Logical reasoning and programming

First-order resolution and equality

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Our problem

Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$ be a set of senteces and φ be a sentence. We know that

$$\Gamma \models \varphi$$
$$\text{iff}$$
$$\Gamma \cup \{\neg\varphi\} \text{ is unsatisfiable}$$
$$\text{iff}$$
$$\forall\psi_1' \cup \cdots \cup \forall\psi_n' \cup \forall(\neg\varphi)' \text{ is unsatisfiable,}$$

where $\psi_1', \ldots, \psi_n', (\neg\varphi)'$ are $\psi_1, \ldots, \psi_n, \neg\varphi$ in CNF (=clauses) and $\forall\Delta = \{\, \forall\chi \mid \chi \in \Delta \,\}$ for a set of clauses Δ.

We say that Δ is a set of clauses assuming that it is implicitly universally quantified.

Resolution

Let $l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}$ be literals and p and q be atomic formulae.

$$\frac{\{l_1, \ldots, l_m, p\} \qquad \{\neg q, l_{m+1}, \ldots, l_{m+n}\}}{\{l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}\}\sigma}$$

where $\sigma = mgu(p, q)$ and $\{l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}\}\sigma$ is equal to $\{l_1\sigma, \ldots, l_m\sigma, l_{m+1}\sigma, \ldots, l_{m+n}\sigma\}$. The clause $\{l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}\}\sigma$ produced by the (binary) resolution rule is called the *resolvent* of the two *input* clauses. We assume that the input clauses do not share variables (renaming away).

Theorem (correctness)

$\{l_1, \ldots, l_m, p\}, \{\neg q, l_{m+1}, \ldots, l_{m+n}\} \models$
$\{l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}\}\sigma$, *where* $\sigma = mgu(p, q)$.

Hence the resolution rule preserves satisfiability.

Factoring

We need to add the factoring rule. Let $l_1, \ldots, l_m, l_{m+1}, l, k$ be literals.

$$\frac{\{l_1, \ldots, l_m, l, k\}}{\{l_1, \ldots, l_m, l\}\sigma}$$

where $\sigma = mgu(l, k)$. Note that l and k are either both positive, or both negative. Moreover, $\{l_1, \ldots, l_m, l, k\} \models \{l_1, \ldots, l_m, l\}\sigma$.

In propositional logic we avoided this problem completely by using sets (of clauses). However, it is still possible to combine both rules into just one rule.

Example

Using only the binary resolution rule, we cannot derive $\square$ from clauses $\{p(X), p(Y)\}$ and $\{\neg p(U), \neg p(V)\}$.

Resolution calculus

The resolution calculus has no axioms and the only deduction rules are the binary resolution rule and the factoring.

Resolution proof

A (resolution) proof of clause φ from clauses $\psi_1, \ldots, \psi_n$ is a finite sequence of clauses $\chi_1, \ldots, \chi_m$ such that

- every χ_i is
  - among $\psi_1, \ldots, \psi_n$, or
  - derived by the binary resolution rule from input clauses χ_j and χ_k, for $1 \leq j < k < i \leq m$, or
  - derived by the factoring rule from an input clause χ_j, for $1 \leq j < i \leq m$.
- $\varphi = \chi_m$.

We say that a clause φ is *provable* (derivable) from a set of clauses $\{\psi_1, \ldots, \psi_n\}$, we write $\{\psi_1, \ldots, \psi_n\} \vdash \varphi$, if there is a proof of φ from $\psi_1, \ldots, \psi_n$.

Resolution proof

Example

We prove $\square$ from a set of clauses

$$\{\{\neg p(X), q(X), r(X)\}, \{p(a), p(b)\}, \{\neg q(Y)\}, \{\neg r(a)\}, \{\neg r(b)\}\}.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\neg p(X), q(X), r(X) \quad \neg q(Y)}{\neg p(X), r(X)} \quad \neg r(a)}{\neg p(a)} \quad p(a), p(b)}{p(b)} \quad \dfrac{\dfrac{\neg p(X), q(X), r(X) \quad \neg q(Y)}{\neg p(X), r(X)} \quad \neg r(b)}{\neg p(b)}}{\square}$$

Strictly speaking the presented derivation is not a sequence, but it is easy to produce a sequence from it. For example, $\{\neg p(X), r(X)\}$ is derived only once in the sequence.

More resolvents

Unlike in propositional logic, it is possible to resolve two clauses in multiple ways and still obtain useful resolvents.

Example

From $\{p(a), p(b)\}$ and $\{\neg p(X), q(X)\}$ we can derive both $\{p(b), q(a)\}$ and $\{p(a), q(b)\}$.

Completeness of resolution calculus

It is not true that we can derive every valid formula in the resolution calculus, e.g., from the empty set we derive nothing. However, it is so called *refutationally complete*.

Theorem (completeness)

Let Γ *be a set of clauses. If* Γ *is unsatisfiable, then* $\Gamma \vdash \square$.

Note that from the correctness theorem we already know the converse implication.

Theorem

Let Γ *be a set of clauses. If* $\Gamma \vdash \square$*, then* Γ *is unsatisfiable.*

Subsumption

A clause φ *subsumes* a clause ψ, denoted $\varphi \sqsubseteq \psi$, if there is a substitution σ such that $\varphi\sigma \subseteq \psi$.

If $\varphi \sqsubseteq \psi$, then $\varphi \models \psi$.

Let Γ and Δ be sets of clauses. We write $\Gamma \sqsubseteq \Delta$ if for every clause $\psi \in \Delta$ exists a clause $\varphi \in \Gamma$ such that $\varphi \sqsubseteq \psi$.

