13. Approximation algorithms for (Min,+)-problems

We still consider the task

$$s^* \in \operatorname*{argmin}_{s\in K^V} U(s) = \operatorname*{argmin}_{s\in K^V}\Big[\sum_{i\in V} u_i(s_i) + \sum_{ij\in E} u_{ij}(s_i,s_j)\Big]$$

A. Iterated descent

- define a family of neighbourhoods $N_m(s)\subset K^V$, $m=1,\dots,M$
- repeatedly solve the restricted problem

$$s^{(t+1)} \in \operatorname*{argmin}_{s\in N_m(s^{(t)})} U(s)$$

until no further improvement is possible, i.e.

$$s^{(t)} \in \operatorname*{argmin}_{s\in N_m(s^{(t)})} U(s) \quad \forall\, m=1,\dots,M$$

α-Expansions (Boykov et al. 2001)

For each label $\alpha\in K$ define the neighbourhood

$$N_\alpha(s)=\{s'\in K^V \mid s'_i=\alpha \text{ if } s'_i\neq s_i \;\forall i\in V\}$$

Their sizes are exponential, i.e. $|N_\alpha(s)|\sim 2^{|V|}$

Is the task $\operatorname*{argmin}_{s\in N_\alpha(s')} U(s)$ solvable in polynomial time?

Yes, if $u_{ij}(k,k')+u_{ij}(\alpha,\alpha')\le u_{ij}(k,\alpha')+u_{ij}(\alpha,k')$ holds $\forall \{i,j\}\in E$ and $\forall\, k,k'\in K\setminus\alpha$. This can be seen by constructing a binary valued (Min,+)-problem that is equivalent to the considered restricted optimisation task

$$V'=\{i\in V \mid s_i\neq\alpha\},\quad E'=\{\{i,j\}\in E \mid i,j\in V'\}$$

$y_i=0,1$ encodes $s\in N_\alpha(s')$, i.e.

$s_i=s'_i \Leftrightarrow y_i=0$ and $s_i=\alpha \Leftrightarrow y_i=1$

The pairwise functions of this equivalent problem are submodular if the condition given above holds.

Example 1 Consider the Potts model $u_{ij}(k,k') = a_{ij}(1-\delta_{kk'})$, $a_{ij} > 0$. It is not submodular if $|K| > 2$. However, it fulfills the above conditions

Theorem 1 (w/o proof)

Let $\bar{s}$ be a fixpoint of α-expansions $\forall \alpha \in K$. Then $U(\bar{s}) \le 2c \min_{s \in K^V} U(s)$, where c is defined by

$$c = \max_{ij \in E} \frac{\max_{k \neq k'} u_{ij}(k,k')}{\min_{k \neq k'} u_{ij}(k,k')} .$$

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αβ Swaps (Boykov et al. 2001)

Define neighbourhoods $N_{\alpha\beta}$ for each pair of labels

$$N_{\alpha\beta}(s') = \left\{ s \in K^V \;\middle|\; s_i = \begin{cases} s'_i & \text{if } s_i \neq \alpha, \beta \\ \alpha, \beta & \text{otherwise} \end{cases} \quad \forall i \in V \right\}$$

The reduced task $\operatorname{argmin}_{s \in N_{\alpha\beta}(s')} U(s)$ is tractable if the restriction of every $u_{ij} : K^2 \to \mathbb{R}$ to $\{\alpha,\beta\}^2 \in K^2$ is submodular $\forall \alpha, \beta \in K$.

Example 2 Consider the truncated metric on $K \subset \mathbb{Z}$

$$u_{ij}(k,k') = a_{ij} \min(c, |k-k'|), \quad a_{ij} > 0 .$$

It is not submodular. It allows αβ-swaps, but does not allo α-expansions.

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Remark 1 Another class of approximation algorithms construct submodular upper bounds (instead of considering restricted problems). I.e. given $s^{(t)}$, construct a submodular upper bound s.t.

$$\hat{U}_t(s) \geq U(s) \;\; \forall s \in K^V \quad \text{and} \quad \hat{U}_t(s^{(t)}) = U(s^{(t)}).$$

Then, solve

$$s^{(t+1)} \in \operatorname*{argmax}_{s \in K^V} \hat{U}_t(s)$$

B. Alogrithms based on LP-relaxations

Loopy belief propagation (aka message passing): Apply equivalent transformations which resemble dynamic programming on trees, until convergence. Not well grounded. See next section.

More principled: Start from an LP-relaxation of the discrete optimisation problem

$$U(s) = {\sum_{i \in V}}' u_i(s_i) + {\sum_{ij \in E}}' u_{ij}(s_i, s_j) \rightarrow \min_{s \in K^V}$$

A lower bound is given by

$$ {\sum_{i \in V}}' \min_{k \in K} u_i(k) + {\sum_{ij \in E}}' \min_{k, k' \in K} u_{ij}(k, k') \leq \min_{s \in K^V} U(s)$$

Combine it with equivalent transformations and maximise the lower bound w.r.t. them

$$B(\varphi) = {\sum_{i \in V}}' \min_{k \in K} \Big[u_i(k) - {\sum_{j \in N_i}}' \varphi_{ij}(k) \Big] +$$

$$+ {\sum_{ij \in E}}' \min_{k, k'} \Big[\varphi_{ij}(k) + u_{ij}(k, k') + \varphi_{ji}(k') \Big] \rightarrow \max_{\varphi}$$

This can be expressed as a linear optimisation task by introducing additional variables

$$\sum'_{i \in V} c_i + \sum'_{ij \in E} c_{ij} \longrightarrow \max_{\psi, c}$$

s.t.

$$c_i + \sum'_{j \in N_i} \psi_{ij}(k) \leq u_i(k) \quad \forall i \in V, \ \forall k \in K$$

$$c_{ij} - \psi_{ij}(k) - \psi_{ji}(k') \leq u_{ij}(k,k') \quad \forall ij \in E, \ \forall k,k' \in K$$

Notice, that this LP-task is dual to the following direct relaxation of the discrete optimisation task. Encode the label $s_i \in K$ by 1-out-of-K encoding with components denoted as $\lambda_i(k) = 0,1$, and similarly for edges, by $\lambda(k,k') = 0,1$

$$\sum'_{i \in V} \sum'_{k \in K} \lambda_i(k) u_i(k) + \sum'_{ij \in E} \sum'_{k,k' \in K} \lambda_{ij}(k,k') u_{ij}(k,k') \longrightarrow \min_{\lambda \geq 0}$$

s.t.

$$\lambda_i(k) = \sum'_{k' \in K} \lambda_{ij}(k,k') \quad \forall ij \in E, \ \forall k \in K$$

$$\sum'_{k \in K} \lambda_i(k) = 1 \quad \forall i \in V$$

$$\sum'_{k,k'} \lambda_{ij}(k,k') = 1 \quad \forall ij \in E$$

Relaxing the integrality constraints $\lambda_i(k) = 0,1$, $\lambda_{ij}(k,k') = 0,1$ makes this an LP task.

Find suitable algorithms for solving the primal task, or the dual task, or both simultaneously.