INTERSECTIONS OF LINE SEGMENTS AND POLYGONS

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Based on [Berg], [Mount], [Kukral], and [Drtina]

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Talk overview

- Intersections of line segments (Bentley-Ottmann)
  - Motivation
  - Sweep line algorithm recapitulation
  - Sweep line intersections of line segments
- Intersection of polygons or planar subdivisions
  - See assignment [21] or [Berg, Section 2.3]
- Intersection of axis parallel rectangles
  - See assignment [26]

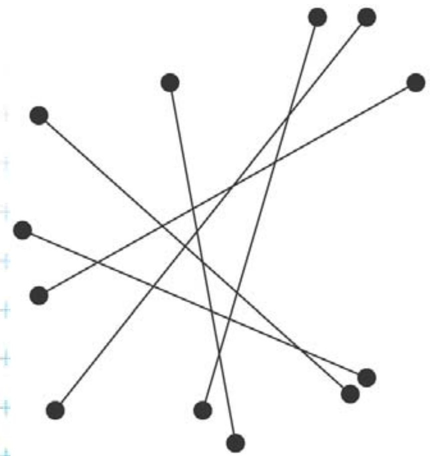
Geometric intersections – what are they for?

One of the most basic problems in computational geometry

- Solid modeling
  - Intersection of object boundaries in CSG
- Overlay of subdivisions, e.g. layers in GIS
  - Bridges on intersections of roads and rivers
  - Maintenance responsibilities (road network X county boundaries)
- Robotics
  - Collision detection and collision avoidance
- Computer graphics
  - Rendering via ray shooting (intersection of the ray with objects)
- …

Line segment intersection

- Intersection of complex shapes is often reduced to simpler and simpler intersection problems
- Line segment intersection is the most basic intersection algorithm
- Problem statement:
  Given n line segments in the plane, report all points where a pair of line segments intersect.
- Problem complexity
  - Worst case – $I = O(n^2)$ intersections
  - Practical case – only some intersections
  - Use an output sensitive algorithm
    - $O(n \log n + I)$ optimal randomized algorithm
    - $O(n \log n + I \log n)$ sweep line algorithm - %



[Berg]

Plane sweep line algorithm recapitulation

- Horizontal line (sweep line, *scan line*) ℓ moves top-down (or vertical line: left to right) over the set of objects
- The move is not continuous, but ℓ jumps from one event point to another
  - Event points are in priority queue or sorted list
  - The left-most event point is removed first
  - New event points may be created (usually as interaction of neighbors on the sweep line) and inserted in the queue
- Scan-line status
  - Stores information about the objects intersected by SL
  - It is updated while stopping on event point

Line segment intersection - Sweep line alg.

- Avoid testing of pairs of segments far apart
- Compute intersections of neighbors on the sweep line only
- O(n log n + I log n) time in O(n) memory

  2n steps for end points, I steps for intersections, log n search the tree

- Ignore "nasty cases" (most of them will be solved later on)
  - No segment is parallel to a sweep line
  - Segments intersect in one point and do not overlap
  - No three segments meet in a common point

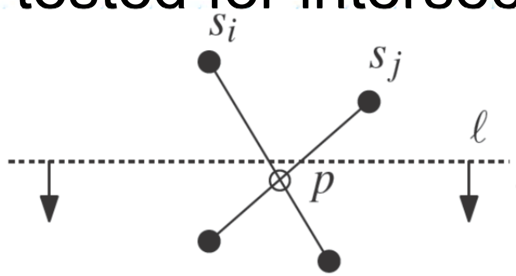
Line segment intersections

- *Status* = ordered sequence of segments intersecting the sweep line ℓ
- *Events* (waiting in the priority queue)

  = points, where the algorithm actually does something
  - Segment *end-points*
    - known at algorithm start
  - Segment *intersections* between neighboring segments along SL
    - Discovered as the sweep executes

Detecting intersections

- Intersection events must be detected and inserted to the event queue before they occur
- Given two segments *a, b* intersecting in a point *p*, there must be a placement of sweep line ℓ prior to *p*, such that segments *a, b* are adjacent along ℓ (only adjacent will be tested for intersection)
  – segments *a, b* are not adjacent when the alg. starts
  – segments *a, b* are adjacent just before *p*
  => there must be an event point when *a,b* become adjacent and therefore are tested for intersection

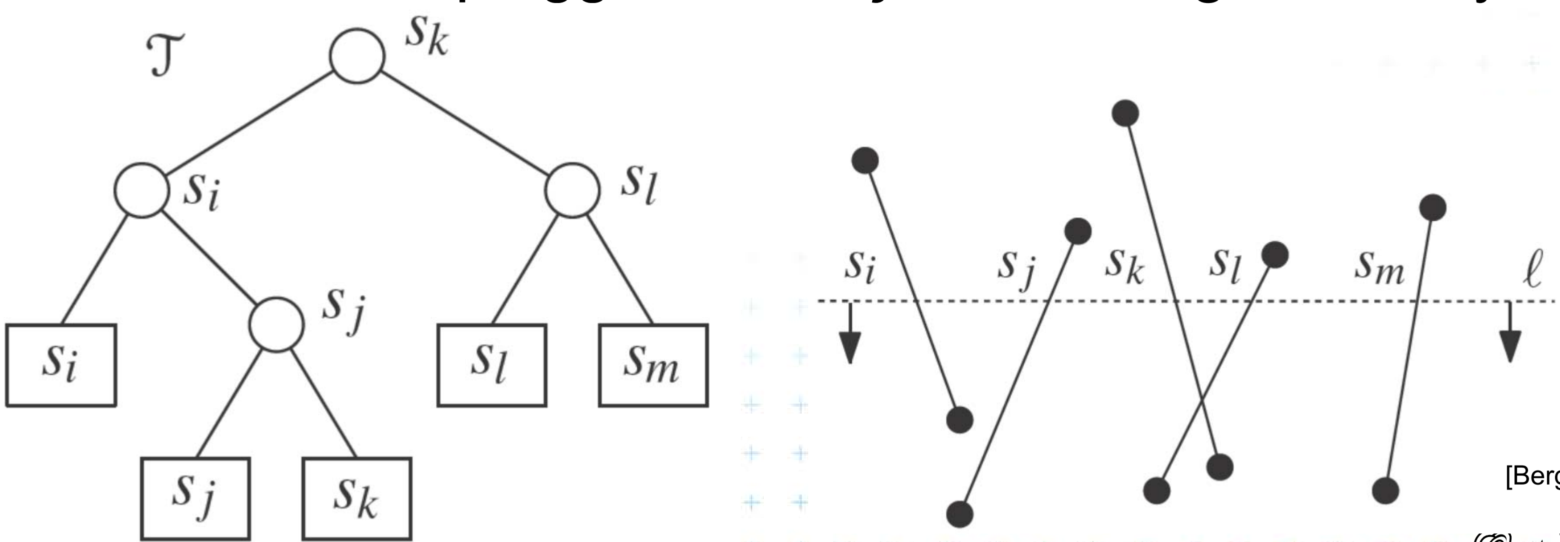


[Berg]

Data structures

Sweep line ℓ status = order of segments along ℓ

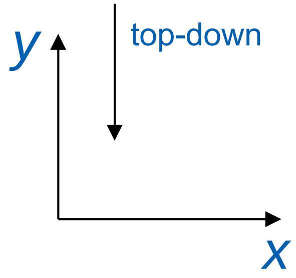
- Balanced binary search tree of segments
- Coords of intersections with ℓ vary as ℓ moves
  => store pointers to line segments in tree nodes
  – Position of ℓ is plugged in the $y=mx+b$ to get the key



[Berg]

Data structures

Event queue (postupový plán, časový plán)



- Define: Order $\prec$ (top-down, lexicographic)

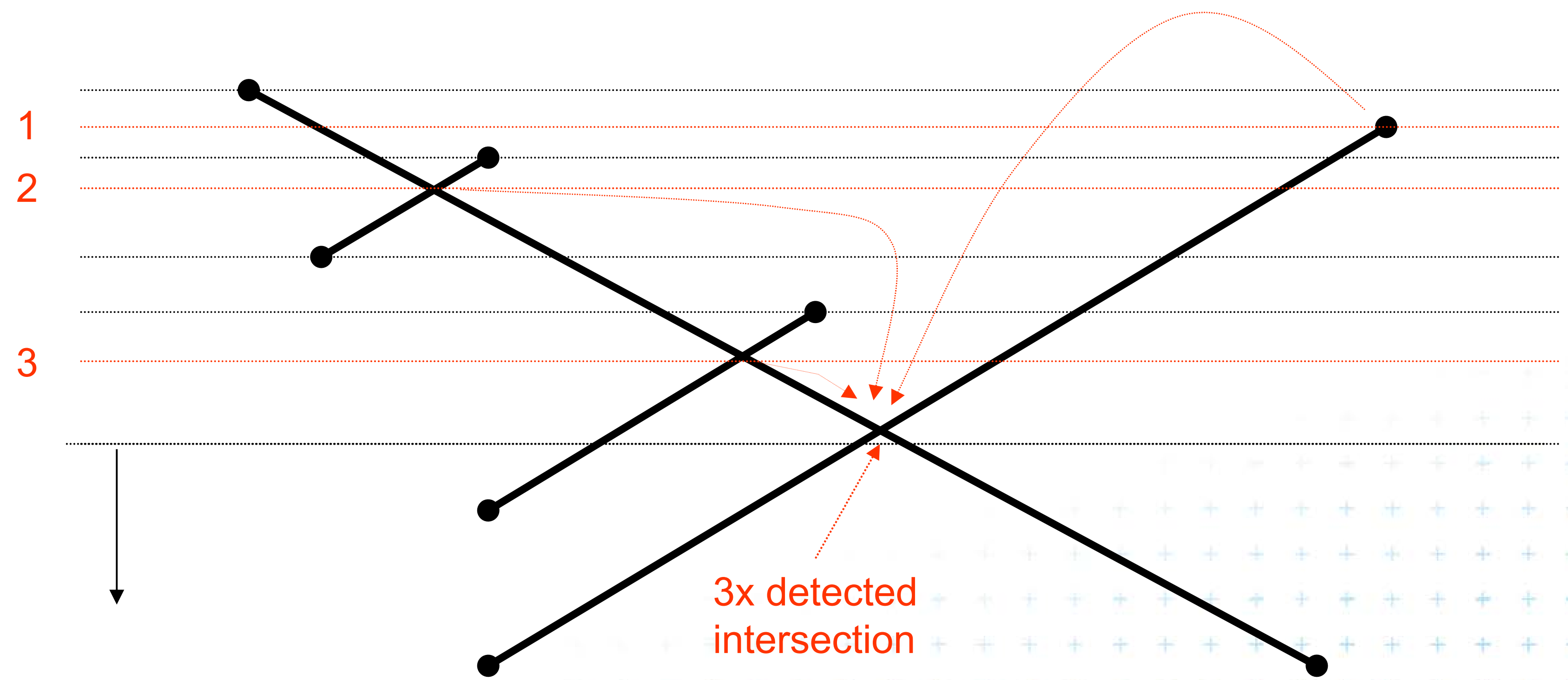
  $p \prec q$ **iff** $p_y > q_y$ **or** $p_y = q_y$ and $p_x < q_x$

  top-down, left-right approach

  (points on ℓ treated left to right)

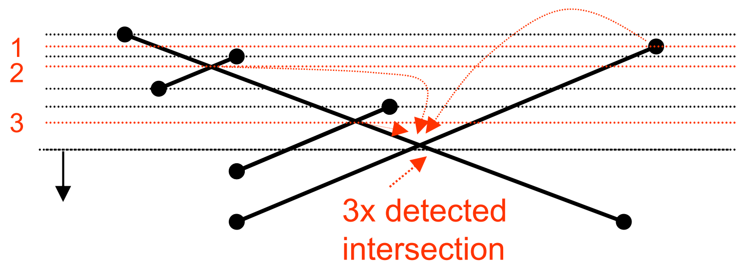
- Operations
  - Insertion of computed intersection points
  - Fetching the next event (highest y below ℓ)
  - Test, if the segment is already present in the queue
  - (Delete intersection event in the queue)

Problem with duplicities of intersections



Data structures

Event queue data structure



- Heap
  - Problem: can not check duplicated intersection events (reinvented more than once)
  - Intersections processed twice or even more
  - Memory complexity up to O(n^2)
- Ordered dictionary (balanced binary tree)
  - Can check duplicated events (adds just constant factor)
  - Nothing inserted twice
  - If non-neighbor intersections are deleted
    i.e., only intersection of neighbors is stored
    then memory complexity just O(n)

Line segment intersection algorithm

FindIntersections(*S*)

Input: A set *S* of line segments in the plane

Output: The set of intersection points + pointers to segments in each

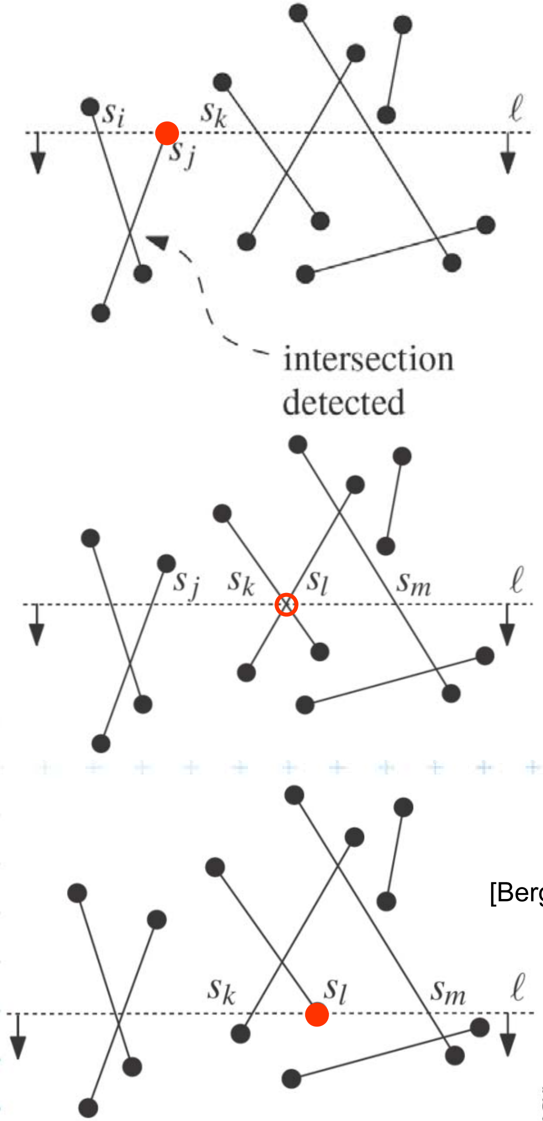
1. init an empty event queue *Q* and insert the segment endpoints
2. init an empty status structure *T*
3. **while** Q in not empty
4. remove next event *p* from *Q*
5. handleEventPoint(*p*)

- Upper endpoint
- Intersection
- Lower endpoint

Note: Upper-end-point events store info about the segment

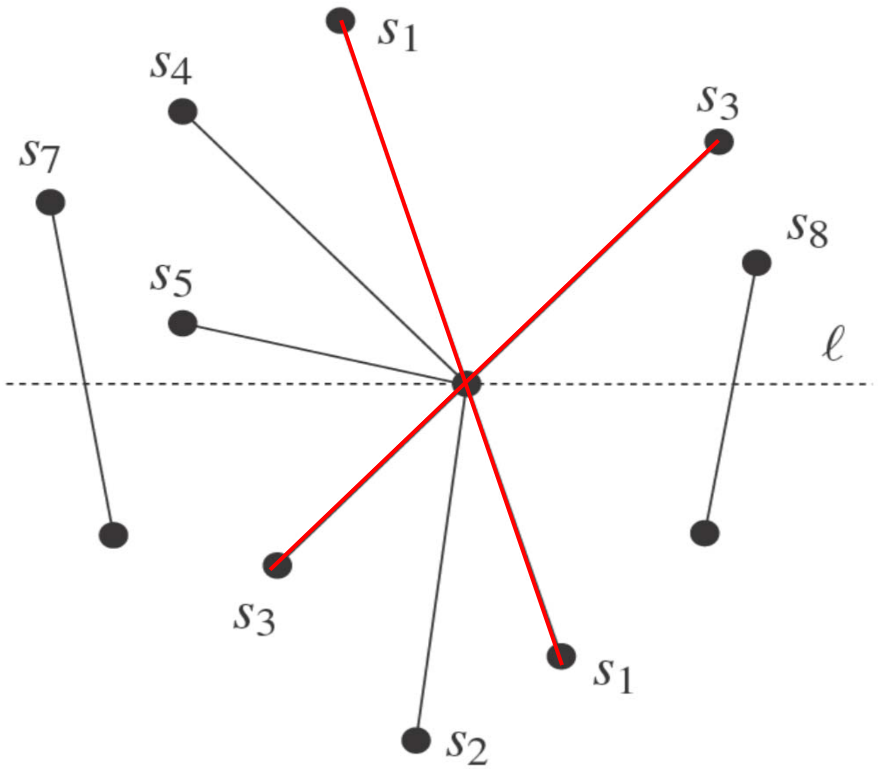
handleEventPoint principle

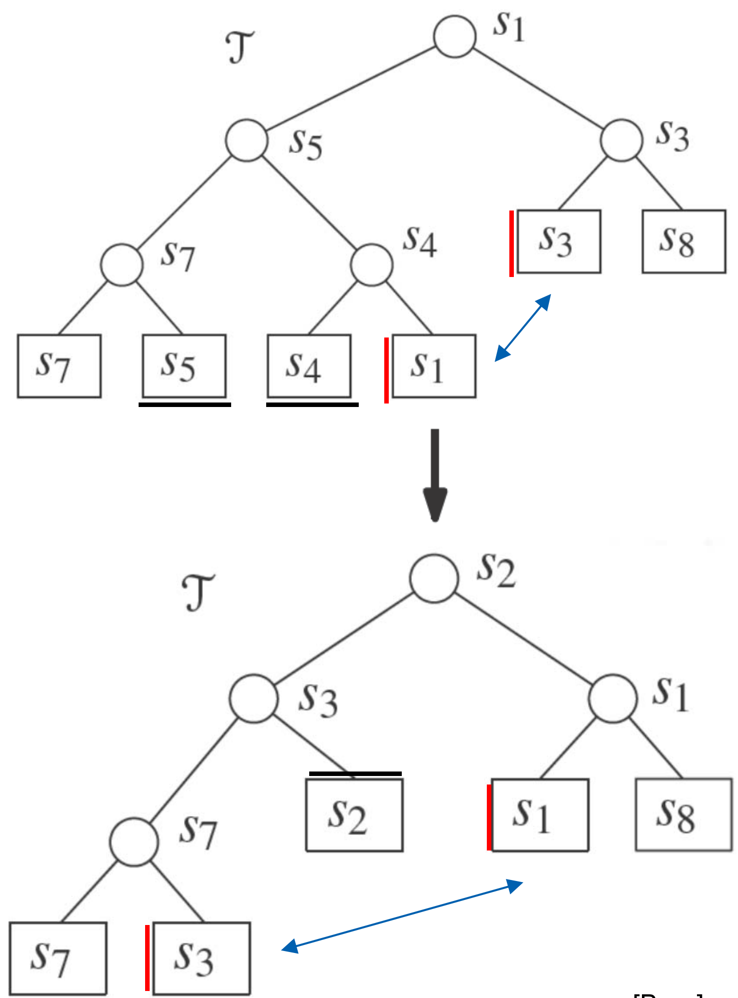
- Upper endpoint $U(p)$
  - insert p (on s_j) to status T
  - add intersections with left and right neighbors to Q
- Intersection $C(p)$
  - switch order of segments in T
  - add intersections of left and right neighbors to Q
- Lower endpoint $L(p)$
  - remove p (on s_l) from T
  - add intersections of left and right neighbors to Q



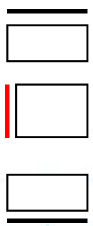
[Berg]

More than two segments incident





[Berg]



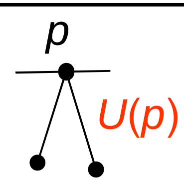
$U(p) = \{s_2\}$ start here

$C(p) = \{s_1, s_3\}$ cross on ℓ

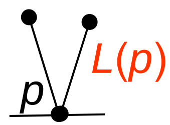
$L(p) = \{s_4, s_5\}$ end here

Handle Events [Berg, page 25]

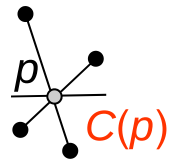
handleEventPoint(p)



1. Let $U(p)$ = set of segments whose upper point is p.
   These segmets are stored with the event point p (will be added to T)
2. Search T for all segments $S(p)$ that contain p (are adjacent in T):
   Let $L(p) \subset S(p)$ = segments whose lower endpoint is p
   Let $C(p) \subset S(p)$ = segments that contains p in interior

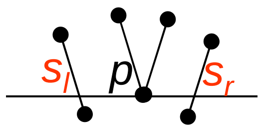


3. **if**($L(p) \cup U(p) \cup C(p)$ contains more than one segment)
4. report p as intersection together with $L(p)$, $U(p)$, $C(p)$



5. Delete the segments in $L(p) \cup C(p)$ from T
6. Insert the segments in $U(p) \cup C(p)$ into T
   (order as below ℓ, horizontal segment as the last)

Steps 5–6: Reverse order of $C(p)$ in T



7. **if**($U(p) \cup C(p) = \emptyset$) then findNewEvent(s_l, s_r, p) // left & right neighbors
8. **else** s' = leftmost segment of $U(p) \cup C(p)$; findNewEvent(s_l , s', p)
   s'' = rightmost segment of $U(p) \cup C(p)$; findNewEvent(s'', s_r , p)

Detection of new intersections

findNewEvent(s_l, s_r, p)   **// with handling of horizontal segments**

Input: two segments (left & right from *p* in *T*) and a current event point *p*

Output: updated event queue *Q* with new intersection

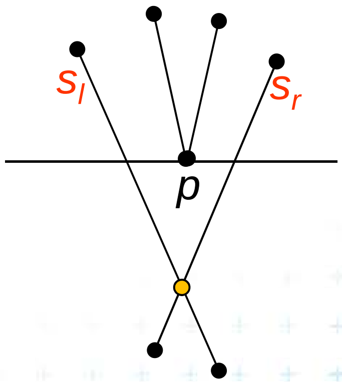
1. **if** [**(** s_l and s_r intersect below the sweep line ℓ) or

   (intersect on ℓ and to the right of *p*)] and   // horizontal segments

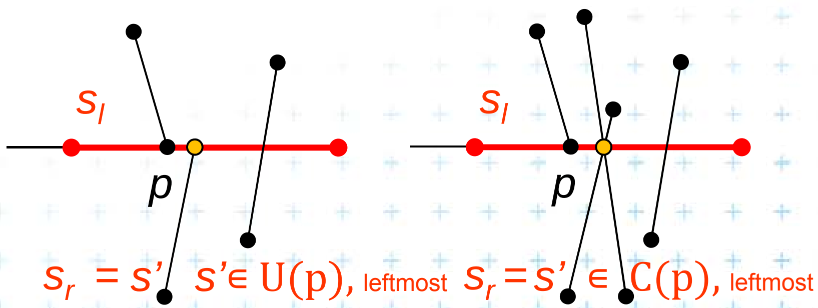
   (the intersection is not present in *Q*)

2. **then**

   insert *p* as an event into *Q*



s_l and s_r intersect below



s_l and s_r = s' intersect on ℓ and to the right of *p*

Line segment intersections

- Memory $O(I) = O(n^2)$ with duplicities in Q
  or $O(n)$ with duplicities in Q deleted
- Operational complexity
  - $n + I$ stops
  - $\log n$ each

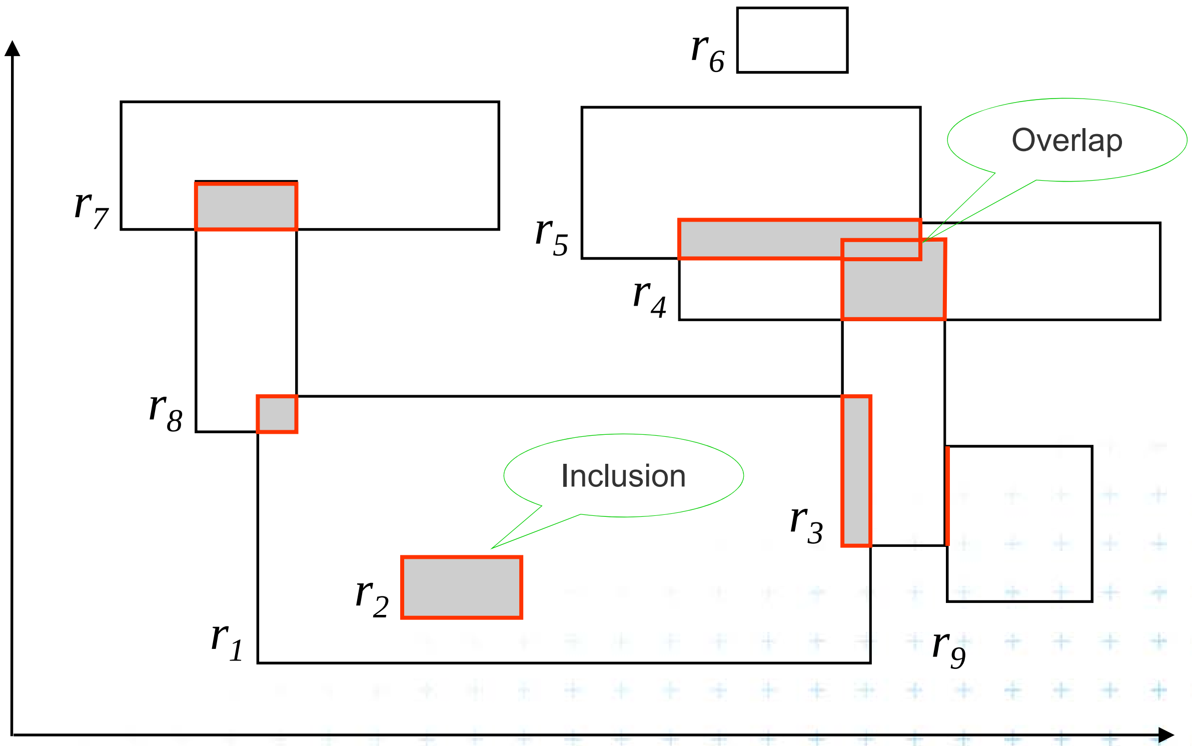
  => $O(I + n) \log n$ total
- The algorithm is by Bentley-Ottmann

  Bentley, J. L.; Ottmann, T. A. (1979), "Algorithms for reporting and counting geometric intersections", *IEEE Transactions on Computers* **C-28** (9): 643-647, doi:10.1109/TC.1979.1675432 .