Lemma

If $\Delta \vdash \square$ and $\Gamma \sqsubseteq \Delta$, then $\Gamma \vdash \square$ and this proof is no longer than $\Delta \vdash \square$.

Example

$\{p(X)\} \sqsubseteq \{p(f(a))\}$, $\{p(X)\} \sqsubseteq \{p(Y), q(Y)\}$, and $\{p(X), q(Y)\} \sqsubseteq \{p(Z), q(Z)\}$, but $\{p(Z), q(Z)\} \not\sqsubseteq \{p(X), q(Y)\}$.

Subsumption example

Assume we have the following resolution refutation

$$\dfrac{\dfrac{\dfrac{p(f(X)), q(X,Y), r(X) \qquad \neg p(f(f(c)))}{q(f(c),Y), r(f(c))} \qquad \neg q(U,V)}{r(f(c))} \qquad \neg r(f(c))}{\square}$$

Then after deriving $\{p(Y), r(Z)\}$, we can simplify the previous proof into

$$\dfrac{\dfrac{p(Y), r(Z) \qquad \neg p(f(f(c)))}{r(Z)} \qquad \neg r(f(c))}{\square}$$

thanks to $\{p(Y), r(Z)\} \sqsubseteq \{p(f(X)), q(X,Y), r(X)\}$.

Forward and backward subsumptions

Forward subsumption

If we derive a clause ψ and we already have a clause φ such that $\varphi \sqsubseteq \psi$, then we can remove ψ, because φ is stronger.

Backward subsumption

If we derive a clause φ and we already have a clause ψ such that $\varphi \sqsubseteq \psi$, then we can remove ψ, because φ is stronger. We can remove all such ψs.

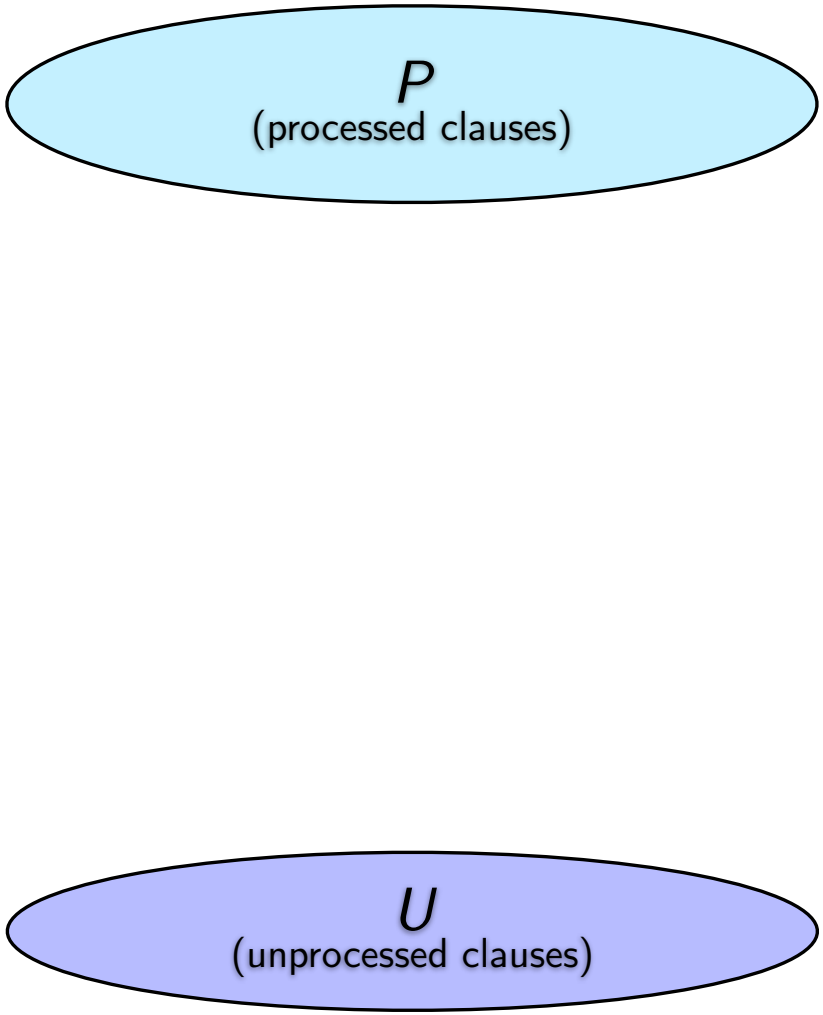
Saturation procedure

We have already seen a saturated set in the propositional case—we systematically process a set of clauses in such a way that if there is no clause to be processed, then it is impossible to derive $\square$ from the original set.

The next slides are from the presentation by Stefan Schultz, the author of the E prover, at the SAT/SMT/AR Summer School 2018.

Note that it is sometimes useful to simplify also U and not to delay such simplifications as in E.

