INTERSECTIONS OF LINE SEGMENTS AND POLYGONS

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FEL CTU PRAGUE

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https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

Based on [Berg], [Mount], [Kukral], and [Drtina]

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Talk overview

- Intersections of line segments (Bentley-Ottmann)
  – Motivation
  – Sweep line algorithm recapitulation
  – Sweep line intersections of line segments
- Intersection of polygons or planar subdivisions
  – See assignment [21] or [Berg, Section 2.3]
- Intersection of axis parallel rectangles
  – See assignment [26]

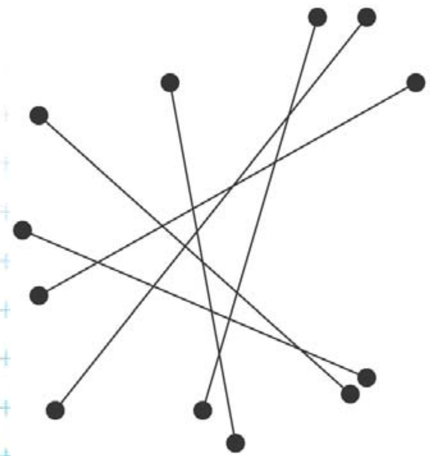
Geometric intersections – what are they for?

One of the most basic problems in computational geometry

- Solid modeling
  - Intersection of object boundaries in CSG
- Overlay of subdivisions, e.g. layers in GIS
  - Bridges on intersections of roads and rivers
  - Maintenance responsibilities (road network X county boundaries)
- Robotics
  - Collision detection and collision avoidance
- Computer graphics
  - Rendering via ray shooting (intersection of the ray with objects)
- …

Line segment intersection

- Intersection of complex shapes is often reduced to simpler and simpler intersection problems
- Line segment intersection is the most basic intersection algorithm
- Problem statement:
  Given *n* line segments in the plane, report all points where a pair of line segments intersect.
- Problem complexity
  - Worst case – $I = O(n^2)$ intersections
  - Practical case – only some intersections
  - Use an output sensitive algorithm
    - $O(n \log n + I)$ optimal randomized algorithm
    - $O(n \log n + I \log n)$ sweep line algorithm - %



[Berg]

Plane sweep line algorithm recapitulation

Postupový plán

- Horizontal line (sweep line, *scan line*) ℓ moves top-down (or vertical line: left to right) over the set of objects
- The move is not continuous, but ℓ jumps from one event point to another
  - Event points are in priority queue or sorted list (~y)
  - The (left) top-most event point is removed first
  - New event points may be created (usually as interaction of neighbors on the sweep line) and inserted into the queue

Status

- Scan-line status
  - Stores information about the objects intersected by ℓ
  - It is updated while stopping on event point

Line segment intersection - Sweep line alg.

- Avoid testing of pairs of segments far apart
- Compute intersections of neighbors on the sweep line only
- $O(n \log n + I \log n)$ time in $O(n)$ memory
  - $2n$ steps for end points,
  - I steps for intersections,
  - $\log n$ search the status tree
- Ignore "nasty cases" (most of them will be solved later on)
  - No segment is parallel to the sweep line
  - Segments intersect in one point and do not overlap
  - No three segments meet in a common point

Line segment intersections

Status = ordered sequence of segments intersecting the sweep line ℓ *Stav*

Events (waiting in the priority queue) *Postupový plán*

= points, where the algorithm actually does something

– Segment *end-points*

  - known at algorithm start

– Segment *intersections* between neighboring segments along SL

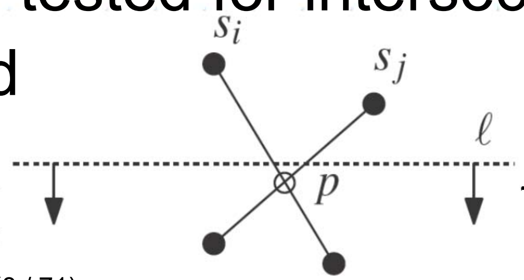
  - discovered as the sweep executes

Detecting intersections

- Intersection events must be detected and inserted to the event queue before they occur
- Given two segments *a, b* intersecting in point *p*, there must be a placement of sweep line ℓ prior to *p*, such that segments *a, b* are adjacent along ℓ (only adjacent will be tested for intersection)
  - segments *a, b* are not adjacent when the alg. starts
  - segments *a, b* are adjacent just before *p*

  => there must be an event point when *a,b* become adjacent and therefore are tested for intersection

  => All intersections are found

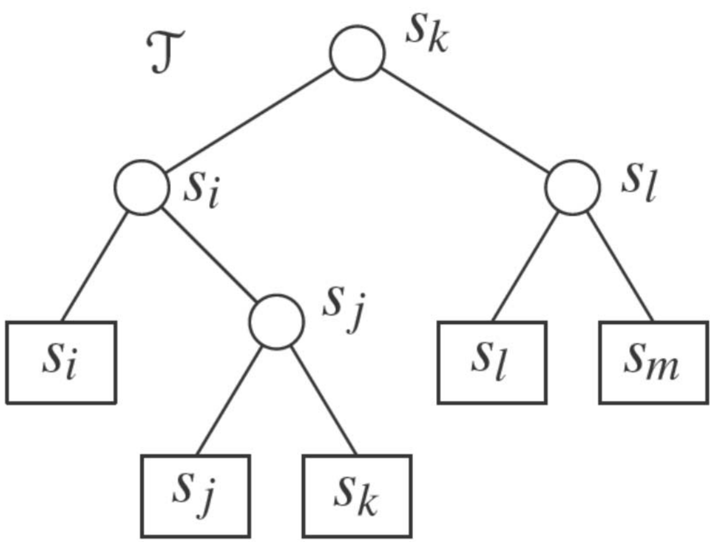


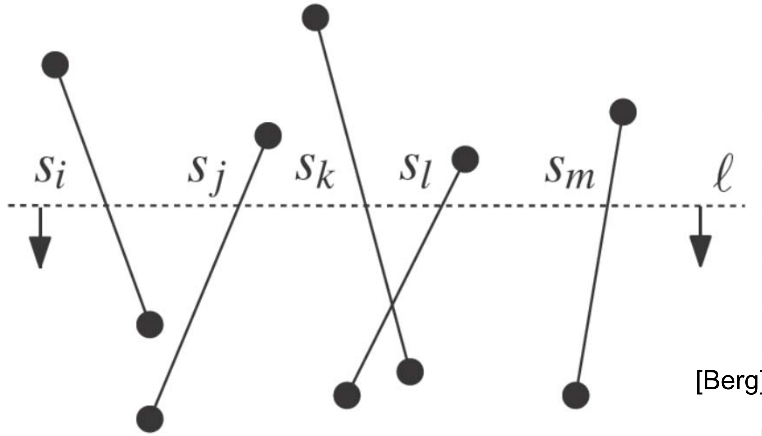
[Berg]

Data structures

Sweep line ℓ status = order of segments along ℓ

- Balanced binary search tree of segments
- Coords of intersections with ℓ vary as ℓ moves
  => store pointers to line segments in tree nodes
  – Position of ℓ is plugged in the *y=mx+b* to get the x-key

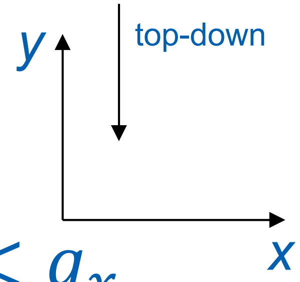




[Berg]

Data structures

Event queue *(postupový plán, časový plán)*



- Define: Order $\succ$ (top-down, lexicographic)

  $p \succ q$ **iff** $p_y > q_y$ **or** $p_y = q_y$ **and** $p_x < q_x$

  top-down, left-right approach

  (points on ℓ treated left to right)

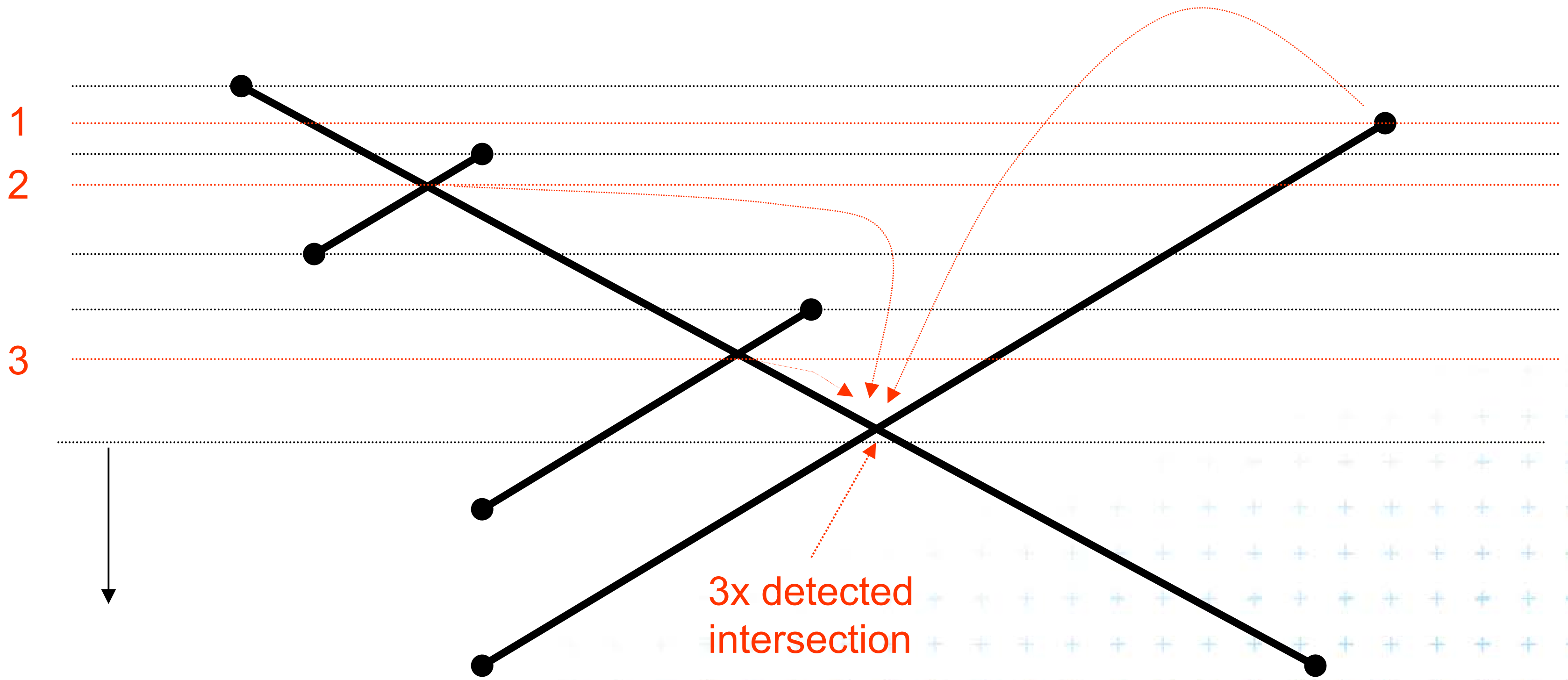
- Operations
  - Insertion of computed intersection points
  - Fetching the next event (highest *y* below ℓ or the leftmost right of e)

    } must have
  - Test, if the segment is already present in the queue (Locate and delete intersection event in the queue)

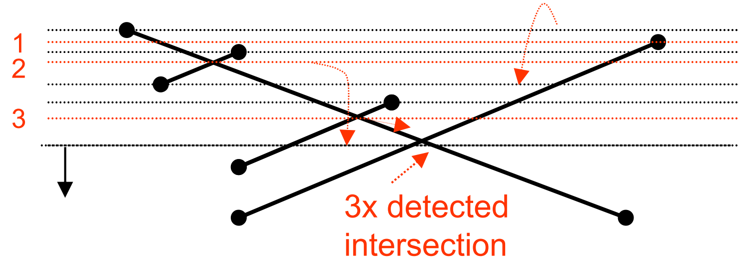
    } may have

Problem with duplicities of intersections

Intersection may be detected many times



Data structures



Event queue data structure

a) Heap

- Problem: can not check duplicated intersection events (reinvented & stired more than once)
- Intersections processed twice or even more times
- Memory complexity up to $O(n^2)$

b) Ordered dictionary (balanced binary tree)

- Can check duplicated events (adds just constant factor)
- Nothing inserted twice
- If non-neighbor intersections are deleted
  i.e., if only intersections of neighbors along ℓ are stored
  then memory complexity just $O(n)$

Line segment intersection algorithm

FindIntersections(*S*)

Input: A set *S* of line segments in the plane

Output: The set of intersection points + pointers to segments in each

1. init an empty event queue *Q* and insert the segment endpoints
2. init an empty status structure *T*
3. **while** Q in not empty
4. remove next event *p* from *Q*
5. handleEventPoint(*p*)

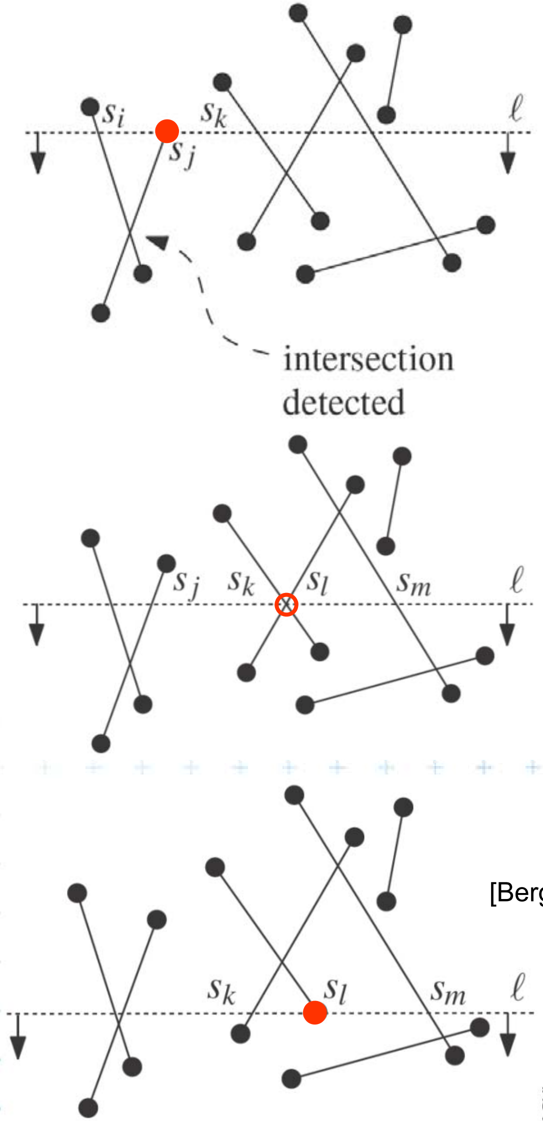
handleEventPoint(*p*) →
- Upper endpoint
- Intersection
- Lower endpoint

