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EXPOSITION OF A NEW THEORY ON THE MEASUREMENT OF RISK[1]

BY DANIEL BERNOULLI

§1. EVER SINCE mathematicians first began to study the measurement of risk there has been general agreement on the following proposition: *Expected values are computed by multiplying each possible gain by the number of ways in which it can occur, and then dividing the sum of these products by the total number of possible cases where, in this theory, the consideration of cases which are all of the same probability is insisted upon.* If this rule be accepted, what remains to be done within the framework of this theory amounts to the enumeration of all alternatives, their breakdown into equi-probable cases and, finally, their insertion into corresponding classifications.

§2. Proper examination of the numerous demonstrations of this proposition that have come forth indicates that they all rest upon one hypothesis: *since there is no reason to assume that of two persons encountering identical risks,*[2] *either*

[1] Translated from Latin into English by Dr. Louise Sommer, The American University, Washington, D. C., from "Specimen Theoriae Novae de Mensura Sortis," *Commentarii Academiae Scientiarum Imperialis Petropolitanae*, Tomus V [*Papers of the Imperial Academy of Sciences in Petersburg*, Vol. V], 1738, pp. 175–192. Professor Karl Menger, Illinois Institute of Technology has written footnotes 4, 9, 10, and 15.

EDITOR'S NOTE: In view of the frequency with which Bernoulli's famous paper has been referred to in recent economic discussion, it has been thought appropriate to make it more generally available by publishing this English version. In her translation Professor Sommer has sought, in so far as possible, to retain the eighteenth century spirit of the original. The mathematical notation and much of the punctuation are reproduced without change. References to some of the recent literature concerned with Bernoulli's theory are given at the end of the article.

TRANSLATOR'S NOTE: I highly appreciate the help of Karl Menger, Professor of Mathematics, Illinois Institute of Technology, a distinguished authority on the Bernoulli problem, who has read this translation and given me expert advice. I am also grateful to Mr. William J. Baumol, Professor of Economics, Princeton University, for his valuable assistance in interpreting Bernoulli's paper in the light of modern econometrics. I wish to thank also Mr. John H. Klingenfeld, Economist, U. S. Department of Labor, for his cooperation in the English rendition of this paper. The translation is based solely upon the original Latin text.

BIOGRAPHICAL NOTE: Daniel Bernoulli, a member of the famous Swiss family of distinguished mathematicians, was born in Groningen, January 29, 1700 and died in Basle, March 17, 1782. He studied mathematics and medical sciences at the University of Basle. In 1725 he accepted an invitation to the newly established academy in Petersburg, but returned to Basle in 1733 where he was appointed professor of physics and philosophy. Bernoulli was a member of the academies of Paris, Berlin, and Petersburg and the Royal Academy in London. He was the first to apply mathematical analysis to the problem of the movement of liquid bodies.

(On Bernoulli see: *Handwörterbuch der Naturwissenschaften*, second edition, 1931, pp. 800–801; "Die Basler Mathematiker Daniel Bernoulli und Leonhard Euler. Hundert Jahre nach ihrem Tode gefeiert von der Naturforschenden Gesellschaft," Basle, 1884 (Annex to part VII of the proceedings of this Society); and *Correspondance mathématique* . . ., edited by Paul Heinrich Fuss, 1843 containing letters written by Daniel Bernoulli to Leonhard Euler, Nicolaus Fuss, and C. Goldbach.)

[2] i.e., risky propositions (gambles). [Translator]

should expect to have his desires more closely fulfilled, the risks anticipated by each must be deemed equal in value. No characteristic of the persons themselves ought to be taken into consideration; only those matters should be weighed carefully that pertain to the terms of the risk. The relevant finding might then be made by the highest judges established by public authority. But really there is here no need for judgment but of deliberation, i.e., rules would be set up whereby anyone could estimate his prospects from any risky undertaking in light of one's specific financial circumstances.

§3. To make this clear it is perhaps advisable to consider the following example: Somehow a very poor fellow obtains a lottery ticket that will yield with equal probability either nothing or twenty thousand ducats. Will this man evaluate his chance of winning at ten thousand ducats? Would he not be ill-advised to sell this lottery ticket for nine thousand ducats? To me it seems that the answer is in the negative. On the other hand I am inclined to believe that a rich man would be ill-advised to refuse to buy the lottery ticket for nine thousand ducats. If I am not wrong then it seems clear that all men cannot use the same rule to evaluate the gamble. The rule established in §1 must, therefore, be discarded. But anyone who considers the problem with perspicacity and interest will ascertain that the concept of *value* which we have used in this rule may be defined in a way which renders the entire procedure universally acceptable without reservation. To do this the determination of the *value* of an item must not be based on its *price*, but rather on the *utility* it yields. The price of the item is dependent only on the thing itself and is equal for everyone; the utility, however, is dependent on the particular circumstances of the person making the estimate. Thus there is no doubt that a gain of one thousand ducats is more significant to a pauper than to a rich man though both gain the same amount.