  See also http://wapedia.mobi/en/Bentley%E2%80%93Ottmann_algorithm

Intersection of axis parallel rectangles

- Given the collection of *n isothetic* rectangles, report all intersecting parts



[?]

Answer: (r_1, r_2) (r_1, r_3) (r_1, r_8) (r_3, r_4) (r_3, r_5) (r_4, r_5) (r_7, r_8) (r_3, r_9)

Brute force intersection

Brute force algorithm

Input: set *S* of axis parallel rectangles
Output: pairs of intersected rectangles

1. For every pair (r_i, r_j) of rectangles $\in S$, $i \neq j$
2. if $(r_i \cap r_j \neq \emptyset)$ then
3. report (r_i, r_j)

Analysis

Preprocessing: None.

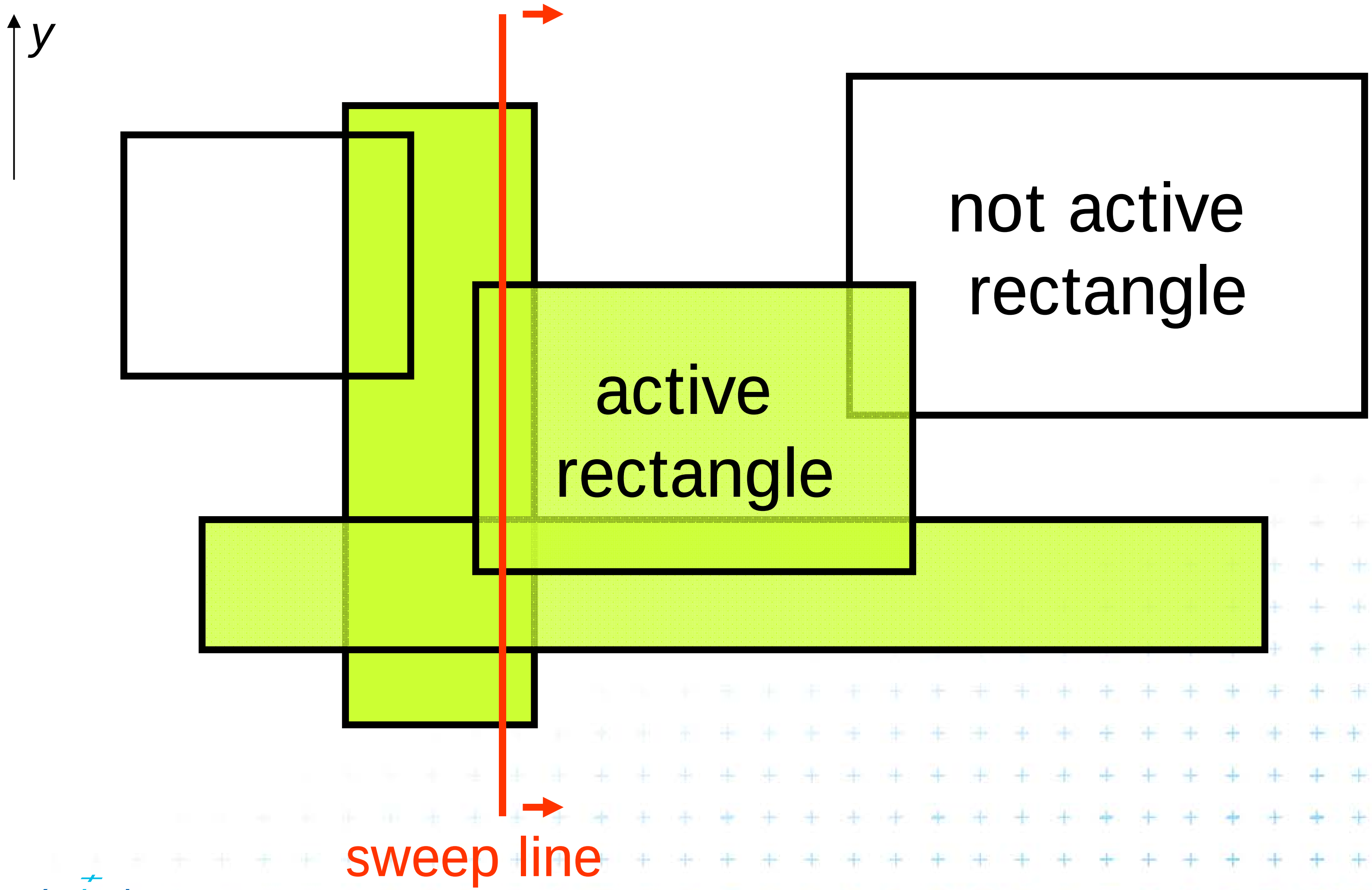
Query: $O(N^2)$; $\binom{N}{2} = (N(N-1))/2 \in O(N^2)$.

Storage: $O(N)$.

Plane sweep intersection algorithm

- Vertical sweep line moves from left to right
- Stops at every x-coordinate of a rectangle (either its left side or its right side).
- active rectangles – a set
  = rectangles currently intersecting the sweep line
  - left side event of a rectangle
    => the rectangle is added to the active set.
  - right side
    => the rectangle is deleted from the active set.
- The active set used to detect rectangle intersection

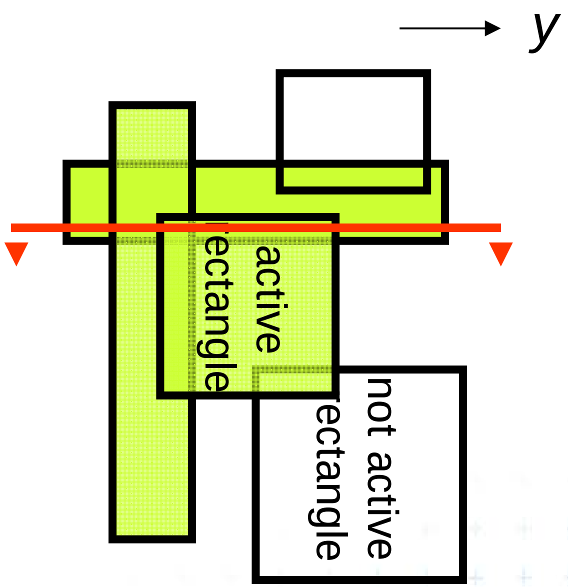
Example rectangles and sweep line



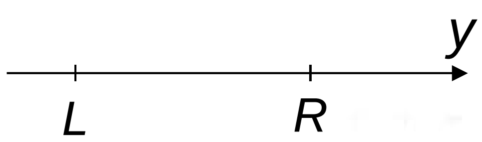
[Drtina]

Interval tree as sweep line status structure

- Vertical sweep-line => Only *y*-coordinates along it
- Turn our view in slides 90° right
- Sweep line (*y*-axis) will be drawn as horizontal



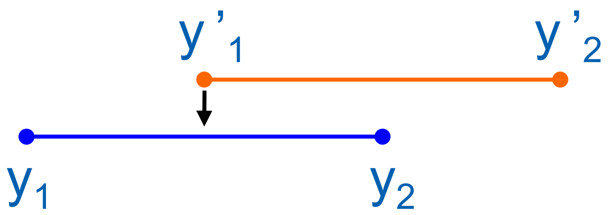
sweep line [Drtina]



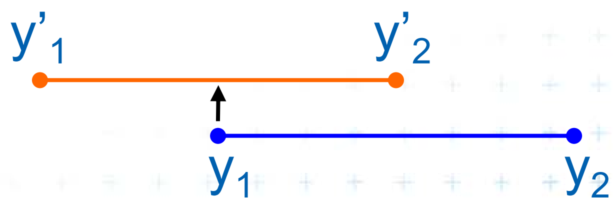
Intersection test – between pair of intervals

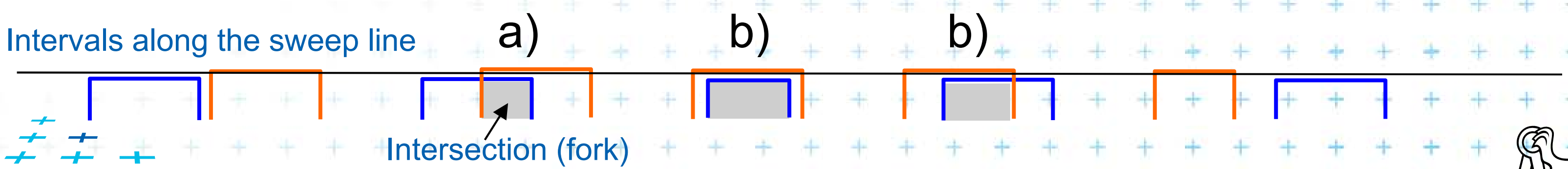
- Given two intervals $R = [y_1, y_2]$ and $R' = [y'_1, y'_2]$ the condition $R \cap R'$ is equivalent to one of these mutually exclusive conditions:

a) $y_1 \le y'_1 \le y_2$



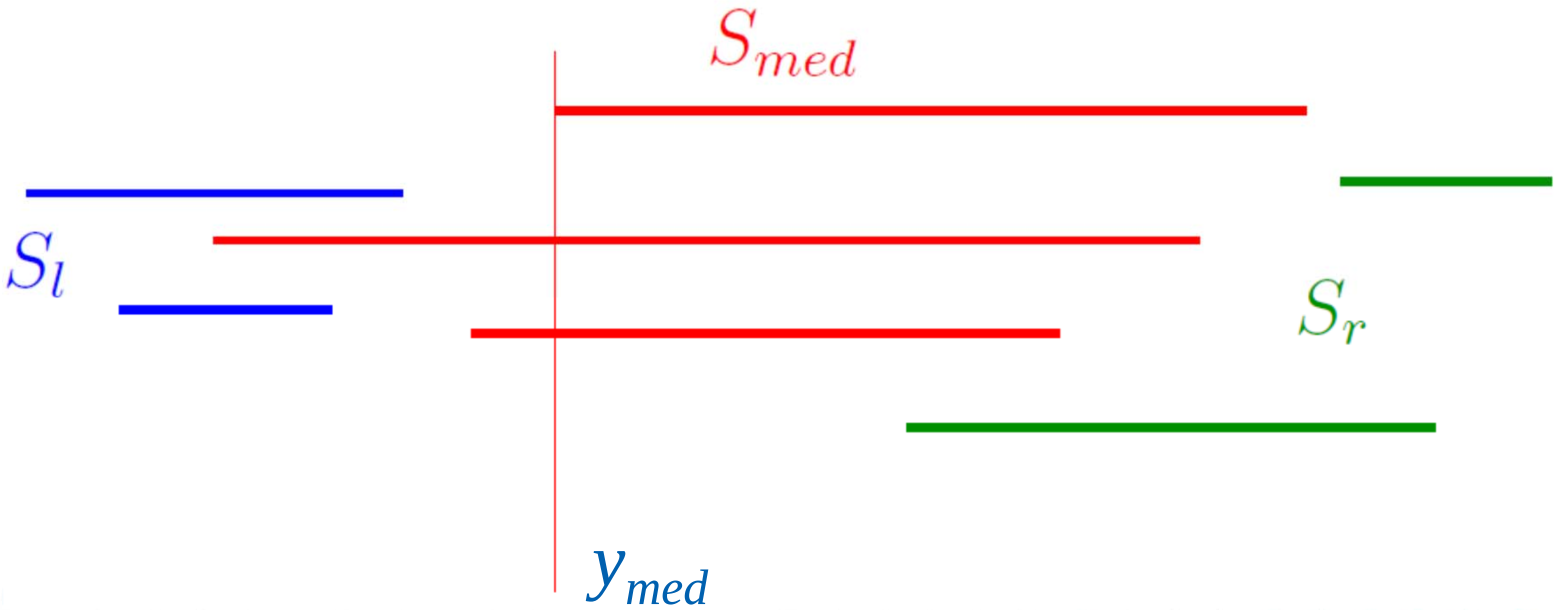
b) $y'_1 \le y_1 \le y'_2$





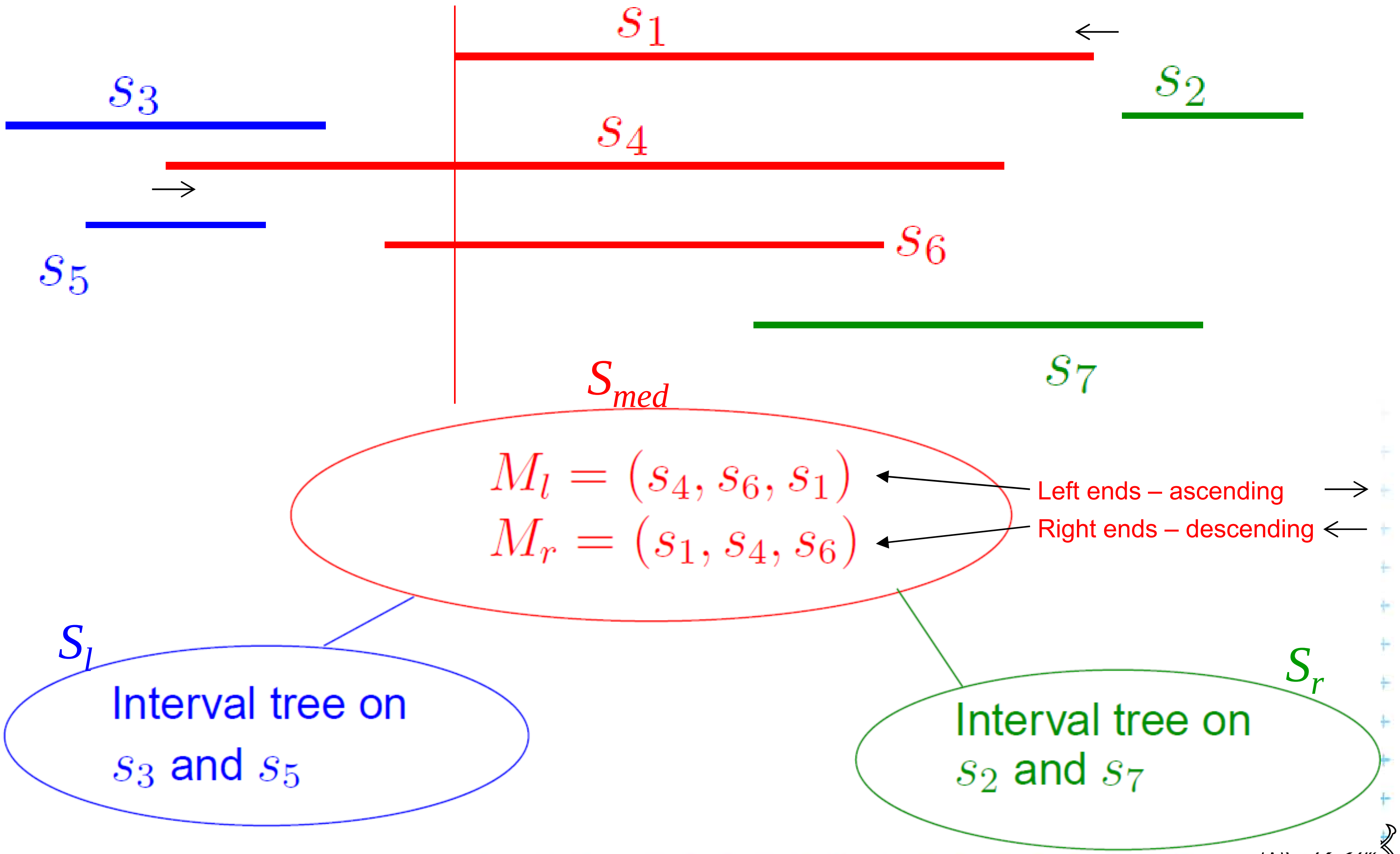
Static interval tree – stores all end points

- Let $v = y_{med}$ be the median of end-points of segments
- S_l : segments of S that are completely to the left of y_{med}
- S_{med} : segments of S that contain y_{med}
- S_r : segments of S that are completely to the right of y_{med}



[Vigneron]

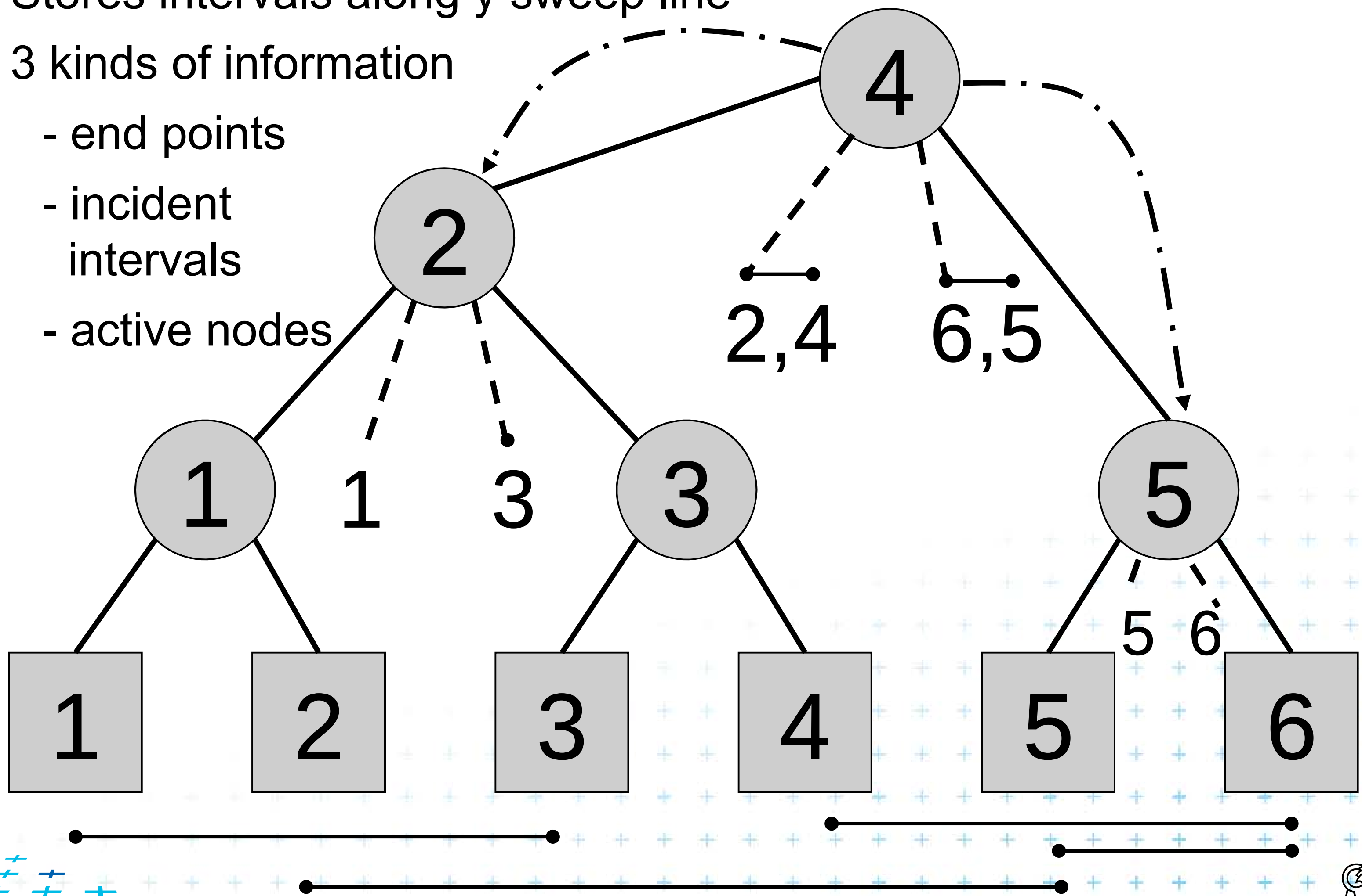
Static interval tree – Example



[Vigneron]

Static interval tree [Edelsbrunner80]

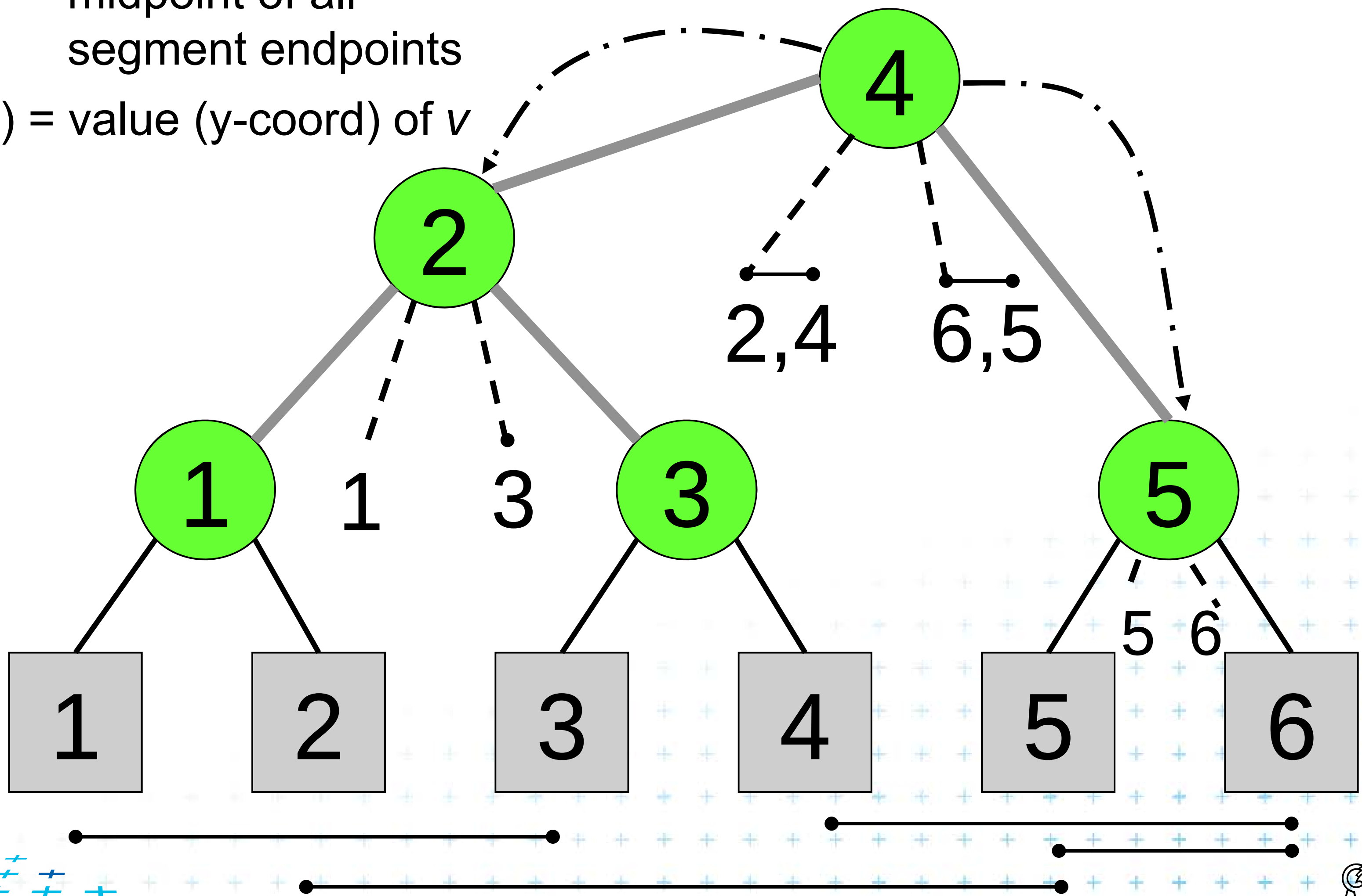
- Stores intervals along y sweep line
- 3 kinds of information
  - end points
  - incident intervals
  - active nodes



Primary structure – static tree for endpoints

v = midpoint of all segment endpoints

H(v) = value (y-coord) of *v*

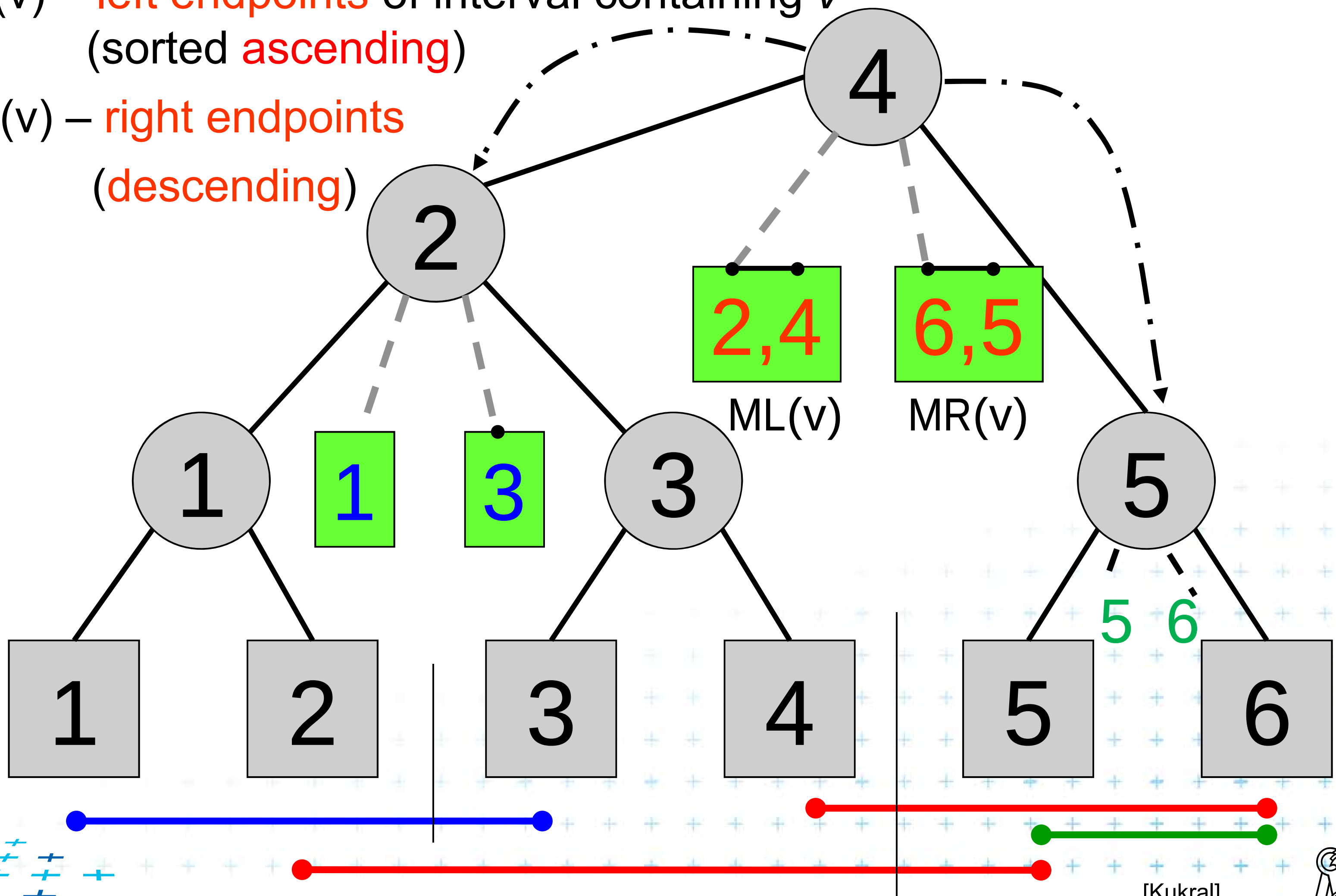


[Kukral]

Secondary lists of incident interval end-pts.