Improved algorithm:
Handles all in *p*
in a single step

Note: Upper-end-point events store info about the segment

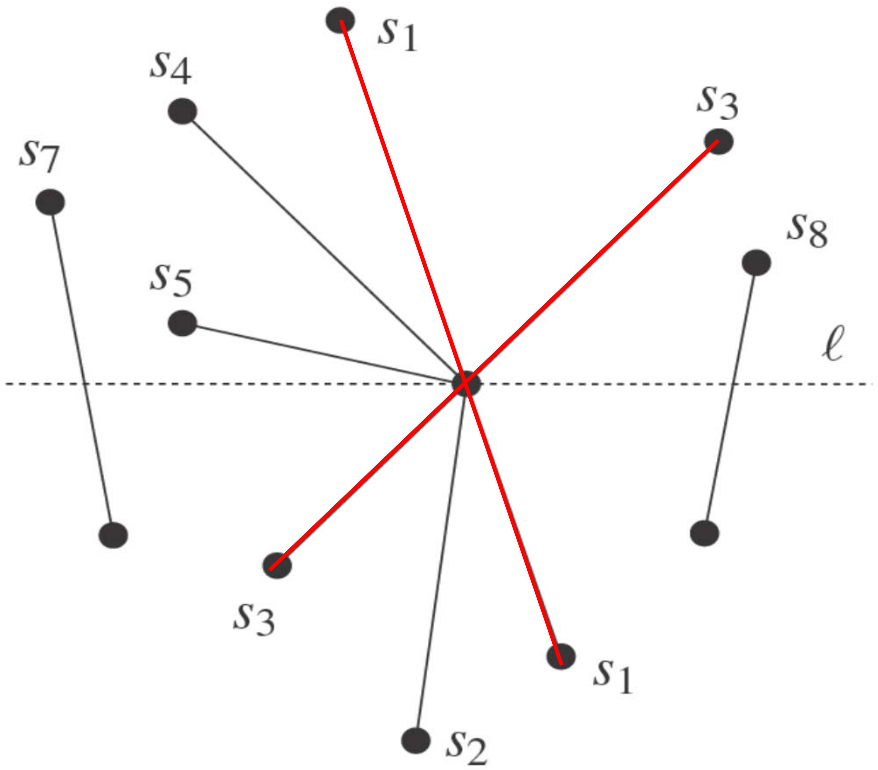
handleEventPoint() principle

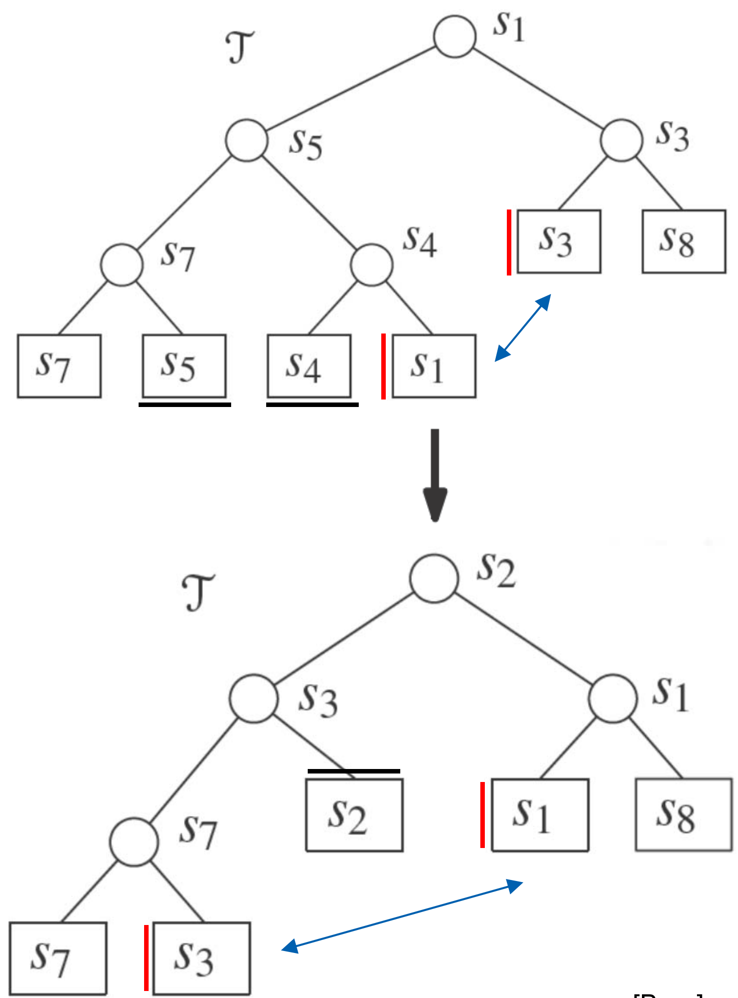
- Upper endpoint $U(p)$
  - insert p (on s_j) to status T
  - add intersections with left and right neighbors to Q
- Intersection $C(p)$
  - switch order of segments in T
  - add intersections with nearest left and nearest right neighbor to Q
- Lower endpoint $L(p)$
  - remove p (on s_l) from T
  - add intersections of left and right neighbors to Q



[Berg]

More than two segments incident





$U(p) = \{s_2\}$ — start here

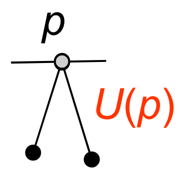
$C(p) = \{s_1, s_3\}$ — cross on ℓ

$L(p) = \{s_4, s_5\}$ — end here

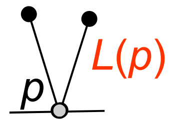
[Berg]

Handle Events [Berg, page 25]

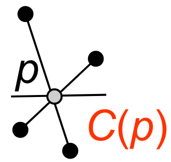
handleEventPoint(p)



1. Let *U*(*p*) = set of segments whose Upper endpoint is *p*.
   These segmets are stored with the event point *p* (will be added to *T*)
2. Search *T* for all segments *S*(*p*) that contain *p* (are adjacent in *T*):
   Let *L*(*p*) ⊂ *S*(*p*) = segments whose Lower endpoint is *p*
   Let *C*(*p*) ⊂ *S*(*p*) = segments that Contain *p* in interior

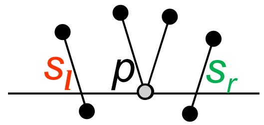


3. **if**(*L*(*p*) ∪ *U*(*p*) ∪ *C*(*p*) contains more than one segment)
4. report *p* as intersection ∘ together with *L*(*p*), *U*(*p*), *C*(*p*)



5. Delete the segments in *L*(*p*) ∪ *C*(*p*) from *T*
6. Insert the segments in *U*(*p*) ∪ *C*(*p*) into *T*
   (order as below ℓ, horizontal segment as the last)

} Reverse order of *C*(*p*) in *T*



7. **if**(*U*(*p*) ∪ *C*(*p*) = ∅) then findNewEvent(s_l, s_r, *p*) // left & right neighbors
8. **else** s' = leftmost segment of *U*(*p*) ∪ *C*(*p*); findNewEvent(s_l , *s'*, *p*)
   s'' = rightmost segment of *U*(*p*) ∪ *C*(*p*); findNewEvent(*s''*, s_r , *p*)

Detection of new intersections

findNewEvent(s_l, s_r, p) **// with handling of horizontal segments**

Input: two segments (left & right from p in T) and a current event point p

Output: updated event queue Q with new intersection

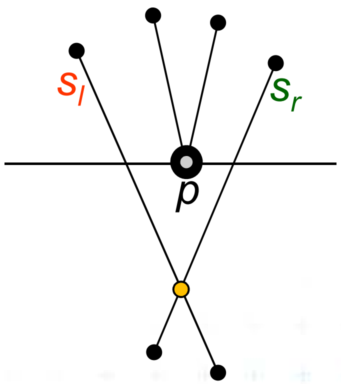
1. **if [(** s_l and s_r intersect below the sweep line ℓ) // line 7. above

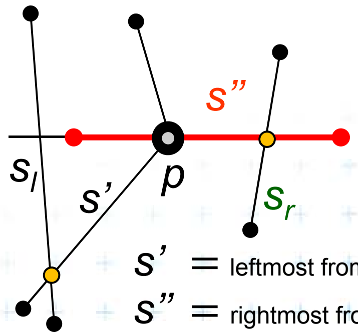
   or (s_r intersect s'' on ℓ and to the right of p)] // Non-overlapping horizontal segm.

   and(the intersection ● is not present in Q)
2. **then**

   insert intersection ● as a new event into Q

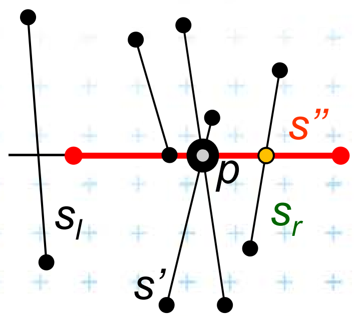
○	Intersection - line 4
●	Intersection - line 7,8





s' = leftmost from $U(p) \cup C(p)$

s'' = rightmost from $U(p) \cup C(p)$



s_l and s_r intersect below

s_r and s'' intersect on ℓ,
s'' is horizontal and to the right of p

Line segment intersections

- Memory $O(I) = O(n^2)$ with duplicities in Q
  or $O(n)$ with duplicities in Q deleted
- Operational complexity
  - $n + I$ stops
  - $\log n$ each

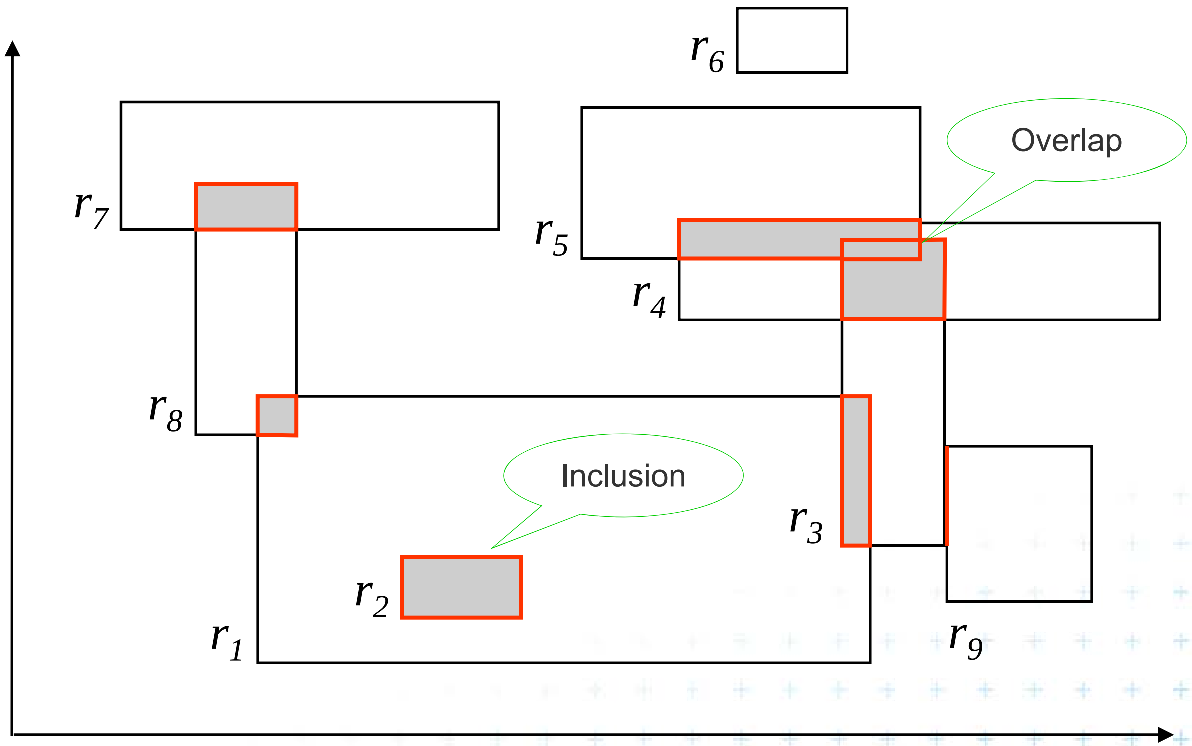
  => $O(I + n) \log n$ total
- The algorithm is by Bentley-Ottmann

  Bentley, J. L.; Ottmann, T. A. (1979), "Algorithms for reporting and counting geometric intersections", *IEEE Transactions on Computers* **C-28** (9): 643-647, doi:10.1109/TC.1979.1675432 .

  See also http://wapedia.mobi/en/Bentley%E2%80%93Ottmann_algorithm

Intersection of axis parallel rectangles

- Given the collection of *n isothetic* rectangles, report all intersecting parts



Alternate sides belong to two pencils of lines
(trsy přímek)
(often used with points in infinity = axis parallel)
2D => 2 pencils

[?]

Answer: (r_1, r_2) (r_1, r_3) (r_1, r_8) (r_3, r_4) (r_3, r_5) (r_3, r_9) (r_4, r_5) (r_7, r_8)

Brute force intersection

Brute force algorithm

Input: set S of axis parallel rectangles
Output: pairs of intersected rectangles

1. For every pair (r_i, r_j) of rectangles $\in S$, $i \neq j$
2. if $(r_i \cap r_j \neq \emptyset)$ then
3. report (r_i, r_j)

Analysis

Preprocessing: None.

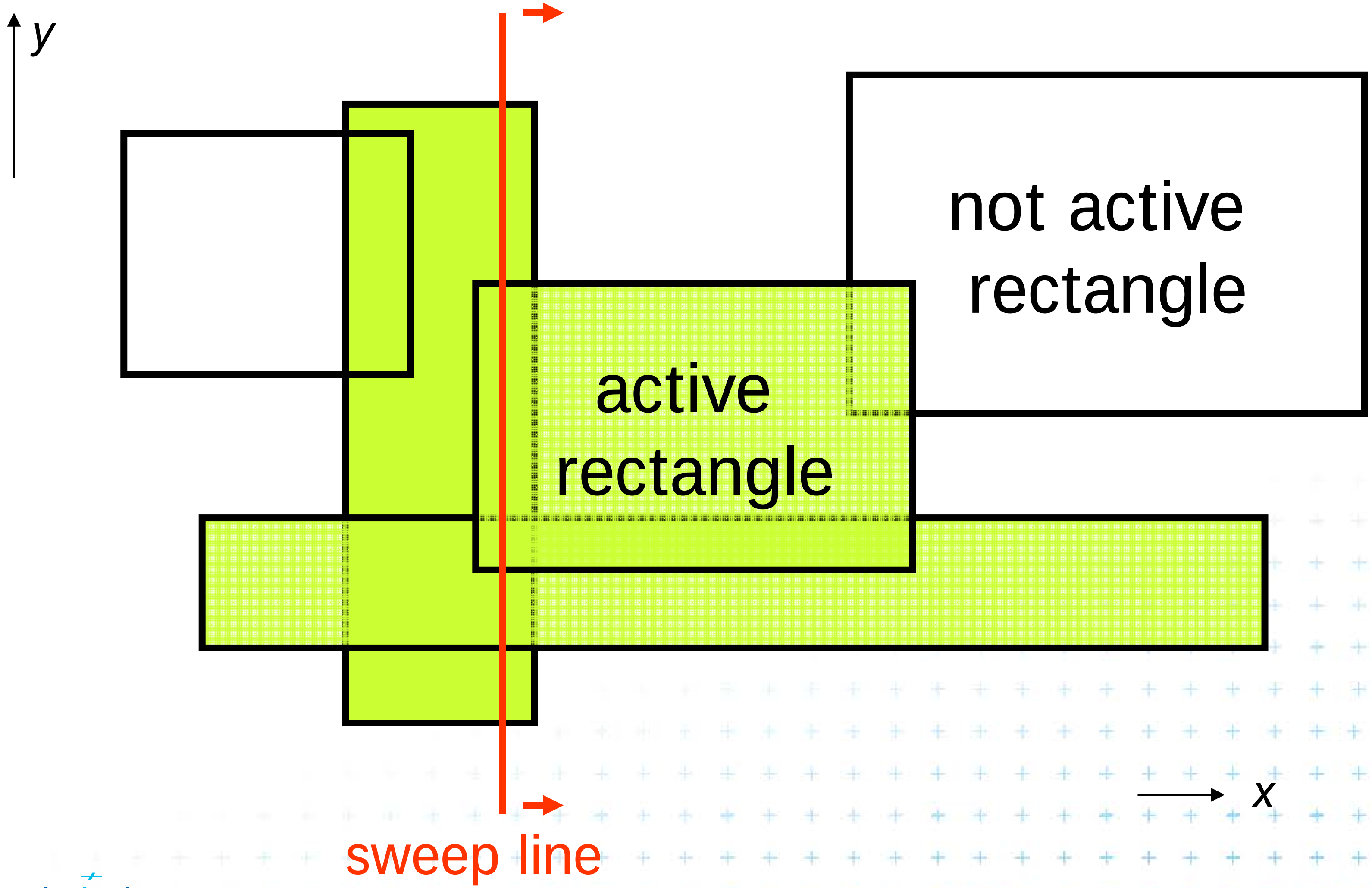
Query: $O(N^2)$ $\binom{N}{2} = \frac{N(N-1)}{2} \in O(N^2)$.

