Ch I Markov Models on chains and acyclic graphs

1. Markov Models on chains

A. Definitions and basic properties

- Sequence $s = (s_1, \dots, s_n)$ of K-valued random variables $s_i \in K$
- K is a finite set, its elements are called states
- $p(s) = p(s_1, \dots, s_n)$ is a joint probability distribution on K^n

w.l.o.g. we may write

$$p(s_1, \dots, s_n) = p(s_n \mid s_1, \dots, s_{n-1})\, p(s_1, \dots, s_{n-1})$$
$$= \dots$$
$$= p(s_n \mid s_1, \dots, s_{n-1})\, p(s_{n-1} \mid s_1, \dots, s_{n-2}) \cdot \dots \cdot p(s_2 \mid s_1)\, p(s_1)$$

<u>Definition 1a</u> A p.d. on K^n is a Markov chain if

$$p(s) = p(s_1) \prod_{i=2}^{n} p(s_i \mid s_{i-1})$$

holds $\forall\, s \in K^n$. ▪

<u>Definition 1b</u> A p.d. on K^n is a Markov chain if

$$p(s) = \prod_{i=2}^{n} g_i(s_{i-1}, s_i)$$

holds $\forall\, s \in K^n$, where $g_i : K^2 \to \mathbb{R}_+$ are some functions. ▪

<u>Equivalence</u>:

a) → b) trivial

b) → a) recursively apply the following step

$$p(s_{n-1}, s_n) = \left\{ \sum_{s_1, \dots, s_{n-2}} \prod_{i=2}^{n-1} g_i(s_{i-1}, s_i) \right\} g_n(s_{n-1}, s_n)$$

↳ $g_n(s_{n-1}, s_n) = p(s_n \mid s_{n-1}) \cdot b_{n-1}(s_{n-1})$ with some b_{n-1}

Therefore, we have

$$p(s_1,\dots,s_n) = \underbrace{\left[\prod_{i=2}^{n-1} g_i(s_{i-1},s_i)\right] b_{n-1}(s_{n-1})}_{p(s_1,\dots,s_{n-1})} \cdot p(s_n \mid s_{n-1})$$

$\blacksquare$

Another useful formula

$$p(s_1,\dots,s_n) = \frac{p(s_1,s_2)\, p(s_2,s_3) \cdot \ldots \cdot p(s_{n-1},s_n)}{p(s_2)\cdot p(s_3)\cdot \ldots \cdot p(s_{n-1})}$$

Example 1 (Ehrenfest model)

The model considers N particles in two containers. At each discrete time $t = 1, 2, \dots$, independently of the past, a particle is selected at random and moved to the other container. Let s_t denote the number of particles in the first container at time t. Then we have

$$p(s_t = k \mid s_{t-1} = \ell) = \begin{cases} \frac{N-\ell}{N} & \text{if } k = \ell+1 \\ \frac{\ell}{N} & \text{if } k = \ell-1 \\ 0 & \text{otherwise} \end{cases}$$

Q: How does $p(s_t = k)$, $k = 0, 1, \dots, N$ behave for $t \to \infty$? $\blacksquare$

Example 2 (Random walk on a graph)

Consider a random walk on an undirected graph V, E

- $K = V$ states, $s_t \in V$ position of the walker at time t
- $p(s_1)$ – some p.d. for the start vertex
- $p(s_t = i \mid s_{t-1} = j) = \begin{cases} W_{ij} & \text{if } \{i,j\} \in E \\ 0 & \text{otherwise} \end{cases}$,

where the $W_{ij} \geq 0$ fulfill $\sum_{i \in \mathcal{N}(j)} W_{ij} = 1 \quad \forall j \in V$. $\blacksquare$

