Fast Fourier Transform

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Based on the texts: Ananth Grama, Anshul Gupta, George Karypis, and Vipin Kumar ``Introduction to Parallel Computing'', Addison Wesley, 2003, and
Paul Heckbert ``Fourier Transforms and the Fast Fourier Transform (FFT) Algorithm'', Carnegie Mellon School of Computer Science, 1998.

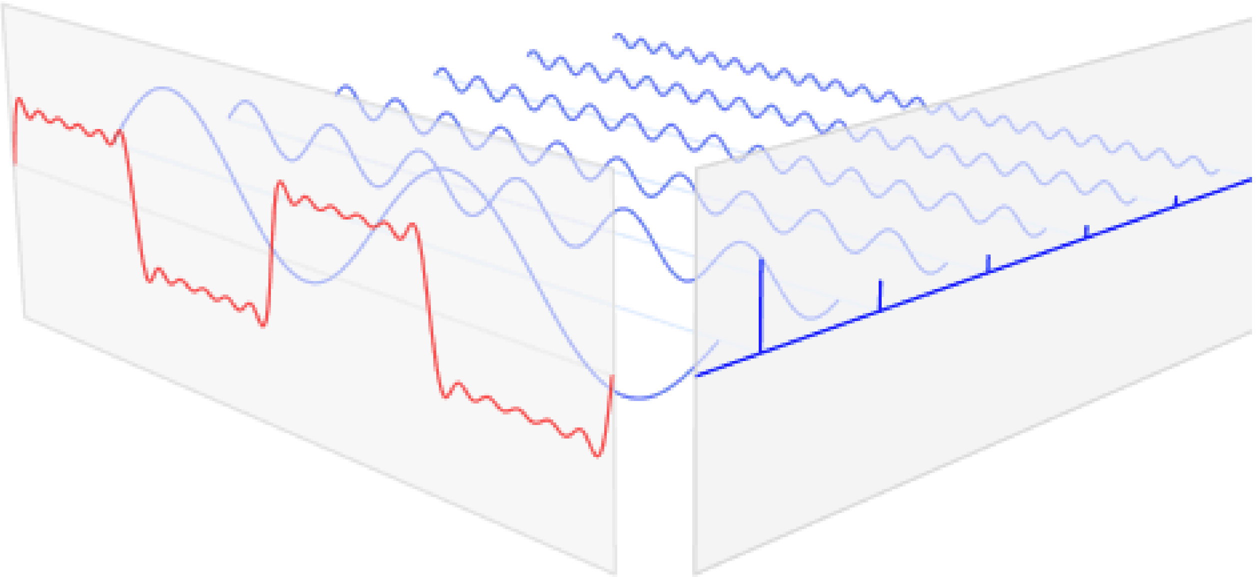
Topic Overview

- Introduction to Fast Fourier Transform
- Binary-Exchange algorithm
- Transpose algorithm

Introduction to Fast Fourier Transform

- The discrete Fourier transform (DFT) plays an important role in many applications, including **digital signal processing**, **image filtering**, solutions to **linear partial differential equations**, convolution …
- The DFT is a linear transformation that **maps** *n* **regularly sampled points** from a cycle of a periodic signal onto an equal number of points representing the **frequency spectrum** of the signal.
- In 1965, Cooley and Tukey devised an algorithm to compute the DFT of an n-point series in O(*n* log *n*) operations. Its variations are referred to as the **Fast Fourier Transform** (FFT).

Fourier Transform



Transformation of a signal (red) onto an equal number of points representing the **frequency spectrum** (blue)**.**