Storage: $O(N)$

Plane sweep intersection algorithm

- Vertical sweep line moves from left to right
- Stops at every x-coordinate of a rectangle (either at its left side or at its right side).
- active rectangles – a set
  = rectangles currently intersecting the sweep line
  - left side event of a rectangle – start
    => the rectangle is added to the active set.
  - right side – end
    => the rectangle is deleted from the active set.
- The active set used to detect rectangle intersection

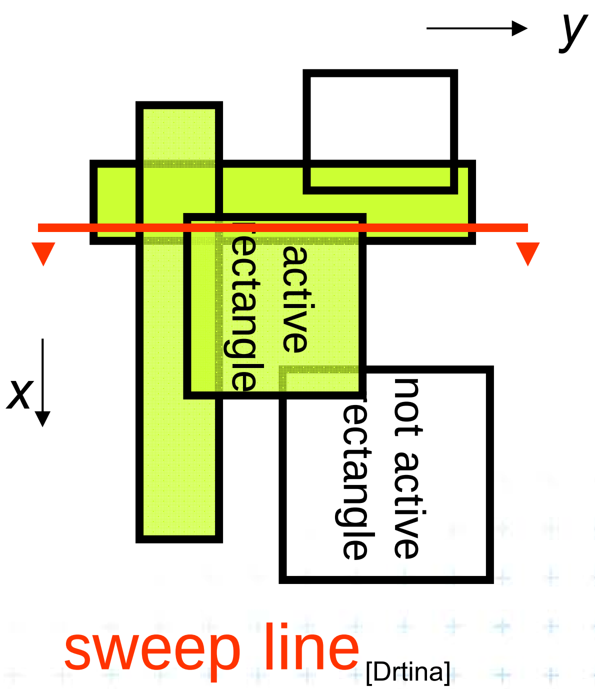
Example rectangles and sweep line



[Drtina]

Interval tree as sweep line status structure

- Vertical sweep-line => only *y*-coordinates along it
- The status tree is drawn horizontal - turn 90° right as if the sweep line (*y*-axis) is horizontal

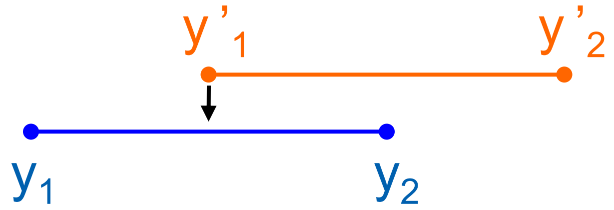


Intersection test – between pair of intervals

- Given two intervals $R = [y_1, y_2]$ and $R' = [y'_1, y'_2]$ the condition $R \cap R'$ is equivalent to one of these mutually exclusive conditions:

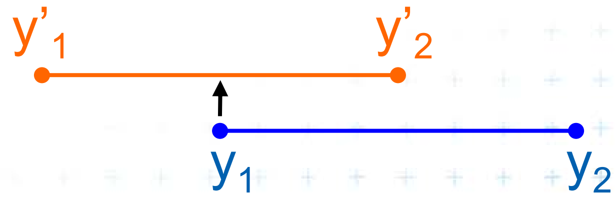
1st variant

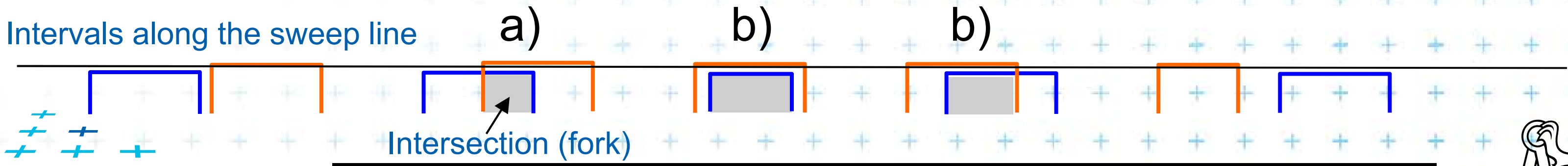
a) $y_1 \le y'_1 \le y_2$



OR

b) $y'_1 \le y_1 \le y'_2$





Intersection test – between pair of intervals

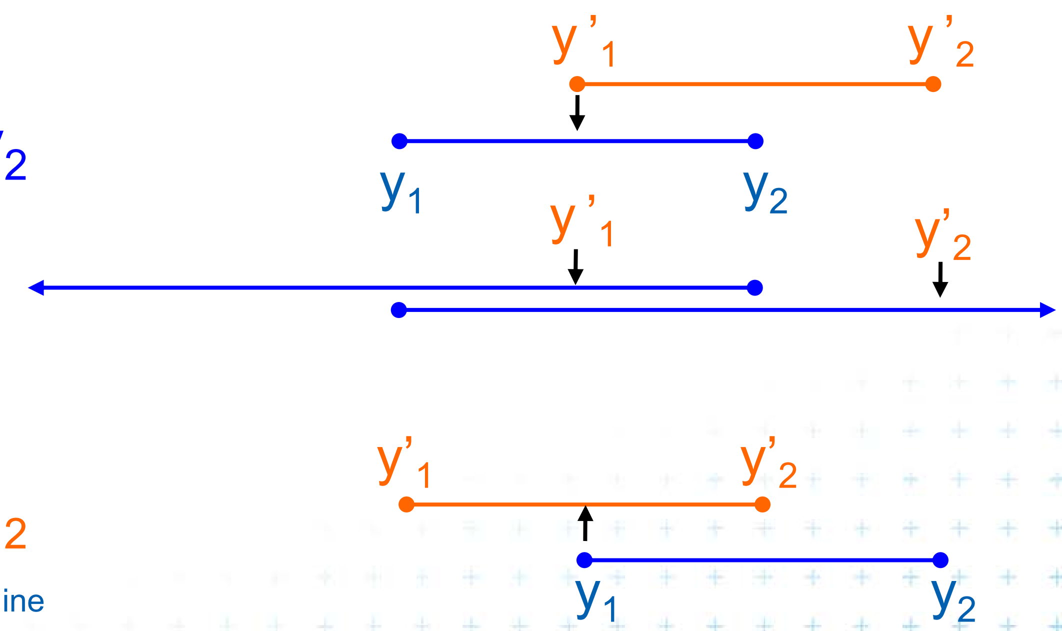
- Given two intervals R = [y_1, y_2] and R' = [y'_1, y'_2] the condition R ∩ R' is equivalent to both of these conditions simultaneously:

2nd variant

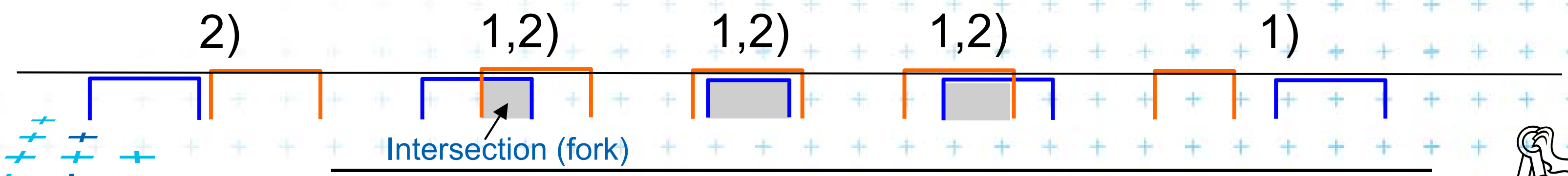
1) $y'_1 \le y_2$

AND

2) $y_1 \le y'_2$

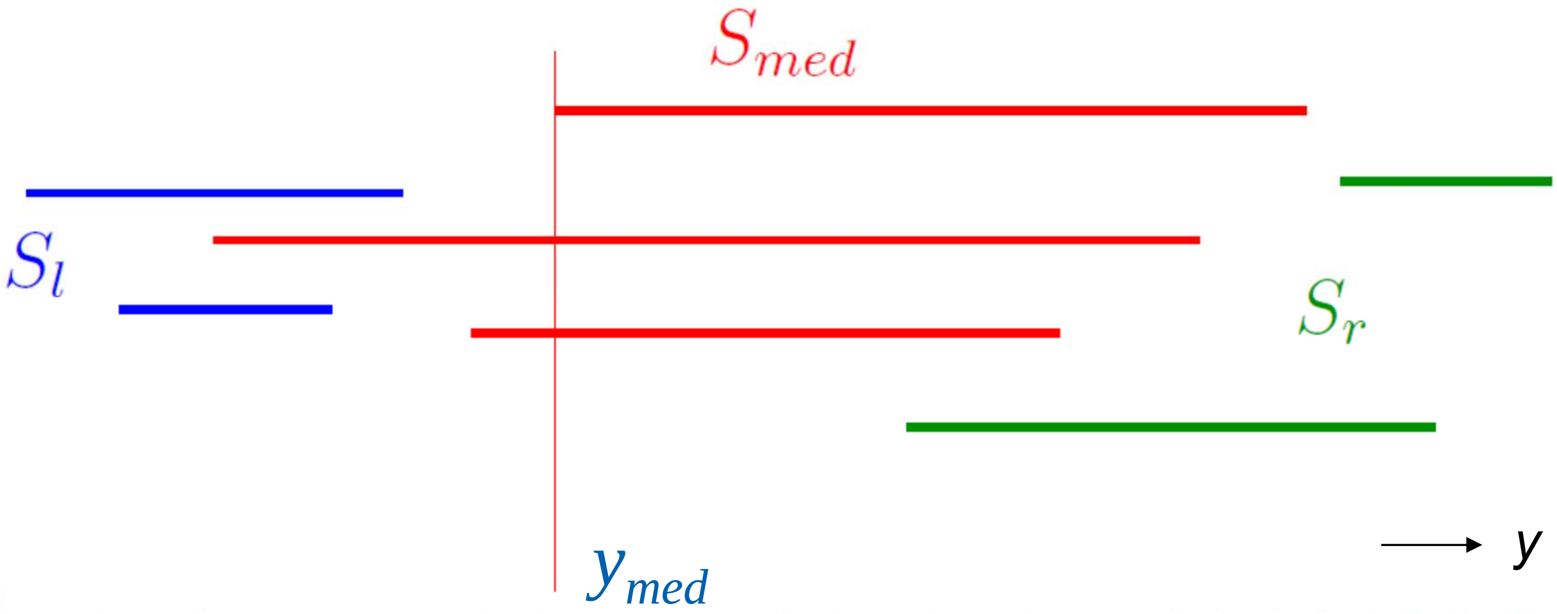


Intervals along the sweep line



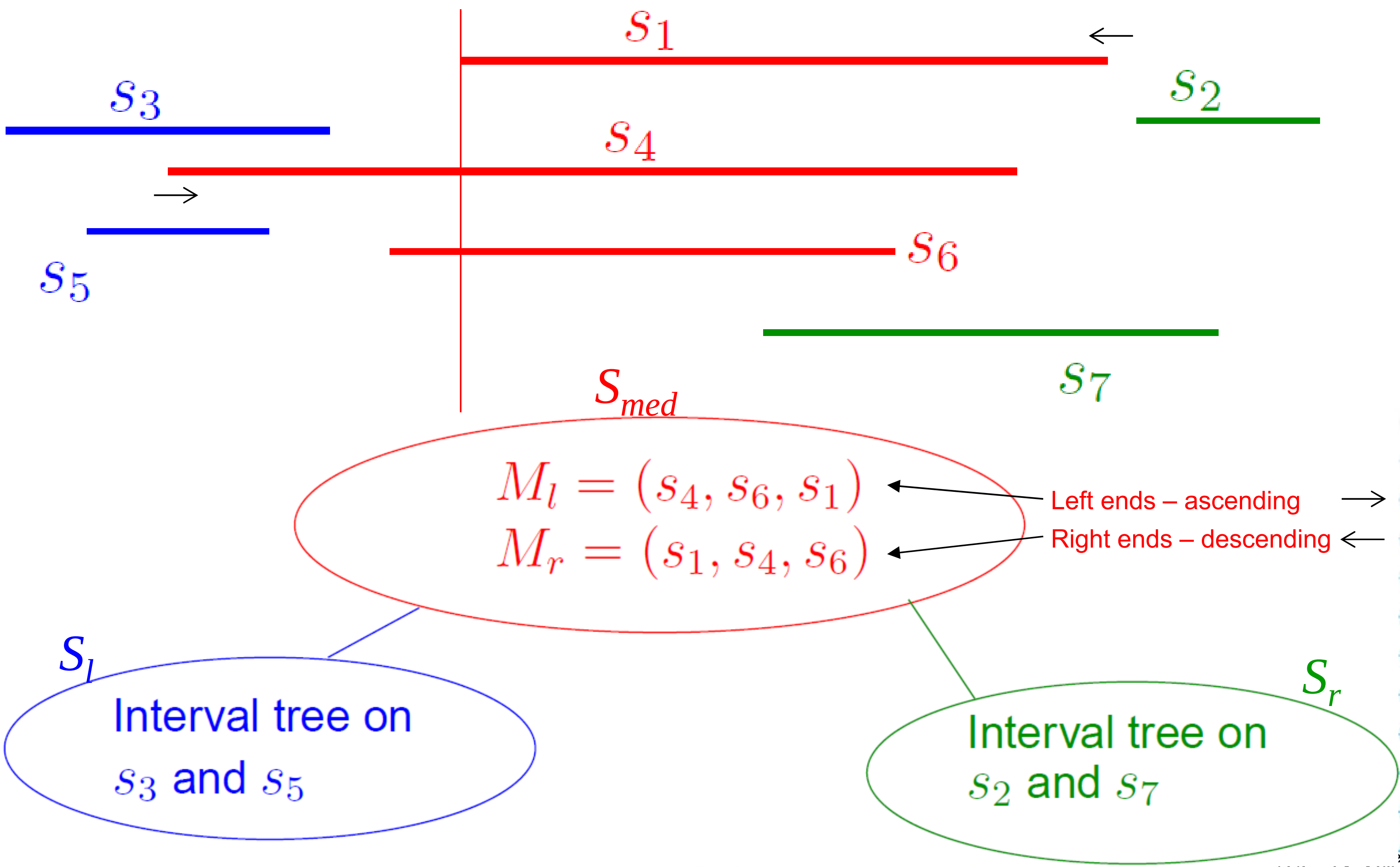
Static interval tree – stores all end point ys

- Let $v = y_{med}$ be the median of end-points of segments
- S_l : segments of S that are completely to the left of y_{med}
- S_{med} : segments of S that contain y_{med}
- S_r : segments of S that are completely to the right of y_{med}



[Vigneron]

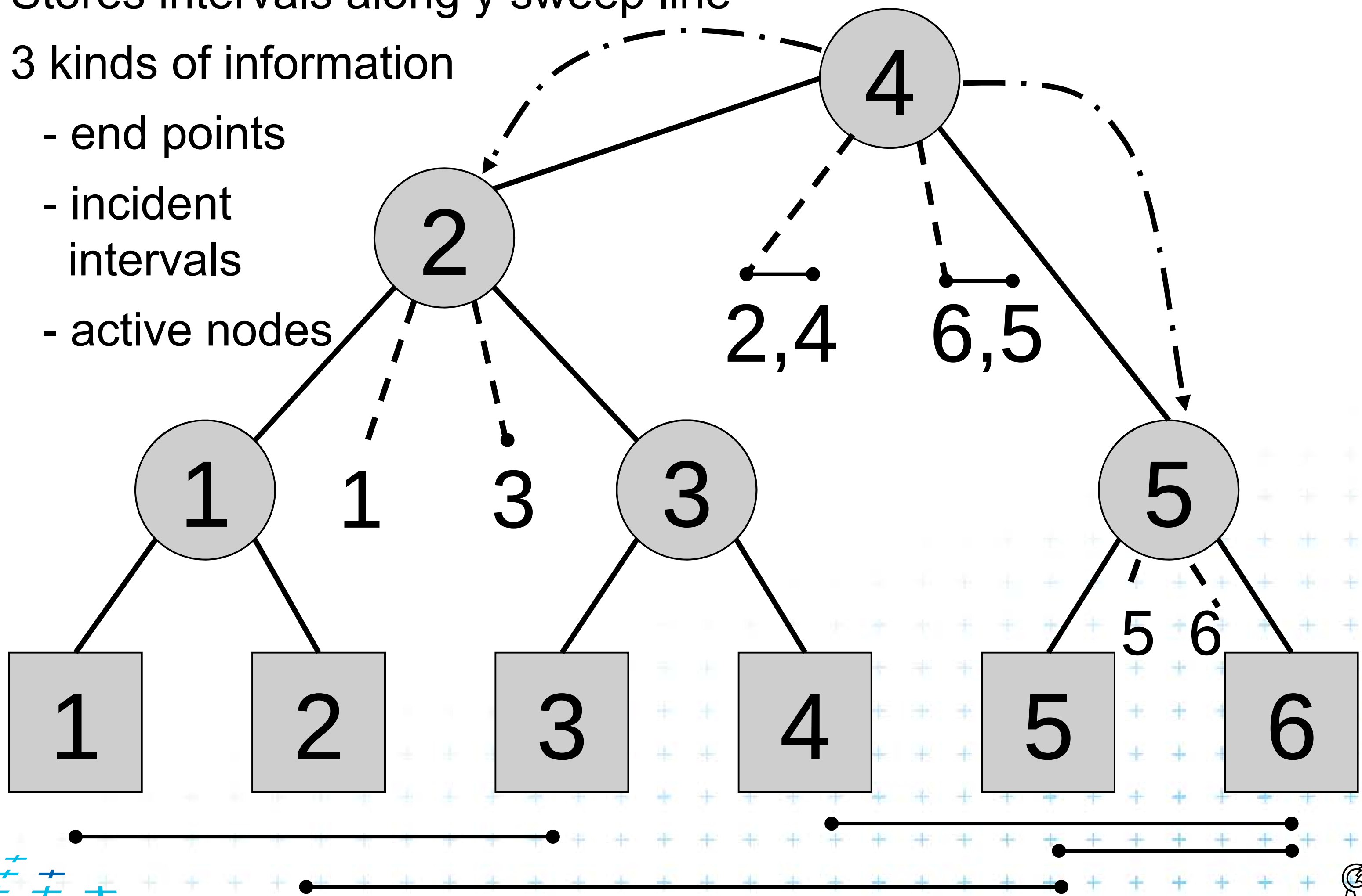
Static interval tree – Example



[Vigneron]

Static interval tree [Edelsbrunner80]

- Stores intervals along y sweep line
- 3 kinds of information
  - end points
  - incident intervals
  - active nodes