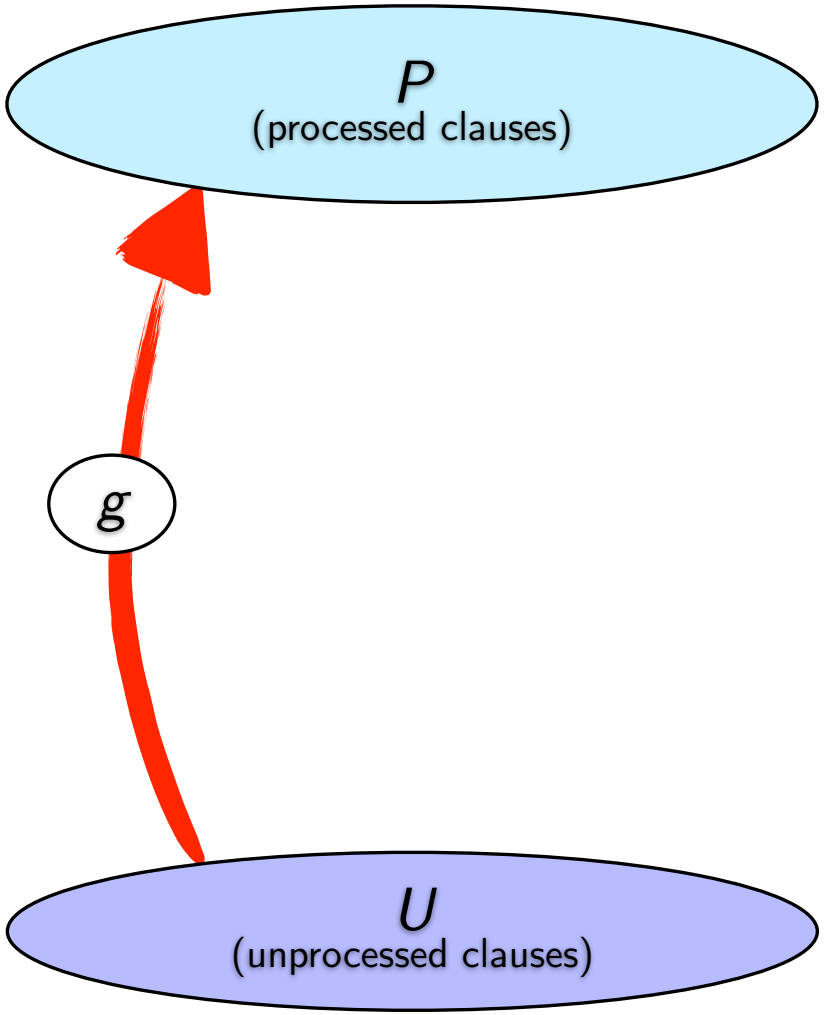
The Given-Clause Algorithm



We represent the proof state S by two sets of clauses:

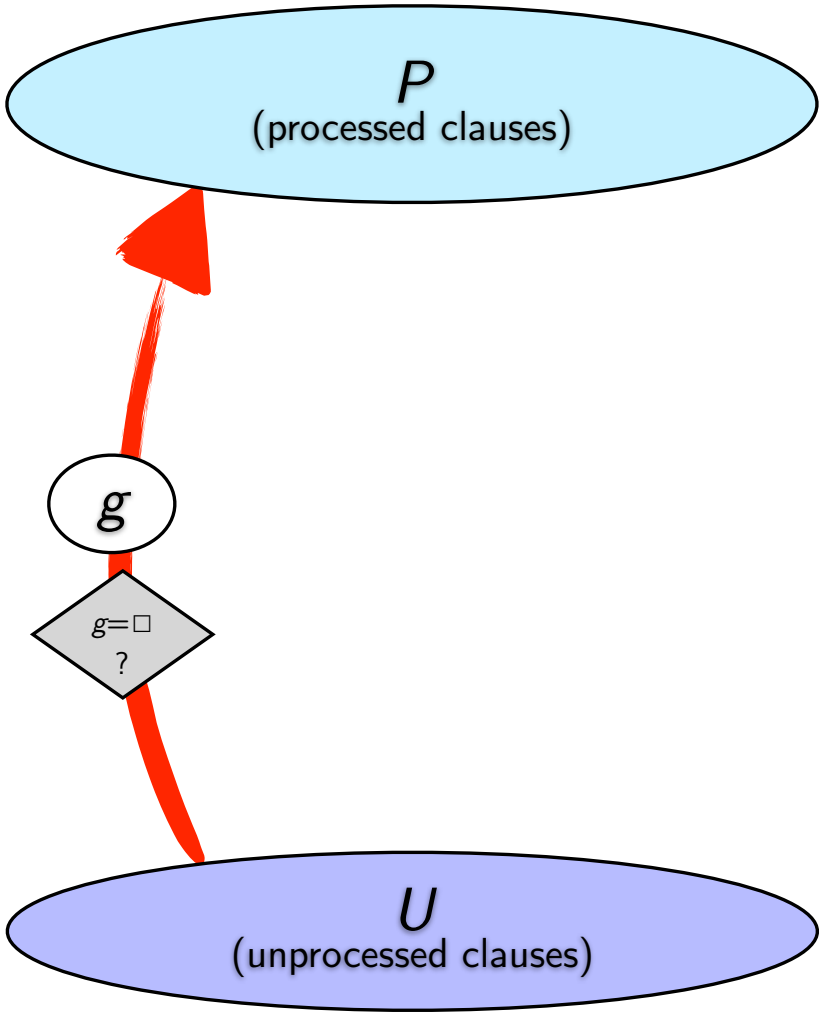
- P holds the processed clauses (originally empty)
- U holds the unprocessed clauses (originally all clauses in S)

