathbf{m}})$$

$$\det \mathbf{E} = 0 \qquad \text{1 cubic constraint}$$

$$\mathbf{E}\mathbf{E}^\top\mathbf{E} - \frac{1}{2}\operatorname{tr}(\mathbf{E}\mathbf{E}^\top)\mathbf{E} = \mathbf{0} \qquad \text{9 cubic constraints, 2 independent}$$

1. estimate $\mathbf{E}$ by SVD from $\underline{\mathbf{v}}_i^\top \mathbf{E}\, \underline{\mathbf{v}}'_i = 0$ by the null-space method, this gives $\mathbf{E} = x\mathbf{E}_1 + y\mathbf{E}_2 + z\mathbf{E}_3 + \mathbf{E}_4$
2. at most 10 (complex) solutions for x, y, z from the cubic constraints

- when all 3D points lie on a plane: at most 2 solutions (twisted-pair)

  can be disambiguated in 3 views

  or by chirality constraint (Slide 83) unless all 3D points are closer to one camera
- 6-point problem for unknown f [Kukelova et al. BMVC 2008]
- resources at `http://cmp.felk.cvut.cz/minimal/5_pt_relative.php`