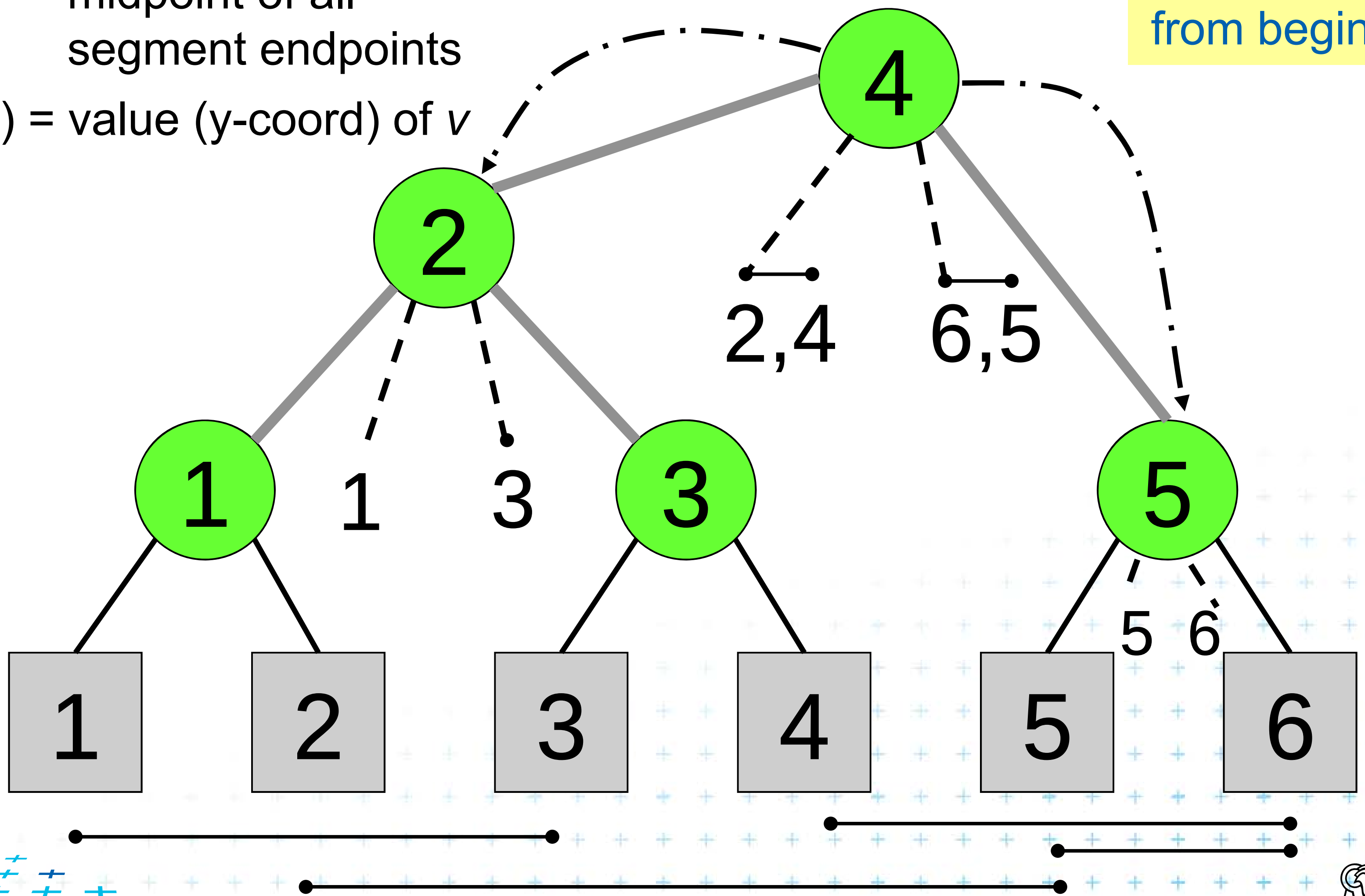
[Kukral]

Primary structure – static tree for endpoints

v = midpoint of all segment endpoints

H(v) = value (y-coord) of *v*

Static – known from beginning

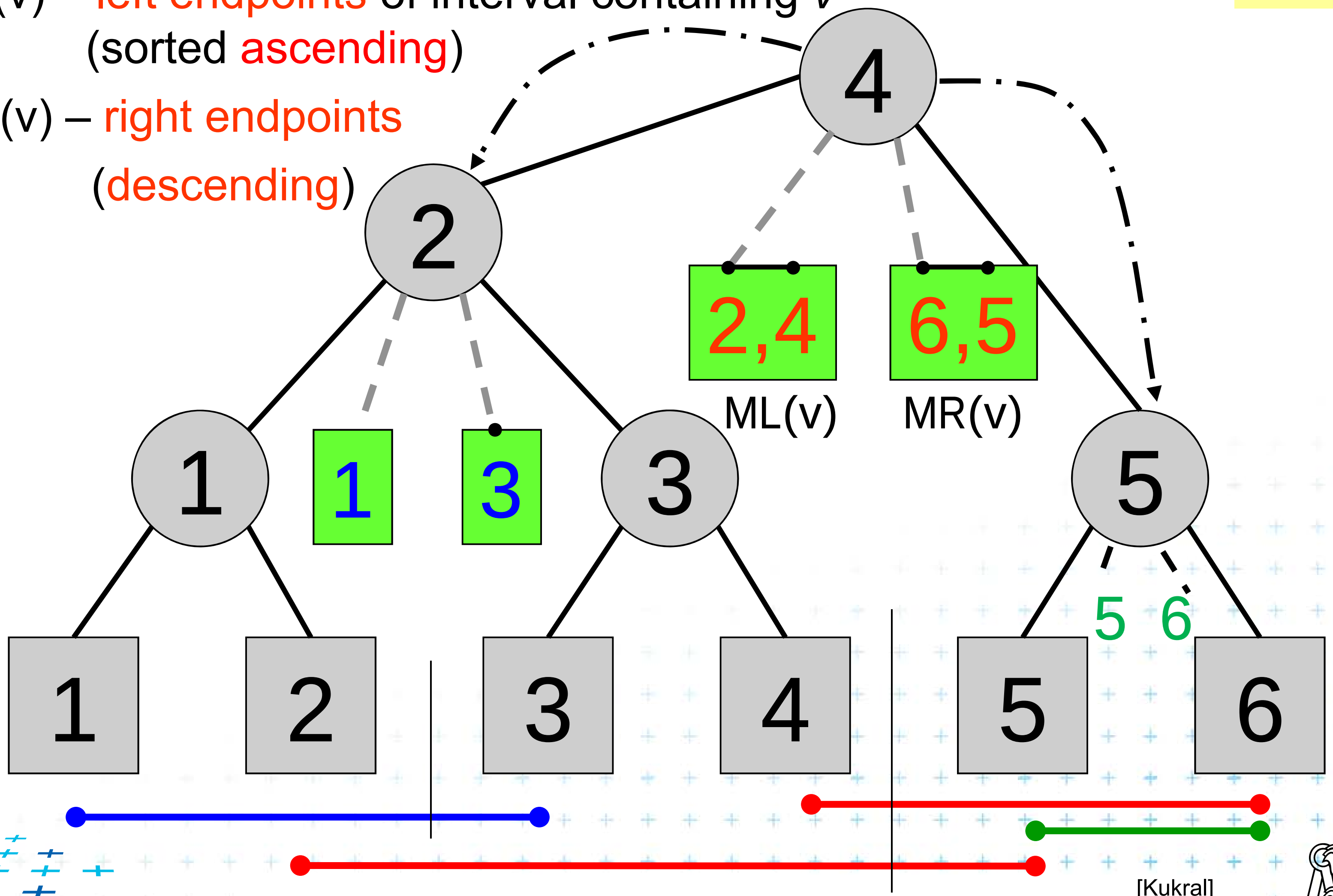


Secondary lists of incident interval end-pts.

Dynamic

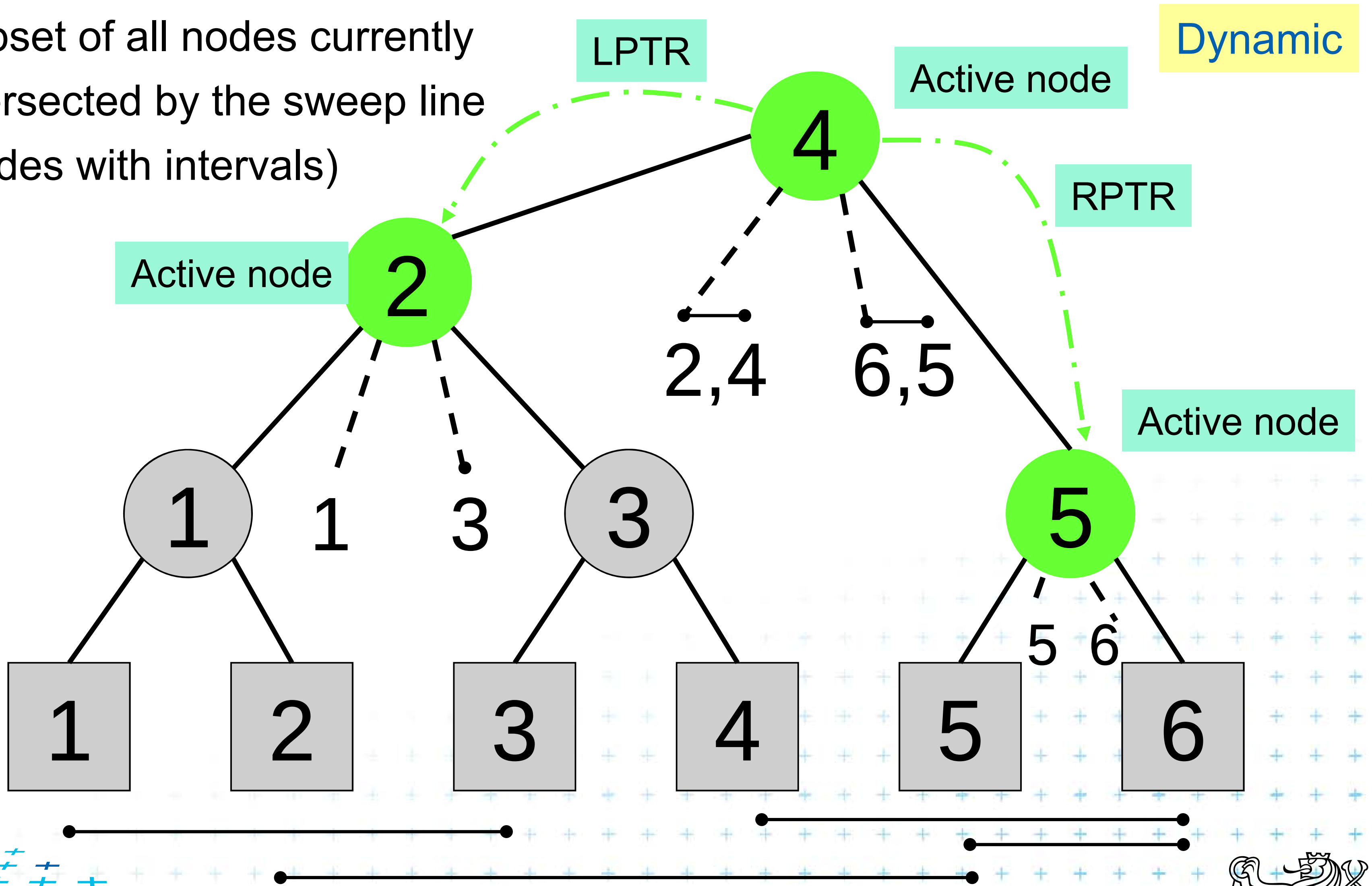
ML(v) – left endpoints of interval containing *v*
(sorted ascending)

MR(v) – right endpoints
(descending)



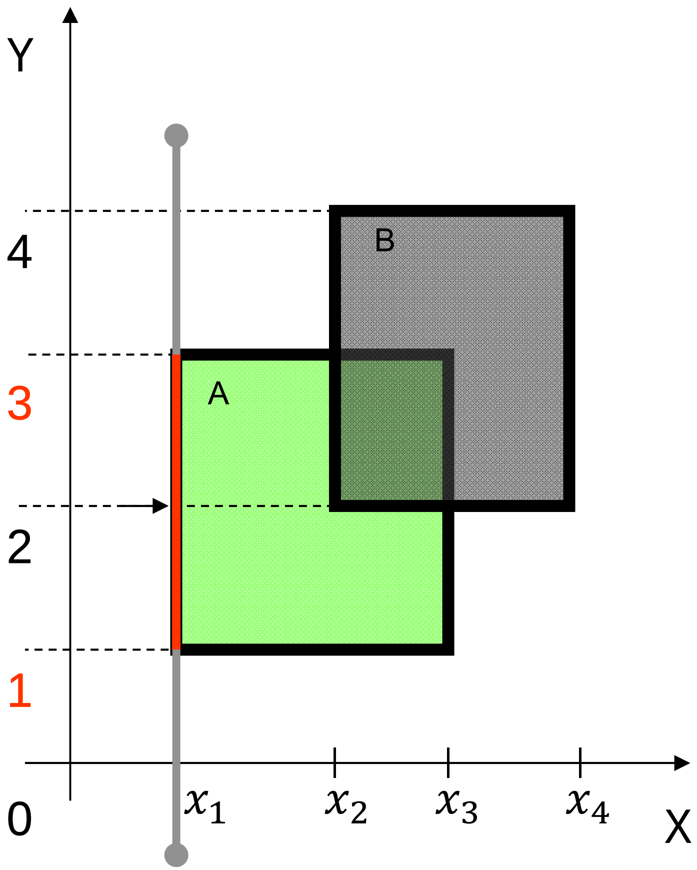
Active nodes – intersected by the sweep line

Subset of all nodes currently intersected by the sweep line (nodes with intervals)



[Kukral]

Entries in the event queue



(x_i, y_{il}, y_{ir}, t)

$(x_1, 1, 3, left)$

$(x_2, 2, 4, left)$

$(x_3, 1, 3, right)$

$(x_4, 2, 4, right)$

Nodes in the SL status tree

1,2,3,4

Query = sweep and report intersections

Rectangle**Intersections****(*S*)**
Input: Set *S* of rectangles
Output: Intersected rectangle pairs

1. Preprocess(S) // create the interval tree *T* (for y-coords)
   // and event queue *Q* (for x-coords)
2. **while** (*Q* $\neq \emptyset$) do
3. Get next entry (x_i, y_{il}, y_{ir}, t) from *Q* // $t \in \{$ *left* | *right* $\}$
4. **if** (t = left) // left edge
5. a) QueryInterval (y_{il}, y_{ir}, root(T)) // report intersections
6. b) InsertInterval (y_{il}, y_{ir}, root(T)) // insert new interval
7. **else** // right edge
8. c) DeleteInterval (y_{il}, y_{ir}, root(*T*))

Preprocessing

Preprocess(S)
Input: Set *S* of rectangles
Output: Primary structure of the interval tree *T* and the event queue *Q*

1. *T* = PrimaryTree(*S*) // Construct the static primary structure // of the interval tree -> sweep line STATUS *T*

2. // Init event queue Q with vertical rectangle edges in ascending order ~x
   // Put the left edges with the same x ahead of right ones
3. for i = 1 to n
4. insert((x_{il} , y_{il} , y_{ir} , left), Q) // left edges of *i-th* rectangle
5. insert((x_{ir}, y_{il} , y_{ir} , right), Q) // right edges

Interval tree – primary structure construction

PrimaryTree(S) // only the y-tree structure, without intervals

Input: Set S of rectangles

Output: Primary structure of an interval tree T

1. S_y = Sort endpoints of all segments in S according to y-coordinate
2. T = BST(S_y)
3. **return** T

BST(S_y)

1. **if**($|S_y| = 0$) **return** null
2. *yMed = median of* S_y // the smaller item for even S_y.size
3. L = endpoints $p_y \leq yMed$
4. R = endpoints $p_y > yMed$
5. t = *new* IntervalTreeNode(*yMed*)
6. *t.left* = BST(L)
7. *t.right* = BST(R)
8. **return** t