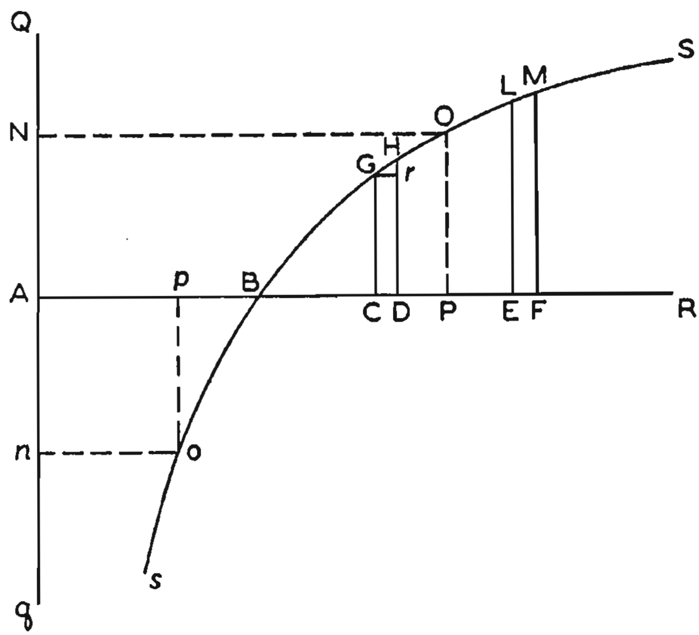
§4. The discussion has now been developed to a point where anyone may proceed with the investigation by the mere paraphrasing of one and the same principle. However, since the hypothesis is entirely new, it may nevertheless require some elucidation. I have, therefore, decided to explain by example what I have explored. Meanwhile, let us use this as a fundamental rule: *If the utility of each possible profit expectation is multiplied by the number of ways in which it can occur, and we then divide the sum of these products by the total number of possible cases, a mean utility*[3] *[moral expectation] will be obtained, and the profit which corresponds to this utility will equal the value of the risk in question.*

§5. Thus it becomes evident that no valid measurement of the value of a risk can be obtained without consideration being given to its *utility*, that is to say, the utility of whatever gain accrues to the individual or, conversely, how much profit is required to yield a given utility. However it hardly seems plausible to make any precise generalizations since the utility of an item may change with circumstances. Thus, though a poor man generally obtains more utility than does a rich man from an equal gain, it is nevertheless conceivable, for example,

[3] Free translation of Bernoulli's "emolumentum medium," literally: "mean utility." [Translator]

that a rich prisoner who possesses two thousand ducats but needs two thousand ducats more to repurchase his freedom, will place a higher value on a gain of two thousand ducats than does another man who has less money than he. Though innumerable examples of this kind may be constructed, they represent exceedingly rare exceptions. We shall, therefore, do better to consider what usually happens, and in order to perceive the problem more correctly we shall assume that there is an imperceptibly small growth in the individual's wealth which proceeds continuously by infinitesimal increments. Now it is highly probable that *any increase in wealth, no matter how insignificant, will always result in an increase in utility which is inversely proportionate to the quantity of goods already possessed.* To explain this hypothesis it is necessary to define what is meant by the *quantity of goods.* By this expression I mean to connote food, clothing, all things which add to the conveniences of life, and even to luxury—anything that can contribute to the adequate satisfaction of any sort of want. There is then nobody who can be said to possess nothing at all in this sense unless he starves to death. For the great majority the most valuable portion of their possessions so defined will consist in their productive capacity, this term being taken to include even the beggar's talent: a man who is able to acquire ten ducats yearly by begging will scarcely be willing to accept a sum of fifty ducats on condition that he henceforth refrain from begging or otherwise trying to earn money. For he would have to live on this amount, and after he had spent it his existence must also come to an end. I doubt whether even those who do not possess a farthing and are burdened with financial obligations would be willing to free themselves of their debts or even to accept a still greater gift on such a condition. But if the beggar were to refuse such a contract unless immediately paid no less than one hundred ducats and the man pressed by creditors similarly demanded one thousand ducats, we might say that the former is possessed of wealth worth one hundred, and the latter of one thousand ducats, though in common parlance the former owns nothing and the latter less than nothing.

§6. Having stated this definition, I return to the statement made in the previous paragraph which maintained that, in the absence of the unusual, the *utility resulting from any small increase in wealth will be inversely proportionate to the quantity of goods previously possessed.* Considering the nature of man, it seems to me that the foregoing hypothesis is apt to be valid for many people to whom this sort of comparison can be applied. Only a few do not spend their entire yearly incomes. But, if among these, one has a fortune worth a hundred thousand ducats and another a fortune worth the same number of semi-ducats and if the former receives from it a yearly income of five thousand ducats while the latter obtains the same number of semi-ducats it is quite clear that to the former a ducat has exactly the same significance as a semi-ducat to the latter, and that, therefore, the gain of one ducat will have to the former no higher value than the gain of a semi-ducat to the latter. Accordingly, if each makes a gain of one ducat the latter receives twice as much utility from it, having been enriched by two semi-ducats. This argument applies to many other cases which, therefore, need not be discussed separately. The proposition is all the more valid for the majority of men who possess no fortune apart from their working capacity which is their only source of livelihood. True, there are men to whom one ducat means more than many ducats do to others who are less rich but more generous than they. But since we shall now concern ourselves only with one individual (in different states of affluence) distinctions of this sort do not concern us. The man who is emotionally less affected by a gain will support a loss with greater patience. Since, however, in special cases things can conceivably occur otherwise, I shall first deal with the most general case and then develop our special hypothesis in order thereby to satisfy everyone.