ML(v) – left endpoints of interval containing *v* (sorted ascending)

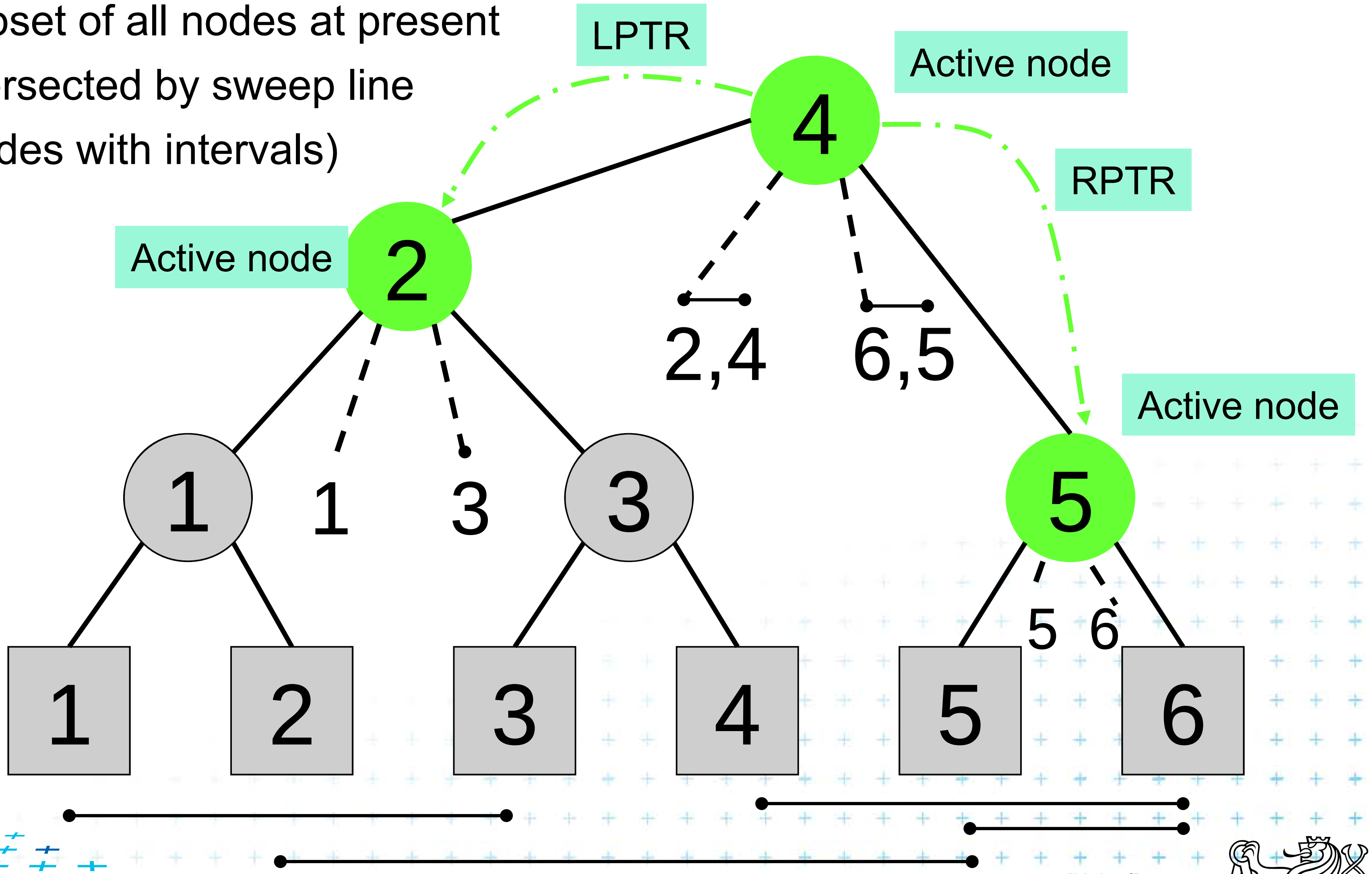
MR(v) – right endpoints (descending)



[Kukral]

Active nodes – intersected by the sweep line

Subset of all nodes at present intersected by sweep line (nodes with intervals)



[Kukral]

Query = sweep and report intersections

RectangleIntersections(*S*)
Input: Set *S* of rectangles
Output: Intersected rectangle pairs

Preprocess(S) // create the interval tree *T* (for y-coords)
// and event queue *Q* (for x-coords)
while (*Q* $\neq \emptyset$) do
  Get next entry (x_i, y_{il}, y_{ir}, t) from *Q* // t = { *left* | *right* }
  if (t = left) // left edge
    a) QueryInterval (y_{il}, y_{ir}, root(T)) // report intersections
    b) InsertInterval (y_{il}, y_{ir}, root(T)) // insert new interval
  else // right edge
    c) DeleteInterval (y_{il}, y_{ir}, root(T))

Preprocessing

Preprocess(S)
Input: Set *S* of rectangles
Output: Primary structure of the interval tree *T* and the event queue *Q*

1. *T* = PrimaryTree(*S*) // Construct the static primary structure
   // of the interval tree -> sweep line STATUS *T*

2. // Init event queue Q with vertical rectangle edges in ascending order.
   // Put the left edges with the same x ahead of right ones.
3. for i = 1 to n
4. insert((x_{il} , y_{il} , y_{ir} , left), Q) // left edges of *i-th* rectangle
5. insert((x_{ir}, y_{il} , y_{ir} , right), Q) // right edges

Interval tree – primary structure construction

PrimaryTree(*S*)

Input: Set *S* of rectangles

Output: Primary structure of an interval tree *T*

1. S_y = Sort endpoints of all segments in *S* according to *y*-coordinate
2. *T* = BST(S_y)
3. **return** *T*

BST(S_y)

1. **if**($|S_y| = 0$) **return** null
2. *yMed = median of* S_y
3. *L* = endpoints $p_y \leq yMed$
4. R = endpoints $p_y > yMed$
5. *t* = *new* IntervalTreeNode(*yMed*)
6. *t.left* = BST(*L*)
7. *t.right* = BST(*R*)
8. **return** *t*

Interval tree – search the intersections

QueryInterval (*b, e, T*)

Input: Interval of the edge and current tree *T*

Output: Report the rectangles that intersect [*b, e*]

if(*T* = null) **return**

i=0; **if**(b < H(v) < e) // forks at node

    while ~~(*MR*(v).[i] >= b) &&~~ (i < Count(v)) .. Report all intervals inM

        ReportIntersection; i++

    QueryInterval(*b,e,T.LPTR*)

    QueryInterval(*b,e,T.RPTR*)

else if (H(v) ≤ b < e) // search RIGHT

    while (*MR*(v).[i] >= b) && (i < Count(v))

        ReportIntersection; i++

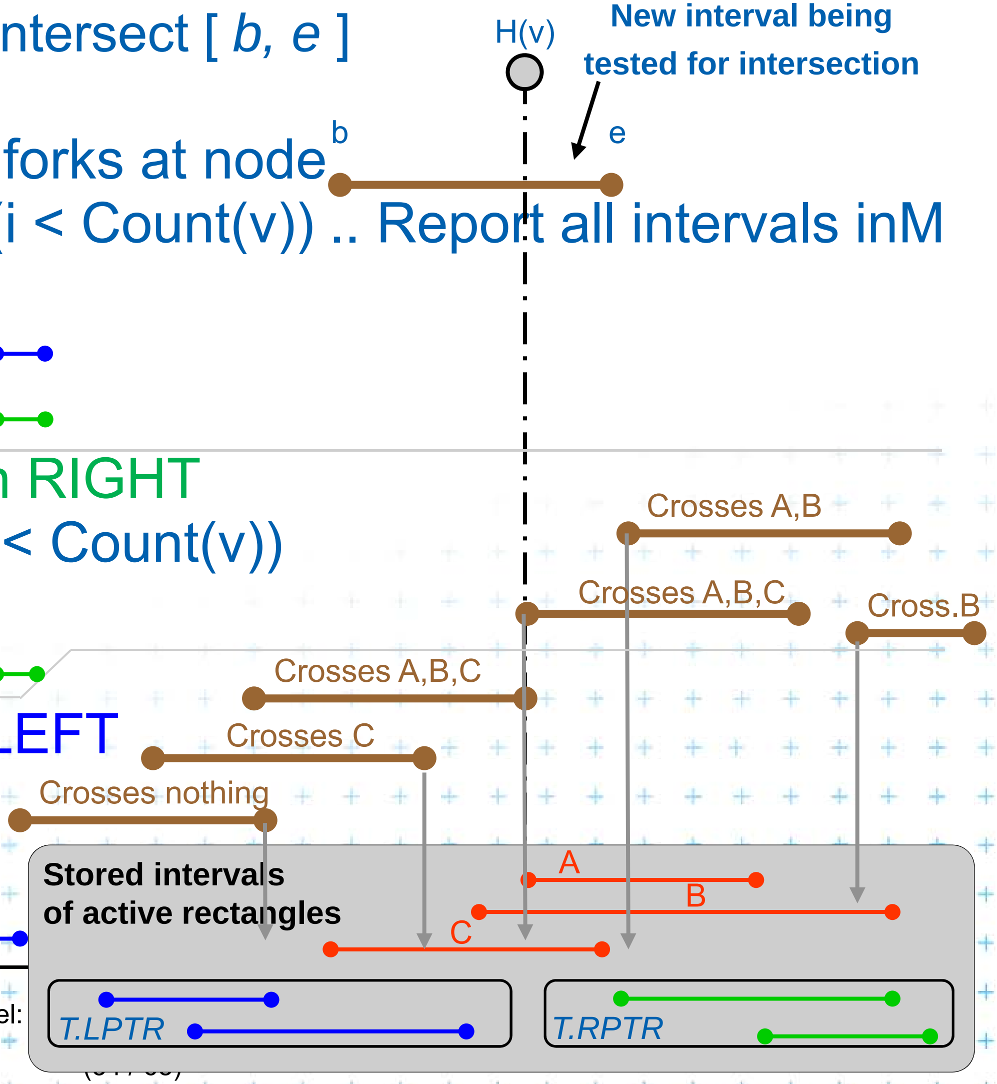
    QueryInterval(*b,e,T.RPTR*)

else // b < e ≤ H(v) //search LEFT

    while (*ML*(v).[i] <= e)

        ReportIntersection; i++

    QueryInterval(*b,e,T.LPTR*)



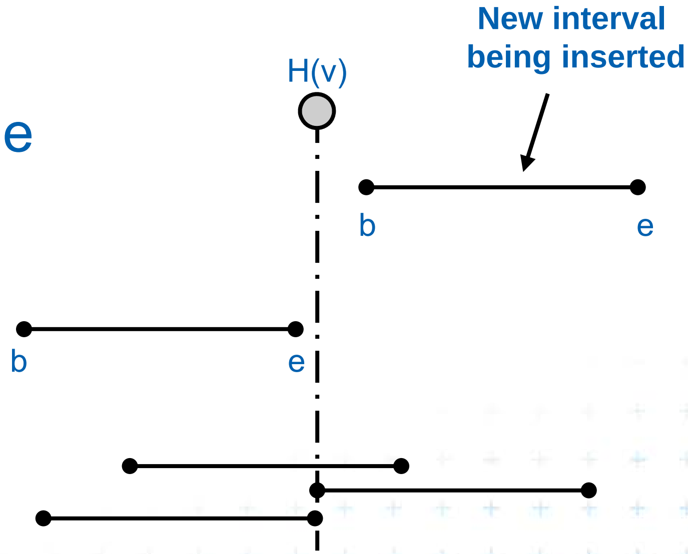
Interval tree - interval insertion

InsertInterval (*b, e, T*)

Input: Interval [*b*,*e*] and interval tree *T*

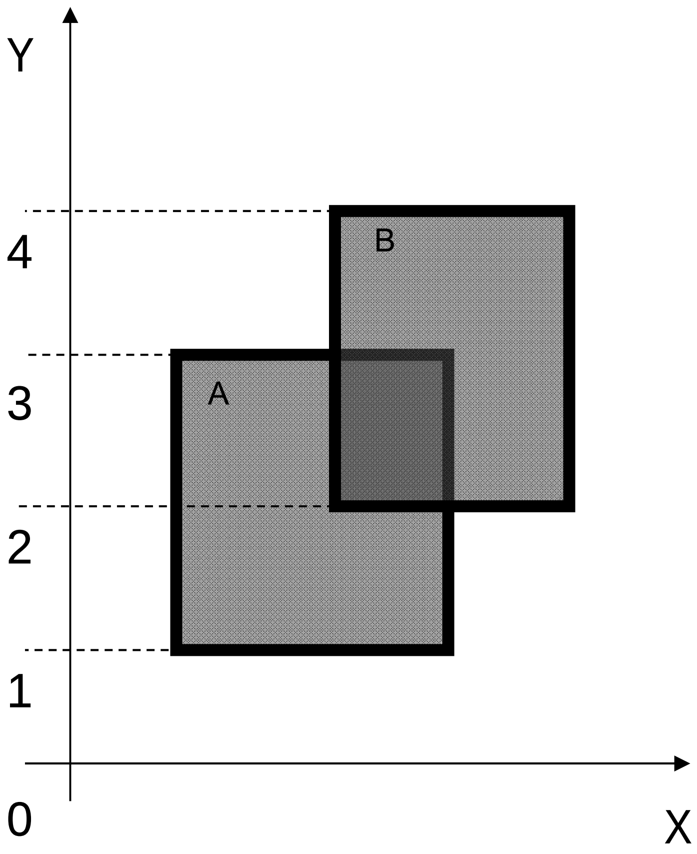
Output: *T* after insertion of the interval

```
v = root(T )
while( v != null )        // find the fork node
    if (H(v) < b < e)
        v = v.right       // continue right
    else if (b < e < H(v))
        v = v.left        // continue left
    else  // b ≤ H(v) ≤ e  // insert interval
        set v node to active
        connect LPTR resp. RPTR to its parent
        insert [b,e] into list ML(v) – sorted in ascending order of b's
        insert [b,e] into list MR(v) – sorted in descending order of e's
        break
endwhile
return T
```

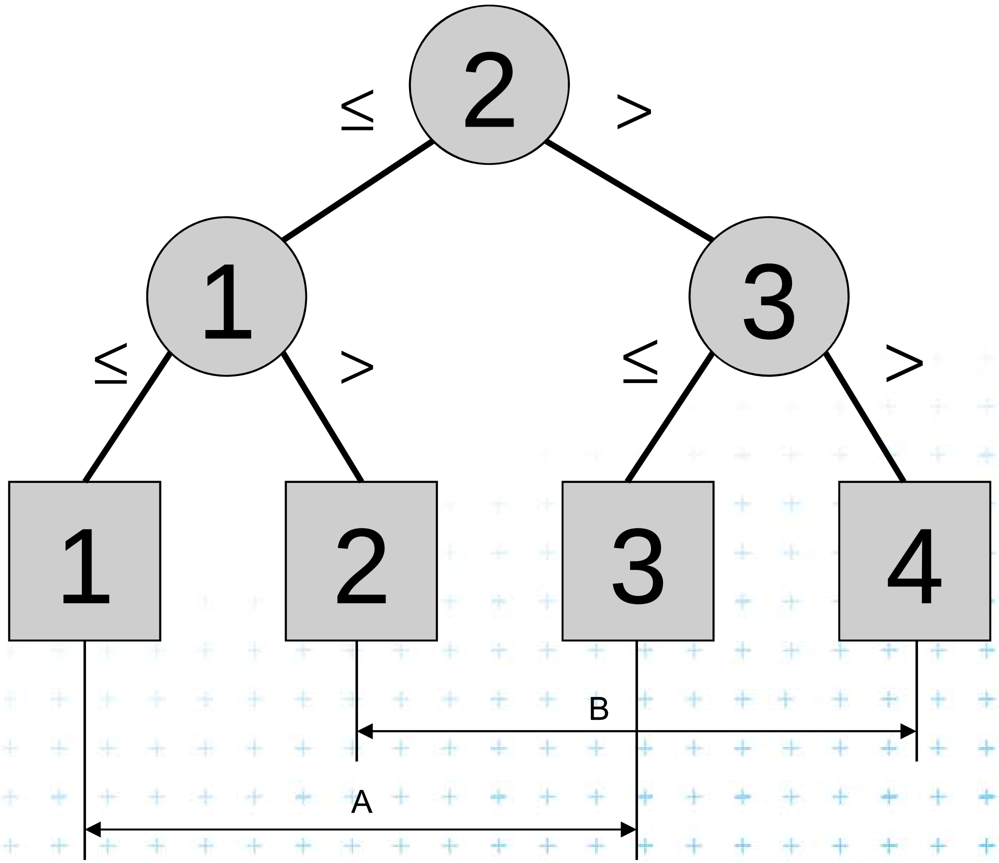


Example 1

Example 1 – static tree on endpoints



H(v) – value of node *v*



[Drtina]

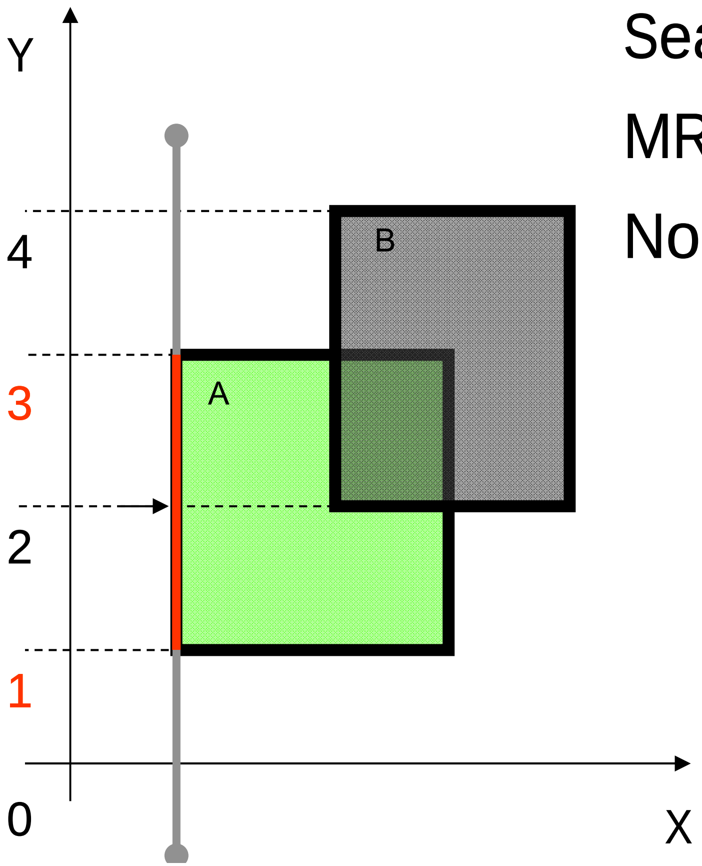
Interval insertion [1,3]   a) Query Interval

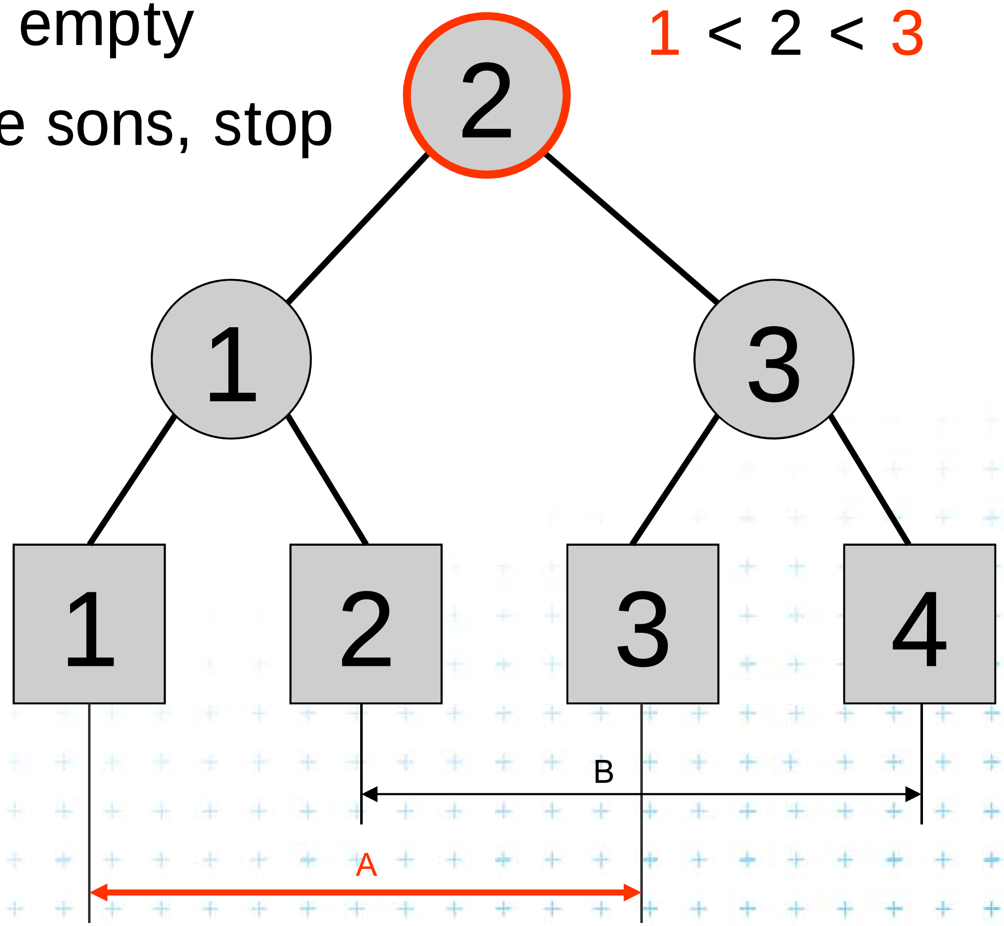
Search MR(v) or ML(v): ← b < H(v) < e

MR(v) is empty

No active sons, stop

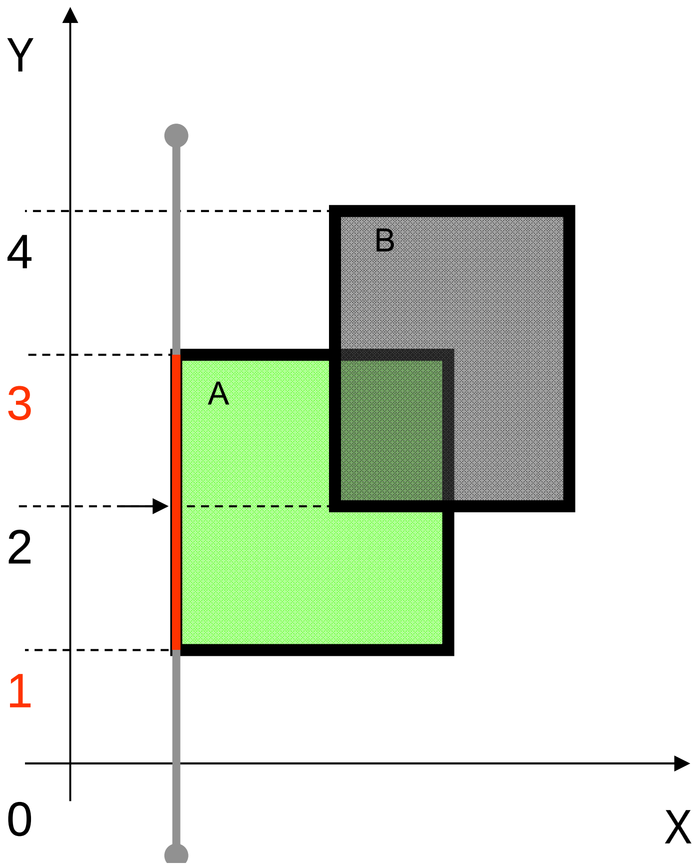
1 < 2 < 3





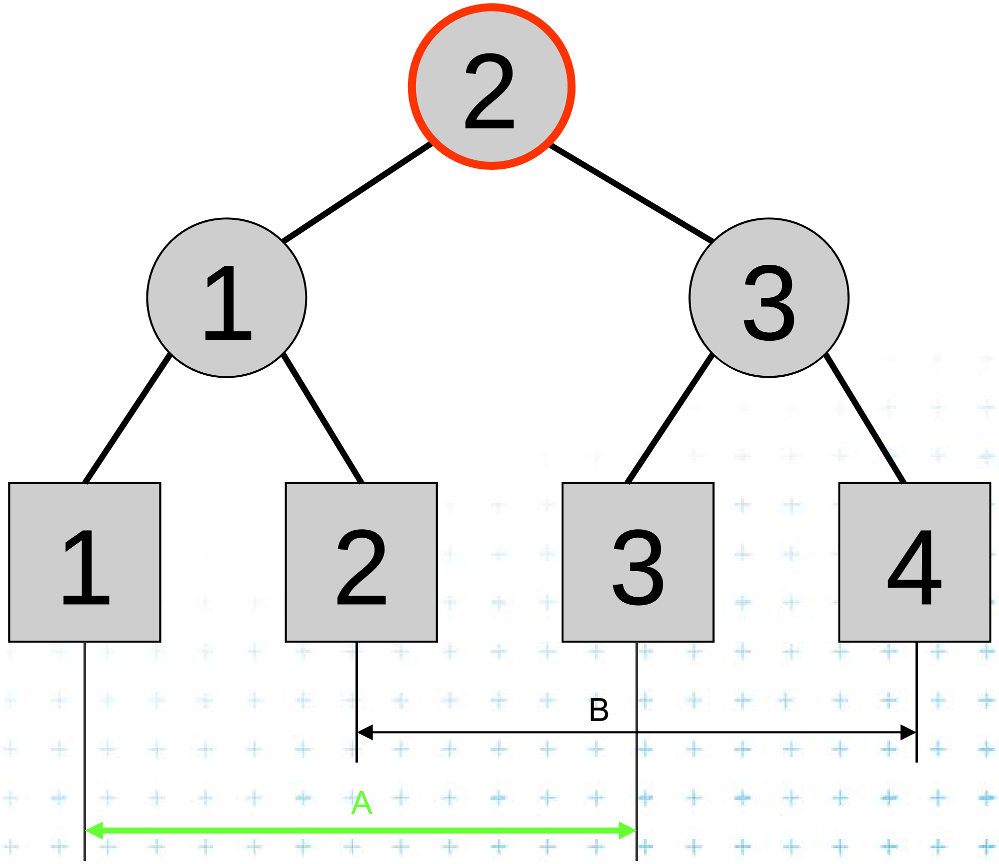
[Drtina]