The Given-Clause Algorithm



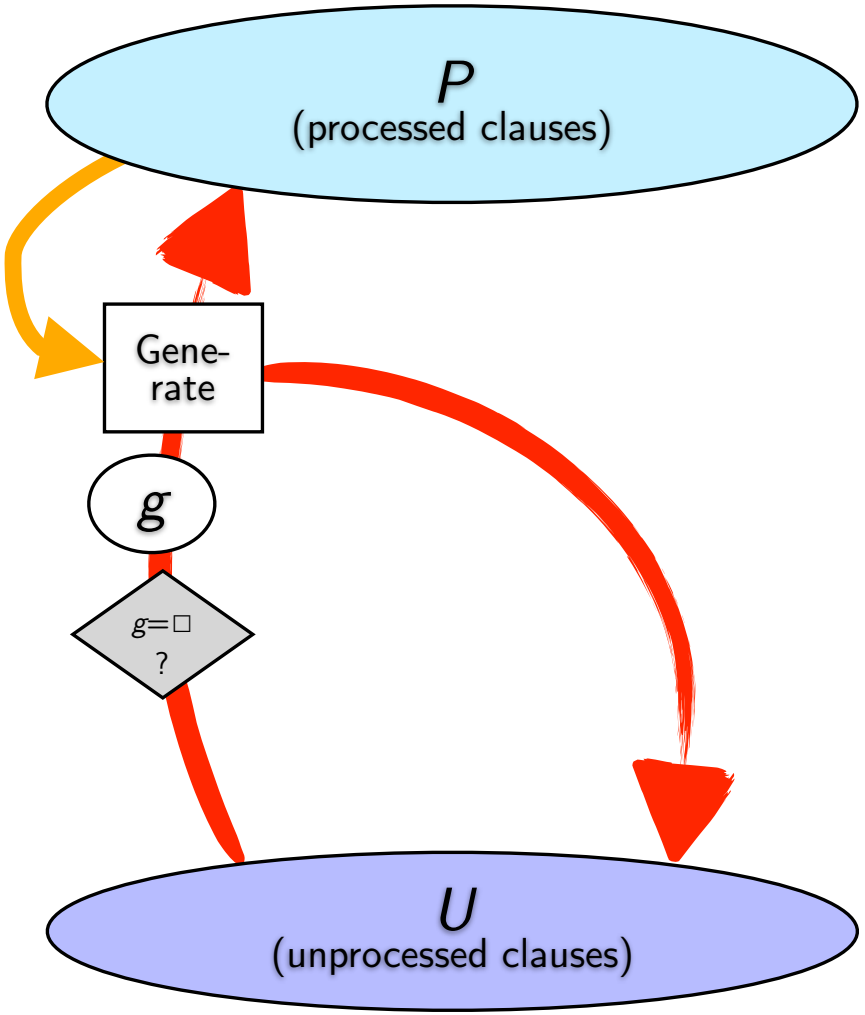
- Aim: Move everything from U to P

The Given-Clause Algorithm



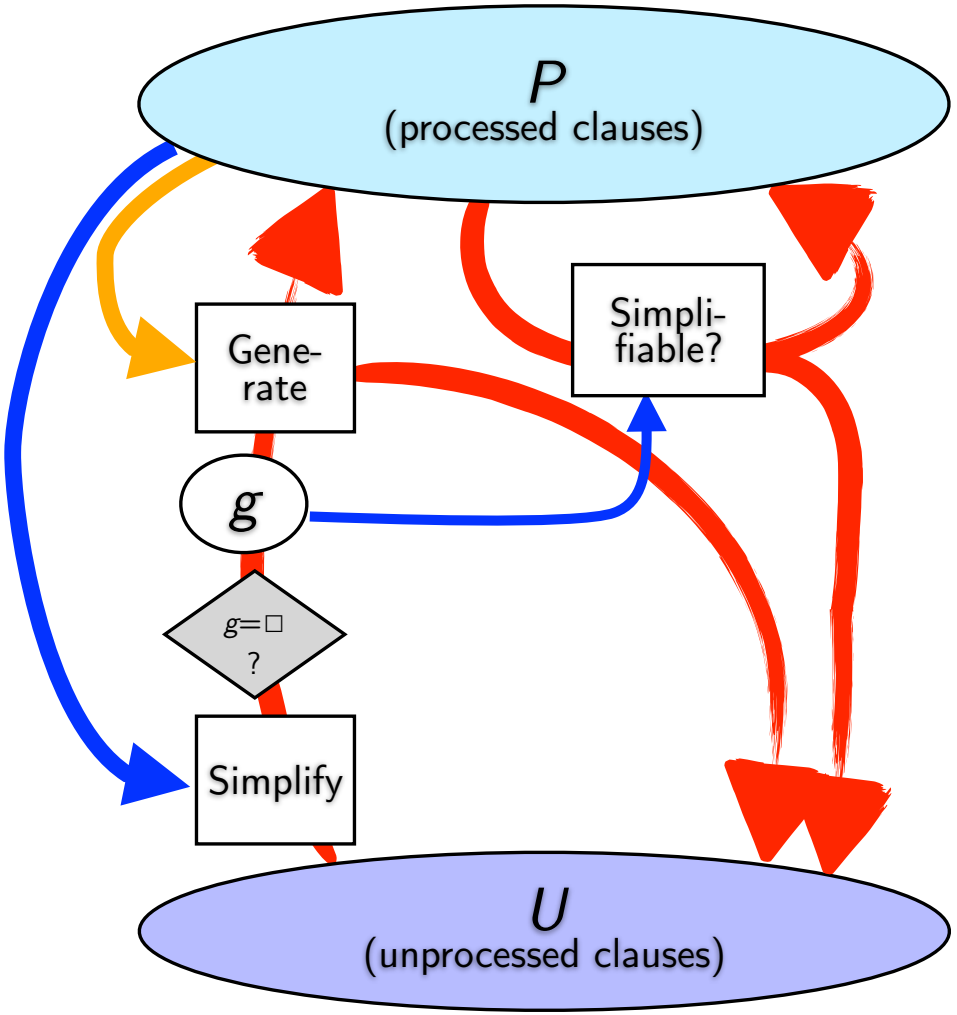
- Aim: Move everything from U to P

The Given-Clause Algorithm



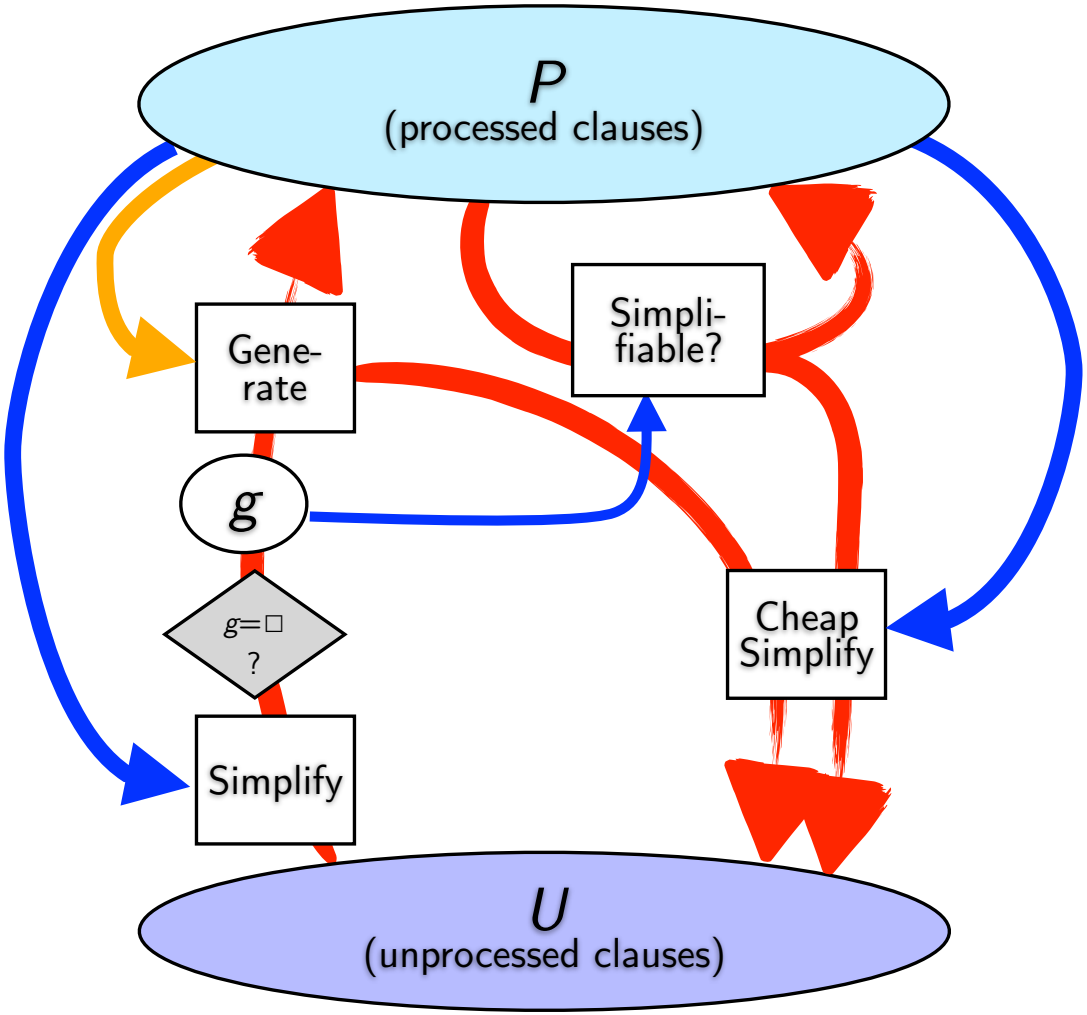
- Aim: Move everything from U to P
- Invariant: All generating inferences with premises from P have been performed

The Given-Clause Algorithm



- Aim: Move everything from U to P
- Invariant: All generating inferences with premises from P have been performed
- Invariant: P is interreduced