§7. Therefore, let AB represent the quantity of goods initially possessed. Then after extending AB, a curve $BGLS$ must be constructed, whose ordinates CG, DH, EL, FM, etc., designate *utilities* corresponding to the abscissas BC, BD, BE, BF, etc., designating gains in wealth. Further, let m, n, p, q, etc., be the numbers which indicate the number of ways in which gains in wealth BC, BD, BE, BF [misprinted in the original as CF], etc., can occur. Then (in accord with §4) the *moral* expectation of the risky proposition referred to is given by:

$$PO = \frac{m.CG + n.DH + p.EL + q.FM + \cdots}{m + n + p + q + \cdots}$$

Now, if we erect AQ perpendicular to AR, and on it measure off $AN = PO$, the straight line $NO - AB$ represents the gain which may properly be expected, or the value of the risky proposition in question. If we wish, further, to know how large a stake the individual should be willing to venture on this risky proposition, our curve must be extended in the opposite direction in such a way that the abscissa Bp now represents a loss and the ordinate po represents the corresponding decline in utility. Since in a fair game the disutility to be suffered by losing must be equal to the utility to be derived by winning, we must assume that $An = AN$, or $po = PO$. Thus Bp will indicate the stake more than which persons who consider their own pecuniary status should not venture.

COROLLARY I

§8. Until now scientists have usually rested their hypothesis on the assumption that all gains must be evaluated exclusively in terms of themselves, i.e., on the basis of their intrinsic qualities, and that these gains will always produce a *utility* directly proportionate to the gain. On this hypothesis the curve BS becomes a straight line. Now if we again have:

$$PO = \frac{m.CG + n.DH + p.EL + q.FM + \cdots}{m + n + p + q + \cdots},$$

and if, on both sides, the respective factors are introduced it follows that:

$$BP = \frac{m.BC + n.BD + p.BE + q.BF + \cdots}{m + n + p + q + \cdots},$$

which is in conformity with the usually accepted rule.

COROLLARY II

§9. If AB were infinitely great, even in proportion to BF, the greatest possible gain, the arc BM may be considered very like an infinitesimally small straight line. Again in this case the usual rule [for the evaluation of risky propositions] is applicable, and may continue to be considered approximately valid in games of insignificant moment.

§10. Having dealt with the problem in the most general way we turn now to the aforementioned particular hypothesis, which, indeed, deserves prior attention to all others. First of all the nature of curve sBS must be investigated under the conditions postulated in §7. Since on our hypothesis we must consider infinitesimally small gains, we shall take gains BC and BD to be nearly equal, so that their difference CD becomes infinitesimally small. If we draw Gr parallel to BR, then rH will represent the infinitesimally small gain in *utility* to a man whose fortune is AC and who obtains the small gain, CD. This *utility*, however, should be related not only to the tiny gain CD, to which it is, other things being equal, proportionate, but also to AC, the fortune previously owned to which it is inversely proportionate. We therefore set: $AC = x$, $CD = dx$, $CG = y$, $rH = dy$ and $AB = \alpha$; and if b designates some constant we obtain $dy = \frac{bdx}{x}$ or $y =$

$b \log \frac{x}{\alpha}$. The curve sBS is therefore a logarithmic curve, the subtangent[4] of which is everywhere b and whose asymptote is Qq.

§11. If we now compare this result with what has been said in paragraph 7, it will appear that: $PO = b \log AP/AB, CG = b \log AC/AB, DH = b \log AD/AB$ and so on; but since we have

$$PO = \frac{m.CG + n.DH + p.EL + q.FM + \cdots}{m + n + p + q + \cdots}$$

it follows that

$$b \log \frac{AP}{AB} = \left(mb \log \frac{AC}{AB} + nb \log \frac{AD}{AB} + pb \log \frac{AE}{AB} + qb \log \frac{AF}{AB} + \cdots\right) : (m + n + p + q + \cdots)$$

and therefore

$$AP = (AC^m.AD^n.AE^p.AF^q. \cdots)^{1/m+n+p+q+\cdots}$$

and if we subtract AB from this, the remaining magnitude, BP, will represent the value of the risky proposition in question.