Interval insertion [1,3]   b) Insert Interval



$$b \le H(v) \le e$$

$$? \; 1 \le 2 \le 3 \; ?$$



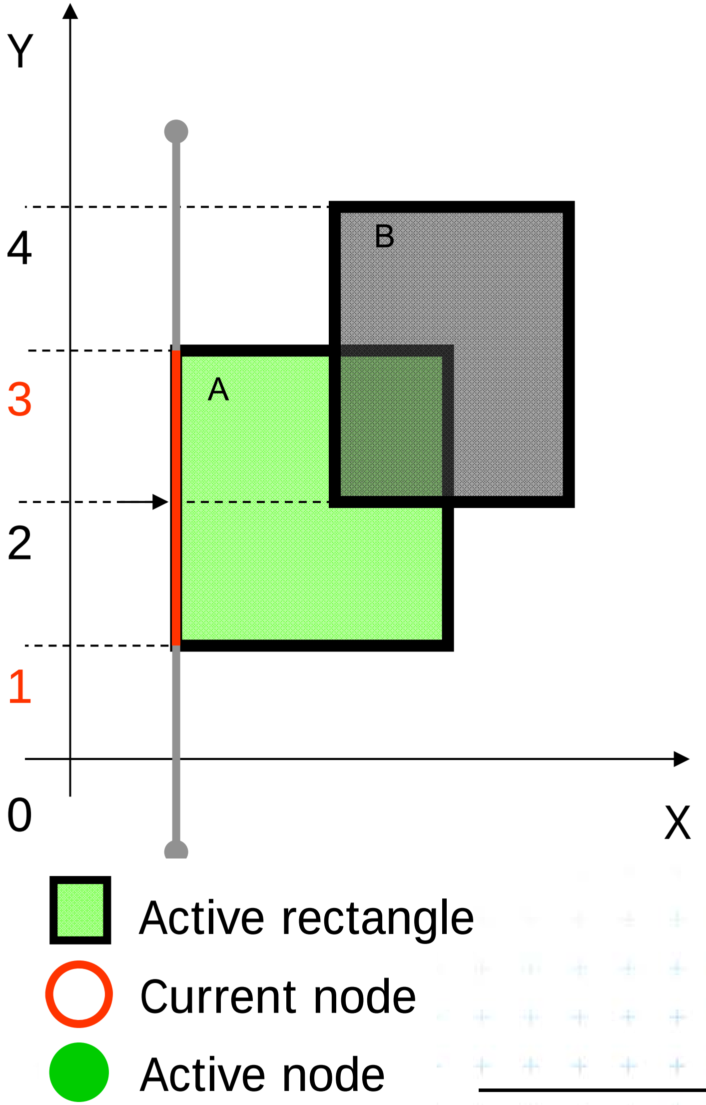
Active rectangle

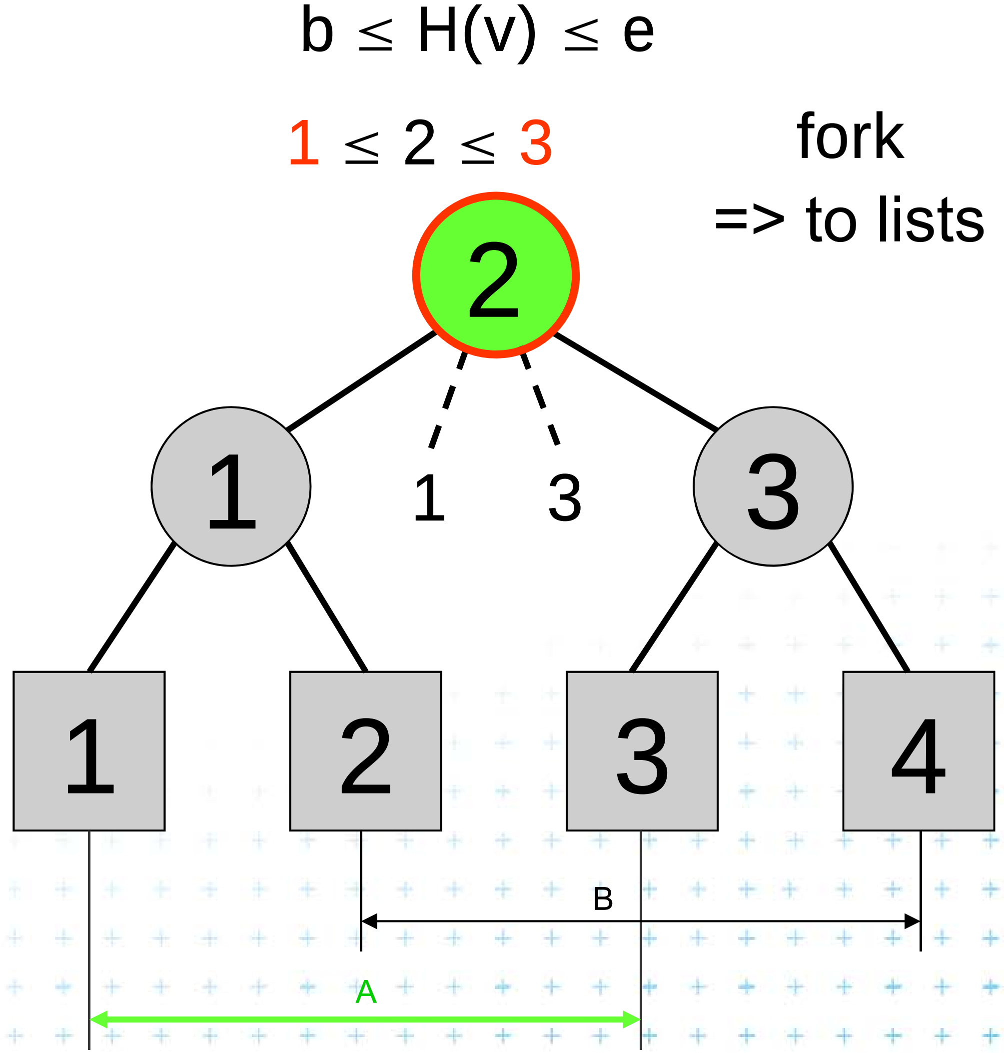
Current node

Active node

[Drtina]

Interval insertion [1,3]   b) Insert Interval





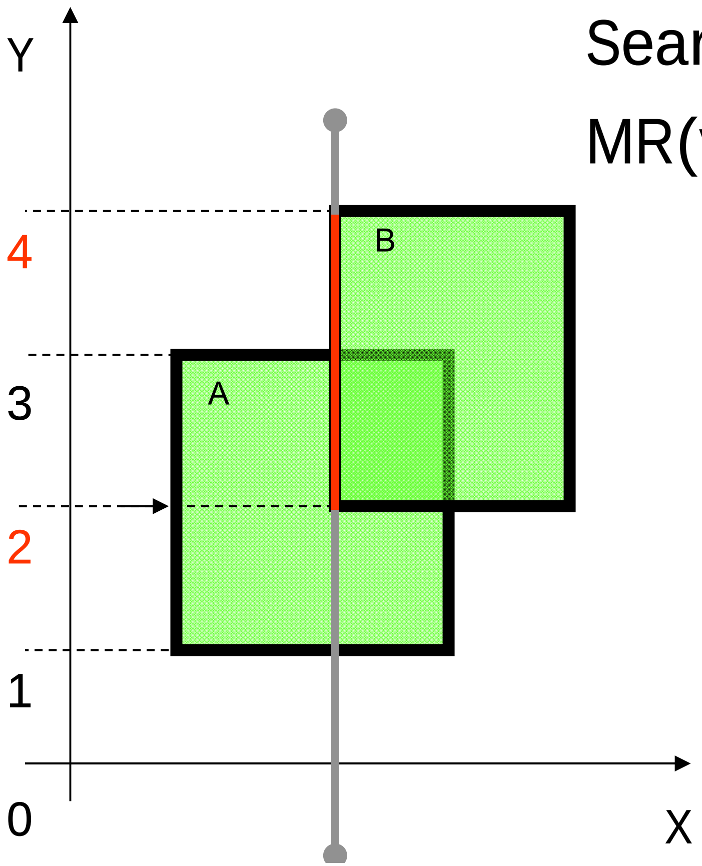
[Drtina]

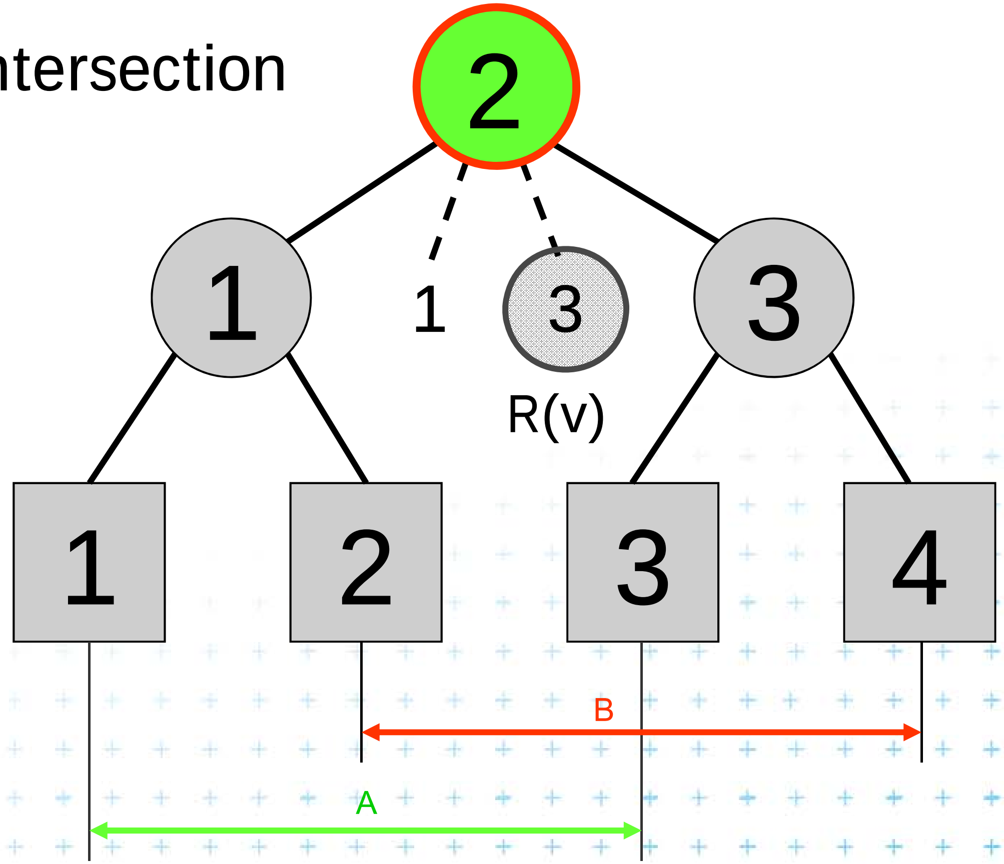
Interval insertion [2,4] a) Query Interval

Search MR(v) only: ← $H(v) \le b < e$

$MR(v)[1] = 3 \ge 2$? $2 \le 2 < 4$

=> intersection





- Active rectangle
- Current node
- Active node

[Drtina]

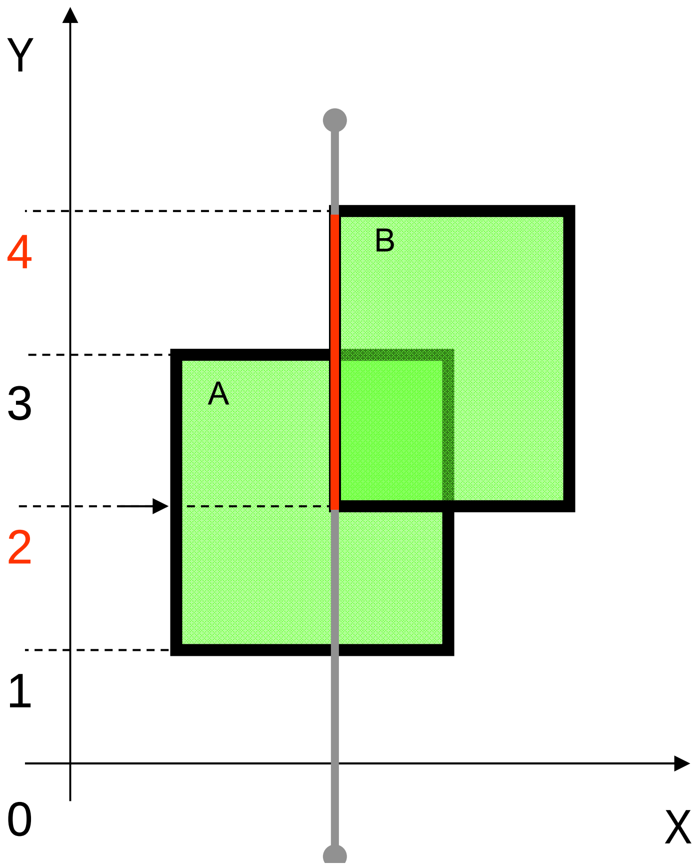
Interval insertion [2,4]   b) Insert Interval

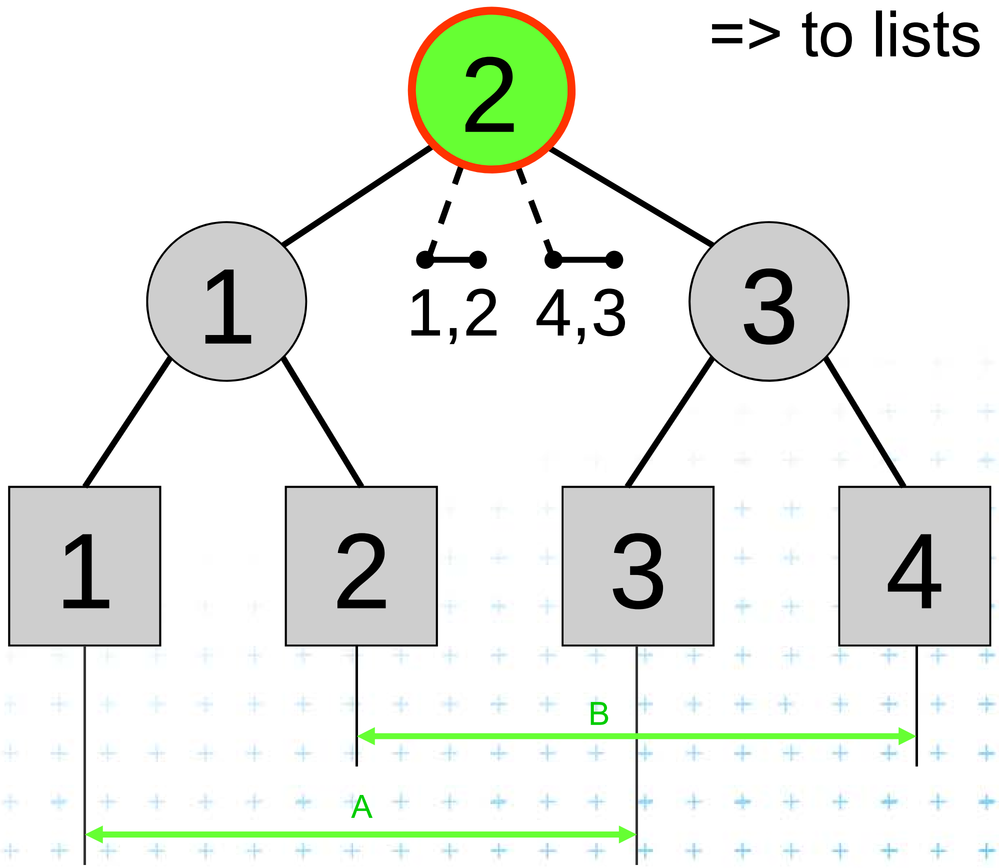
$b \leq H(v) \leq e$

$2 \leq 2 \leq 4$

fork

=> to lists

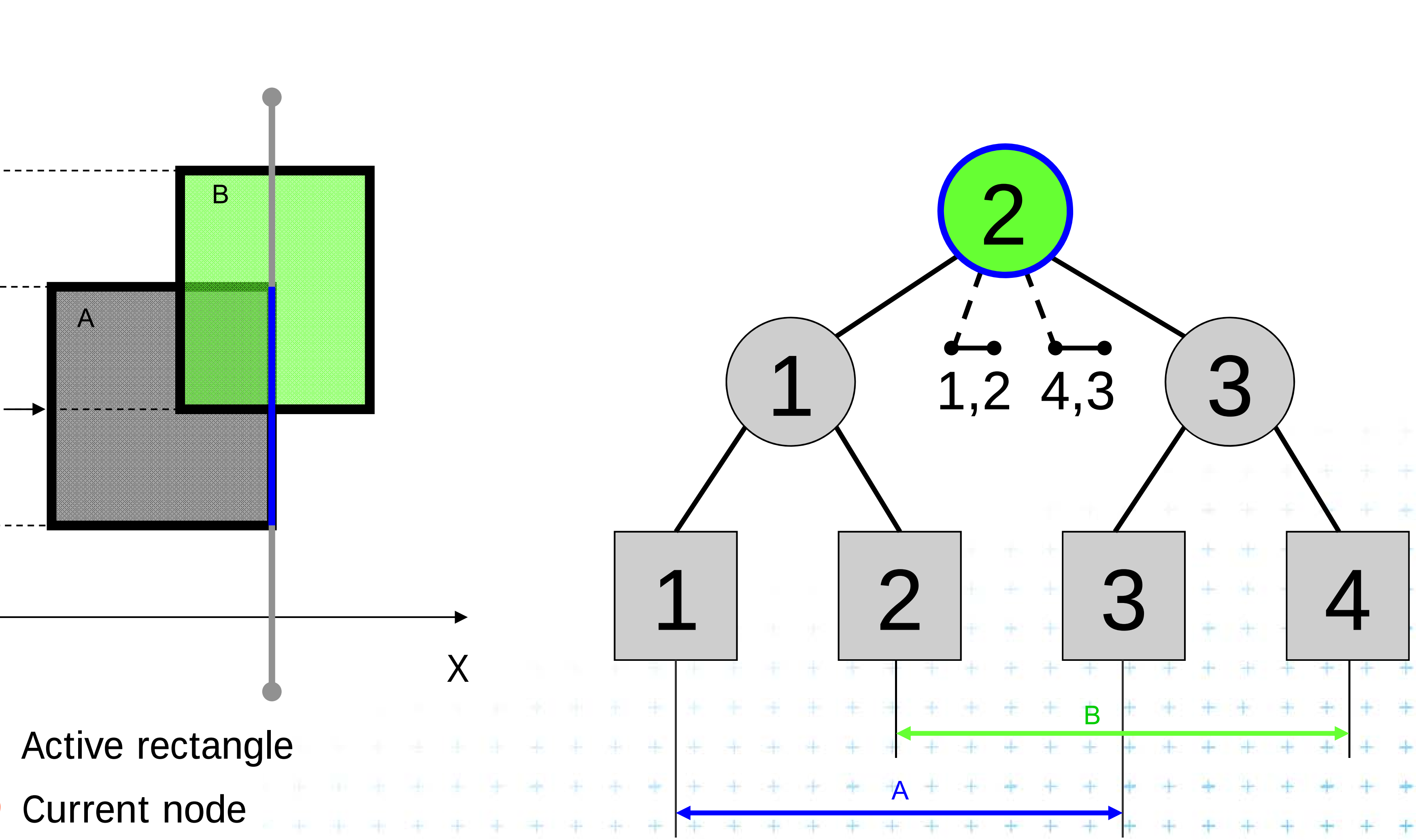




[Drtina]

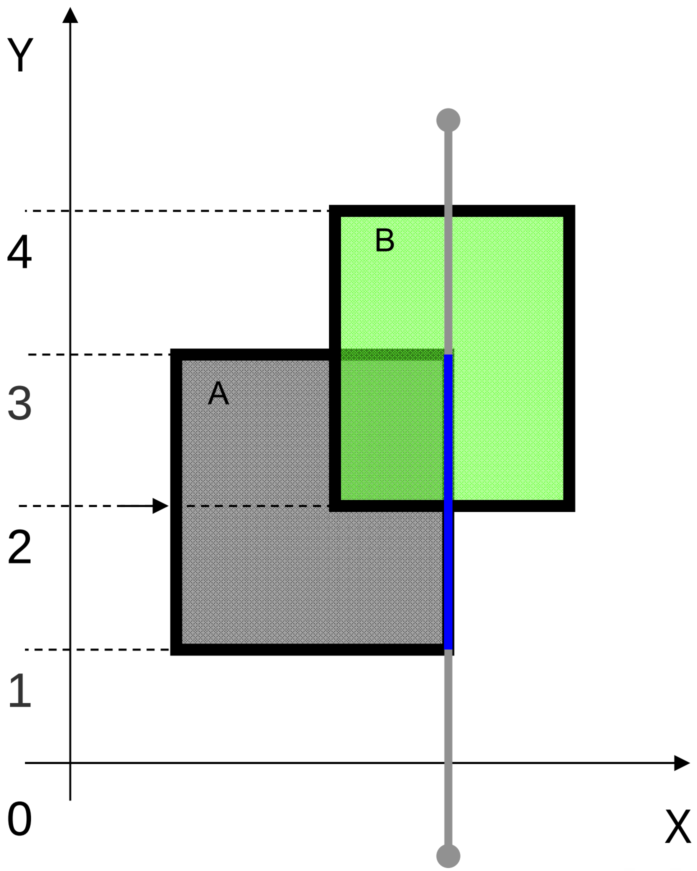
- Active rectangle
- Current node
- Active node

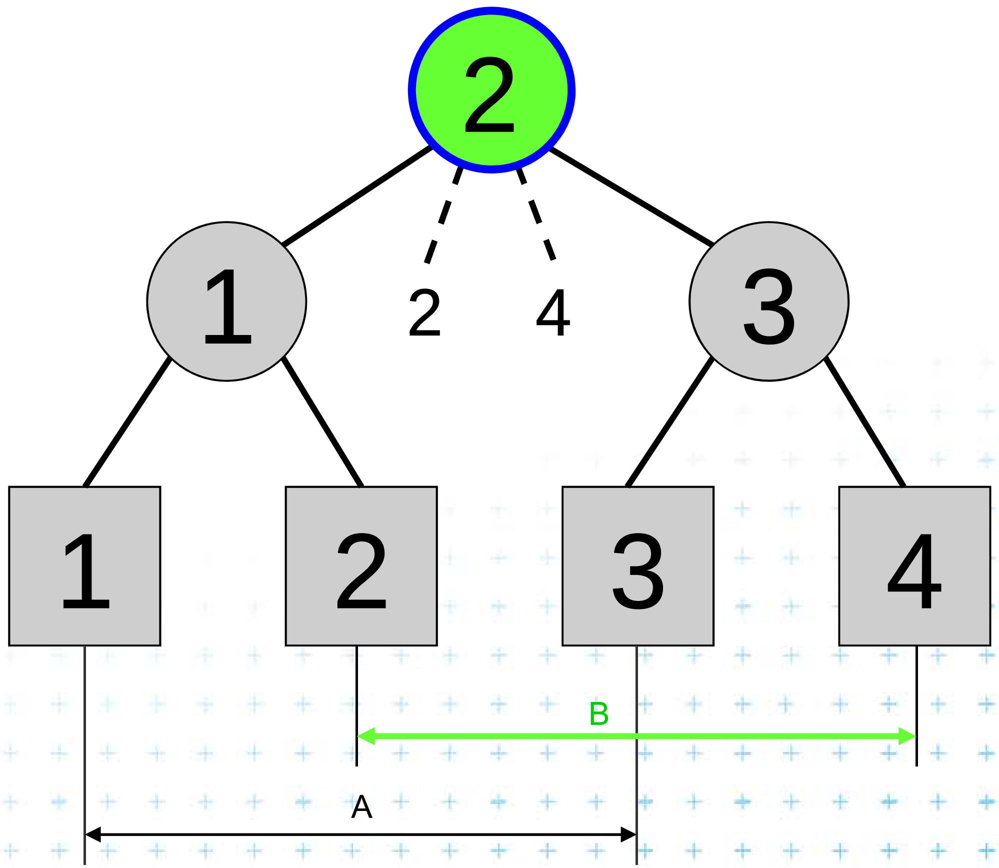
Interval delete [1,3]



[Drtina]

Interval delete [1,3]