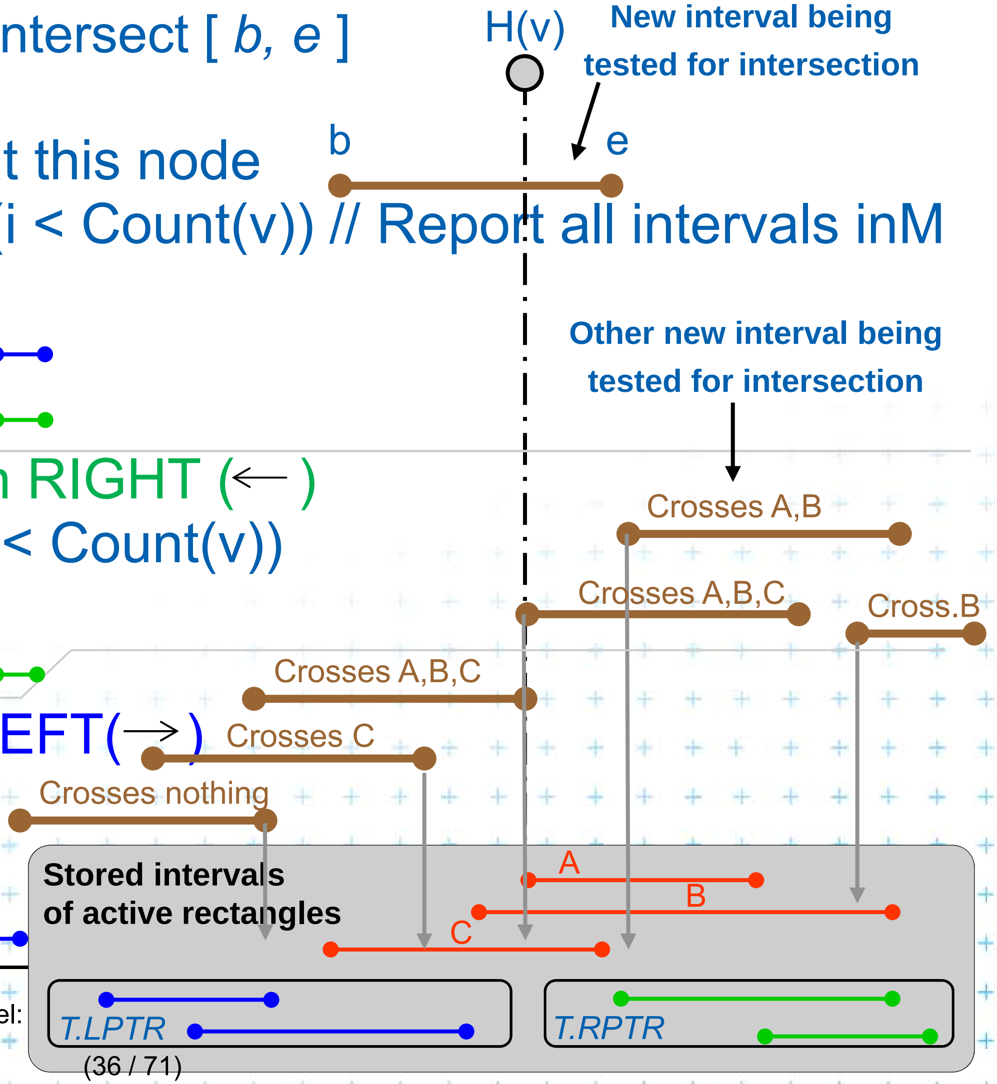
Interval tree – search the intersections

QueryInterval (*b, e, T*)

Input: Interval of the edge and current tree *T*

Output: Report the rectangles that intersect [*b, e*]

```
if( T =  null ) return
i=0; if( b < H(v) < e )  // forks at this node
    while ~~( MR(v).[i] >= b ) &&~~ (i < Count(v)) // Report all intervals inM
        ReportIntersection; i++
    QueryInterval( b,e,T.LPTR )
    QueryInterval( b,e,T.RPTR )
else if (H(v) ≤ b < e)  // search RIGHT (← )
    while (MR(v).[i] >= b) && (i < Count(v))
        ReportIntersection; i++
    QueryInterval( b,e,T.RPTR )
else  // b < e ≤ H(v) //search LEFT(→ )
    while (ML(v).[i] <= e)
        ReportIntersection; i++
    QueryInterval( b,e,T.LPTR )
```



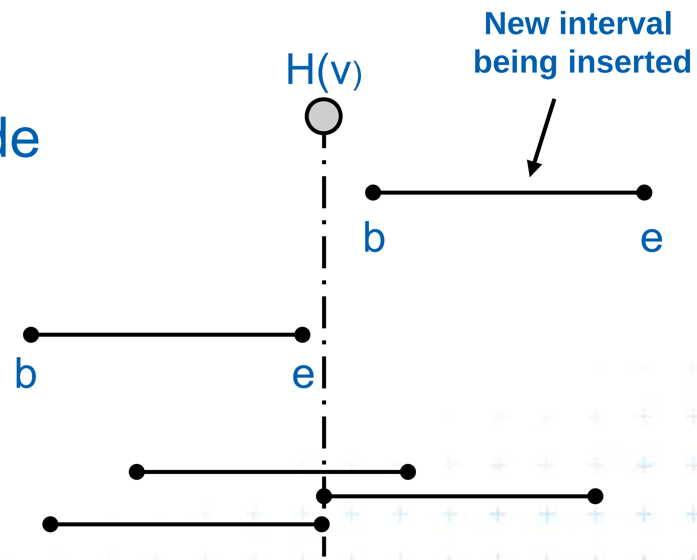
Interval tree - interval insertion

InsertInterval (*b, e, T*)

Input: Interval [*b*,*e*] and interval tree *T*

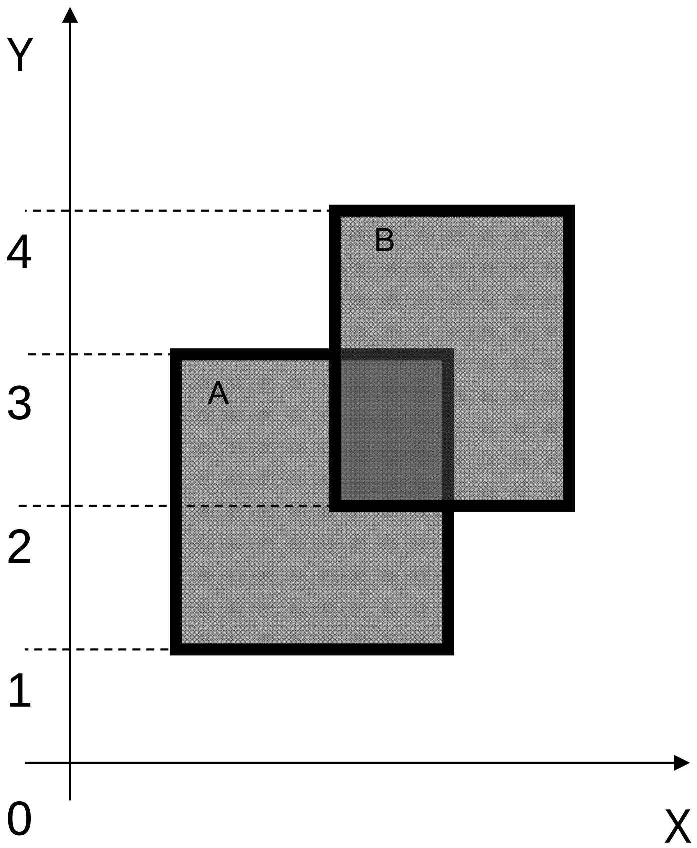
Output: *T* after insertion of the interval

```
v = root(T )
while( v != null )        // find the fork node
    if (H(v) < b < e)
        v = v.right       // continue right
    else if (b < e < H(v))
        v = v.left        // continue left
    else  // b ≤ H(v) ≤ e  // insert interval
        set v node to active
        connect LPTR resp. RPTR to its parent
        insert [b,e] into list ML(v) – sorted in ascending order of b's
        insert [b,e] into list MR(v) – sorted in descending order of e's
        break
endwhile
return T
```

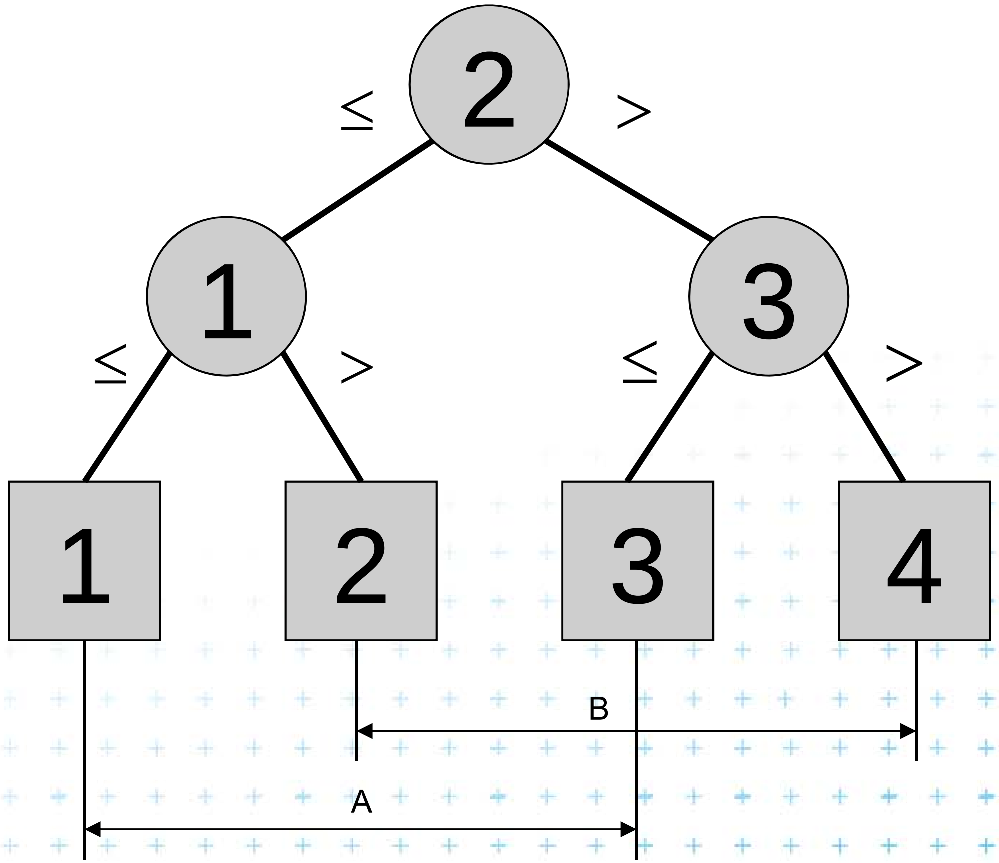


Example 1

Example 1 – static tree on endpoints

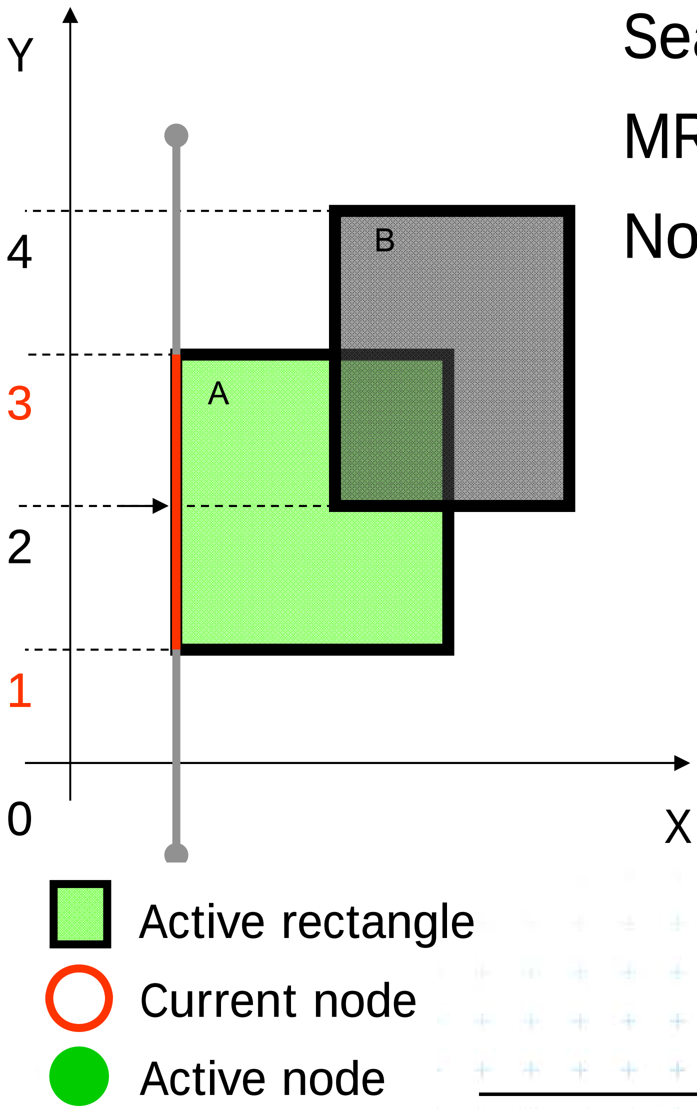


H(v) – value of node *v*



[Drtina]