B. Homogeneous Markov chains, stationary distributions

Definition 2 A Markov chain is <u>homogeneous</u> if the conditional prob's $p(s_i \mid s_{i-1})$ do not depend on the position i, i.e

$$p(s_i = k \mid s_{i-1} = k') = q(k, k') \quad \forall\, i = 2, \dots, n.$$

We know that

$$p(s_i = k) = \sum_{k' \in K} p(s_i = k \mid s_{i-1} = k')\, p(s_{i-1} = k').$$

Consider $p(s_i = k)$, $k \in K$ as components of a vector $\vec{\pi}_i \in \mathbb{R}^K_+$ and $p(s_i = k \mid s_{i-1} = k')$, $k, k' \in K$ as elements of a $K \times K$ matrix P. The previous eq. reads

$$\vec{\pi}_i = P \cdot \vec{\pi}_{i-1}$$

and, more general, we have $\vec{\pi}_i = P^{i-1} \vec{\pi}_1$. It may happen that there exists a p.d. $\vec{\pi}^*$ on K s.t. $P \cdot \vec{\pi}^* = \vec{\pi}^*$. We call it a <u>stationary</u> p.d.

Definition 3 A homogeneous Markov chain is <u>irreducible</u> if for each pair $k, k' \in K$ there is an $m > 0$ s.t. $P^m_{kk'} > 0$. I.e. there is a non-zero probability to reach state k starting from state k' (after m transitions).

A condition somewhat stronger than irreducibility ensures existence and uniqueness of a stationary p.d.

Theorem 1 (w/o proof) If for some $m > 0$ all elements of the matrix P^m are strictly positive, then the Markov chain has a <u>unique</u> stationary distribution $\vec{\pi}^*$, which is a fixpoint

$$P^n \vec{\pi} \xrightarrow{n \to \infty} \vec{\pi}^* \quad \forall\, \vec{\pi}$$

Moreover,

$$P^n = \vec{\pi}^* \otimes \vec{e} + \mathcal{E}(n),$$

where $\vec{e} = (1, \ldots, 1)$ and $\mathcal{E}_{kk'}(n) = \mathcal{O}(h^n)$ with some $0 < h < 1$. ∎

Remark 1 (Infinite Markov chains)

Consider infinite sequences $s = (s_1, s_2, \ldots)$, $s_i \in K$. $K^{\mathbb{N}}$ is uncountable infinite. Any probability on it will assign zero probability to almost every sequence $s \in K^{\mathbb{N}}$.

However, a finite sequence $(k_1, k_2, \ldots, k_n) \in K^n$ can be seen as a set of infinite sequences

$$(k_1, \ldots, k_n) \longmapsto \{ s \in K^{\mathbb{N}} \mid s_1 = k_1, \ldots, s_n = k_n \}.$$

A Markov model on $K^{\mathbb{N}}$ assigns probabilities to such sets in the same way as described for finite sequences. ∎

C. Hidden Markov models on chains

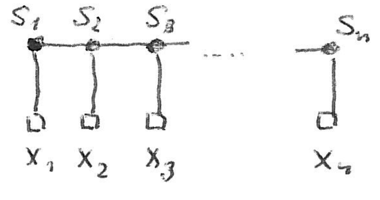
Common situation in pattern recognition:

$X = (X_1, \ldots, X_n)$ sequence of features (observable)

$S = (S_1, \ldots, S_n)$ sequence of states (hidden)

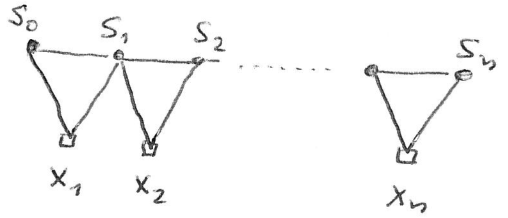
Hidden Markov model (HMM): a p.d. on pairs (x, s) s.t.

a) $$p(x,s) = \underbrace{\prod_{i=1}^{n} p(x_i \mid s_i)}_{p(x \mid s)} \cdot \underbrace{p(s_1) \cdot \prod_{i=2}^{n} p(s_i \mid s_{i-1})}_{p(s)\text{ - Markov model}}$$



b) or, slightly more general

$$p(x,s) = p(s_0) \prod_{i=1}^{n} p(x_i, s_i \mid s_{i-1})$$



Remark 2 This describes a stochastic regular language! ∎