The Serial Algorithm

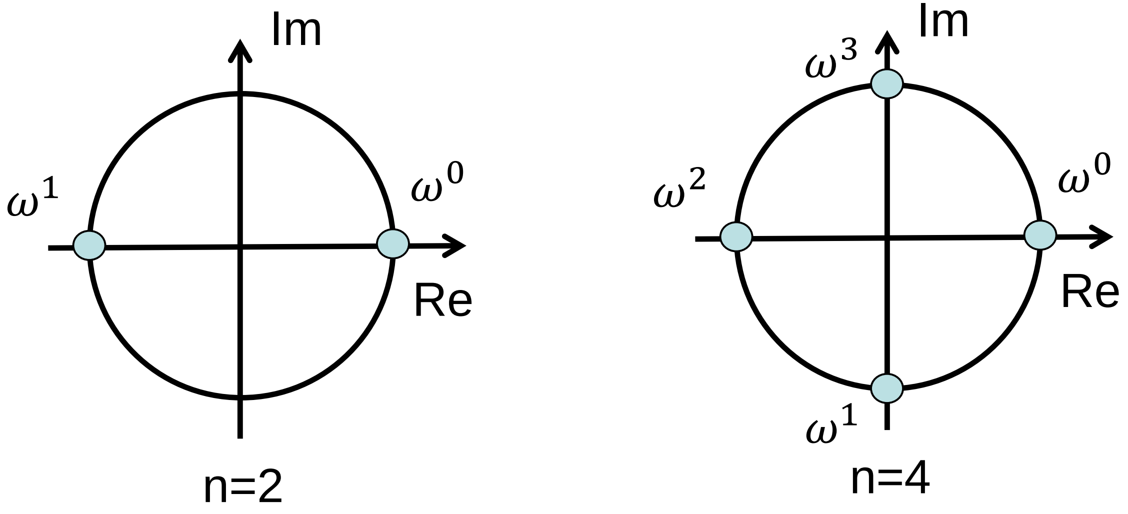
- Consider a sequence X = <X [0], X [1], ..., X [*n* - 1]> of length n. The discrete Fourier transform of the sequence X is the sequence Y = <Y [0], Y [1], ..., Y [*n* - 1]>, where

$$Y[l] = \sum_{k=0}^{n-1} X[k]\,\omega^{kl}, 0 \leq l < n$$

- ω is the ***n*-th root of unity** in the complex plane; that is $\omega = e^{2\pi\sqrt{-1}/n}$.

The Serial Algorithm

- The powers of ω used in an FFT computation are also known as **twiddle factors**.
- Note: $\omega = e^{2\pi\sqrt{-1}/n} = \cos\left(\frac{2\pi}{n}\right) + i \sin\left(\frac{2\pi}{n}\right)$



- Power of roots of unity are **periodic** with period *n*.

The Serial Algorithm

- The **sequential complexity** of computing the entire sequence Y of length n is O(n^2).
- The **fast Fourier transform** algorithm reduces this complexity to O(n log n).
- The FFT algorithm is based on the idea that permits an n-point DFT computation to be **split into two (n/2)-point DFT** computations.

Two-point DFT (n=2)

- For *n*=2 the twiddle factor is $\omega = e^{2\pi\sqrt{-1}/n} = e^{-\pi i} = -1^{(*}$.
- Then DFT is:

$$Y[l] = \sum_{k=0}^{n-1} X[k]\,(-1)^{kl} = X[0](-1)^{l0} + X[1](-1)^{l1} =$$
$$= X[0]+X[1](-1)^{l}$$

so

$$Y[0] = X[0] + X[1] \qquad and \qquad Y[1] = X[0] - X[1]$$

$^{(*}$ $e^{i\theta} = \cos\theta + i\sin\theta$

Four-point DFT (n=4)

- For *n*=4 the twiddle factor is $\omega = e^{-i\pi/2} = -\mathrm{i}$.
- Then DFT is:

$$Y[l] = \sum_{k=0}^{n-1} X[k]\,(-i)^{kl} =$$

$$= X[0] + X[1](-i)^{l} + X[2](-1)^{l} + X[3]i^{l}$$

so

$$\begin{gathered} Y[0] = X[0] + X[1] + X[2] + X[3], \\ Y[1] = X[0] - iX[1] - X[2] + iX[3], \\ Y[2] = X[0] - X[1] + X[2] - X[3], \\ Y[3] = X[0] + iX[1] - X[2] - iX[3]. \end{gathered}$$