Interval insertion [1,3]   a) Query Interval

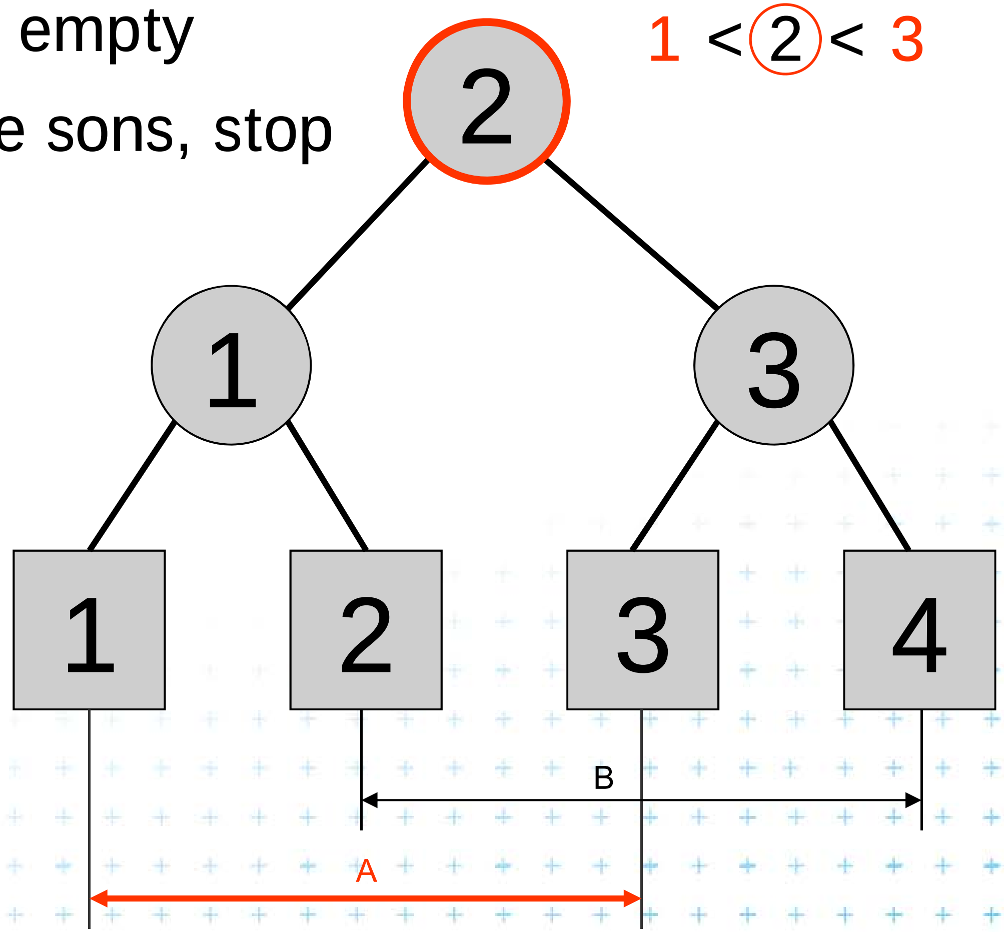


Search MR(v) or ML(v): ← b < H(v) < e

MR(v) is empty

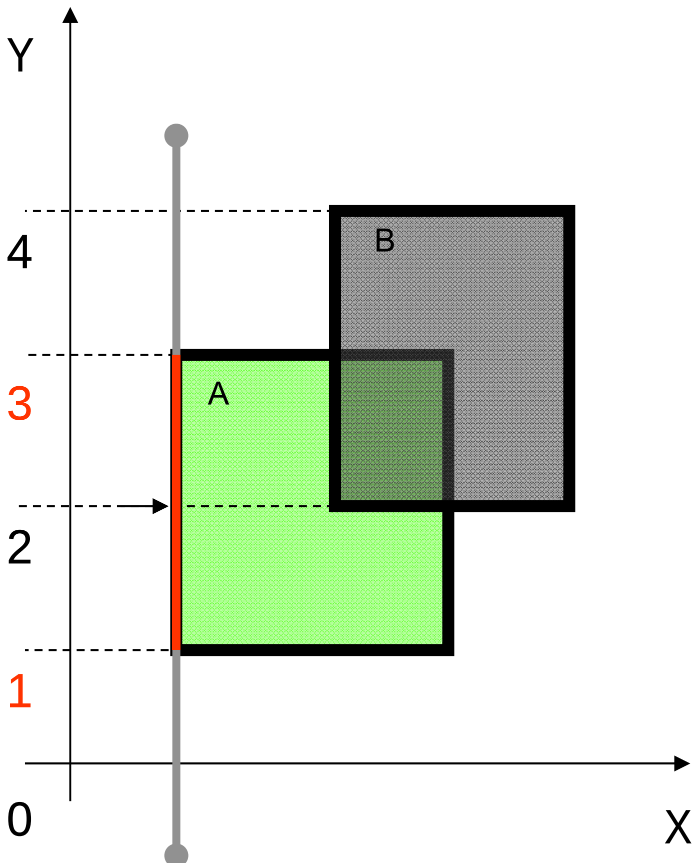
No active sons, stop

1 < 2 < 3



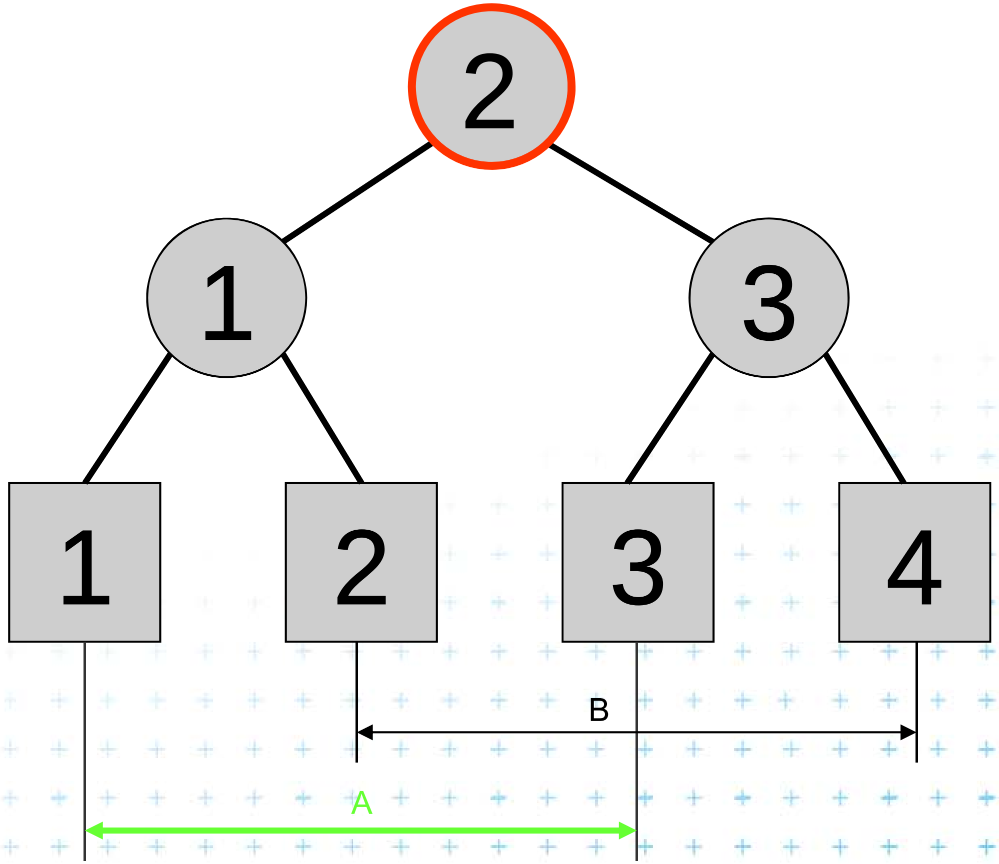
[Drtina]

Interval insertion [1,3]   b) Insert Interval



$$b \leq H(v) \leq e$$

? 1 ≤ 2 ≤ 3 ?



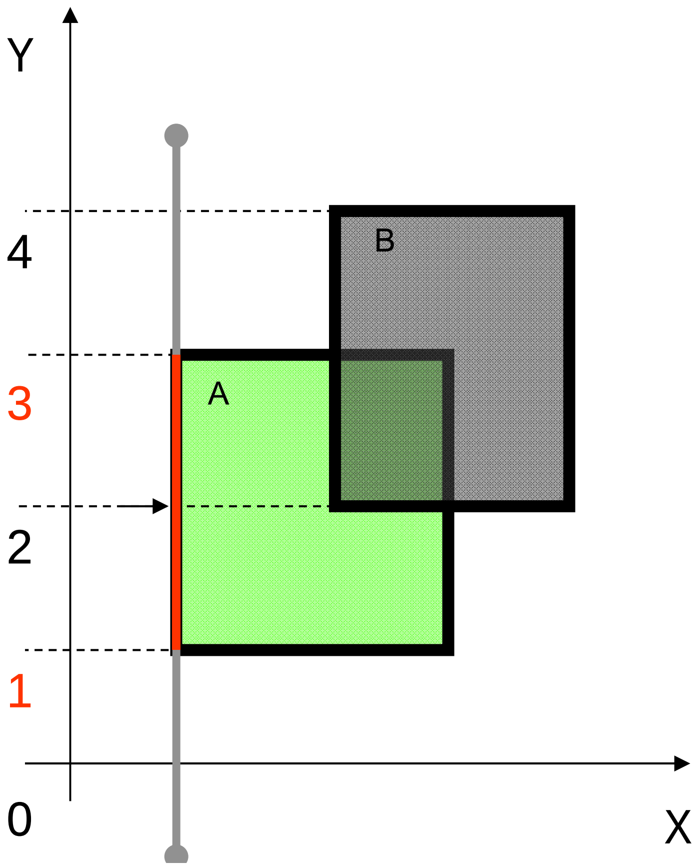
Active rectangle

Current node

Active node

[Drtina]

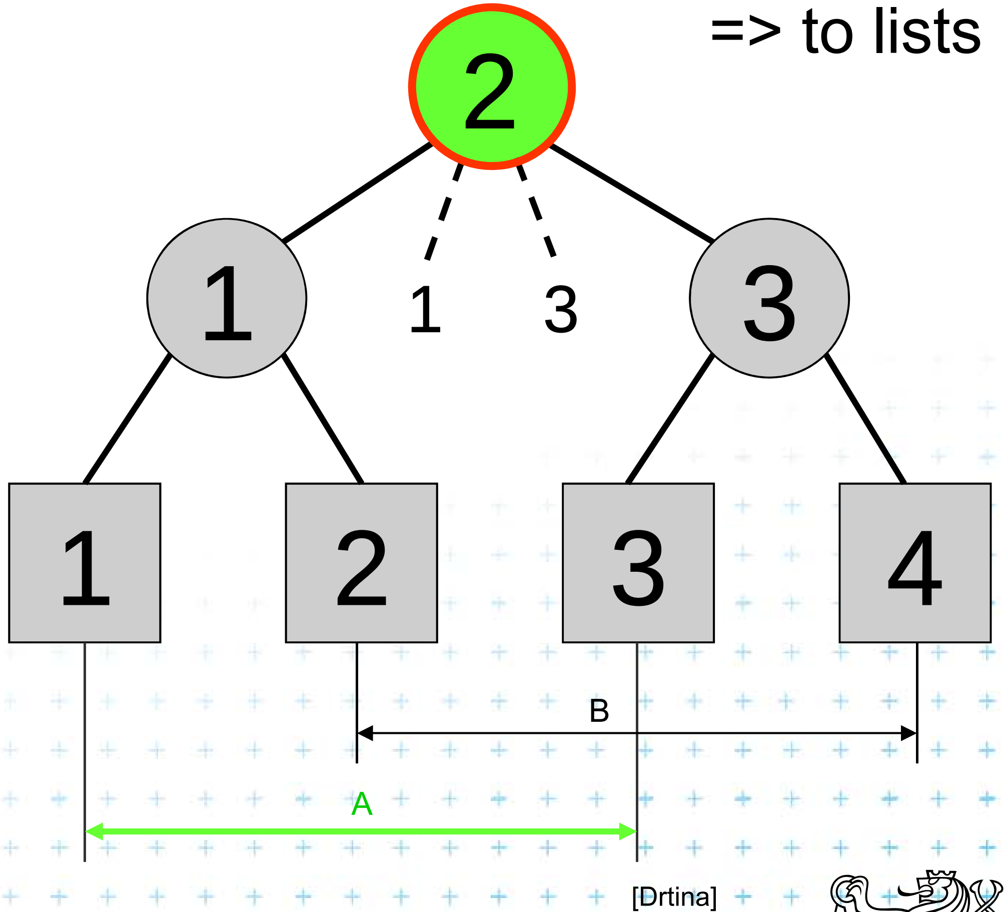
Interval insertion [1,3] b) Insert Interval



$b \leq H(v) \leq e$

$1 \leq 2 \leq 3$

fork
=> to lists



- Active rectangle
- Current node
- Active node

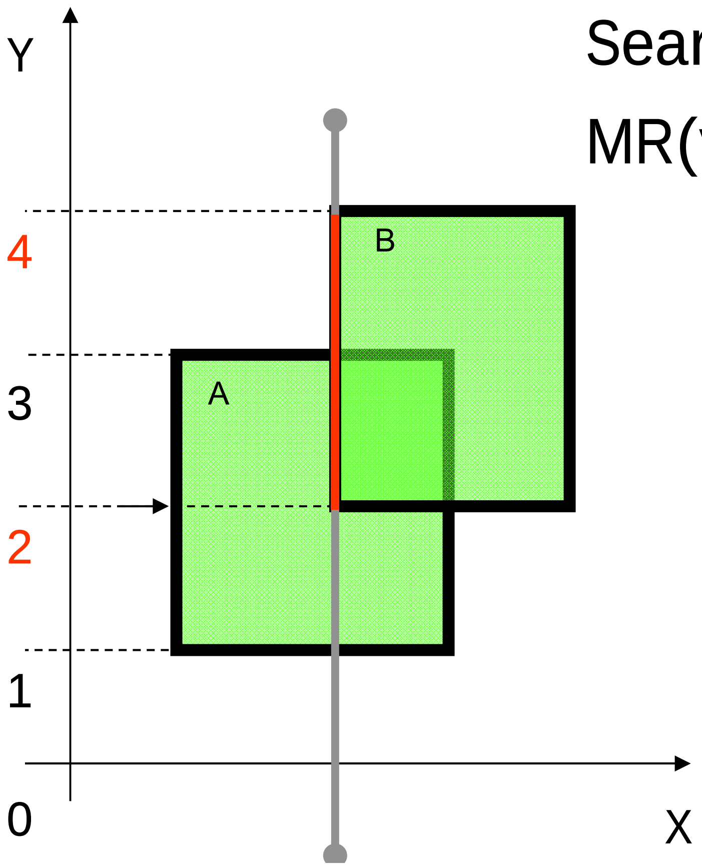
Interval insertion [2,4] a) Query Interval

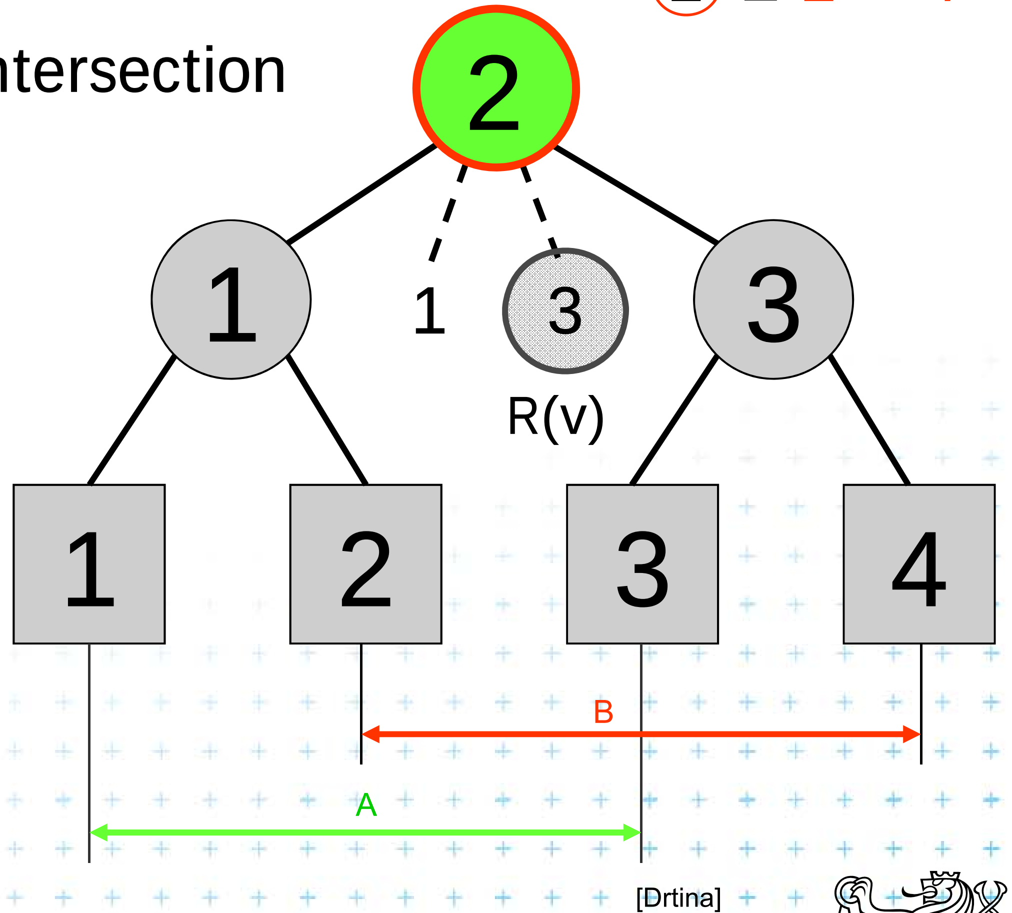
Search MR(v) only: ← H(v) ≤ b < e

MR(v)[1] = 3 ≥ 2?