The Given-Clause Algorithm

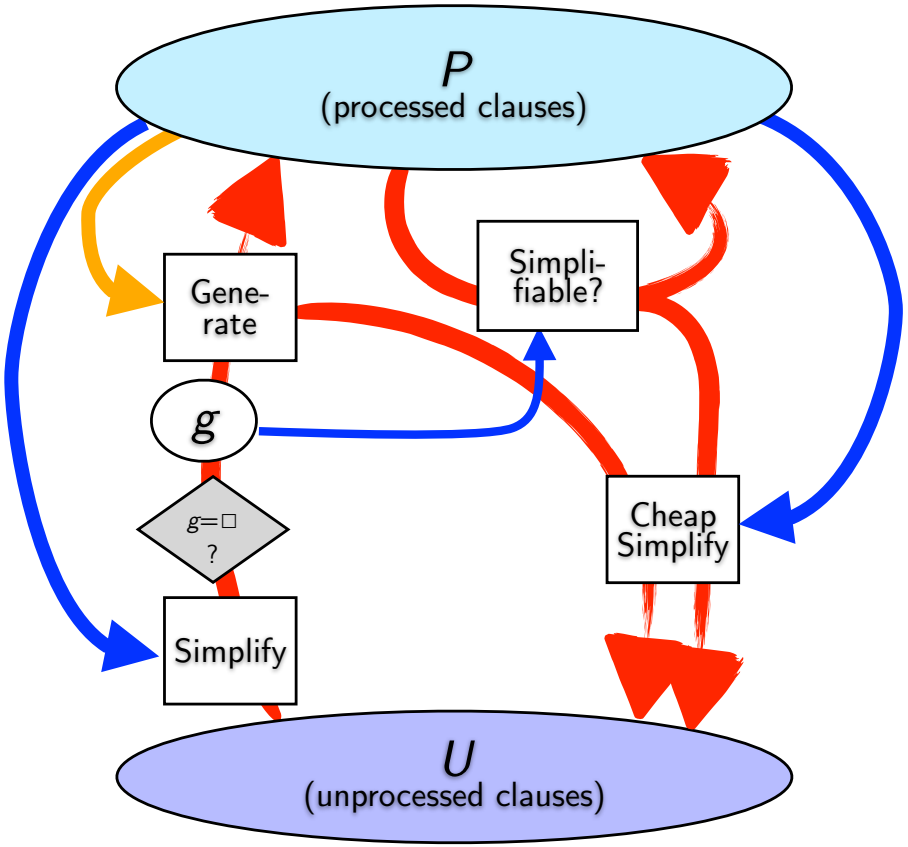


- Aim: Move everything from U to P
- Invariant: All generating inferences with premises from P have been performed
- Invariant: P is interreduced
- Clauses added to U are simplified with respect to P

The Given-Clause Loop in Fewer Words

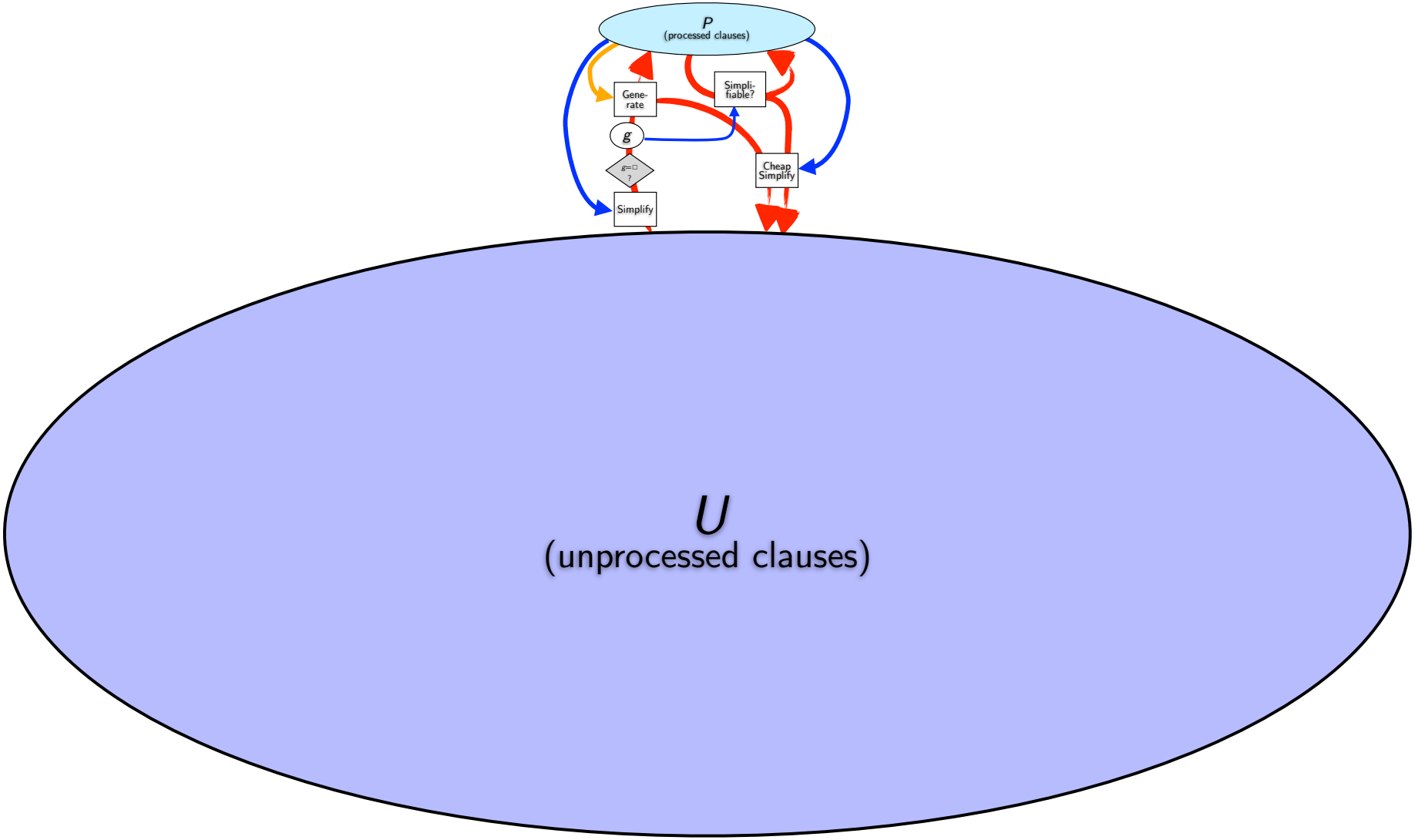
while $U \neq \{\}$
  $g = \text{delete\_best}(U)$
  $g = \text{simplify}(g, P)$
  if $g == \square$
    SUCCESS, Proof found
  if g is not subsumed by any clause in P (or otherwise redundant w.r.t. P)
    $P = P \backslash \{c \in P \mid c \text{ subsumed by (or otherwise redundant w.r.t.) } g\}$
    $T = \{c \in P \mid c \text{ can be simplified with } g\}$
    $P = (P \backslash T) \cup \{g\}$
    $T = T \cup \text{generate}(g, P)$
    foreach $c \in T$
      $c = \text{cheap\_simplify}(c, P)$
      if c is not trivial
        $U = U \cup \{c\}$
SUCCESS, original U is satisfiable