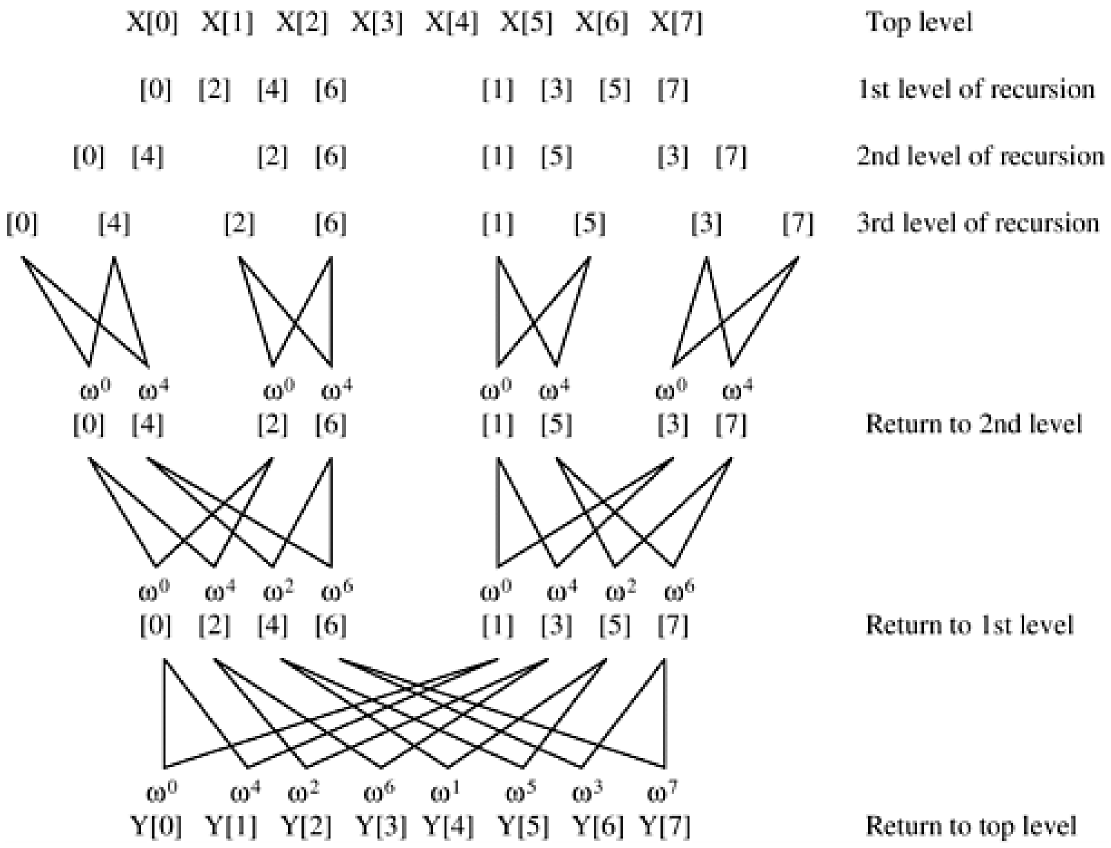
Four-point DFT (n=4)

- To compute Y faster, we can precompute common subexpressions:

$$
\begin{aligned}
Y[0] &= (X[0] + X[2]) + (X[1] + X[3]),\\
Y[1] &= (X[0] - X[2]) - i(X[1] - X[3]),\\
Y[2] &= (X[0] + X[2]) - (X[1] + X[3]),\\
Y[3] &= (X[0] - X[2]) + i(X[1] - X[3]).
\end{aligned}
$$

- Pre-computation of brackets in two-point DFT can **save a lot of addition operations**.

A recursive unordered FFT



If *n* is a power of two (e.g. 8 in the figure above), each of these DFT **computations can be divided similarly into smaller computations** in a **recursive manner**. This leads to the recursive FFT algorithm.

The Serial Algorithm

- This FFT algorithm is called the **radix-2 algorithm** because at each level of recursion, the **input sequence is split into two equal halves**.

```
procedure R_FFT(X, Y, n, w)
if (n = 1) then Y[0] := X[0] else
begin
   R_FFT(<X[0], X[2], ..., X[n - 2]>, <Q[0], Q[1], ..., Q[n/2]>, n/2, $w^2$);
   R_FFT(<X[1], X[3], ..., X[n - 1]>, <T[0], T[1], ..., T[n/2]>, n/2, $w^2$);
   for i := 0 to n - 1 do
       Y[i] :=Q[i mod (n/2)] + $w^i$ T[i mod (n/2)];
end R_FFT
```

The Serial Algorithm

- The maximum **number of levels of recursion is log *n*** for an initial sequence of length *n*.
- The total number of **arithmetic operations** (line 7) at each level is O(*n*).
- The overall **sequential complexity** of the algorithm is O(*n* log *n*).