=> intersection

② ≤ 2 < 4



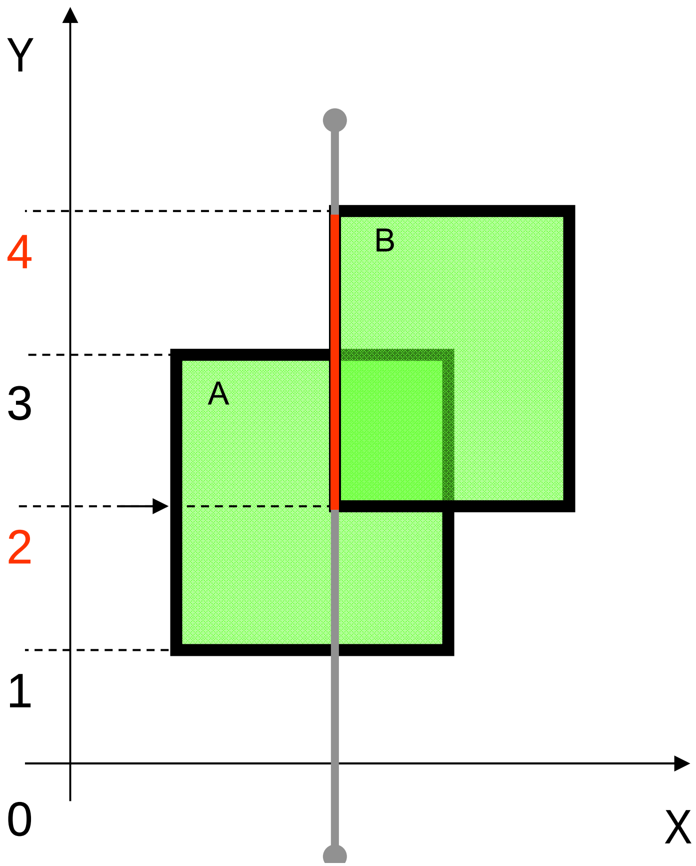


Active rectangle

Current node

Active node

Interval insertion [2,4]   b) Insert Interval



Active rectangle

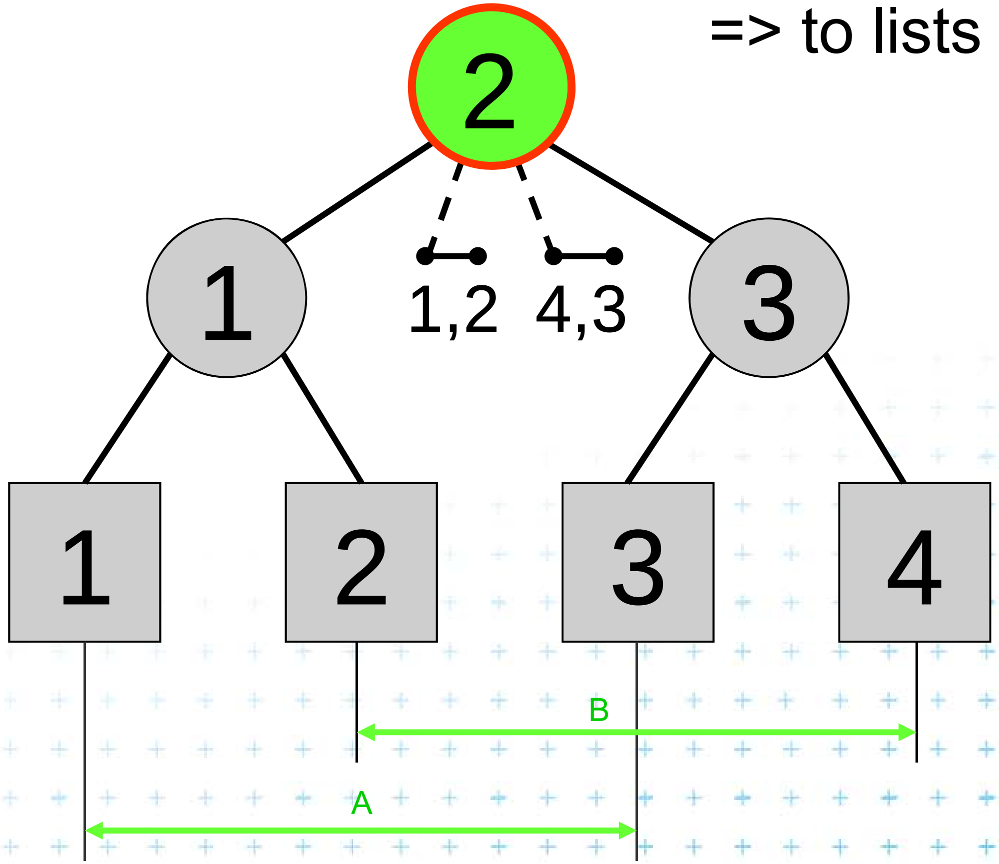
Current node

Active node

$b \le H(v) \le e$

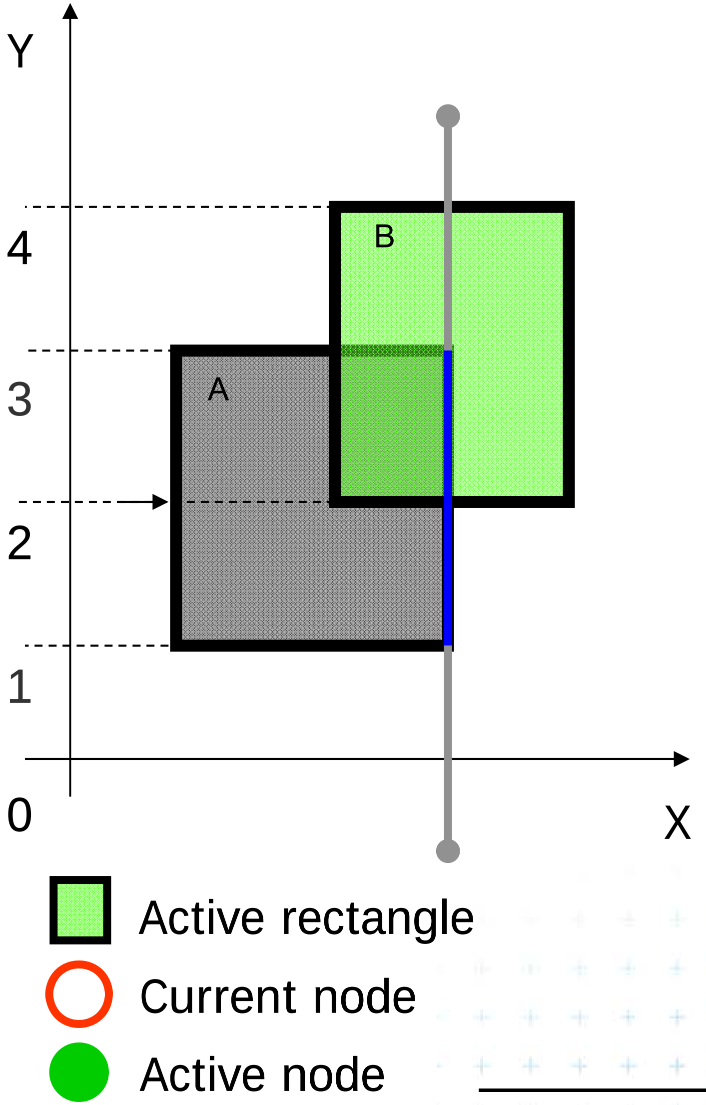
$2 \le 2 \le 4$

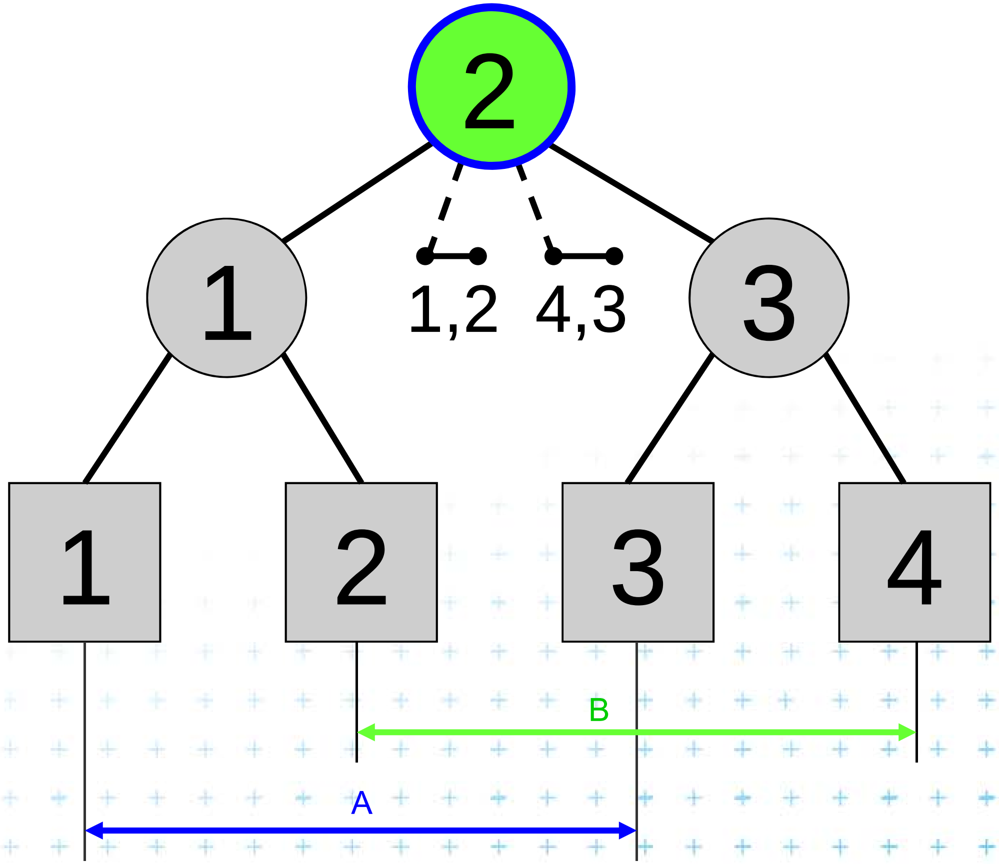
fork
=> to lists



[Drtina]

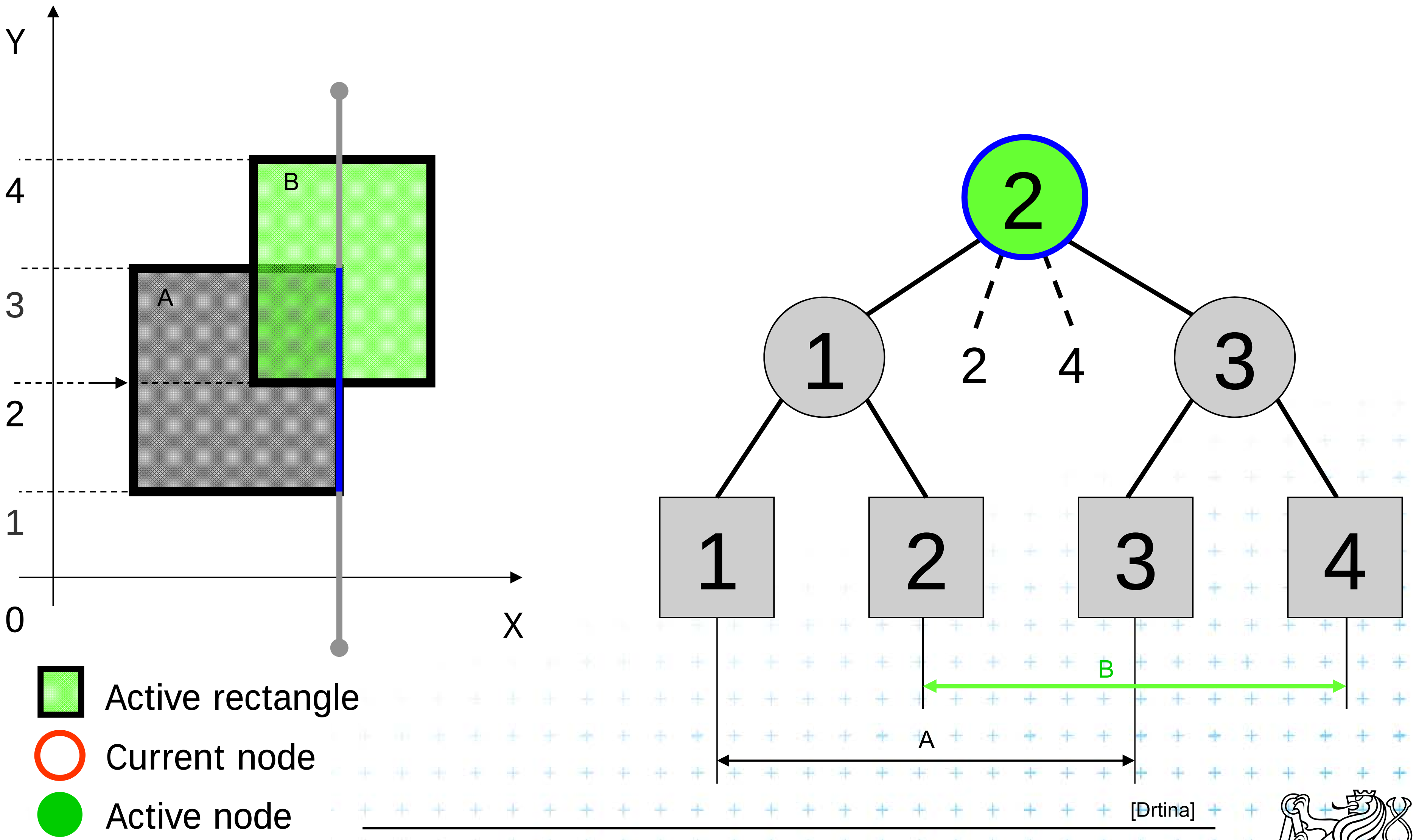
Interval delete [1,3]





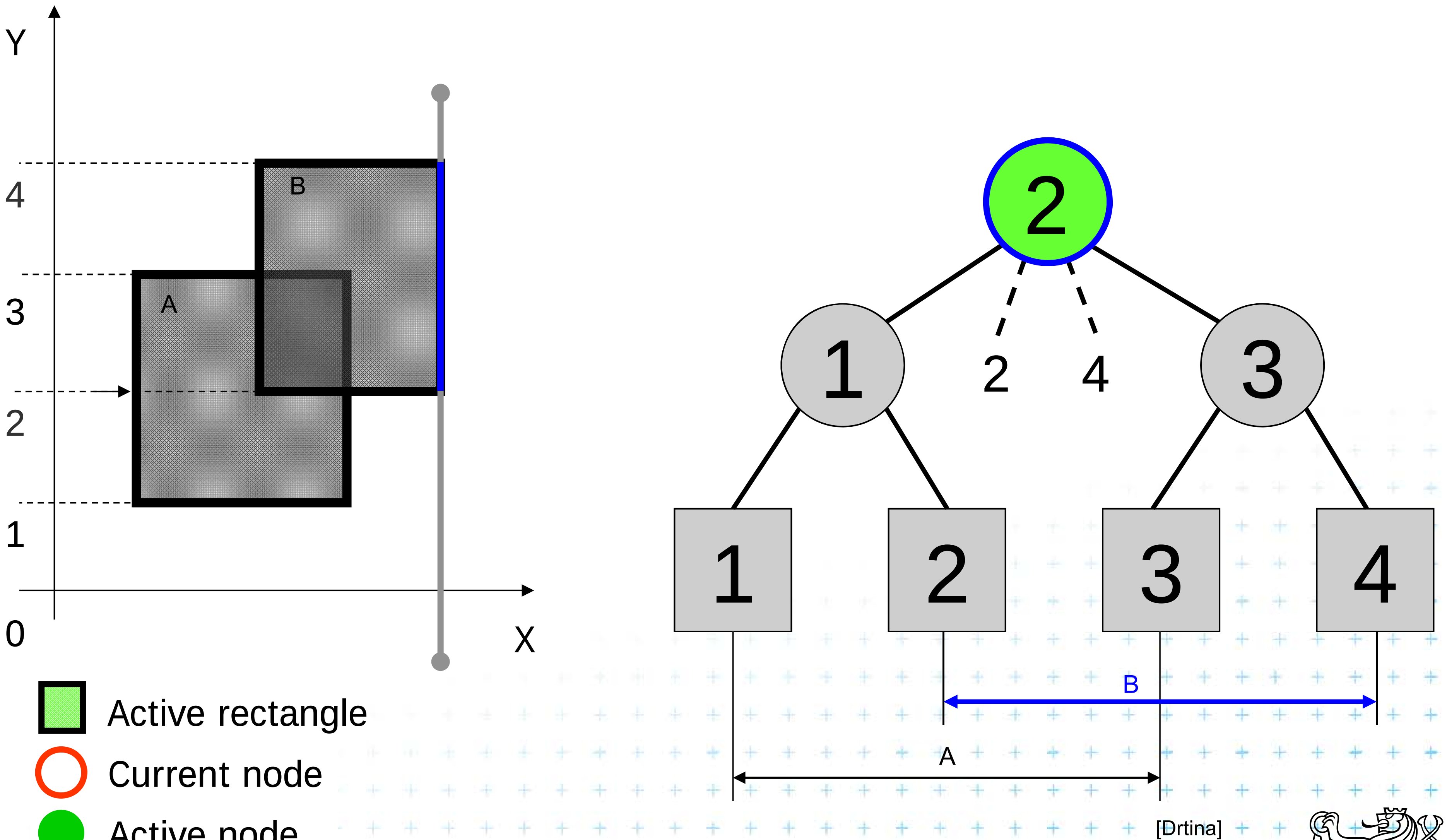
[Drtina]

Interval delete [1,3]



[Drtina]

Interval delete [2,4]

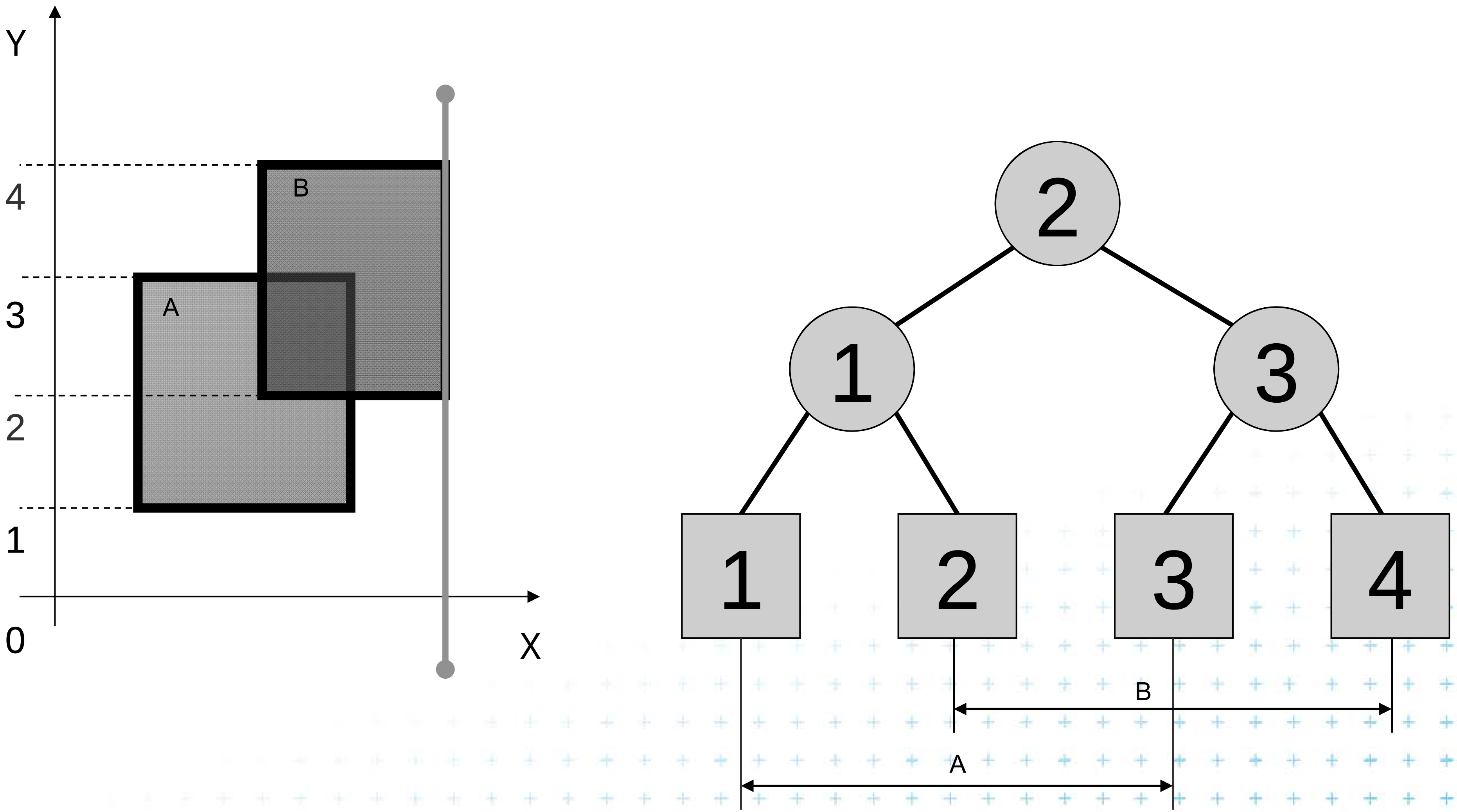


Active rectangle

Current node

Active node

Interval delete [2,4]