§12. Thus the preceding paragraph suggests the following rule: *Any gain must be added to the fortune previously possessed, then this sum must be raised to the power given by the number of possible ways in which the gain may be obtained; these terms should then be multiplied together. Then of this product a root must be extracted the degree of which is given by the number of all possible cases, and finally the value of the initial possessions must be subtracted therefrom; what then remains indicates the value of the risky proposition in question.* This principle is essential for the measurement of the value of risky propositions in various cases. I would elaborate it into a complete theory as has been done with the traditional analysis, were it not that, despite its usefulness and originality, previous obligations do not permit me to undertake this task. I shall therefore, at this time, mention only the more significant points among those which have at first glance occurred to me.

[4] The tangent to the curve $y = b \log \frac{x}{\alpha}$ at the point $\left(x_0, \log \frac{x_0}{\alpha}\right)$ is the line $y - b \log \frac{x_0}{\alpha} = \frac{b}{x_0}(x - x_0)$. This tangent intersects the Y-axis ($x = 0$) at the point with the ordinate $b \log \frac{x_0}{\alpha} - b$. The point of contact of the tangent with the curve has the ordinate $b \log \frac{x_0}{\alpha}$. So also does the projection of this point on the Y-axis. The segment between the two points on the Y-axis that have been mentioned has the length b. That segment is the projection of the segment on the tangent between its intersection with the Y-axis and the point of contact. The length of this projection (which is b) is what Bernoulli here calls the "subtangent." Today, by the subtangent of the curve $y = f(x)$ at the point $(x_0, f(x_0))$ is meant the length of the segment on the X-axis (and not the Y-axis) between its intersection with the tangent and the projection of the point of contact. This length is $f(x_0)/f'(x_0)$. In the case of the logarithmic curve it equals $x_0 \log \frac{x_0}{\alpha}$.—Karl Menger.

§13. First, it appears that in many games, even those that are absolutely fair, both of the players may expect to suffer a loss; indeed this is Nature's admonition to avoid the dice altogether. . . . This follows from the concavity of curve sBS to BR. For in making the stake, Bp, equal to the expected gain, BP, it is clear that the disutility po which results from a loss will always exceed the expected gain in utility, PO. Although this result will be quite clear to the mathematician, I shall nevertheless explain it by example, so that it will be clear to everyone. Let us assume that of two players, both possessing one hundred ducats, each puts up half this sum as a stake in a game that offers the same probabilities to both players. Under this assumption each will then have fifty ducats plus the expectation of winning yet one hundred ducats more. However, the sum of the values of these two items amounts, by the rule of §12, to only $(50^1.150^1)^{\frac{1}{2}}$ or $\sqrt{50.150}$, i.e., less than eighty-seven ducats, so that, though the game be played under perfectly equal conditions for both, either will suffer an expected loss of more than thirteen ducats. We must strongly emphasize this truth, although it be self evident: the imprudence of a gambler will be the greater the larger the part of his fortune which he exposes to a game of chance. For this purpose we shall modify the previous example by assuming that one of the gamblers, before putting up his fifty ducat stake possessed two hundred ducats. This gambler suffers an expected loss of $200 - \sqrt{150.250}$, which is not much greater than six ducats.

§14. Since, therefore, everyone who bets any part of his fortune, however small, on a mathematically fair game of chance acts irrationally, it may be of interest to inquire how great an advantage the gambler must enjoy over his opponent in order to avoid any expected loss. Let us again consider a game which is as simple as possible, defined by two equiprobable outcomes one of which is favorable and the other unfavorable. Let us take a to be the gain to be won in case of a favorable outcome, and x to be the stake which is lost in the unfavorable case. If the initial quantity of goods possessed is α we have $AB = \alpha$; $BP = a$; $PO = b \log \frac{\alpha + a}{\alpha}$ (see §10), and since (by §7) $po = PO$ it follows by the nature of a logarithmic curve that $Bp = \frac{\alpha a}{\alpha + a}$. Since however Bp represents the stake x, we have $x = \frac{\alpha a}{\alpha + a}$ a magnitude which is always smaller than a, the expected gain. It also follows from this that a man who risks his entire fortune acts like a simpleton, however great may be the possible gain. No one will have difficulty in being persuaded of this if he has carefully examined our definitions given above. Moreover, this result sheds light on a statement which is universally accepted in practice: it may be reasonable for some individuals to invest in a doubtful enterprise and yet be unreasonable for others to do so.