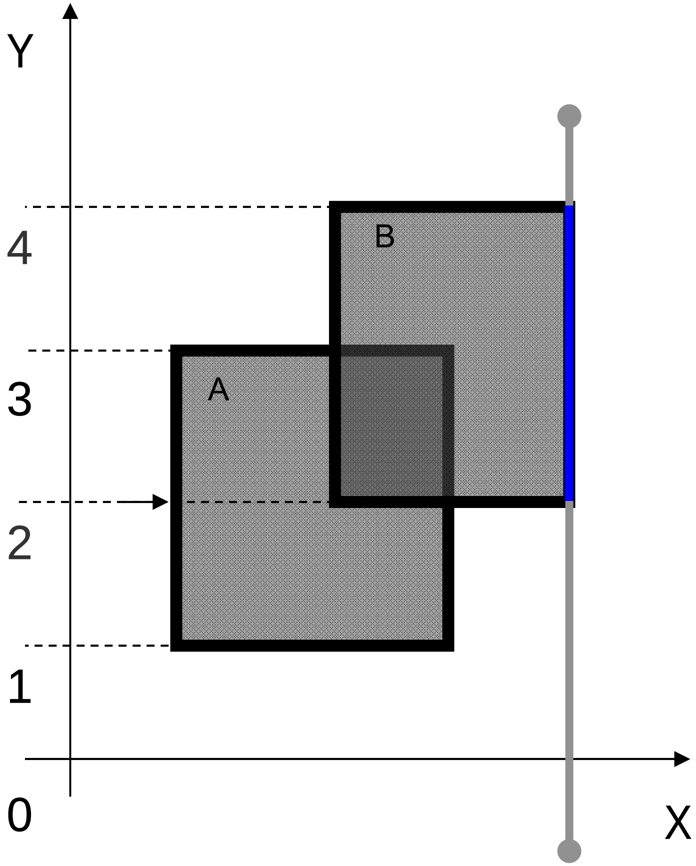


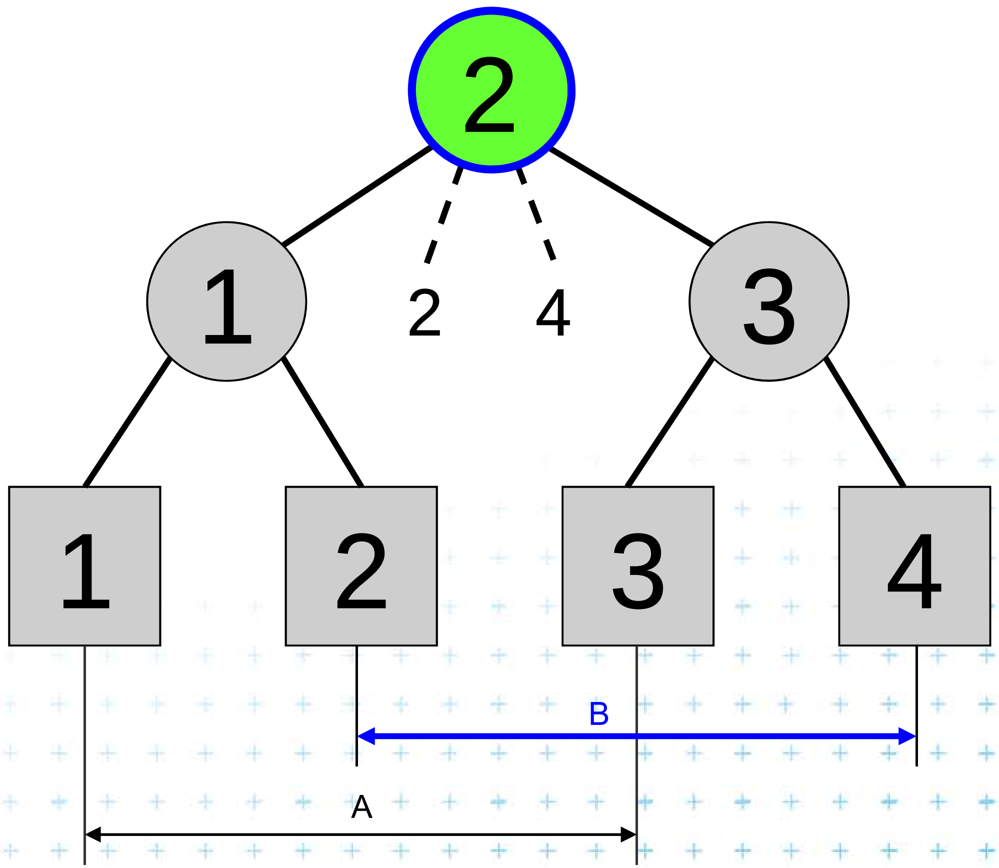


Active rectangle
Current node
Active node

[Drtina]

Interval delete [2,4]

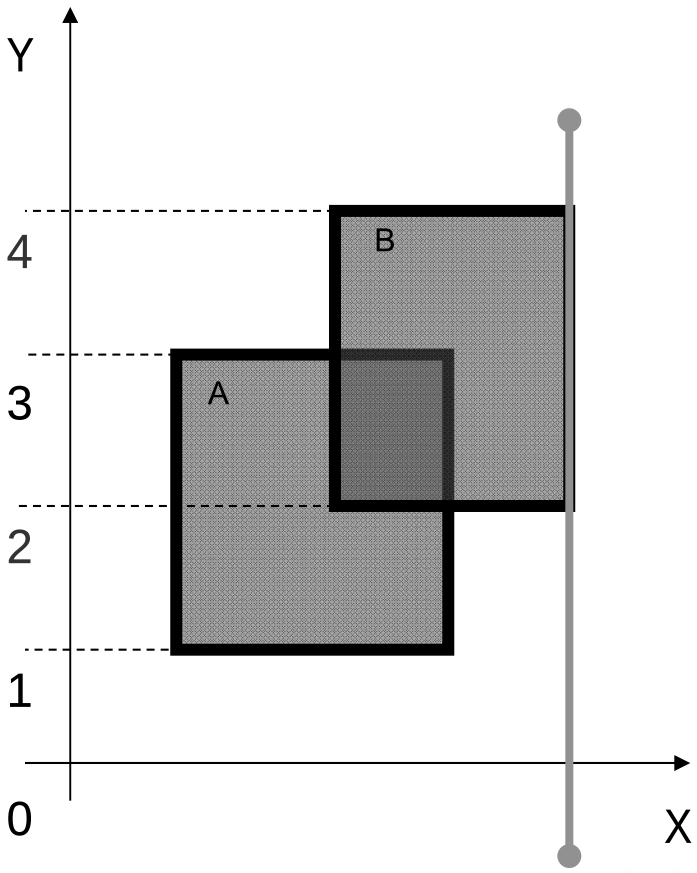


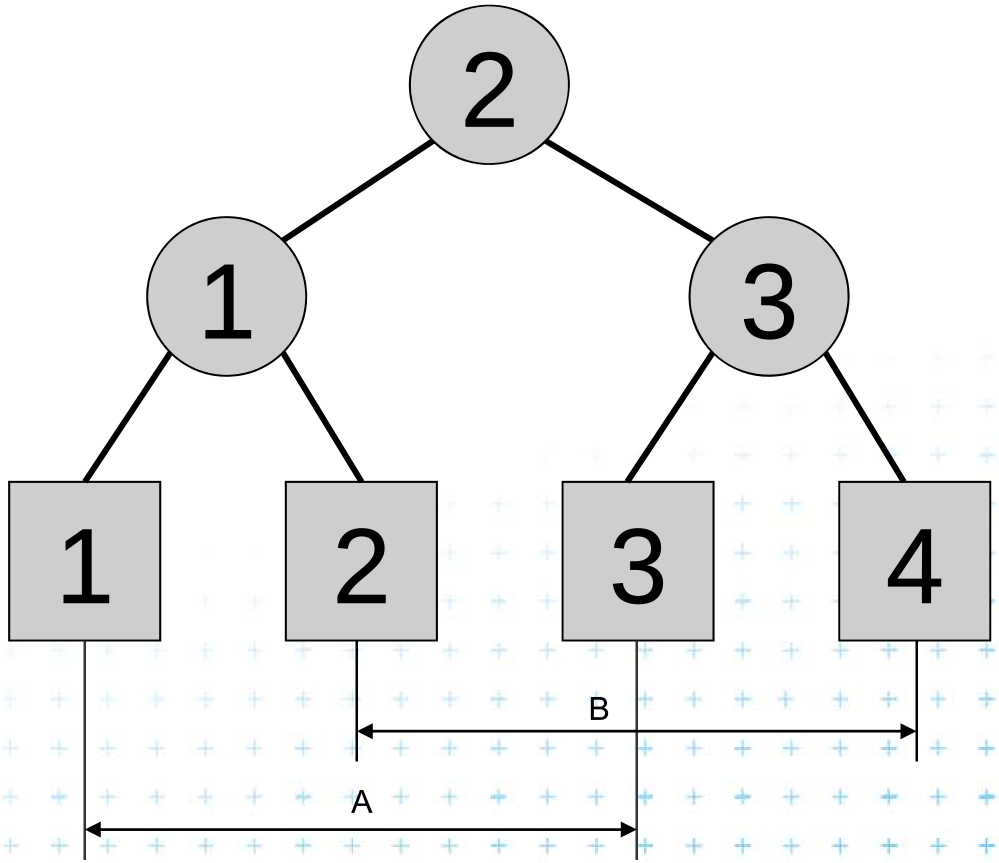


- Active rectangle
- Current node
- Active node

[Drtina]

Interval delete [2,4]





[Drtina]

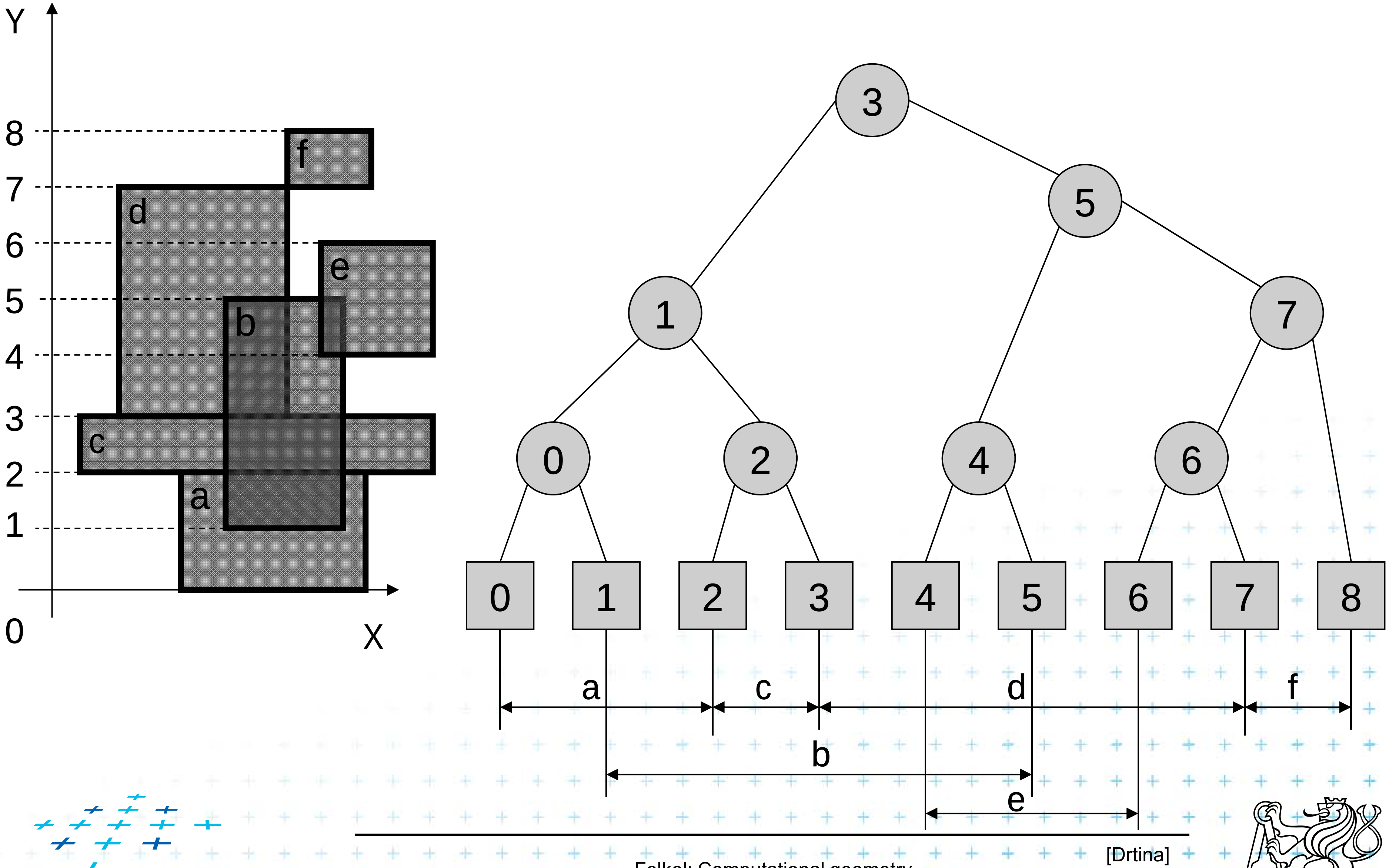
Example 2

RectangleIntersections(*S*) **// this is copy of the slide before**
Input: Set *S* of rectangles **// just to remember the algorithm**
Output: Intersected rectangle pairs

1. Preprocess(S) // create the interval tree *T* and event queue *Q*

2. **while** ($Q \neq \emptyset$) do
3. Get next entry ($x_{il}, y_{il}, y_{ir}, t$) from *Q* // $t = \{$ *left* | *right* $\}$
4. **if** (t = left) // left edge
5. a) QueryInterval ($y_{il}, y_{ir}, \mathrm{root}(T)$) // report intersections
6. b) InsertInterval ($y_{il}, y_{ir}, \mathrm{root}(T)$) // insert new interval
7. **else** // right edge
8. c) DeleteInterval (y_{il}, y_{ir}, root(*T*))

Example 2



Insert [2,3] – empty => b) Insert Interval

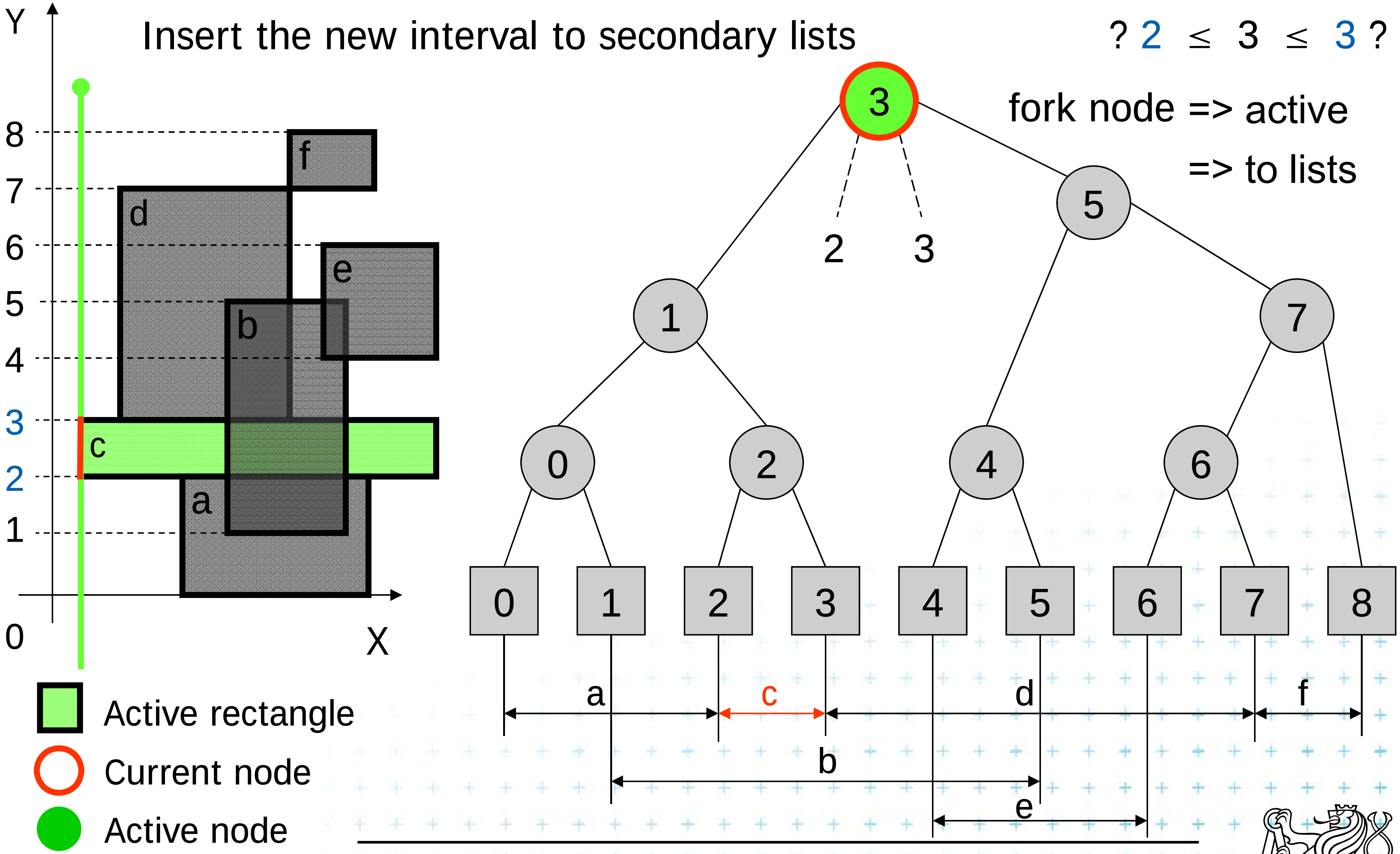
$b \le H(v) \le e$

Insert the new interval to secondary lists

? 2 $\le$ 3 $\le$ 3 ?

fork node => active

=> to lists

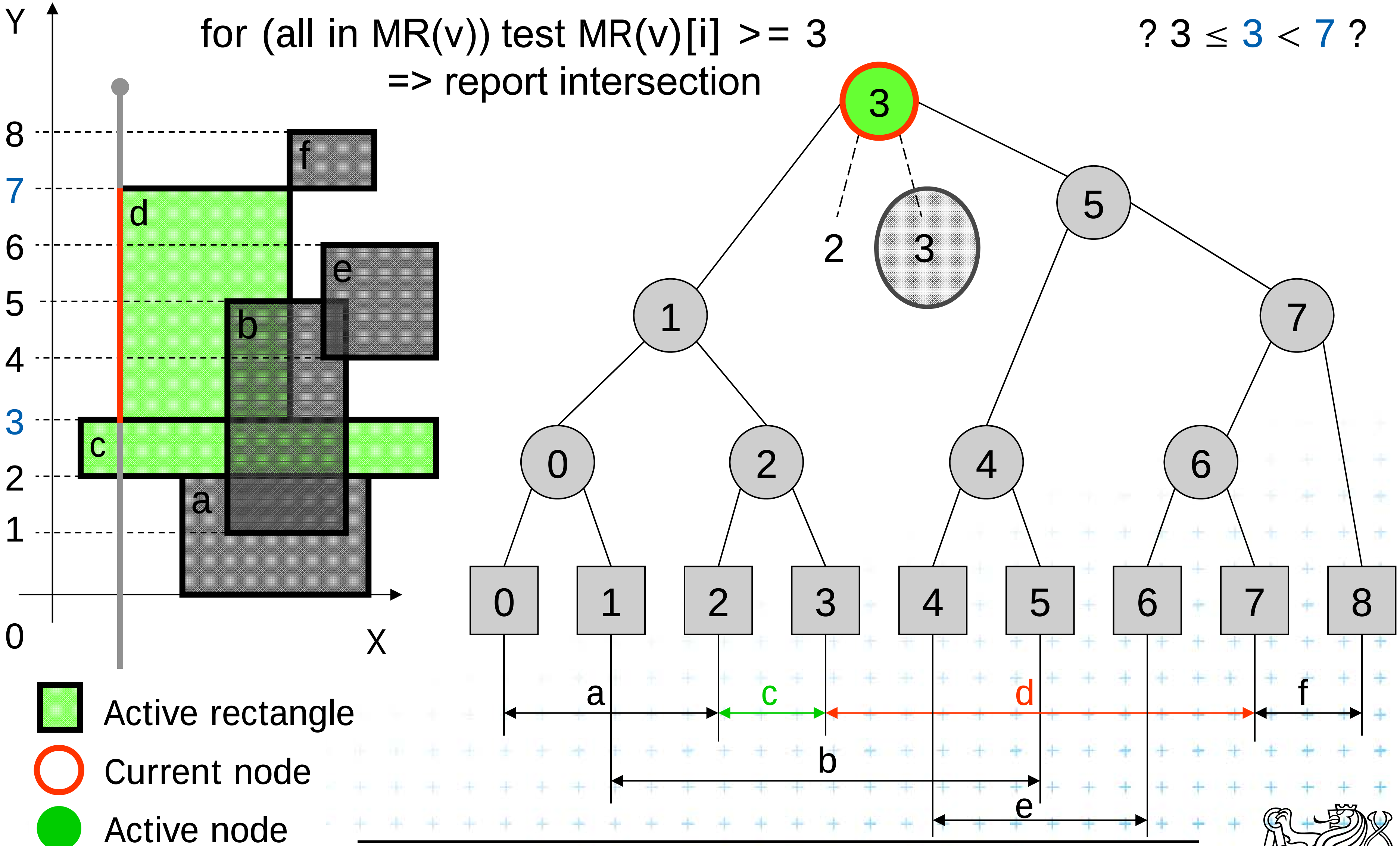


Insert [3,7] a) Query Interval

$H(v) \le b < e$

for (all in MR(v)) test MR(v)[i] >= 3
=> report intersection

? $3 \le 3 < 7$?



Insert [3,7] b) Insert Interval

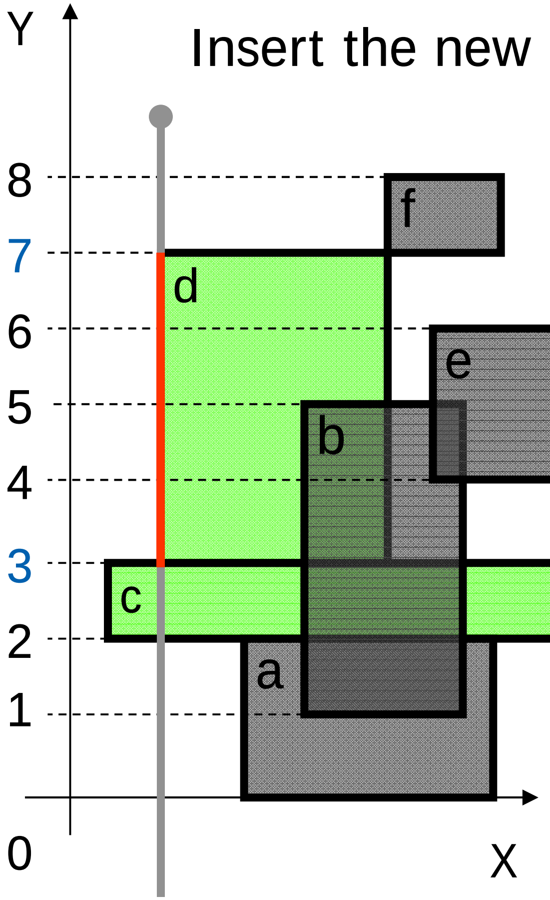
$b \leq H(v) \leq e$

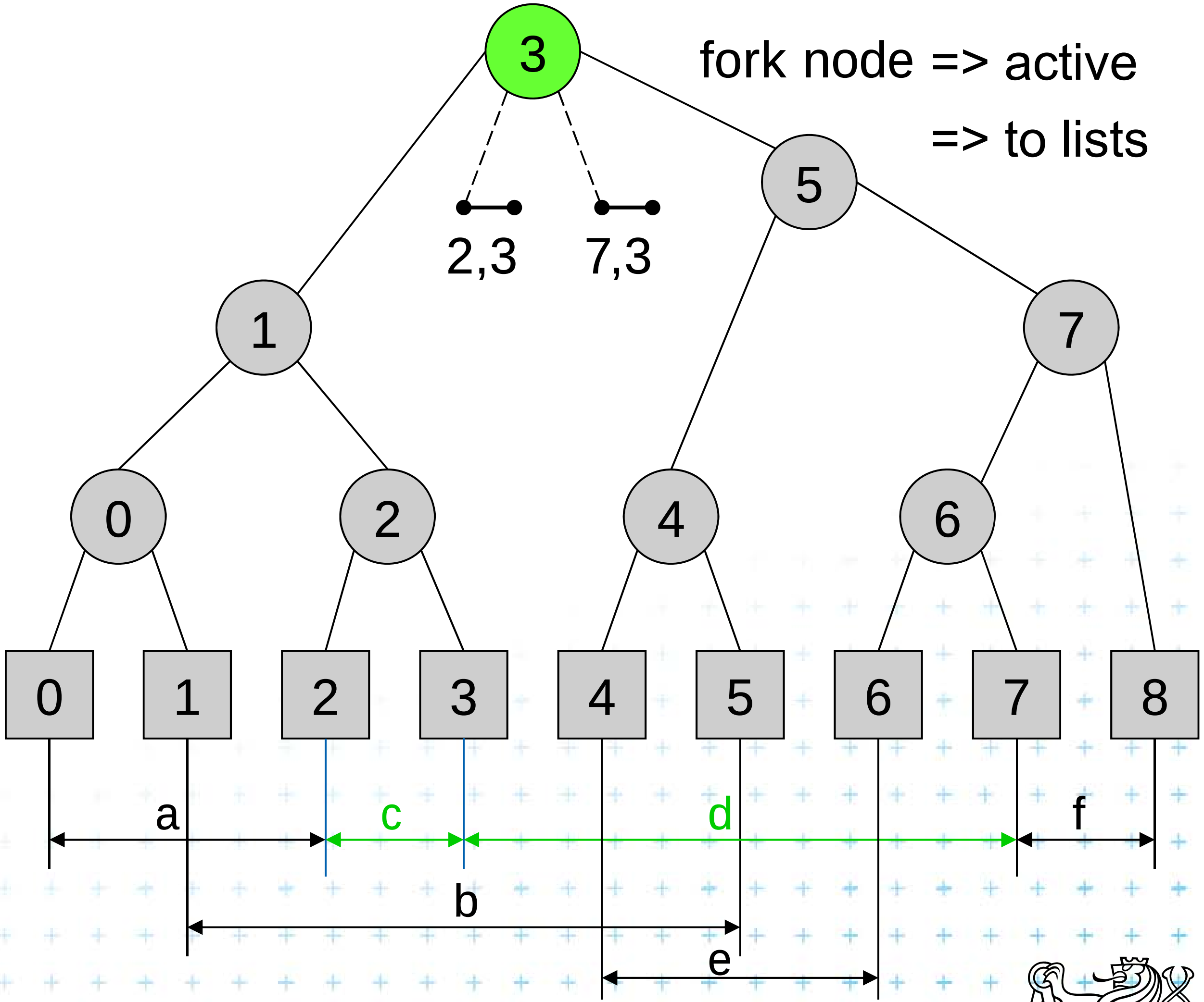
Insert the new interval to secondary lists