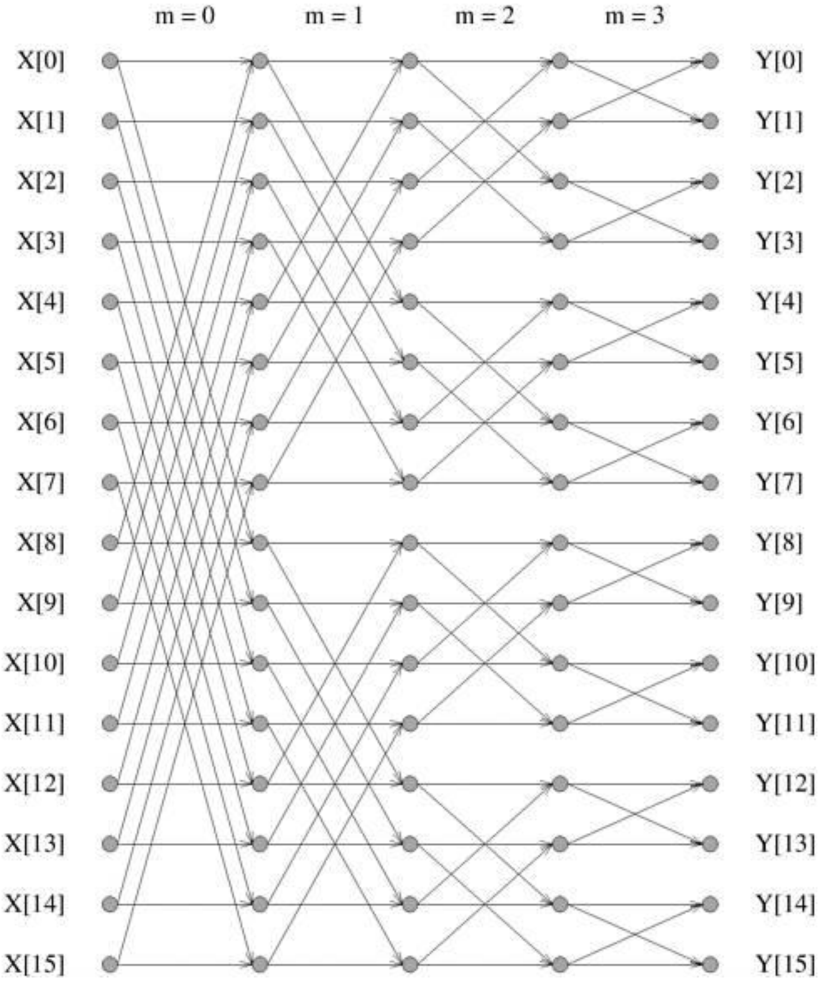
- The serial FFT algorithm can also be cast in an **iterative form**.
- An iterative FFT algorithm is derived by casting each level of recursion, starting with the deepest level, as an iteration.

Cooley-Tukey algorithm

- The **outer loop** (line 5) is executed log *n* times for an n-point FFT, and the **inner loop** (line 8) is executed *n* times during each iteration of the outer loop.

```
1.  procedure ITERATIVE_FFT(X, Y, n)
2.  begin
3.     r := log n;
4.     for i := 0 to n - 1 do R[i] := X[i];
5.     for m := 0 to r - 1 do        /* Outer loop */
6.         begin
7.            for i := 0 to n - 1 do S[i]:= R[i];
8.            for i := 0 to n - 1 do /* Inner loop */
9.               begin
10.                 /* Let (b0b1 ··· br -1) be the binary representation of i */
11.                 j := (b0...bm-1 0 bm+1···br -1);
12.                 k := (b0...bm-1 1 bm+1···br -1);
13.                 R[i] := S[j] + S[k] x $\omega^{(bm, bm-1, b0, 0, …, 0)}$;
14.           endfor;   /* Inner loop */
15.    endfor;          /* Outer loop */
16.    for i := 0 to n - 1 do Y[i] := R[i];
17. end ITERATIVE_FFT
```