§15. The procedure customarily employed by merchants in the insurance of commodities transported by sea seems to merit special attention. This may again be explained by an example. Suppose Caius,[5] a Petersburg merchant, has pur-

[5] Caius is a Roman name, used here in the sense of our "Mr. Jones." Caius is the older form; in the later Roman period it was spelled "Gaius." [Translator]

chased commodities in Amsterdam which he could sell for ten thousand rubles if he had them in Petersburg. He therefore orders them to be shipped there by sea, but is in doubt whether or not to insure them. He is well aware of the fact that at this time of year of one hundred ships which sail from Amsterdam to Petersburg, five are usually lost. However, there is no insurance available below the price of eight hundred rubles a cargo, an amount which he considers outrageously high. The question is, therefore, how much wealth must Caius possess apart from the goods under consideration in order that it be sensible for him to abstain from insuring them? If x represents his fortune, then this together with the value of the expectation of the safe arrival of his goods is given by $\sqrt[100]{(x + 10000)^{95}x^5} = \sqrt[20]{(x + 10000)^{19}x}$ in case he abstains. With insurance he will have a certain fortune of $x + 9200$. Equating these two magnitudes we get: $(x + 10000)^{19}x = (x + 9200)^{20}$ or, approximately, $x = 5043$. If, therefore, Caius, apart from the expectation of receiving his commodities, possesses an amount greater than 5043 rubles he will be right in not buying insurance. If, on the contrary, his wealth is less than this amount he should insure his cargo. And if the question be asked "What minimum fortune should be possessed by the man who offers to provide this insurance in order for him to be rational in doing so?" We must answer thus: let y be his fortune, then

$$\sqrt[20]{(y + 800)^{19}.(y - 9200)} = y$$

or approximately, $y = 14243$, a figure which is obtained from the foregoing without additional calculation. A man less wealthy than this would be foolish to provide the surety, but it makes sense for a wealthier man to do so. From this it is clear that the introduction of this sort of insurance has been so useful since it offers advantages to all persons concerned. Similarly, had Caius been able to obtain the insurance for six hundred rubles he would have been unwise to refuse it if he possessed less than 20478 rubles, but he would have acted much too cautiously had he insured his commodities at this rate when his fortune was greater than this amount. On the other hand a man would act unadvisedly if he were to offer to sponsor this insurance for six hundred rubles when he himself possesses less than 29878 rubles. However, he would be well advised to do so if he possessed more than that amount. But no one, however rich, would be managing his affairs properly if he individually undertook the insurance for less than five hundred rubles.

§16. Another rule which may prove useful can be derived from our theory. This is the rule that it is advisable to divide goods which are exposed to some danger into several portions rather than to risk them all together. Again I shall explain this more precisely by an example. Sempronius owns goods at home worth a total of 4000 ducats and in addition possesses 8000 ducats worth of commodities in foreign countries from where they can only be transported by sea. However, our daily experience teaches us that of ten ships one perishes. Under these conditions I maintain that if Sempronius trusted all his 8000 ducats of goods to one ship his expectation of the commodities is worth 6751 ducats. That is

$$\sqrt[10]{12000^9.4000^1} - 4000.$$

If, however, he were to trust equal portions of these commodities to two ships the value of his expectation would be

$$\sqrt[100]{12000^{81}.8000^{18}.4000} - 4000, \text{ i.e., } 7033 \text{ ducats.}$$

In this way the value of Sempronius' prospects of success will grow more favorable the smaller the proportion committed to each ship. However, his expectation will never rise in value above 7200 ducats. This counsel will be equally serviceable for those who invest their fortunes in foreign bills of exchange and other hazardous enterprises.

§17. I am forced to omit many novel remarks though these would clearly not be unserviceable. And, though a person who is fairly judicious by natural instinct might have realized and spontaneously applied much of what I have here explained, hardly anyone believed it possible to define these problems with the precision we have employed in our examples. Since all our propositions harmonize perfectly with experience it would be wrong to neglect them as abstractions resting upon precarious hypotheses. This is further confirmed by the following example which inspired these thoughts, and whose history is as follows: My most honorable cousin the celebrated *Nicolas Bernoulli*, Professor utriusque iuris[6] at the University of Basle, once submitted five problems to the highly distinguished[7] mathematician *Montmort*.[8] These problems are reproduced in the work *L'analyse sur les jeux de hazard* de *M. de Montmort*, p. 402. The last of these problems runs as follows: *Peter tosses a coin and continues to do so until it should land "heads" when it comes to the ground. He agrees to give Paul one ducat if he gets "heads" on the very first throw, two ducats if he gets it on the second, four if on the third, eight if on the fourth, and so on, so that with each additional throw the number of ducats he must pay is doubled. Suppose we seek to determine the value of Paul's expectation.* My aforementioned cousin discussed this problem in a letter to me asking for my opinion. Although the standard calculation shows[9] that the value of Paul's expectation is infinitely great, it has, he said, to be admitted that any fairly reasonable man would sell his chance, with great pleasure, for twenty ducats. The accepted method of calculation does, indeed, value Paul's prospects at infinity though no one would be willing to purchase it at a moderately high price.

[6] Faculties of law of continental European universities bestow up to the present time the title of a Doctor utriusque juris, which means Doctor of both systems of laws, the Roman and the canon law. [Translator]

[7] Cl., i.e., Vir Clarissimus, a title of respect. [Translator]

[8] Montmort, Pierre Rémond, de (1678–1719). The work referred to here is the then famous "Essai d'analyse sur les jeux de hazard," Paris, 1708. Appended to the second edition, published in 1713, is Montmort's correspondence with Jean and Nicolas Bernoulli referring to the problems of chance and probabilities. [Translator].