Compare and Contrast



while $U \neq \{\}$
  $g = \text{delete\_best}(U)$
  $g = \text{simplify}(g, P)$
  if $g == \square$
    SUCCESS, Proof found
  if g is not redundant w.r.t. P
    $P = P \setminus \{c \in P \mid c \text{ redundant w.r.t. } g\}$
    $T = \{c \in P \mid c \text{ simplifiable with } g\}$
    $P = (P \setminus T) \cup \{g\}$
    $T = T \cup \text{generate}(g, P)$
    foreach $c \in T$
      $c = \text{cheap\_simplify}(c, P)$
      if c is not trivial
        $U = U \cup \{c\}$
SUCCESS, original U is satisfiable

“You can’t handle the truth!”



Acctual problem

Martin Suda's PUZ001+1.p example.

Our situation

We have

- the resolution calculus for FOL (resolution + factoring),
- simplifications
  - pure literal deletion — clauses containing a literal that occurs only positively or negatively can be removed
  - tautology eliminations — tautologous clauses can be removed
  - subsumptions

However, the resolution calculus can produce many clauses that are useless or produced in multiple ways. Hence we would like to guide our proof search.

We can restrict our proof search in many ways, for example:

- select only some clauses,
- select only some literals,
- use different term orderings.

Ordered resolution

We know that we can impose an ordering on propositional atoms and still be complete. Hence we can do a similar thing for ground instances, however, for non-ground instances it is much more involved, see later.

$$\frac{\{l_1, \ldots, l_m, p\} \qquad \{\neg q, l_{m+1}, \ldots, l_{m+n}\}}{\{l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}\}\sigma}$$

where $\sigma = mgu(p, q)$ and p and $\neg q$ are maximal in their respective clauses.

$$\frac{\{l_1, \ldots, l_m, l, k\}}{\{l_1, \ldots, l_m, l\}\sigma}$$

where $\sigma = mgu(l, k)$ and l is maximal in the parent clause.

Selection function

We can overrule ordering restrictions in individual clauses by selecting non-maximal negative literals in clauses. A selection function selects a subset of literals and we compute inferences involving only selected literals (or maximal if no selected).

Clause selection heuristics

We usually sort clauses by some criteria:

- clause age — the maximal age of its parents + 1 (breadth-first search)
- clause weight — different types of weighting (best-first search)
  - small clauses are nice, but selecting only them leads to incompleteness
  - different types of symbols can have different weights (predicates, functions, variables)
  - symbols occurring in the conjecture can be prioritized

In E there are 5 priority queues and you can count different things, in Vampire you have just age and weight. Otter's approach is to combine the best-first and breadth-first search in a given ratio (e.g., 10:1).

Positive resolution

It is possible to restrict resolution in such a way that at least one parent is always a positive clause (=contains no negative literal).

Let Γ be a set of clauses. We split it into the positive part Γ^+ and $\Gamma' = \Gamma \setminus \Gamma^+$. If $\Gamma^+ = \emptyset$, then making all atomic predicates false satisfies $\Gamma' = \Gamma$. If $\Gamma' = \emptyset$, then making all atomic predicates true satisfies $\Gamma^+ = \Gamma$.

Proof is done for the ground case and we use the lifting argument.

Semantic resolution

A generalization of positive resolution, we have an interpretation I and we always select at least one clause that is not valid in I.

Set of support

We assume that some clauses must occur in any refutation. For example, if we want to prove $\Gamma \vdash \varphi$, we can assume that Γ is consistent and to refute $\Gamma \cup \{\neg\varphi\}$, it is necessary to use $\neg\varphi$. Hence we only allow derivations where a clause obtained from $\neg\varphi$ is involved.

A special case is the input resolution strategy, where at least one clause involved was from the input. It is complete for Horn clauses (Prolog) and it is linear; we can see a proof as a linear sequence where every clause is obtained from the previous one.

Watchlist

It is a set of clauses we feed into the prover that can be used for guiding the proof search (lemmata, hints). For example, clauses that led to proofs in previous similar problems.

In E, whenever a clause is generated, it is tested against the watchlist by subsumption. Every clause on the watchlist that is subsumed by a generated clause is removed (optional) and we can use this fact in our strategy (boost the generated clause).

Clause splitting

If we can split a clause into two (or more parts), which do not share variables, then we can do that and solve two (simpler) cases.

If we want to refute $\Gamma \cup \{l_1, \ldots, l_m, l_{m+1}, \ldots, l_{m+n}\}$, where $\{l_1, \ldots, l_m\}$ and $\{l_{m+1}, \ldots, l_{m+n}\}$ do not share variables, then we can do that by refuting both $\Gamma \cup \{l_1, \ldots, l_m\}$ and $\Gamma \cup \{l_{m+1}, \ldots, l_{m+n}\}$.

It is possible to go a step further and assume that these splits are like propositional cases and we can solve them by a SAT solver. It is called AVATAR in Vampire. Moreover, if theories are involved, we can use an SMT solver.

How to select correct parameters?

See Vampire's CASC mode at GitHub. It is a competition mode, where the standard timelimit is 300s.

TPTP and TSTP

The TPTP (Thousands of Problems for Theorem Provers) is a library of test problems for ATP systems. The TSTP (Thousands of Solutions from Theorem Provers) is a library of solutions to TPTP problems.

Language

Prolog like language both for input (problems) and output (solutions). For details see TPTP and TSTP Quick Guide.