Cooley-Tukey algorithm

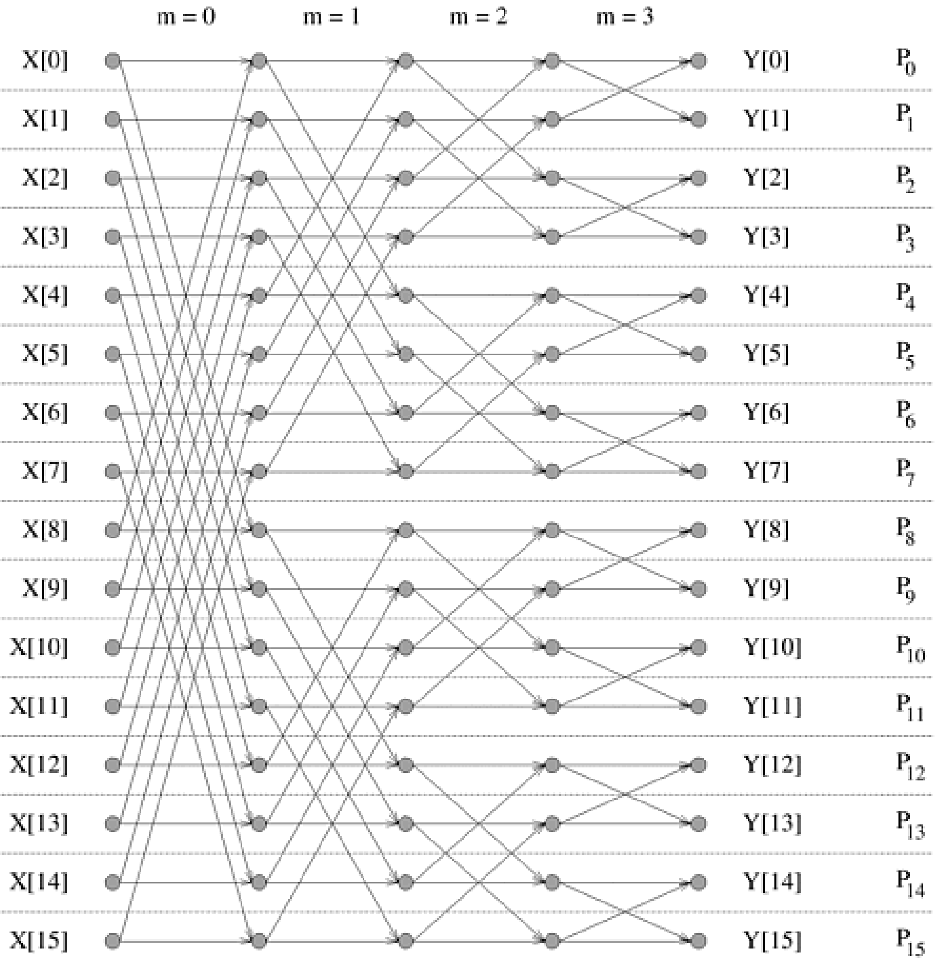


The pattern of combination of elements of the input and the intermediate sequences during a **16-point** unordered **FFT** computation.

Binary-Exchange algorithm

- The decomposition for the parallel algorithm is induced by **partitioning the input or the output vector**.
- We first consider a simple mapping in which one task is assigned to each process.
- Each **task starts with one element of the input vector** and computes the corresponding element of the output. Process i ($0 \leq i < n$) initially stores X [i] and finally generates Y [i].
- In each of the log n iterations of the outer loop, **process P_i updates the value of $R[i]$** by executing line 13 of Cooley-Tukey algorithm.
- All n updates are performed in parallel.

16-point FFT on 16 processes



Parallel mapping where one task is assigned to each process.

Binary-Exchange algorithm

- To perform the updates, process P_i requires an element of S from a process whose label **differs from *i* at only one bit**.
- Parallel FFT computation maps naturally onto a **hypercube** with a one-to-one mapping of processes to nodes.
- In each of the log n iterations of this algorithm, every process performs one **complex multiplication and addition**, and **exchanges** one complex number with another process.
- Now consider a mapping in which the n are mapped onto p processes.