[9] The probability of heads turning up on the 1st throw is 1/2. Since in this case Paul receives one ducat, this probability contributes $1/2 \cdot 1 = 1/2$ ducats to his expectation. The probability of heads turning up on the 2nd throw is 1/4. Since in this case Paul receives 2 ducats, this possibility contributes $1/4 \cdot 2 = 1/2$ to his expectation. Similarly, for every integer n, the possibility of heads turning up on the n-th throw contributes $1/2^n \cdot 2^{n-1} = 1/2$ ducats to his expectation. Paul's total expectation is therefore $1/2 + 1/2 + \cdots + 1/2 + \cdots$, and that is infinite.—Karl Menger.

If, however, we apply our new rule to this problem we may see the solution and thus unravel the knot. The solution of the problem by our principles is as follows.

§18. The number of cases to be considered here is infinite: in one half of the cases the game will end at the first throw, in one quarter of the cases it will conclude at the second, in an eighth part of the cases with the third, in a sixteenth part with the fourth, and so on.[10] If we designate the number of cases through infinity by N it is clear that there are $\frac{1}{2}N$ cases in which Paul gains one ducat, $\frac{1}{4}N$ cases in which he gains two ducats, $\frac{1}{8}N$ in which he gains four, $\frac{1}{16}N$ in which he gains eight, and so on, ad infinitum. Let us represent Paul's fortune by α; the proposition in question will then be worth

$$\sqrt[N]{(\alpha+1)^{N/2}.(\alpha+2)^{N/4}.(\alpha+4)^{N/8}.(\alpha+8)^{N/16}\cdots} - \alpha$$
$$= \sqrt{(\alpha+1)}.\sqrt[4]{(\alpha+2)}.\sqrt[8]{(\alpha+4)}.\sqrt[16]{(\alpha+8)}\cdots - \alpha.$$

§19. From this formula which evaluates Paul's prospective gain it follows that this value will increase with the size of Paul's fortune and will never attain an infinite value unless Paul's wealth simultaneously becomes infinite. In addition we obtain the following corollaries. If Paul owned nothing at all the value of his expectation would be

$$\sqrt[2]{1}.\sqrt[4]{2}.\sqrt[8]{4}.\sqrt[16]{8}\cdots$$

which amounts to two ducats, precisely. If he owned ten ducats his opportunity would be worth approximately three ducats; it would be worth approximately four if his wealth were one hundred, and six if he possessed one thousand. From this we can easily see what a tremendous fortune a man must own for it to make sense for him to purchase Paul's opportunity for twenty ducats. The amount which the buyer ought to pay for this proposition differs somewhat from the amount it would be worth to him were it already in his possession. Since, however, this difference is exceedingly small if α (Paul's fortune) is great,

[10] Since the number of cases is infinite, it is impossible to speak about one half of the cases, one quarter of the cases, etc., and the letter N in Bernoulli's argument is meaningless. However, Paul's expectation on the basis of Bernoulli's hypothesis concerning evaluation can be found by the same method by which, in footnote 9, Paul's classical expectation was determined. If Paul's fortune is α ducats, then, according to Bernoulli, he attributes to a gain of 2^{n-1} ducats the value $b \log \frac{\alpha + 2^{n-1}}{\alpha}$. If the probability of this gain is $1/2^n$, his expectation is $b/2^n \log \frac{\alpha+2^{n-1}}{\alpha}$. Paul's expectation resulting from the game is therefore

$$\frac{b}{2}\log\frac{\alpha+1}{\alpha} + \frac{b}{4}\log\frac{\alpha+2}{\alpha} + \cdots + \frac{b}{2^n}\log\frac{\alpha+2^{n-1}}{\alpha} + \cdots$$
$$= b\log\left[(\alpha+1)^{1/2}(\alpha+2)^{1/4}\cdot\ldots\cdot(\alpha+2^{n-1})^{1/2^n}\cdot\ldots\right] - b\log\alpha.$$

What addition D to Paul's fortune has the same value for him? Clearly, $b \log \frac{\alpha+D}{\alpha}$ must equal the above sum. Therefore

$$D = (\alpha+1)^{1/2}(\alpha+2)^{1/4}\cdot\ldots\cdot(\alpha+2^{n-1})^{1/2^n}\cdot\ldots - \alpha.$$

—Karl Menger.

we can take them to be equal. If we designate the purchase price by x its value can be determined by means of the equation

$$\sqrt[2]{(\alpha + 1 - x)} \cdot \sqrt[4]{(\alpha + 2 - x)} \cdot \sqrt[8]{(\alpha + 4 - x)} \cdot \sqrt[16]{(\alpha + 8 - x)} \cdots = \alpha$$

and if α is a large number this equation will be approximately satisfied by

$$x = \sqrt[2]{\alpha + 1} \cdot \sqrt[4]{\alpha + 2} \cdot \sqrt[8]{\alpha + 4} \cdot \sqrt[16]{\alpha + 8} \cdots - \alpha.$$