$3 \leq 3 \leq 7$

fork node => active

=> to lists





Active rectangle

Current node

Active node

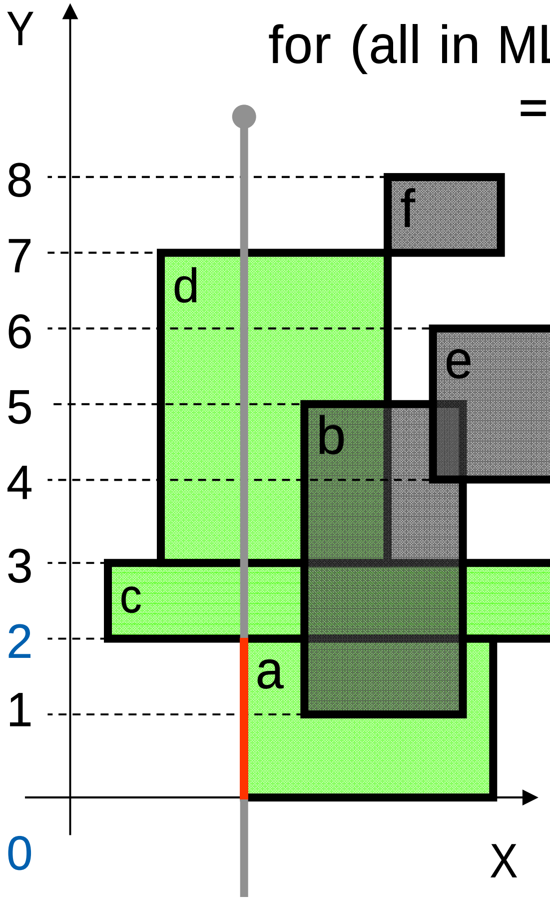
[Drtina]

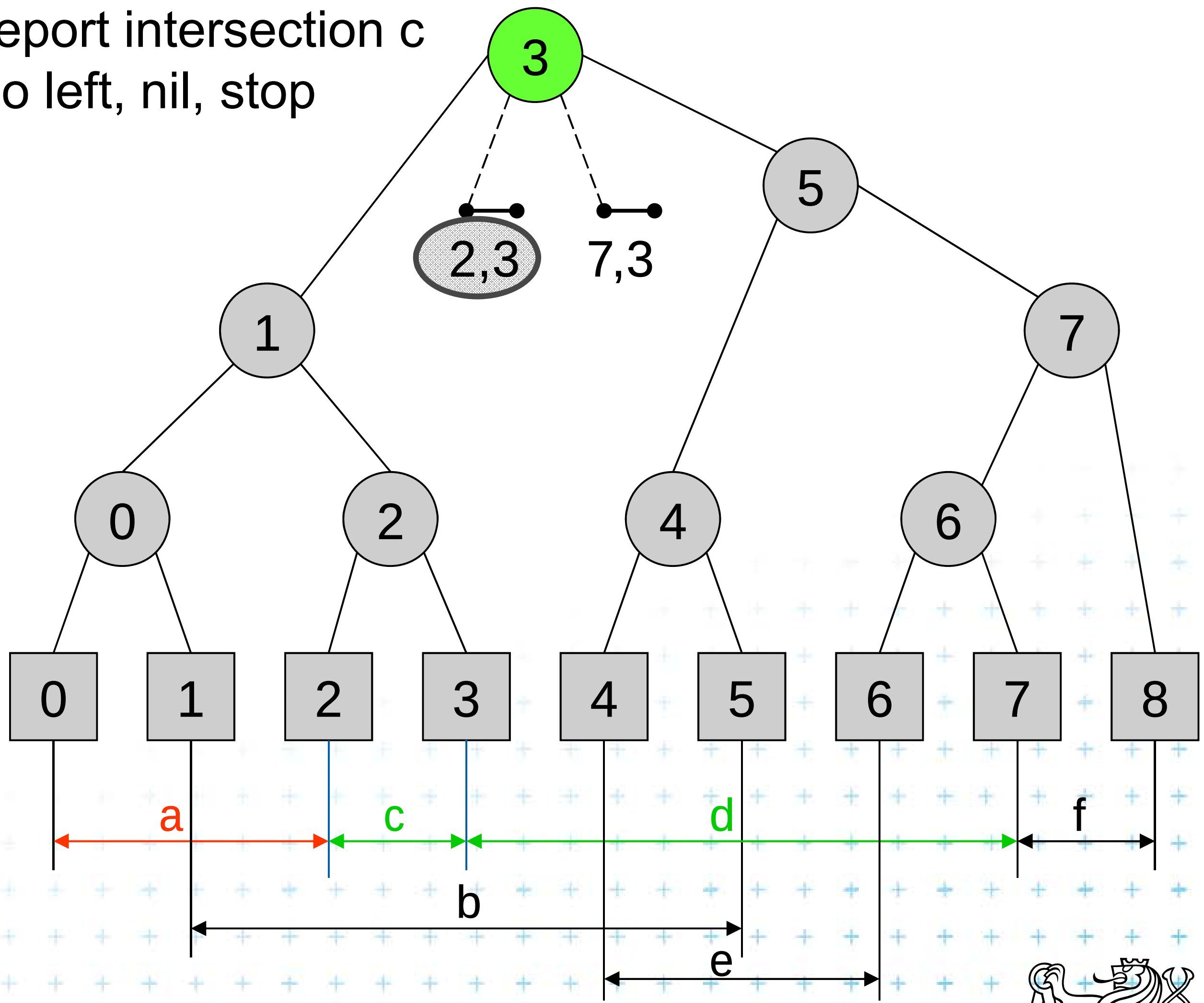
Insert [0,2] a) Query Interval

$b < e \leq H(v)$

for (all in ML(v)) test ML(v).[i] $\leq$ 2
=> report intersection c
go left, nil, stop

? $0 < 2 \leq 3$?





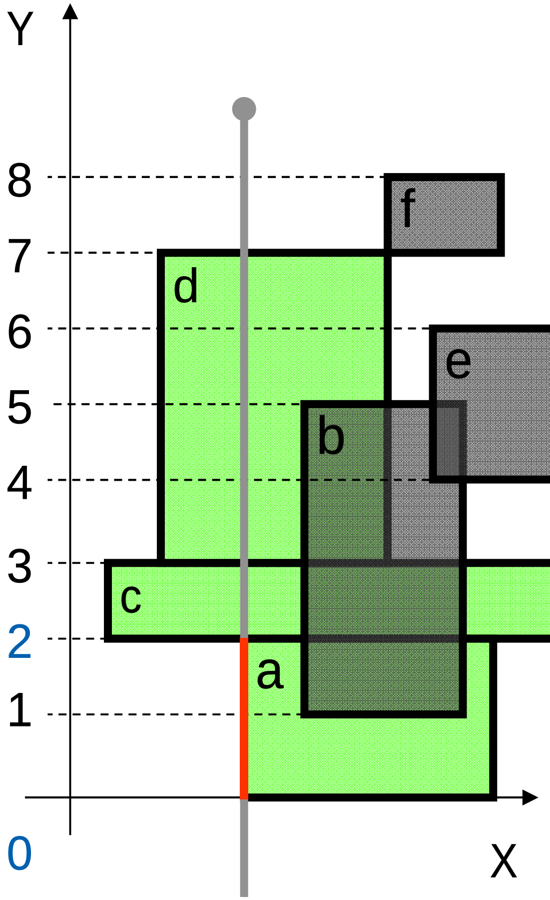
Active rectangle
Current node
Active node

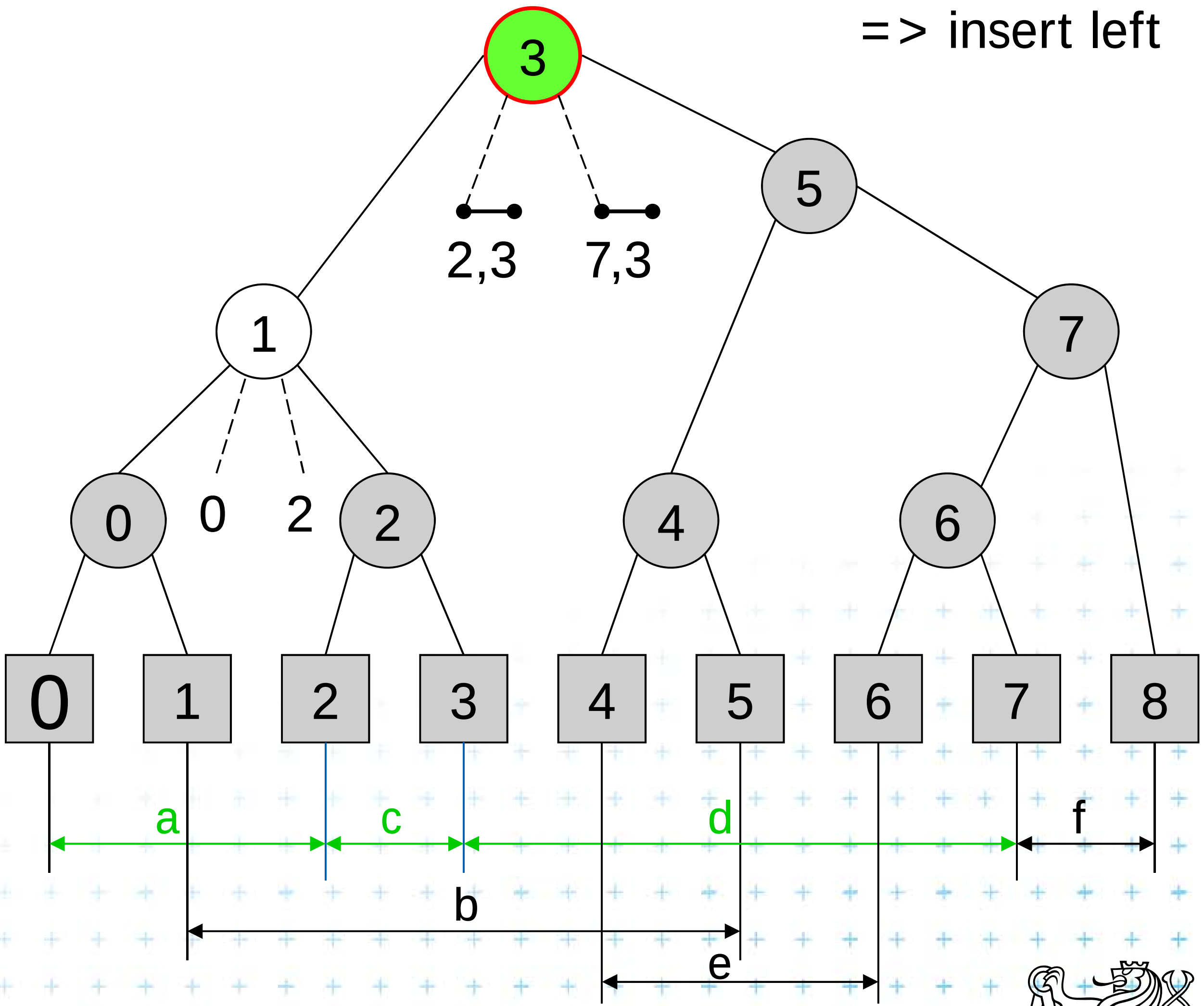
[Drtina]

Insert [0,2] b) Insert Interval 1/2

b < e <H(v)

? 0 < 2 < 3?
=> insert left





- Active rectangle
- Current node
- Active node

[Drtina]

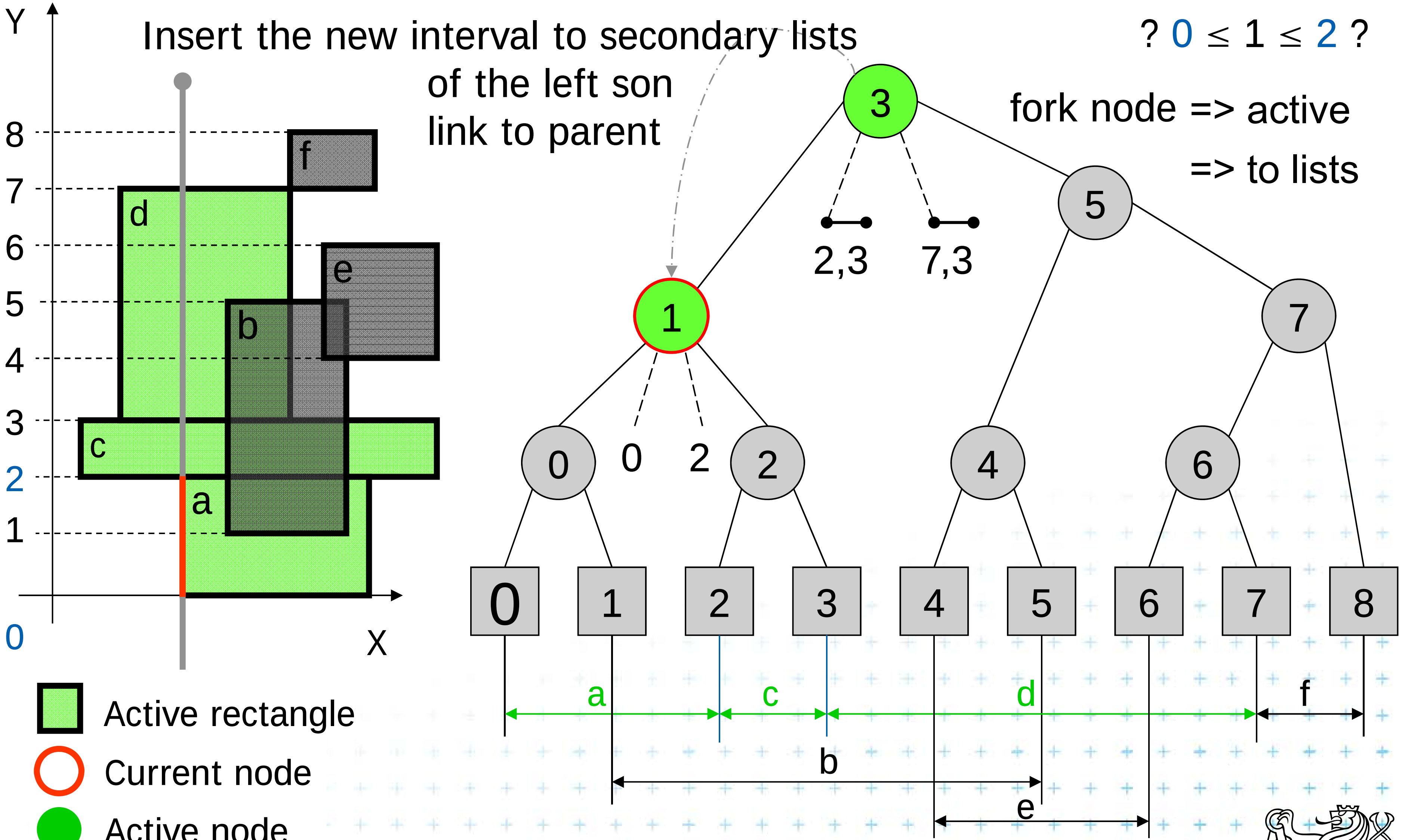
Insert [0,2] b) Insert Interval 2/2

$b \leq H(v) \leq e$

Insert the new interval to secondary lists of the left son link to parent

? $0 \leq 1 \leq 2$?

fork node => active => to lists



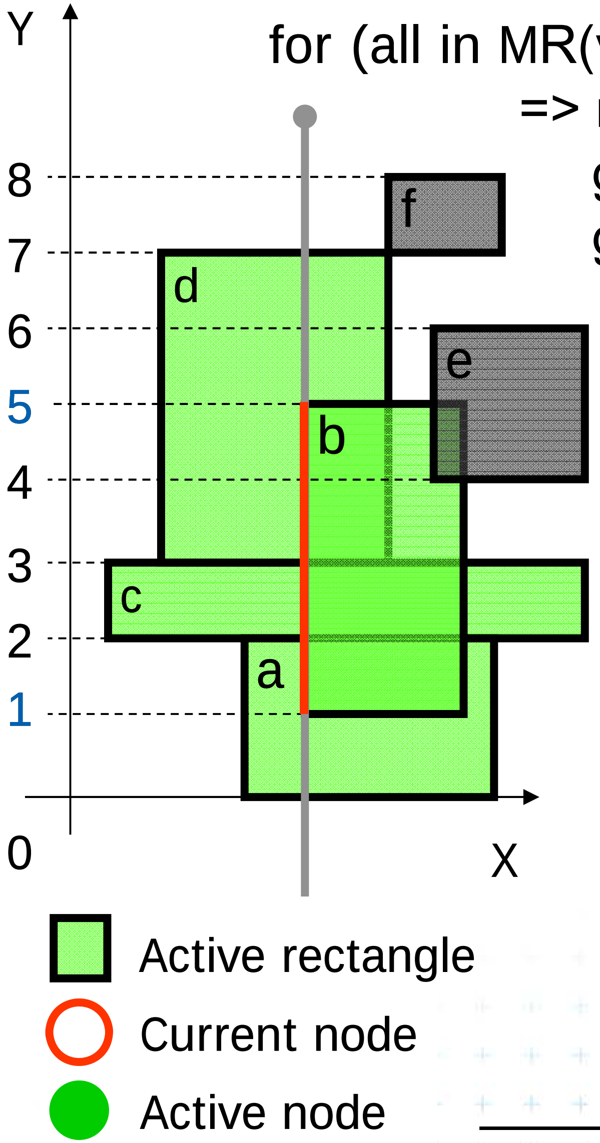
[Drtina]

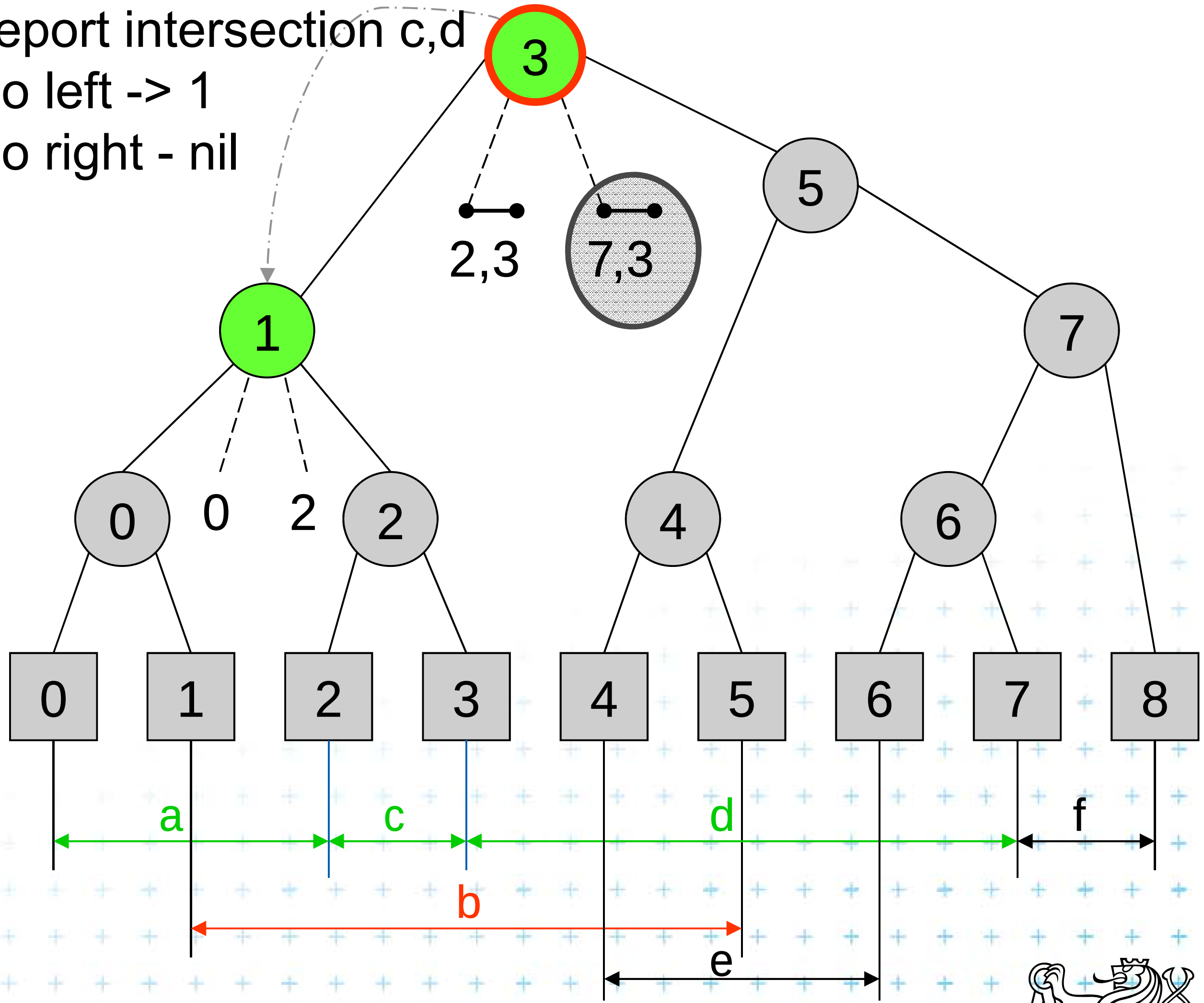
Insert [1,5] a) Query Interval 1/2

b < H(v) < e

for (all in MR(v)) test MR(v)[i] $\geq$ 1
=> report intersection c,d
go left -> 1
go right - nil

? 1 < 3 < 5 ?





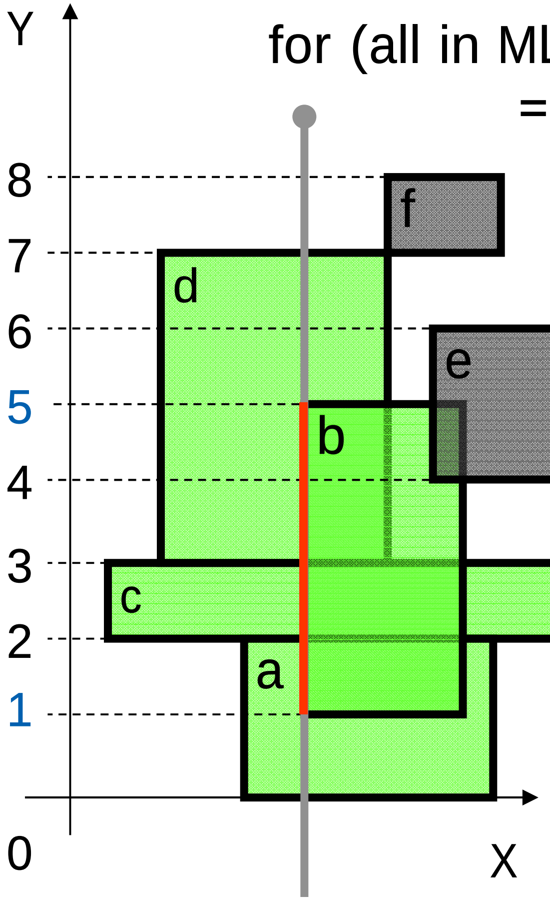
[Drtina]

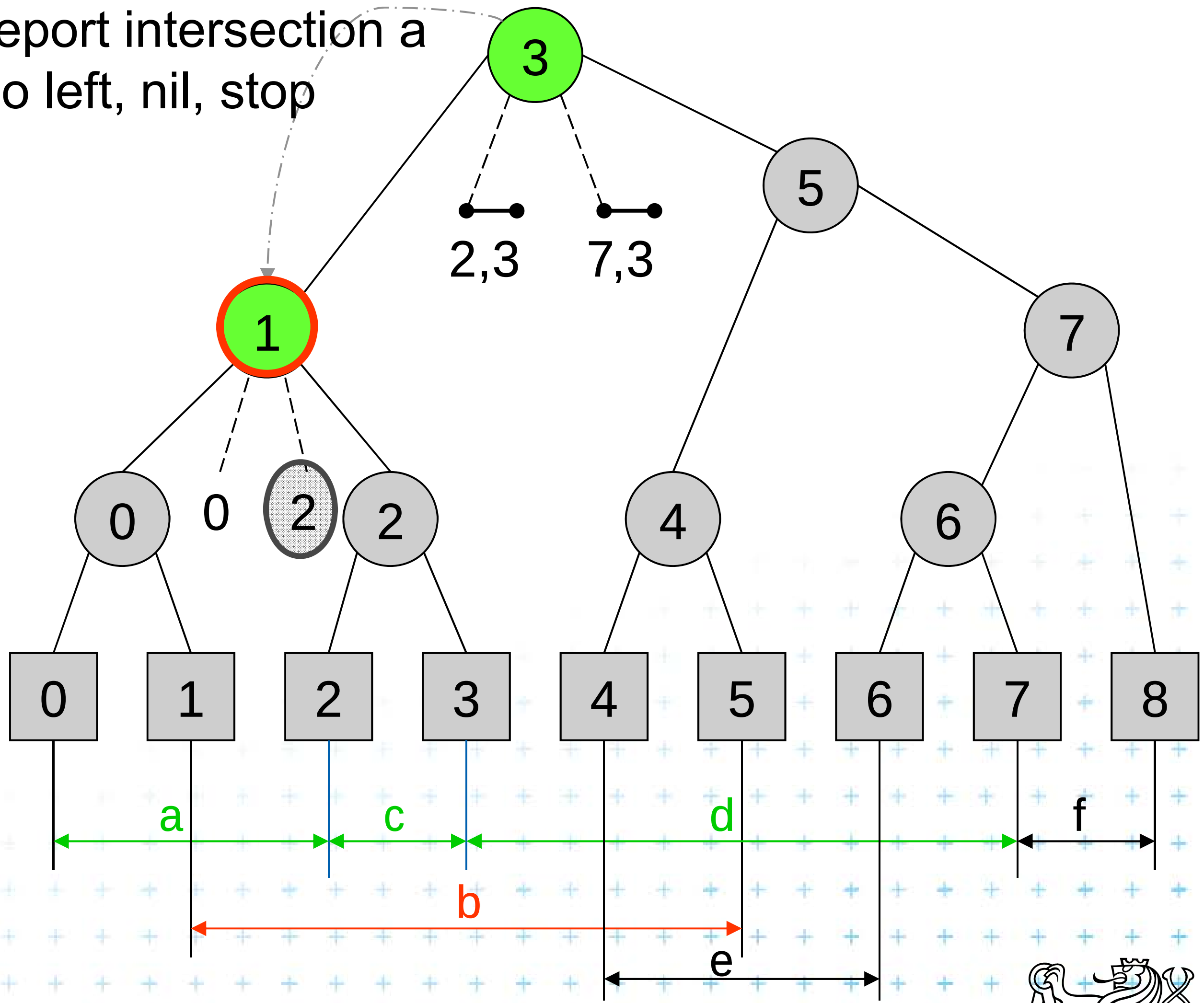
Insert [1,5] a) Query Interval 2/2

H(v) ≤ b < e

for (all in ML(v)) test ML(v)[i] ≥ 1
=> report intersection a
go left, nil, stop

? 1 ≤ 1 < 5 ?