Binary-Exchange algorithm

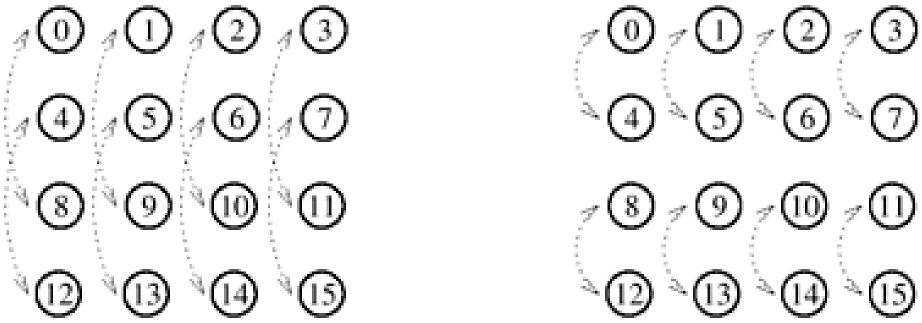
- For the sake of simplicity, let us assume that both n and p are powers of two, i.e., $n = 2^r$ and $p = 2^d$.
- Elements with indices differing at their d (= 2) most significant bits are mapped onto different processes, i.e. there is **no interaction during the last r - d iterations**.
- Each interaction operation **exchanges n/p words of data**. The time spent in communication in the entire algorithm is $t_s \log p + t_w(n/p) \log p$.
- The **parallel run time** of the algorithm on a p-node hypercube network is

$$T_P = t_c \frac{n}{p} \log n + t_s \log p + t_w \frac{n}{p} \log p.$$

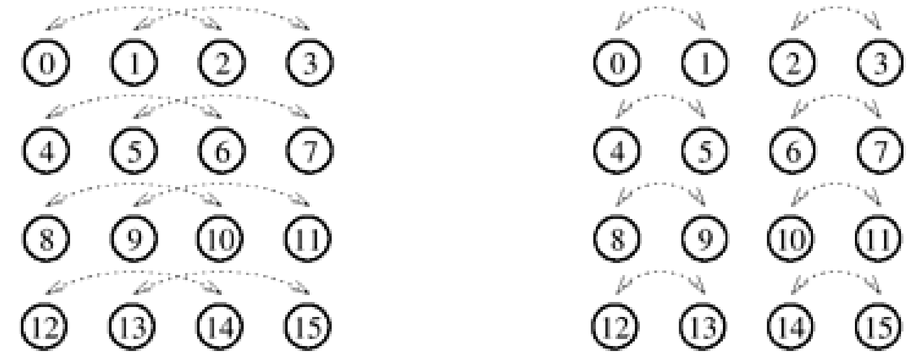
Transpose Algorithm

- The binary-exchange algorithm yields good performance on parallel computers with sufficiently **high communication bandwidth** with respect to the processing speed of the CPUs.
- The simplest transpose algorithm requires a single **transpose operation** over a **two-dimensional array**; we call this algorithm the **two-dimensional transpose** algorithm.
- Assume that $\sqrt{n}$ is a power of 2, and that the input sequence of size *n* is arranged in a $\sqrt{n} \times \sqrt{n}$ two-dimensional square array.

Two-dimensional transpose



(a) Iteration m = 0 (b) Iteration m = 1



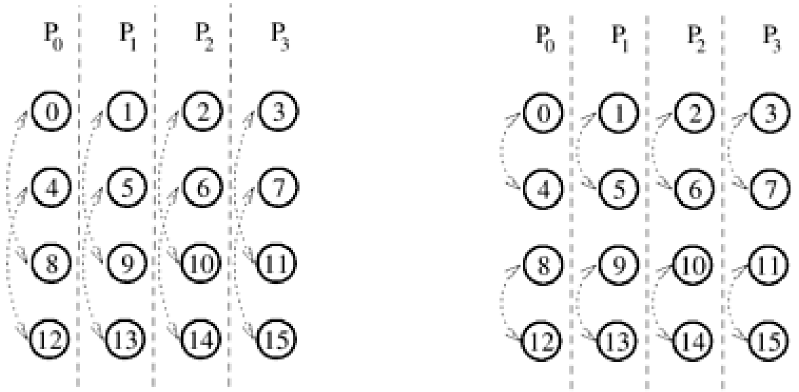
(c) Iteration m = 2 (d) Iteration m = 3

The pattern of combination of elements in a **16-point FFT** when the data are arranged in a **4 x 4 two-dimensional square array**.