After having read this paper to the Society[11] *I sent a copy to the aforementioned Mr. Nicolas Bernoulli, to obtain his opinion of my proposed solution to the difficulty he had indicated. In a letter to me written in* 1732 *he declared that he was in no way dissatisfied with my proposition on the evaluation of risky propositions when applied to the case of a man who is to evaluate his own prospects. However, he thinks that the case is different if a third person, somewhat in the position of a judge, is to evaluate the prospects of any participant in a game in accord with equity and justice. I myself have discussed this problem in* §2. *Then this distinguished scholar informed me that the celebrated mathematician, Cramer,*[12] *had developed a theory on the same subject several years before I produced my paper. Indeed I have found his theory so similar to mine that it seems miraculous that we independently reached such close agreement on this sort of subject. Therefore it seems worth quoting the words with which the celebrated Cramer himself first described his theory in his letter of* 1728 *to my cousin. His words are as follows:*[13]

"Perhaps I am mistaken, but I believe that I have solved the extraordinary"
problem which you submitted to M. *de Montmort*, in your letter of September 9,"
1713, (problem 5, page 402). For the sake of simplicity I shall assume that A"
tosses a coin into the air and B commits himself to give A 1 ducat if, at the"
first throw, the coin falls with its cross upward; 2 if it falls thus only at the"
second throw, 4 if at the third throw, 8 if at the fourth throw, etc. The paradox"
consists in the infinite sum which calculation yields as the equivalent which"
A must pay to B. This seems absurd since no reasonable man would be willing"
to pay 20 ducats as equivalent. You ask for an explanation of the discrepancy"
between the mathematical calculation and the vulgar evaluation. I believe"
that it results from the fact that, *in their theory*, mathematicians evaluate"
money in proportion to its quantity while, *in practice*, people with common"
sense evaluate money in proportion to the utility they can obtain from it. The"
mathematical expectation is rendered infinite by the enormous amount which"
I can win if the coin does not fall with its cross upward until rather late, perhaps"
at the hundredth or thousandth throw. Now, as a matter of fact, if I reason"
as a sensible man, this sum is worth no more to me, causes me no more pleasure"

[11] Bernoulli's paper had been submitted to the Imperial Academy of Sciences in Petersburg. [Translator]

[12] Cramer, Gabriel, famous mathematician, born in Geneva, Switzerland (1704–1752). [Translator]

[13] The following passage of the original text is in French. [Translator]

"and influences me no more to accept the game than does a sum amounting "only to ten or twenty million ducats. Let us suppose, therefore, that any "amount above 10 millions, or (for the sake of simplicity) above $2^{24} = 166777216$ "ducats be deemed by him equal in value to 2^{24} ducats or, better yet, that I "can never win more than that amount, no matter how long it takes before the "coin falls with its cross upward. In this case, my expectation is $\frac{1}{2}.1 + \frac{1}{4}.2 +$ "$\frac{1}{8}.4 \cdots + \frac{1}{2^{25}}.2^{24} + \frac{1}{2^{26}}.2^{24} + \frac{1}{2^{27}}.2^{24} + \cdots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$ "(24 times) $\cdots + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 12 + 1 = 13$. Thus, my moral ex-"pectation is reduced in value to 13 ducats and the equivalent to be paid for "it is similarly reduced—a result which seems much more reasonable than does "rendering it infinite."

Thus far[14] *the exposition is somewhat vague and subject to counter argument. If it, indeed, be true that the amount* 2^{25} *appears to us to be no greater than* 2^{24}, *no attention whatsoever should be paid to the amount that may be won after the twenty-fourth throw, since just before making the twenty-fifth throw I am certain to end up with no less than* $2^{24} - 1$,[15] *an amount that, according to this theory, may be considered equivalent to* 2^{24}. *Therefore it may be said correctly that my expectation is only worth twelve ducats, not thirteen. However, in view of the coincidence between the basic principle developed by the aforementioned author and my own, the foregoing is clearly not intended to be taken to invalidate that principle. I refer to the proposition that reasonable men should evaluate money in accord with the utility they derive therefrom. I state this to avoid leading anyone to judge that entire theory adversely. And this is exactly what Cl. C.*[16] *Cramer states, expressing in the following manner precisely what we would ourselves conclude. He continues thus:*[17]

"The equivalent can turn out to be smaller yet if we adopt some alternative "hypothesis on the moral value of wealth. For that which I have just assumed "is not entirely valid since, while it is true that 100 millions yield more satis-"faction than do 10 millions, they do not give ten times as much. If, for example, "we suppose the moral value of goods to be directly proportionate to the square "root of their mathematical quantities, e.g., that the satisfaction provided by "40000000 is double that provided by 10000000, my psychic expectation "becomes

$$"\tfrac{1}{2}\sqrt{1} + \tfrac{1}{4}\sqrt{2} + \tfrac{1}{8}\sqrt{4} + \tfrac{1}{16}\sqrt{8} + \cdots = \frac{1}{2 - \sqrt{2}}.$$

"However this magnitude is not the equivalent we seek, for this equivalent "need not be equal to my moral expectation but should rather be of such a "magnitude that the pain caused by its loss is equal to the moral expectation "of the pleasure I hope to derive from my gain. Therefore, the equivalent must,

[14] From here on the text is again translated from Latin. [Translator]

[15] This remark of Bernoulli's is obscure. Under the conditions of the game a gain of $2^{24} - 1$ ducats is impossible.—Karl Menger.