TPTP example

```
fof(usa,axiom,( country(usa) )).

fof(country_big_city,axiom,( ! [C] : ( country(C)
     => ( big_city(capital_of(C))
        & beautiful(capital_of(C)) ) ) )).

fof(usa_capital_axiom,axiom,( ? [C] : ( city(C)
      & C = capital_of(usa) ) )).

fof(crime_axiom,axiom,( ! [C] : ( big_city(C)
     => has_crime(C) ) )).

fof(big_city_city,axiom,( ! [C] : ( big_city(C)
     => city(C) ) )).

fof(some_beautiful_crime,conjecture,( ? [C] : ( city(C)
      & beautiful(C) & has_crime(C) ) )).
```

TPTP roles (official definitions)

- axioms are accepted, without proof. There is no guarantee that the axioms of a problem are consistent.
- hypothesiss are assumed to be true for a particular problem, and are used like axioms.
- definitions are intended to define symbols. They are either universally quantified equations, or universally quantified equivalences with an atomic lefthand side. They can be treated like axioms.
- assumptions can be used like axioms, but must be discharged before a derivation is complete.
- lemmas and theorems have been proven from the axioms. They can be used like axioms in problems, and a problem containing a non-redundant lemma or theorem is ill-formed. They can also appear in derivations. theorems are more important than lemmas from the user perspective.
- conjectures are to be proven from the axiom(-like) formulae. A problem is solved only when all conjectures are proven.
- negated_conjectures are formed from negation of a conjecture (usually in a FOF to CNF conversion).
- plains have no specified user semantics.

Morever, there are fi_domain, fi_functors, fi_predicates, type, and unknown roles.

System before TPTP

System before TPTP is an interface for preprocessing systems.

```
cnf(i_0_4,plain, ( capital_of(usa) = esk1_0 )).
cnf(i_0_1,plain, ( country(usa) )).
cnf(i_0_5,plain, ( city(esk1_0) )).
cnf(i_0_7,plain, ( city(X1) | ~ big_city(X1) )).
cnf(i_0_6,plain, ( has_crime(X1) | ~ big_city(X1) )).
cnf(i_0_3,plain,
    ( big_city(capital_of(X1)) | ~ country(X1) )).
cnf(i_0_2,plain,
    ( beautiful(capital_of(X1)) | ~ country(X1) )).
cnf(i_0_8,negated_conjecture,
    ( ~ beautiful(X1) | ~ city(X1) | ~ has_crime(X1) )).
```

System on TPTP

System on TPTP is an interface for solvers.

```
# Proof found!
# SZS status Theorem
# SZS output start CNFRefutation
fof(some_beautiful_crime, conjecture, ?[X1]:((city(X1)&beautiful(X1))&has_crime(X1)), file('/tmp/SystemOn
fof(crime_axiom, axiom, ![X1]:(big_city(X1)=>has_crime(X1)), file('/tmp/SystemOnTPTPFormReply111578/SOT_m
fof(country_big_city, axiom, ![X1]:(country(X1)=>(big_city(capital_of(X1))&beautiful(capital_of(X1)))), f
fof(usa_capital_axiom, axiom, ?[X1]:(city(X1)&X1=capital_of(usa)), file('/tmp/SystemOnTPTPFormReply111578/
fof(usa, axiom, country(usa), file('/tmp/SystemOnTPTPFormReply111578/SOT_mQKyc4', usa)).
fof(c_0_5, negated_conjecture, ~(?[X1]:((city(X1)&beautiful(X1))&has_crime(X1))), inference(assume_negati
fof(c_0_6, negated_conjecture, ![X6]:(~city(X6)|~beautiful(X6)|~has_crime(X6)), inference(variable_rename
fof(c_0_7, plain, ![X4]:(~big_city(X4)|has_crime(X4)), inference(variable_rename,[status(thm)],[inference
fof(c_0_8, plain, ![X2]:((big_city(capital_of(X2))|~country(X2))&(beautiful(capital_of(X2))|~country(X2))
fof(c_0_9, plain, (city(esk1_0)&esk1_0=capital_of(usa)), inference(skolemize,[status(esa)],[inference(var
cnf(c_0_10, negated_conjecture, (~city(X1)|~beautiful(X1)|~has_crime(X1)), inference(split_conjunct,[stat
cnf(c_0_11, plain, (has_crime(X1)|~big_city(X1)), inference(split_conjunct,[status(thm)],[c_0_7])).
cnf(c_0_12, plain, (beautiful(capital_of(X1))|~country(X1)), inference(split_conjunct,[status(thm)],[c_0_
cnf(c_0_13, plain, (esk1_0=capital_of(usa)), inference(split_conjunct,[status(thm)],[c_0_9])).
cnf(c_0_14, plain, (country(usa)), inference(split_conjunct,[status(thm)],[usa])).
cnf(c_0_15, plain, (big_city(capital_of(X1))|~country(X1)), inference(split_conjunct,[status(thm)],[c_0_8
cnf(c_0_16, negated_conjecture, (~city(X1)|~beautiful(X1)|~big_city(X1)), inference(spm,[status(thm)],[c_
cnf(c_0_17, plain, (city(esk1_0)), inference(split_conjunct,[status(thm)],[c_0_9])).
cnf(c_0_18, plain, (beautiful(esk1_0)), inference(cn,[status(thm)],[inference(rw,[status(thm)],[inference
cnf(c_0_19, plain, (big_city(esk1_0)), inference(cn,[status(thm)],[inference(rw,[status(thm)],[inference(
cnf(c_0_20, negated_conjecture, ($false), inference(cn,[status(thm)],[inference(rw,[status(thm)],[inferen
```

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