Active rectangle

Current node

Active node

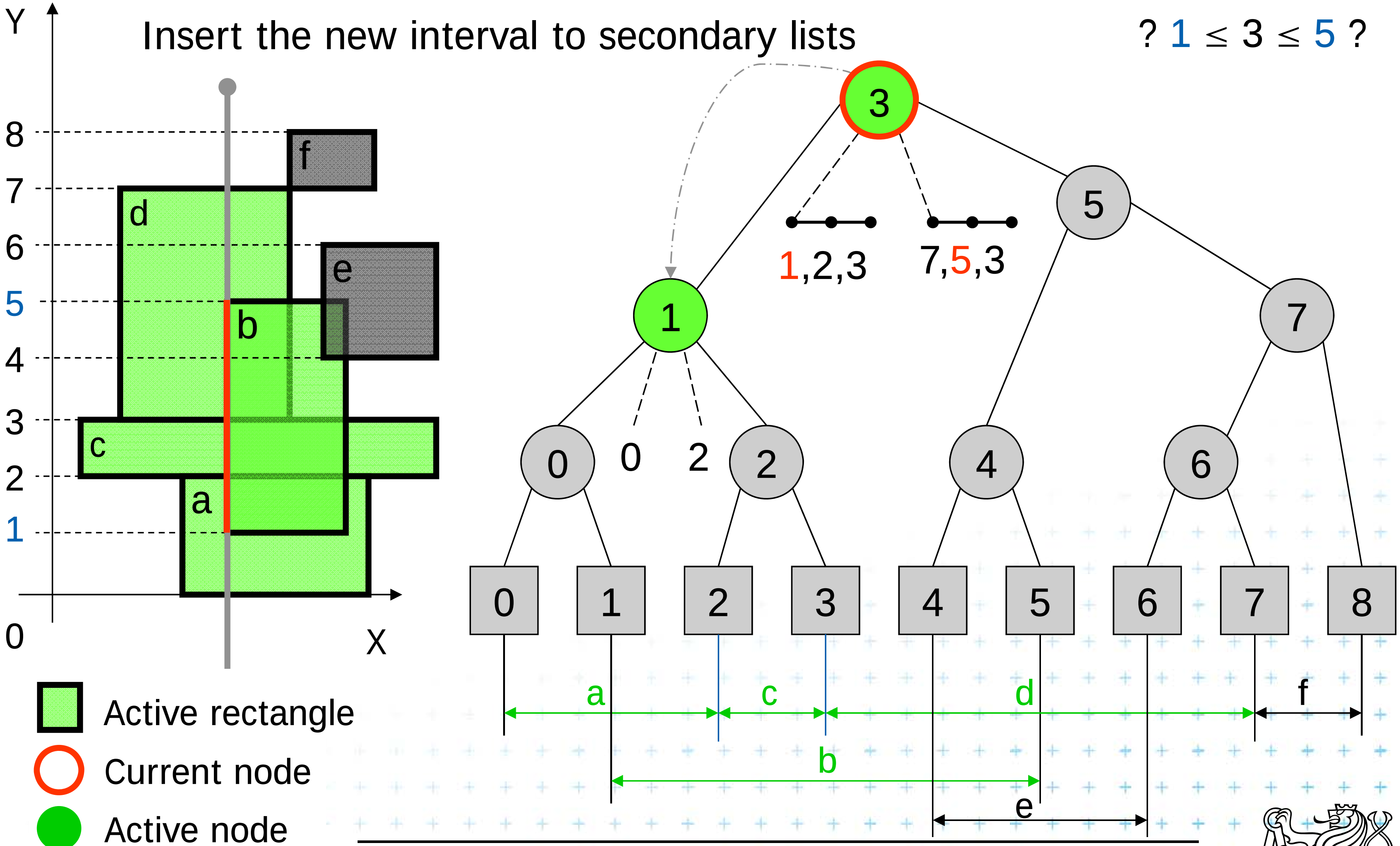
[Drtina]

Insert [1,5] b) Insert Interval

$b \le H(v) \le e$

Insert the new interval to secondary lists

? 1 ≤ 3 ≤ 5 ?



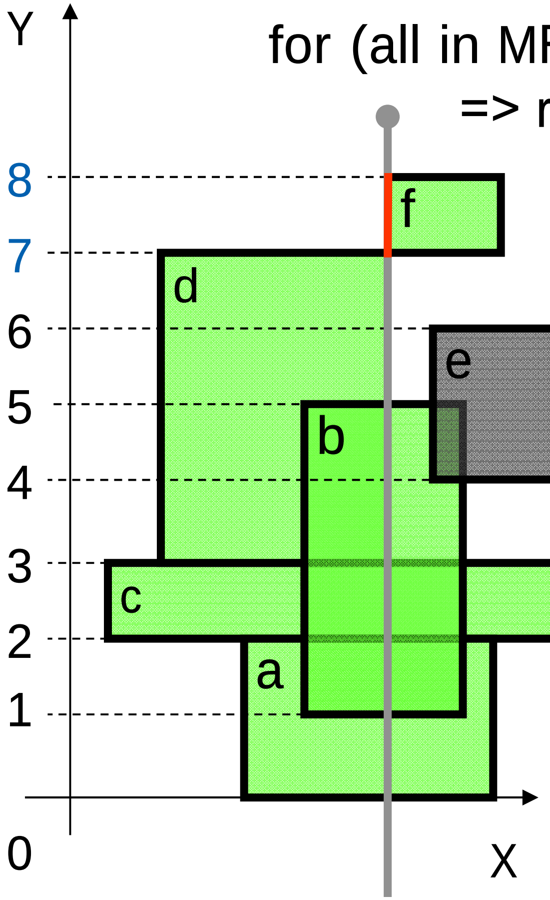
[Drtina]

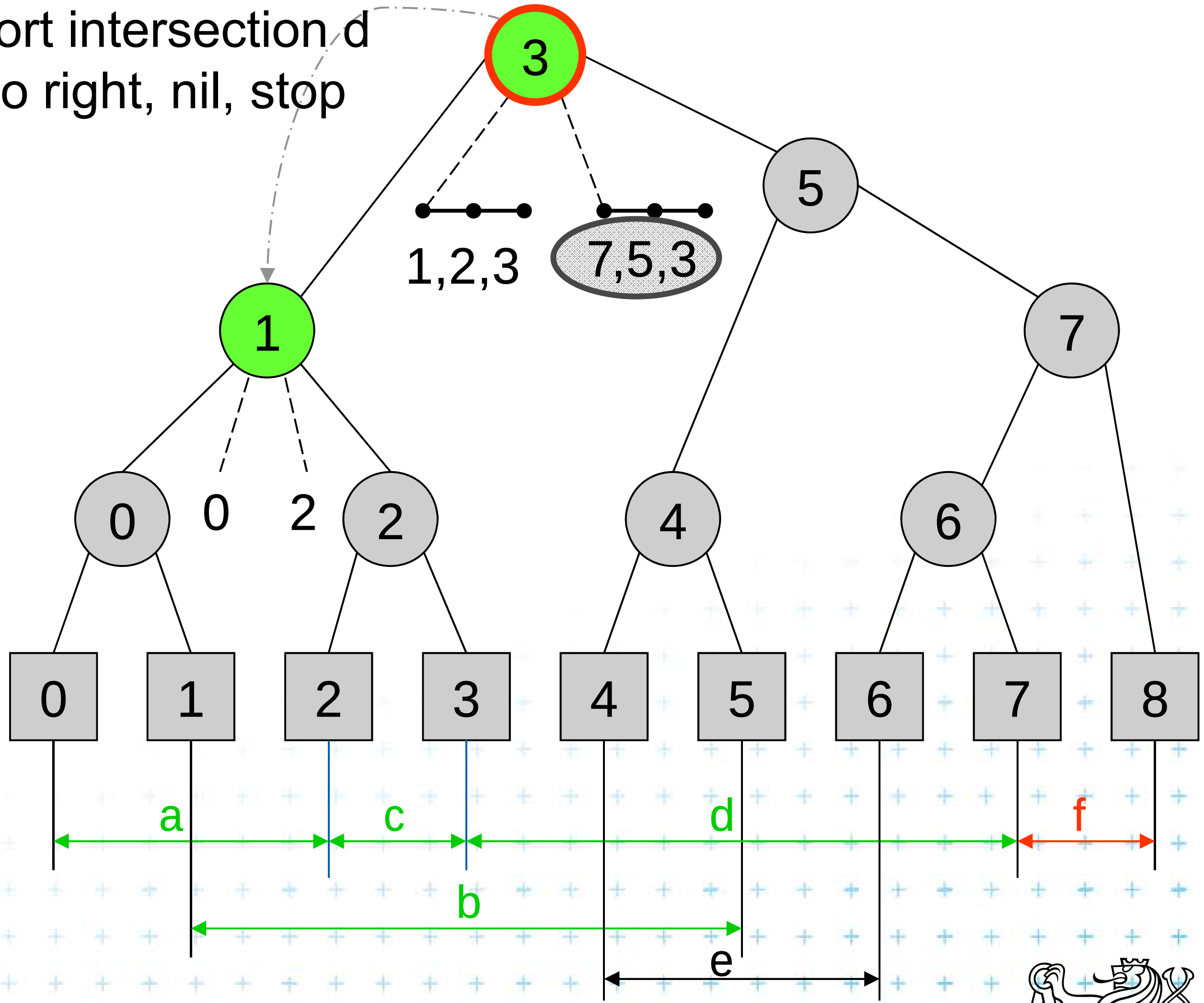
Insert [7,8] a) Query Interval

H(v) ≤ b < e

for (all in MR(v)) test MR(v).[i] ≥ 7
=> report intersection d
go right, nil, stop

? 3 ≤ 7 < 8 ?





- Active rectangle
- Current node
- Active node

[Drtina]

Insert [7,8] b) Insert Interval

$b \le H(v) \le e$

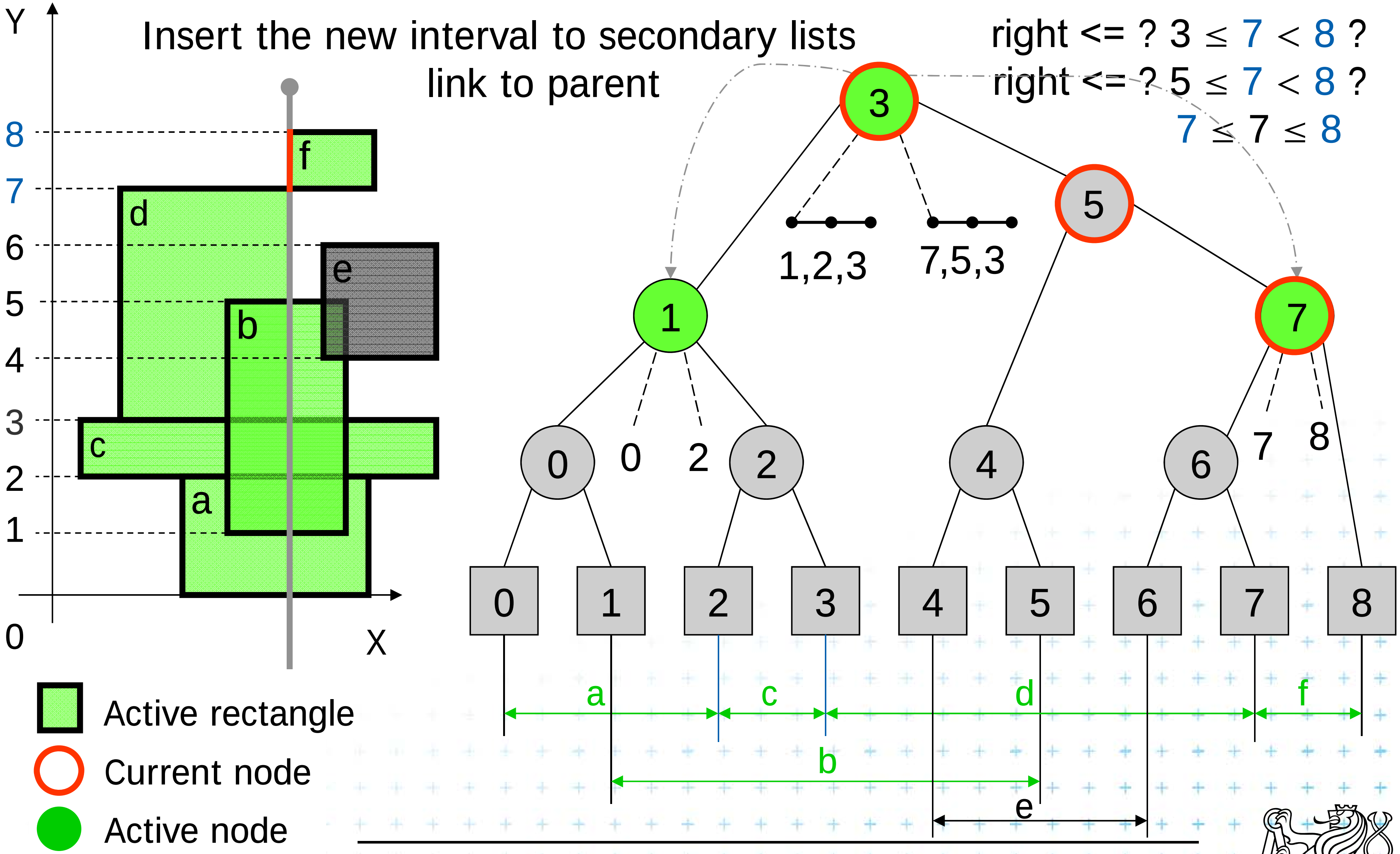
Insert the new interval to secondary lists

link to parent

right <= ? $3 \le 7 < 8$?

right <= ? $5 \le 7 < 8$?

$7 \le 7 \le 8$

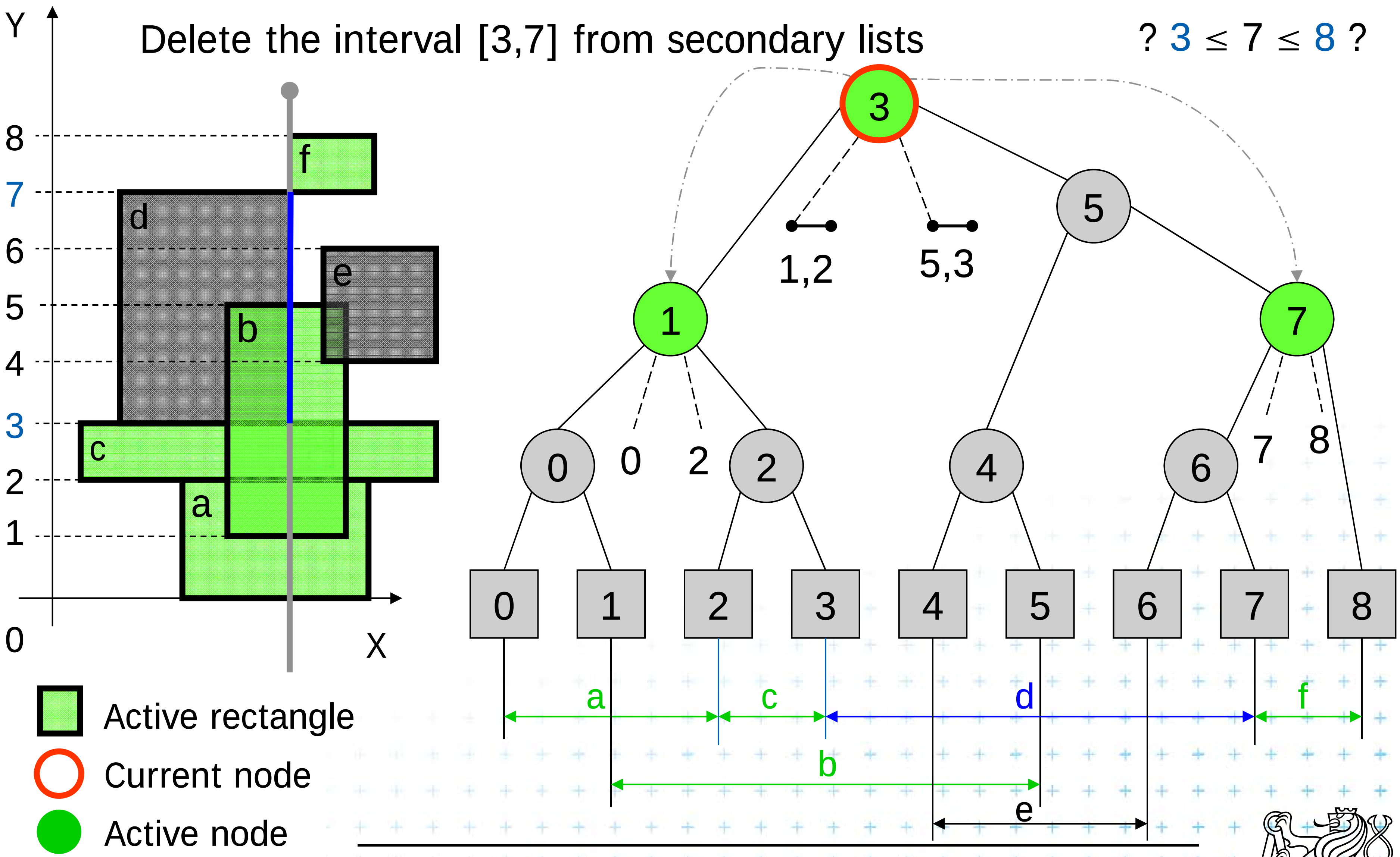


Delete [3,7] Delete Interval

$b \le H(v) \le e$

Delete the interval [3,7] from secondary lists

? $3 \le 7 \le 8$?



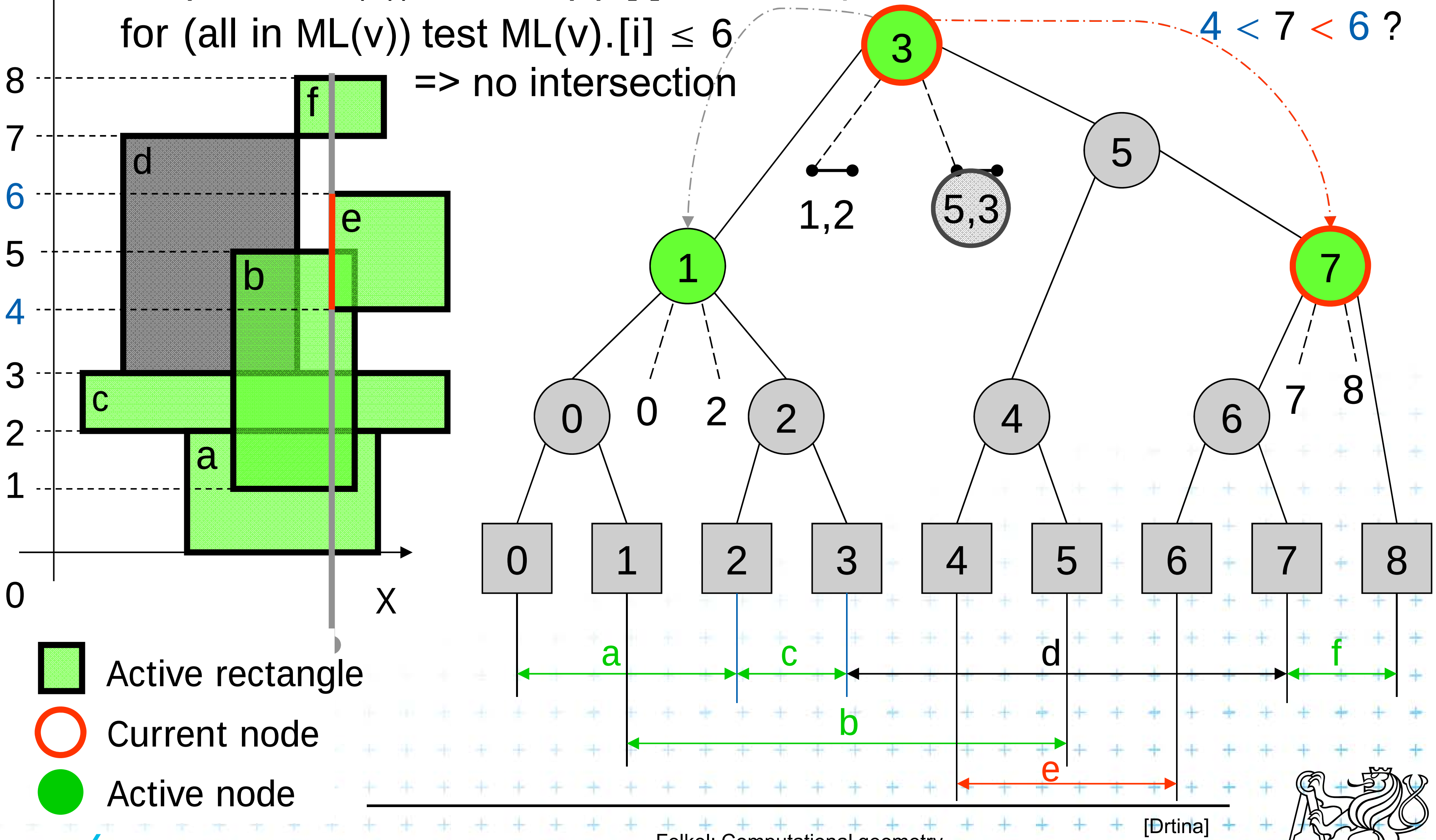
Insert [4,6] a) Query Interval

$H(v) \leq b < e$

for (all in MR(v)) test MR(v).[i] ≥ 4 => report intersection b $3 \leq 4 < 6$?

for (all in ML(v)) test ML(v).[i] ≤ 6 $4 < 7 < 6$?

=> no intersection



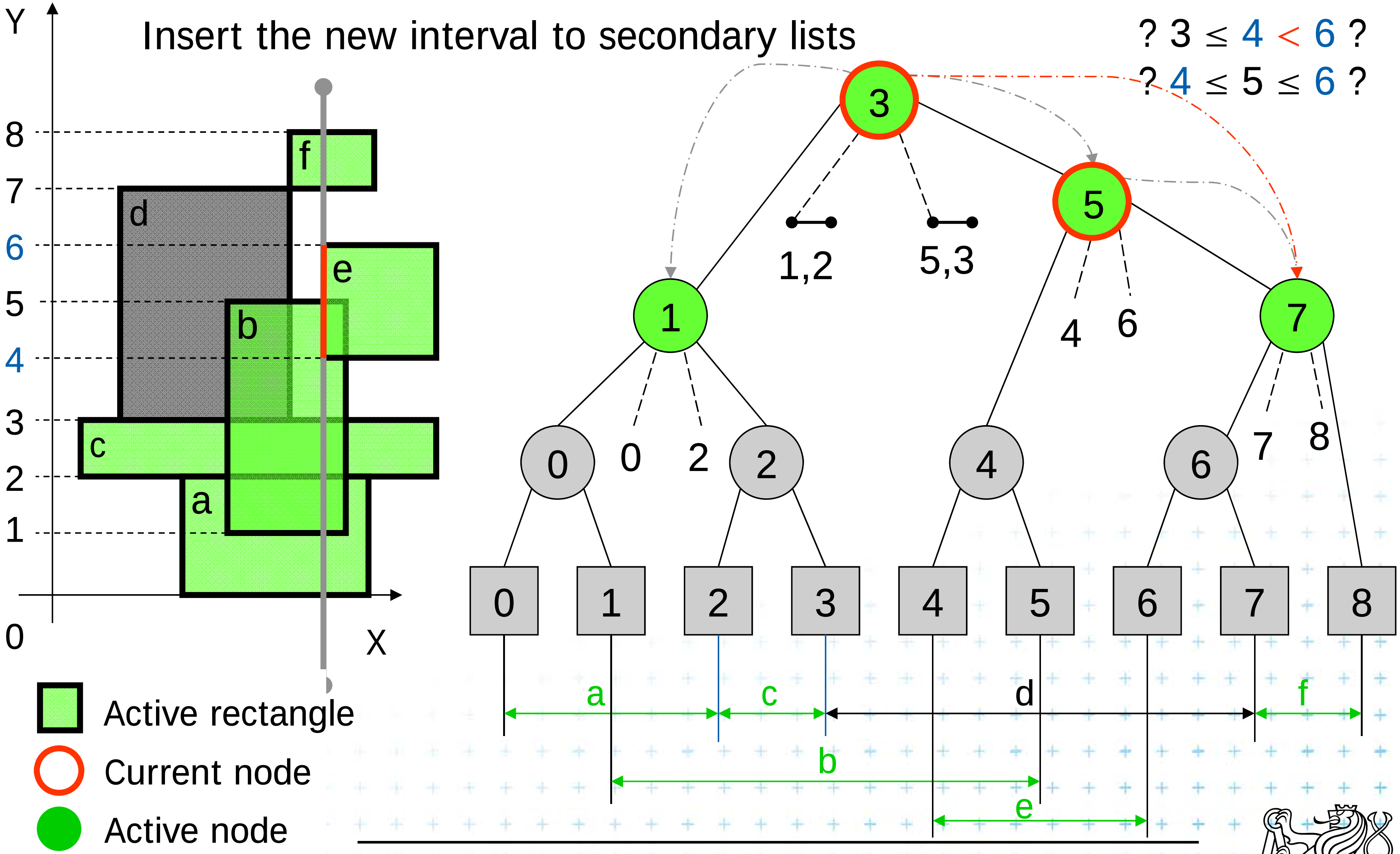
Insert [4,6] b) Insert Interval

s

Insert the new interval to secondary lists

? 3 ≤ 4 < 6 ?