[16] To be translated as "the distinguished Gabriel." [Translator]

[17] Text continues in French. [Translator]

on our hypothesis, amount to $\left(\frac{1}{2 - \sqrt{2}}\right)^2 = \left(\frac{1}{6 - 4\sqrt{2}}\right) = 2.9 \cdots$, which" is consequently less than 3, truly a trifling amount, but nevertheless, I believe," closer than is 13 to the vulgar evaluation."

REFERENCES

There exists only one other translation of Bernoulli's paper:

Pringsheim, Alfred, *Die Grundlage der modernen Wertlehre: Daniel Bernoulli, Versuch einer neuen Theorie der Wertbestimmung von Glücksfällen* (Specimen Theoriae novae de Mensura Sortis). Aus dem Lateinischen übersetzt und mit Erläuterungen versehen von Alfred Pringsheim. Leipzig, Duncker und Humblot, 1896, Sammlung älterer und neuerer staatswissenschaftlicher Schriften des In- und Auslandes hrsg. von L. Brentano und E. Leser, No. 9.

For an early discussion of the Bernoulli problem, reference is made to

Malfatti, Gianfrancesco, "Esame critico di un problema di probabilita del Signor Daniele Bernoulli, e soluzione d'un altro problema analogo al Bernoulliano" in "*Memorie di Matematica e Fisica della Societa italiana*" Vol. I, Verona, 1782, pp. 768–824.

For more on the "St. Petersburg Paradox," including material on later discussions, see

Menger, Karl, "Das Unsicherheitsmoment in der Wertlehre. Betrachtungen im Anschluss an das sogenannte Petersburger Spiel," *Zeitschrift für Nationalökonomie*, Vol. 5, 1934.

This paper by Professor Menger, is the most extensive study on the literature of the problem, and the problem itself.

Recent interest in the Bernoulli hypothesis was aroused by its appearance in

von Neumann, John, and Oskar Morgenstern, *The Theory of Games and Economic Behavior*, second edition, Princeton: Princeton University Press, 1947, Ch. III and Appendix: "The Axiomatic Treatment of Utility."

Many contemporary references and a discussion of the utility maximization hypothesis are to be found in

Arrow, Kenneth J., "Alternative Approaches to the Theory of Choice in Risk-Taking Situations," ECONOMETRICA, Vol. 19, October, 1951.

More recent writings in the field include

Alchian, A. A., "The Meaning of Utility Measurement," *American Economic Review*, Vol. XLIII, March, 1953.

Friedman, M., and Savage, L. J., "The Expected Utility-Hypothesis and the Measurability of Utility," *Journal of Political Economy*, Vol. LX, December, 1952.

Herstein, I. N., and John Milnor, "An Axiomatic Approach to Measurable Utility," ECONOMETRICA, Vol. 21, April, 1953.

Marschak, J., "Why 'Should' Statisticians and Businessmen Maximize 'Moral Expectation'?", *Second Berkeley Symposium on Mathematical Statistics and Probability*, 1953.

Mosteller, Frederick, and Philip Nogee, "An Experimental Measurement of Utility," *Journal of Political Economy*, lix, 5, Oct., 1951.

Samuelson, Paul A., "Probability, Utility, and the Independence Axiom," ECONOMETRICA, Vol. 20, Oct. 1952.

Strotz, Robert H., "Cardinal Utility," *Papers and Proceedings of the Sixty-Fifth Annual Meeting of the American Economic Association, American Economic Review*, Vol. 43, May, 1953, and the comment by W. J. Baumol.

For dissenting views, see:

Allais, M., "Les Theories de la Psychologie du Risque de l'Ecole Americaine", *Revue d'Economie Politique*, Vol. 63, 1953.

——— "Le Comportement de l'Homme Rationnel devant le Risque: Critique des Postu-

lats et Axiomes de l'Ecole Americaine," ECONOMETRICA, Oct., 1953
and
Edwards, Ward, "Probability-Preferences in Gambling," *The American Journal of Psychology*, Vol. 66, July, 1953.

Textbooks dealing with Bernoulli:

Anderson, Oskar, *Einführung in die mathematische Statistik*, Wien: J. Springer, 1935.

Davis, Harold, *The Theory of Econometrics*, Bloomington, Ind.: Principia Press, 1941.

Loria, Gino, *Storia delle Matematiche, dall'alba della civiltà al secolo XIX*, Second revised edition, Milan: U. Hopli, 1950.