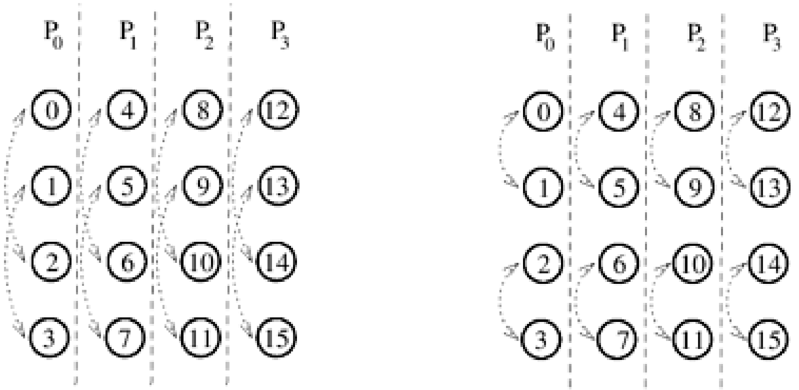
Transpose Algorithm

- The FFT **computation in each column** can proceed **independently** for $\log \sqrt{n}$ iterations without any column requiring data from any other column.
- Similarly, in the remaining $\log \sqrt{n}$ iterations, computation proceeds independently in **each row without any row requiring data from any other row**.

Two-dimensional (2D) transpose



(a) Steps in phase 1 of the transpose algorithm (before transpose)



(b) Steps in phase 3 of the transpose algorithm (after transpose)

The 2D transpose algorithm for a **16-point FFT** on **four processes**.

Transpose Algorithm (p$\leq \sqrt{n}$)

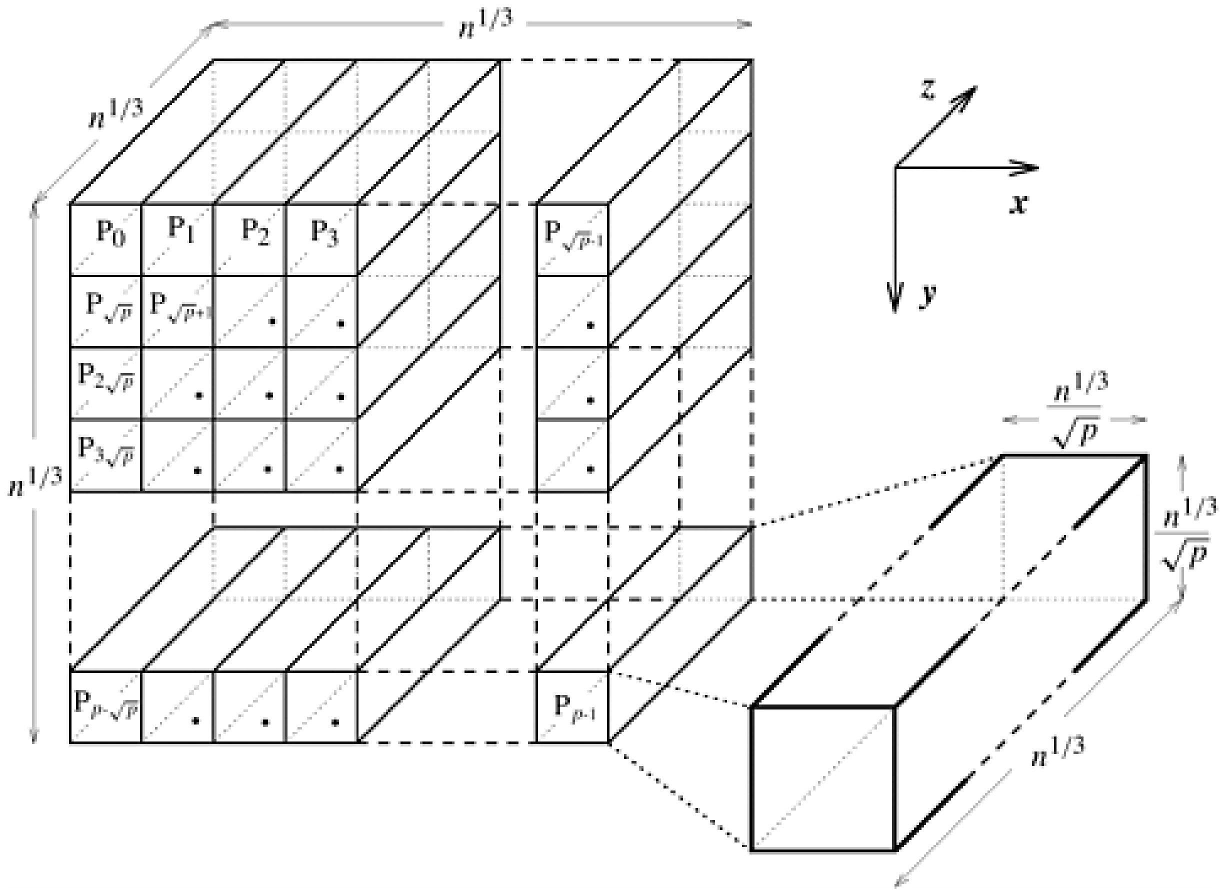
- The 2D array of data is striped into blocks, and one block of $\sqrt{n}/p$ rows is assigned to each process.
- In the **first and third phases** of the algorithm, each process computes $\sqrt{n}/p$ FFTs of size $\sqrt{n}$ each.
- The **second phase** transposes the $\sqrt{n} \times \sqrt{n}$ matrix (all-to-all personalized communication).
- The parallel run time of the **transpose** algorithm on a **hypercube** is:

$$\begin{aligned} T_P &= 2t_c \frac{\sqrt{n}}{p} \sqrt{n} \log \sqrt{n} + t_s(p-1) + t_w \frac{n}{p} \\ &= t_c \frac{n}{p} \log n + t_s(p-1) + t_w \frac{n}{p} \end{aligned}$$

Three-dimensional transpose algorithm

- As an extension of the two-dimensional transpose algorithm, consider the n data points to be arranged in an $n^{1/3} \times n^{1/3} \times n^{1/3}$ **three-dimensional array** mapped onto a logical $\sqrt{p} \times \sqrt{p}$ **two-dimensional mesh** of processes.
- Each process has $(\frac{n^{\frac{1}{3}}}{\sqrt{p}})(\frac{n^{\frac{1}{3}}}{\sqrt{p}})n^{\frac{1}{3}}$ elements of data.

Three-dimensional transpose algorithm



Data distribution in the three-dimensional transpose algorithm for an *n*-point FFT on p processes ($\sqrt{p} \leq n^{1/3}$)

Three-dimensional transpose algorithm

- This algorithm can be divided into the following five phases:
  1. In the first phase, $n^{1/3}$-point FFTs are computed on all the rows **along the *z*-axis**.
  2. In the second phase, all the $n^{1/3}$ cross-sections of size $n^{1/3}$ x $n^{1/3}$ along the ***y-z* plane are transposed**.
  3. In the third phase, $n^{1/3}$ -point FFTs are computed on all the rows of the modified array **along the *z*-axis.**
  4. In the fourth phase, each of the $n^{1/3}$ x $n^{1/3}$ cross-sections along the ***x-z* plane is transposed**.
  5. In the fifth and final phase, $n^{1/3}$ -point FFTs of all the **rows along the *z*-axis are computed again**.

Binary-Exchange vs. Transpose Algorithm

- Parallel runtime of the **transpose** algorithm ($T_P = t_c \frac{n}{p} \log n + t_s (p-1) + t_w \frac{n}{p}$) has a much higher overhead than the **binary-exchange** algorithm ($T_P = t_c \frac{n}{p} \log n + t_s \log p + t_w \frac{n}{p} \log p.$) due to the **message startup** time t_s, but has a lower overhead due to **per-word transfer** time t_w.

- If the **latency** t_s **is very low**, then the **transpose algorithm** may be the algorithm of choice.
- The **binary-exchange algorithm** may perform better on a parallel computer with a **high communication bandwidth** but a significant startup time.