? 4 ≤ 5 ≤ 6 ?



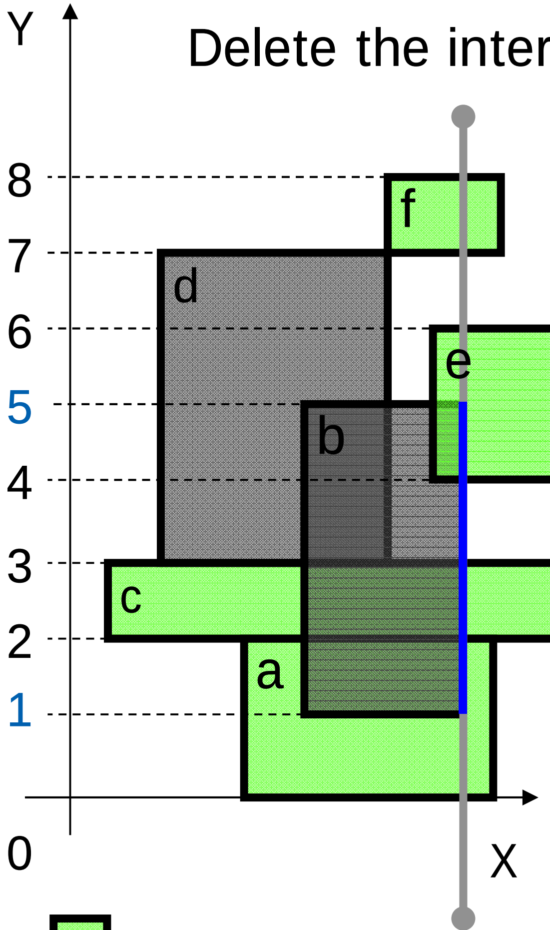
[Drtina]

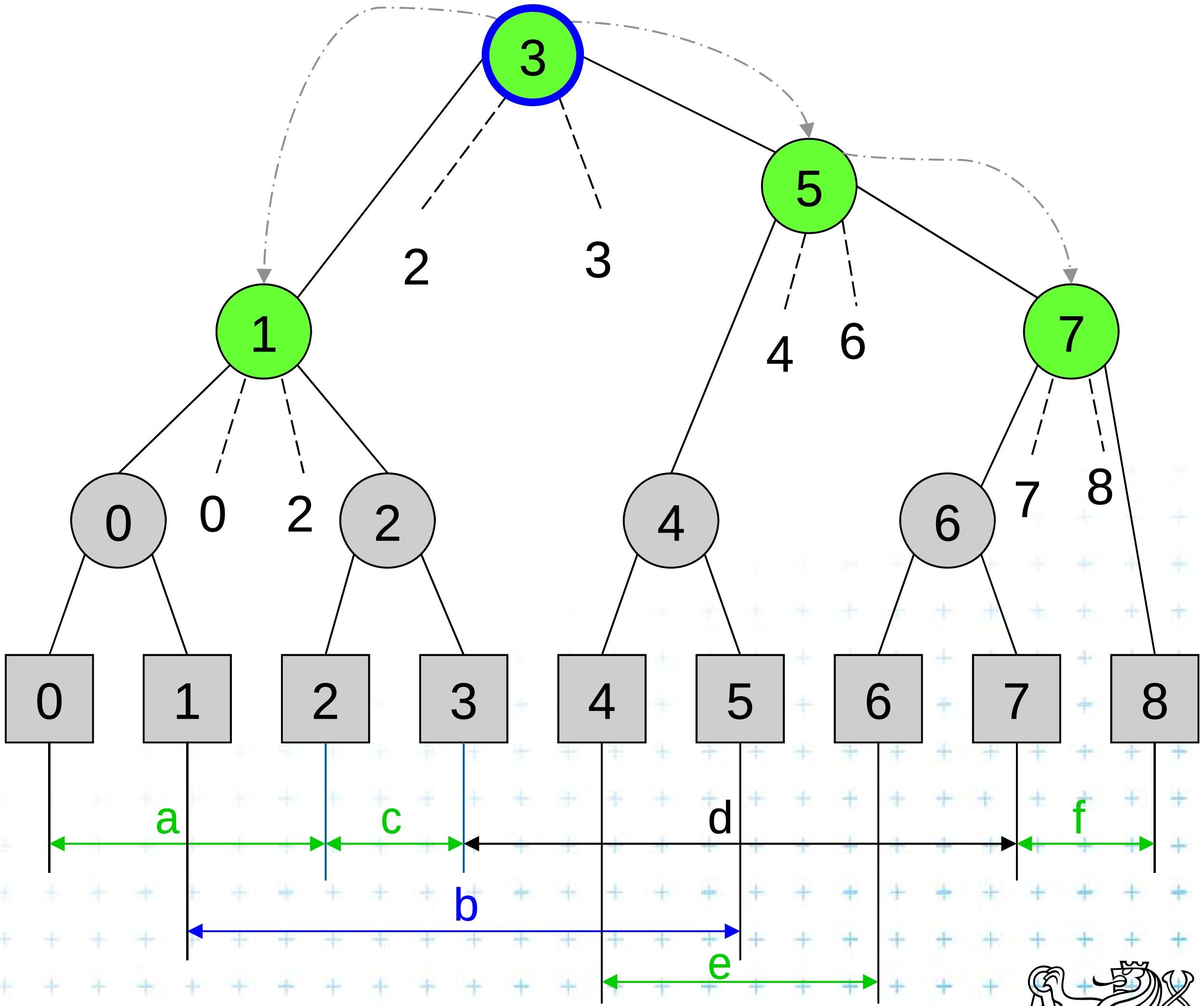
Delete [1,5] Delete Interval

$b \le H(v) \le e$

Delete the interval [1,5] from secondary lists

? $1 \le 3 \le 5$?





Active rectangle

Current node

Active node

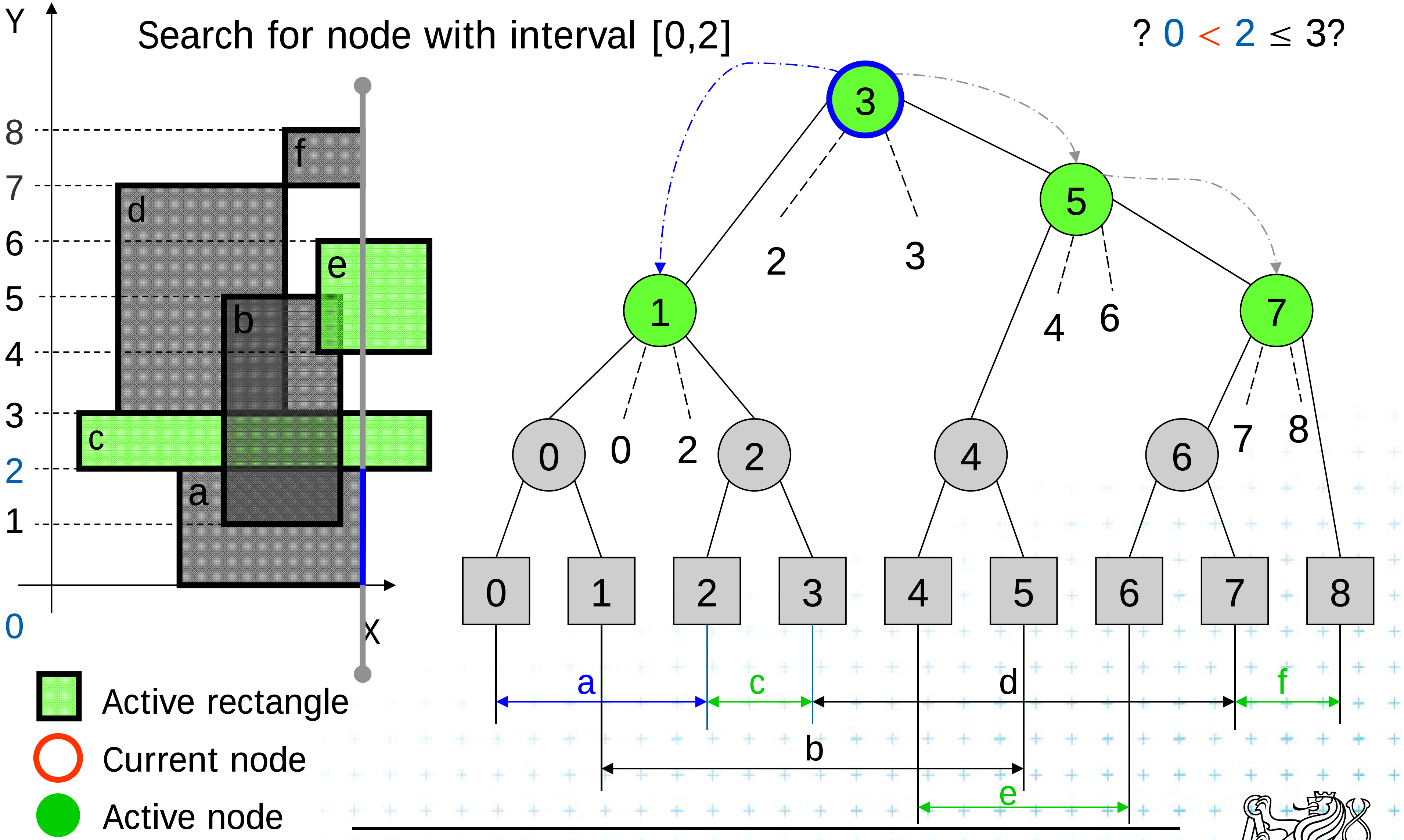
[Drtina]

Delete [0,2] Delete Interval 1/2

$b < e \leq H(v)$

Search for node with interval [0,2]

$? \; 0 < 2 \leq 3?$

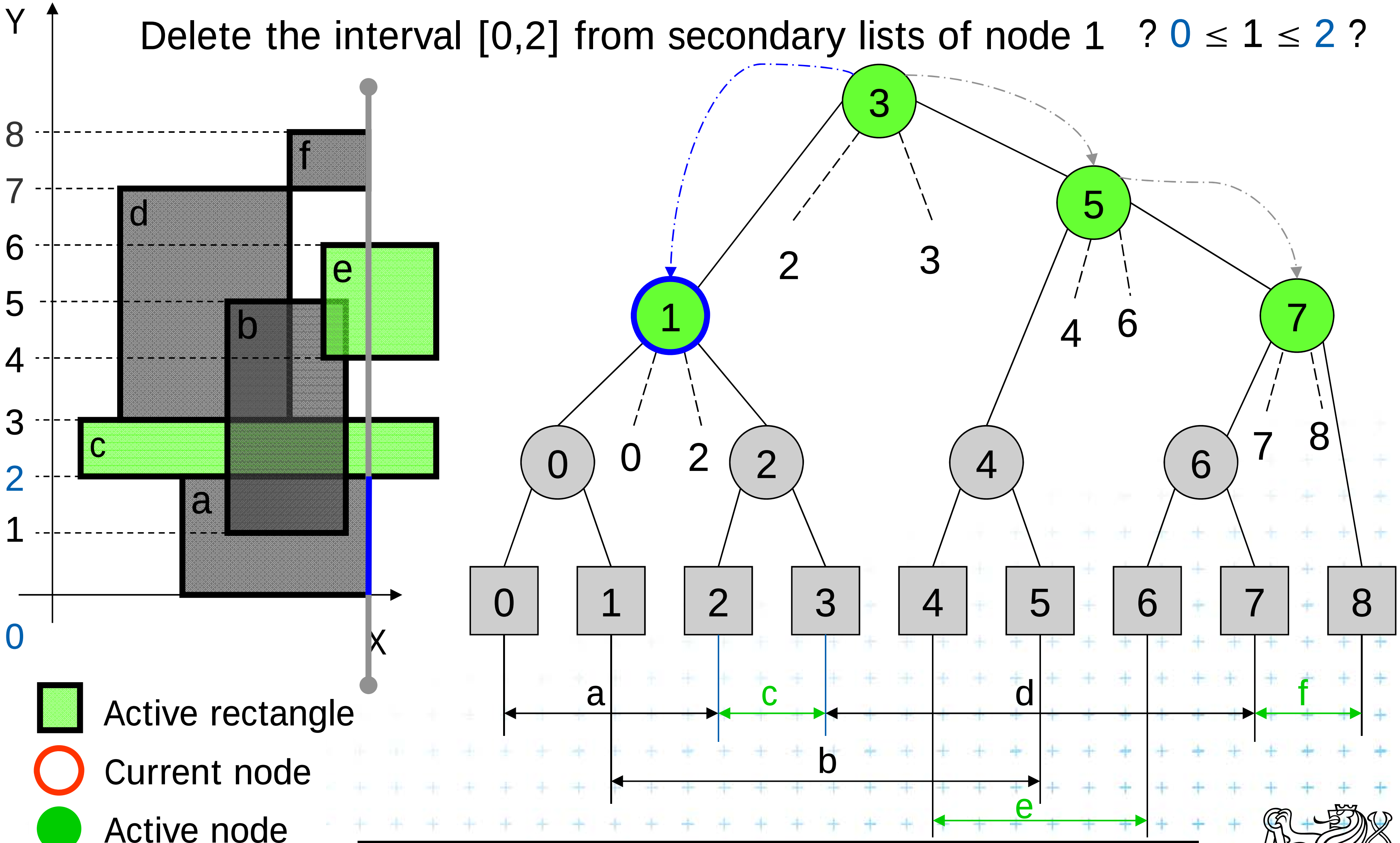


[Drtina]

Delete [0,2] Delete Interval 2/2

$b \le H(v) \le e$

Delete the interval [0,2] from secondary lists of node 1 ? $0 \le 1 \le 2$?



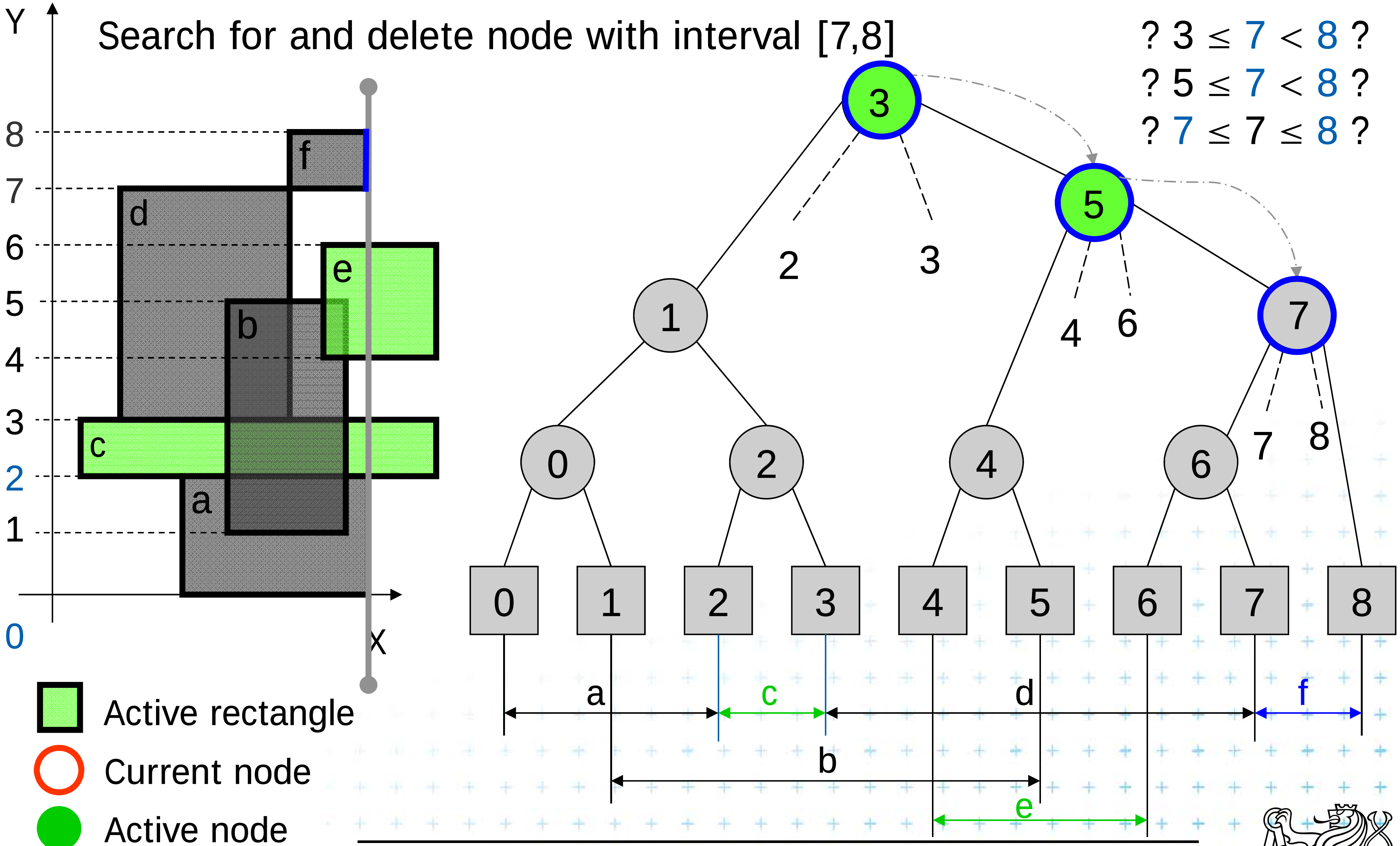
[Drtina]

Delete [7,8] Delete Interval

$b \leq H(v) \leq e$

Search for and delete node with interval [7,8]

? $3 \leq 7 < 8$?
? $5 \leq 7 < 8$?
? $7 \leq 7 \leq 8$?



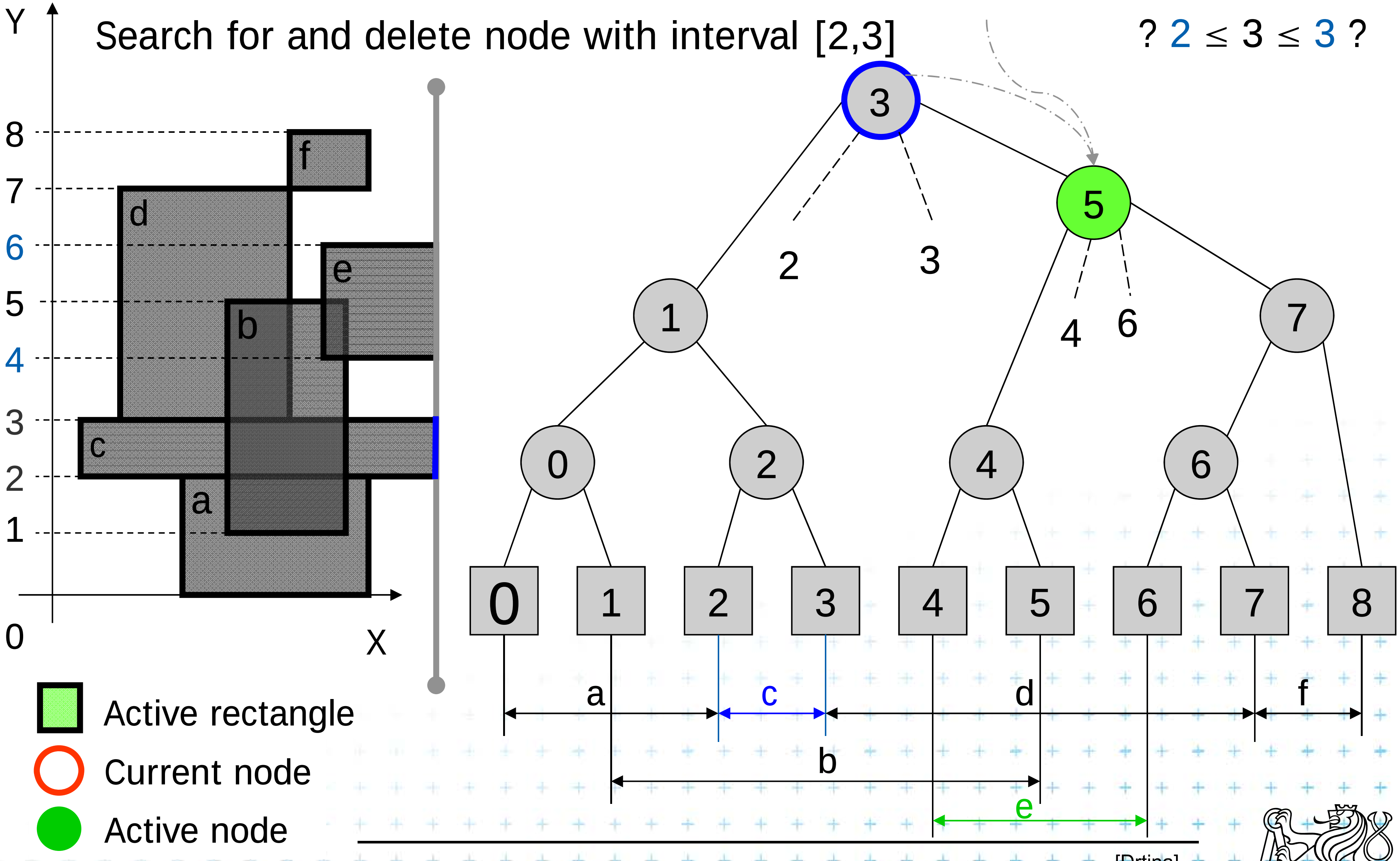
[Drtina]

Delete [2,3] Delete Interval

b ≤ H(v) ≤ e

Search for and delete node with interval [2,3]

? 2 ≤ 3 ≤ 3 ?

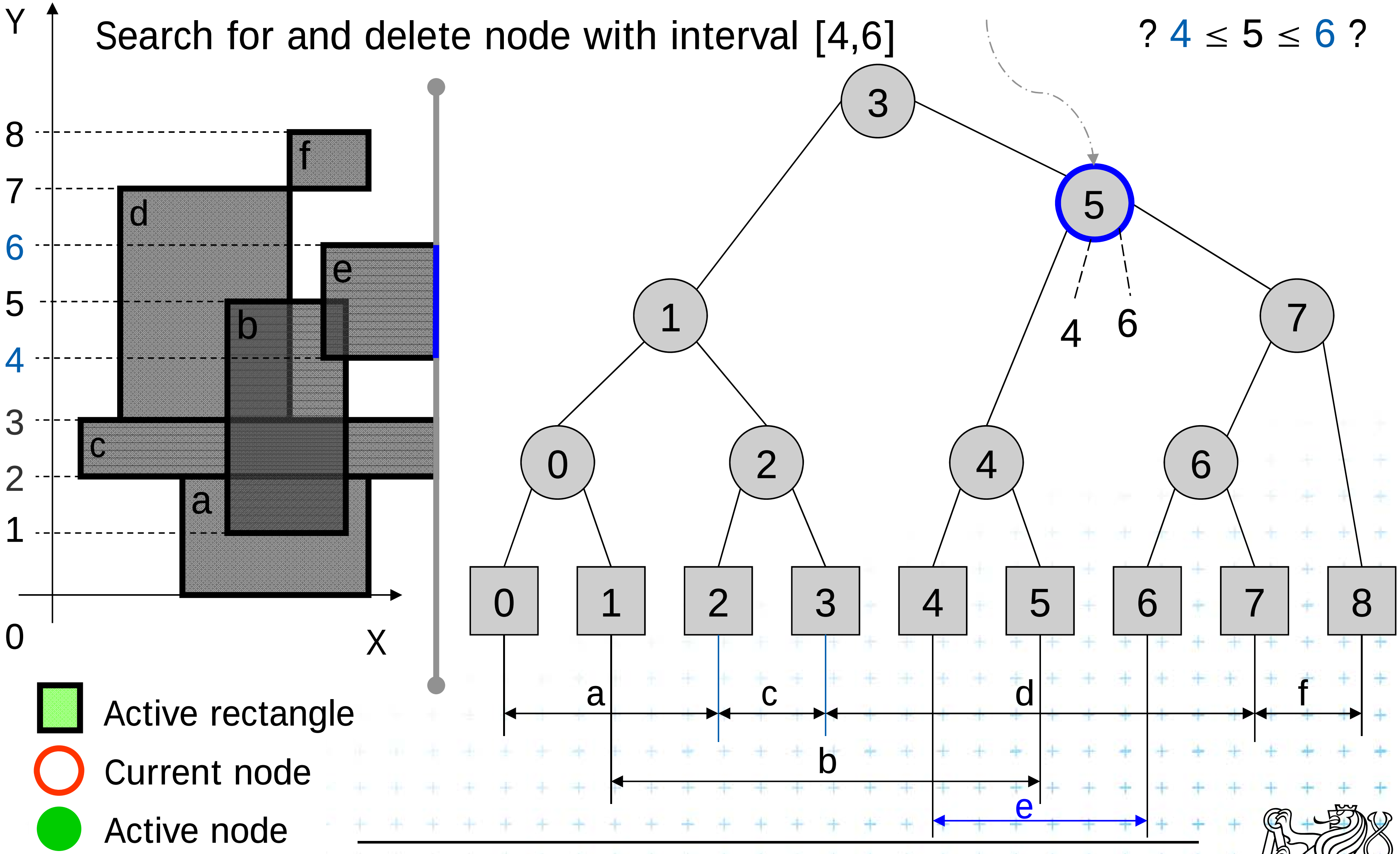


Delete [4,6] Delete Interval

b ≤ H(v) ≤ e

Search for and delete node with interval [4,6]

? 4 ≤ 5 ≤ 6 ?



Complexities of rectangle intersections

- n rectangles, s intersected pairs found
- $O(n \log n)$ preprocessing time to separately sort
  - – x-coordinates of the rectangles for the plane sweep
  - – the y-coordinates for initializing the interval tree.
- The plane sweep itself takes $O(n \log n + s)$ time, so the overall time is $O(n \log n + s)$
- $O(n)$ space
- This time is optimal for a decision-tree algorithm (i.e., one that only makes comparisons between rectangle coordinates).

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