[Drtina]

Example 2

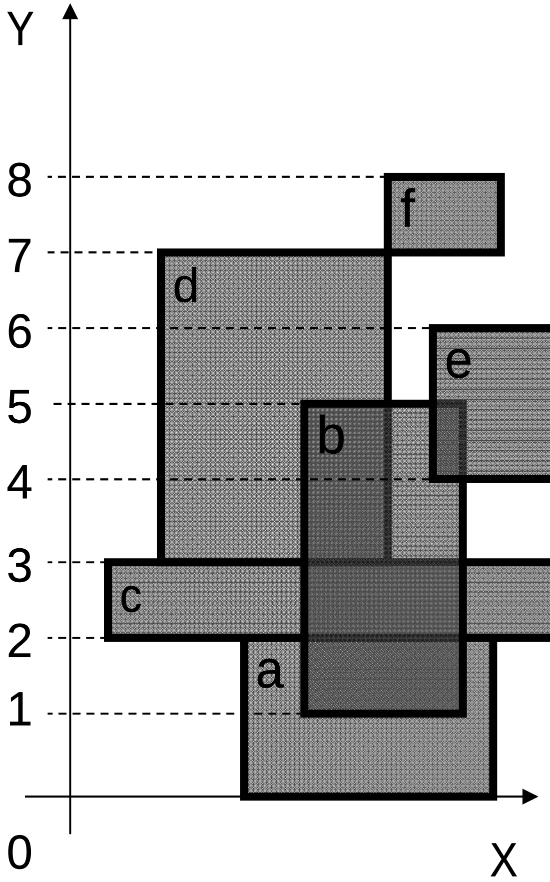
RectangleIntersections(*S*)
Input: Set *S* of rectangles
Output: Intersected rectangle pairs

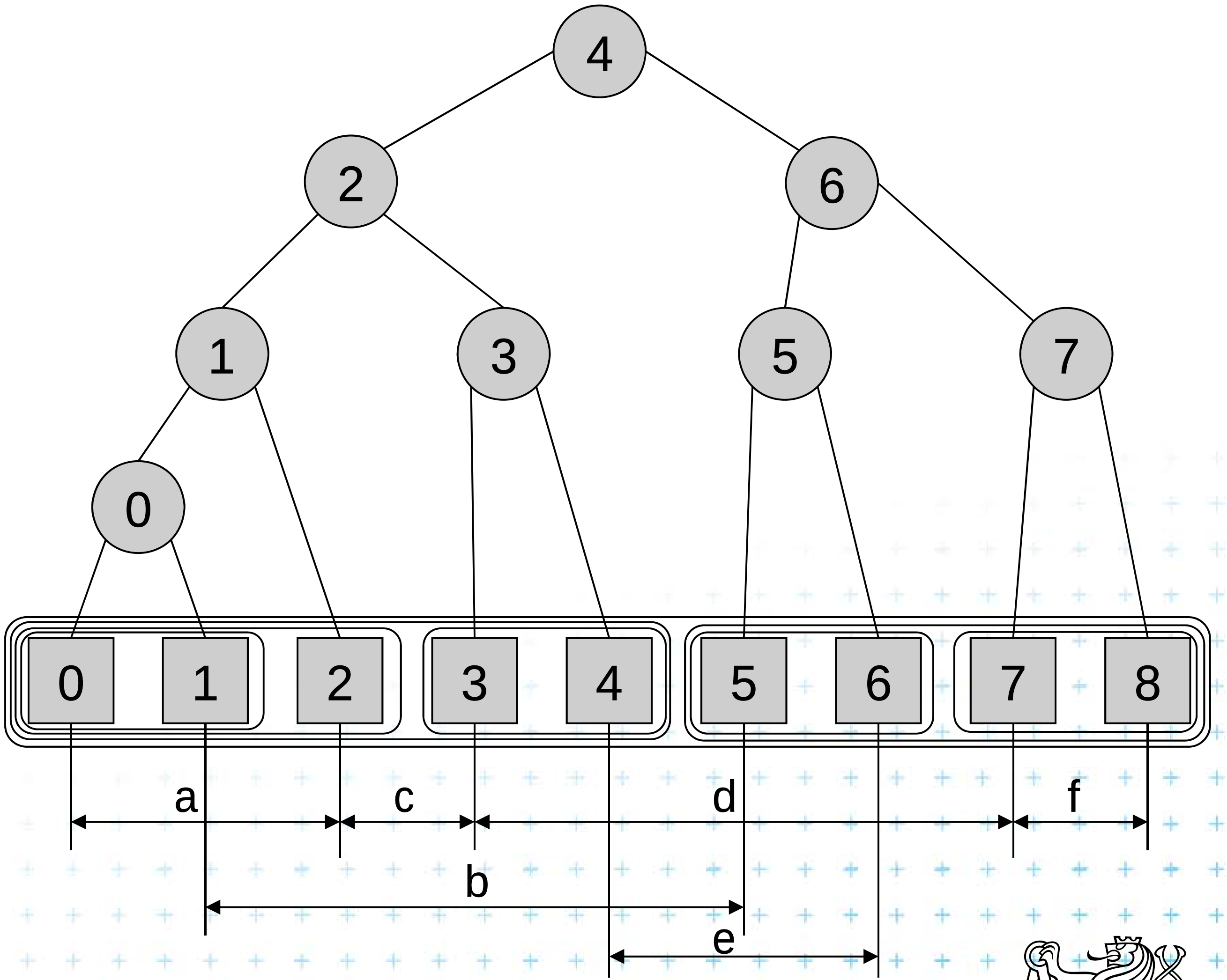
// this is a copy of the slide before
// just to remember the algorithm

1. Preprocess(S) // create the interval tree *T* and event queue *Q*

2. **while** ($Q \neq \emptyset$) do
3. Get next entry ($x_{il}, y_{il}, y_{ir}, t$) from *Q* // $t \in \{ left \mid right \}$
4. **if** (t = left) // left edge
5. a) QueryInterval ($y_{il}, y_{ir}, \mathrm{root}(T)$) // report intersections
6. b) InsertInterval ($y_{il}, y_{ir}, \mathrm{root}(T)$) // insert new interval
7. **else** // right edge
8. c) DeleteInterval (y_{il}, y_{ir}, root(*T*))

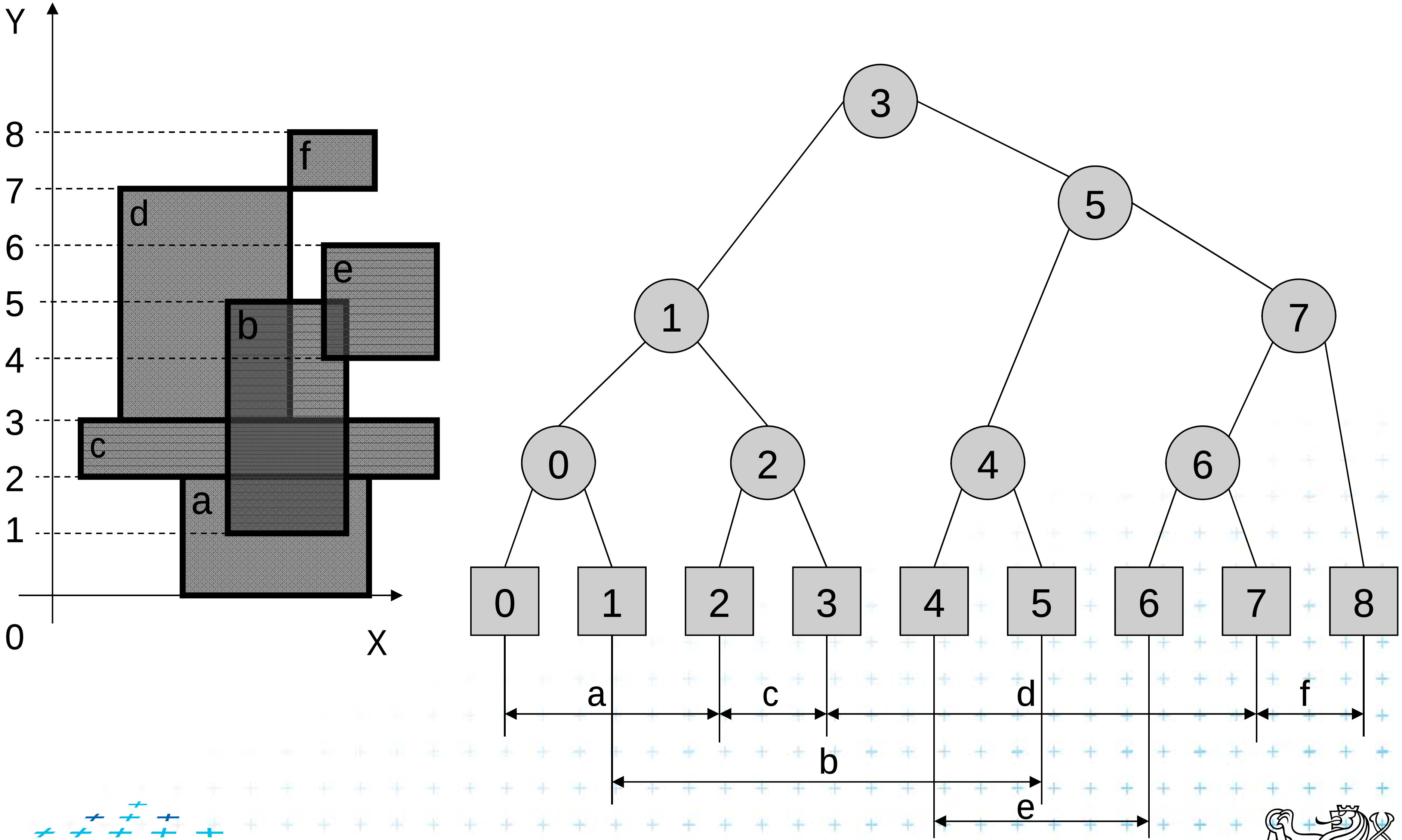
Example 2 – tree from PrimaryTree(*S*)





[Drtina]

Example 2 – slightly unbalanced tree



[Drtina]

Insert [2,3] – empty => b) Insert Interval

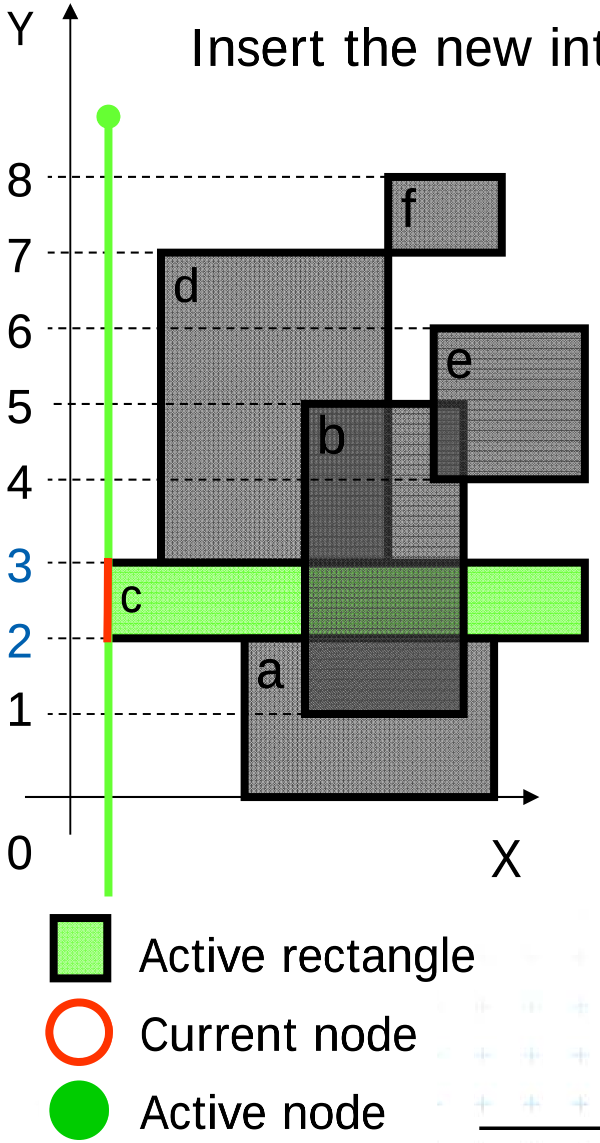
b ≤ H(v) ≤ e

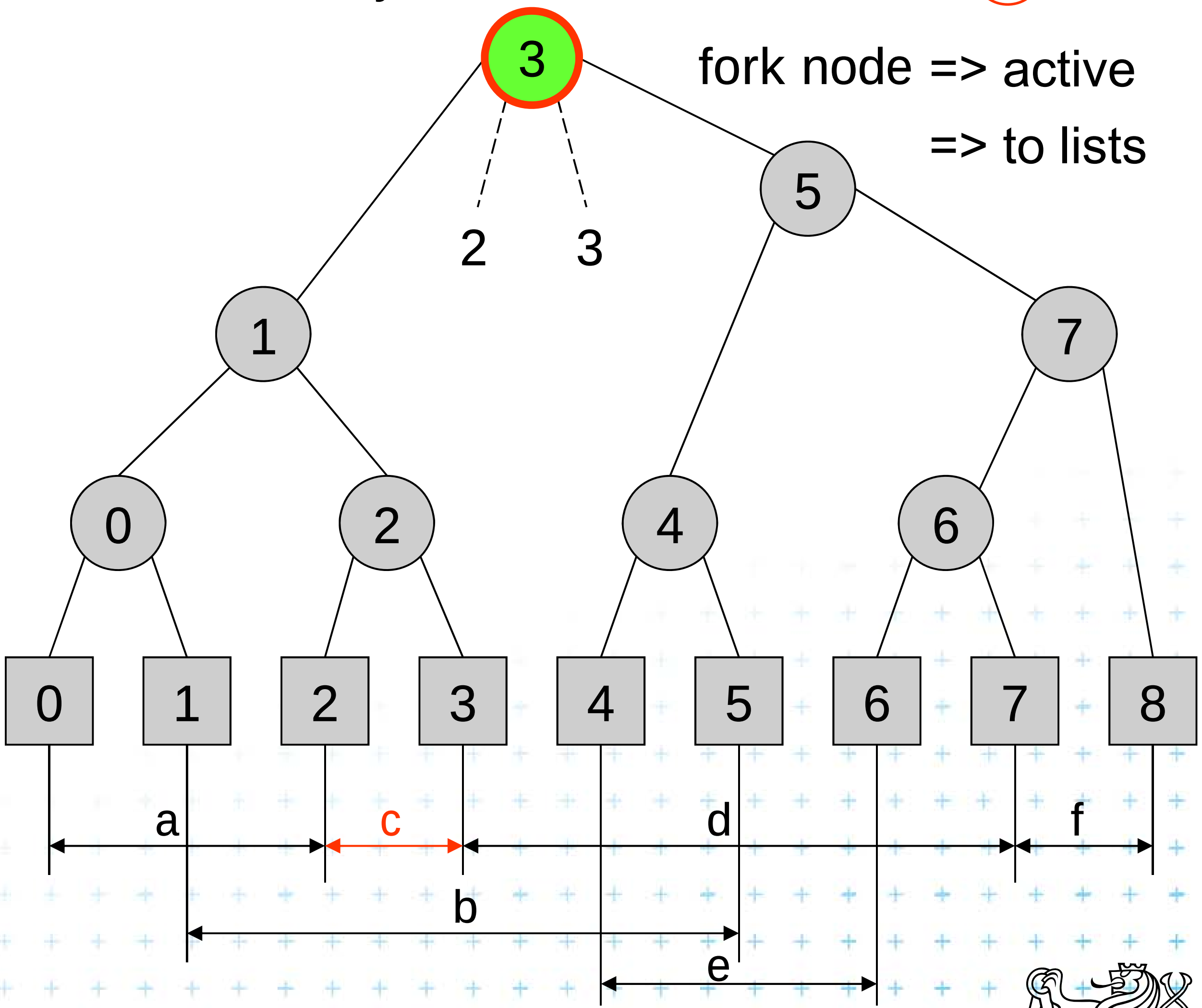
Insert the new interval to secondary lists

? 2 ≤ 3 ≤ 3 ?

fork node => active

=> to lists





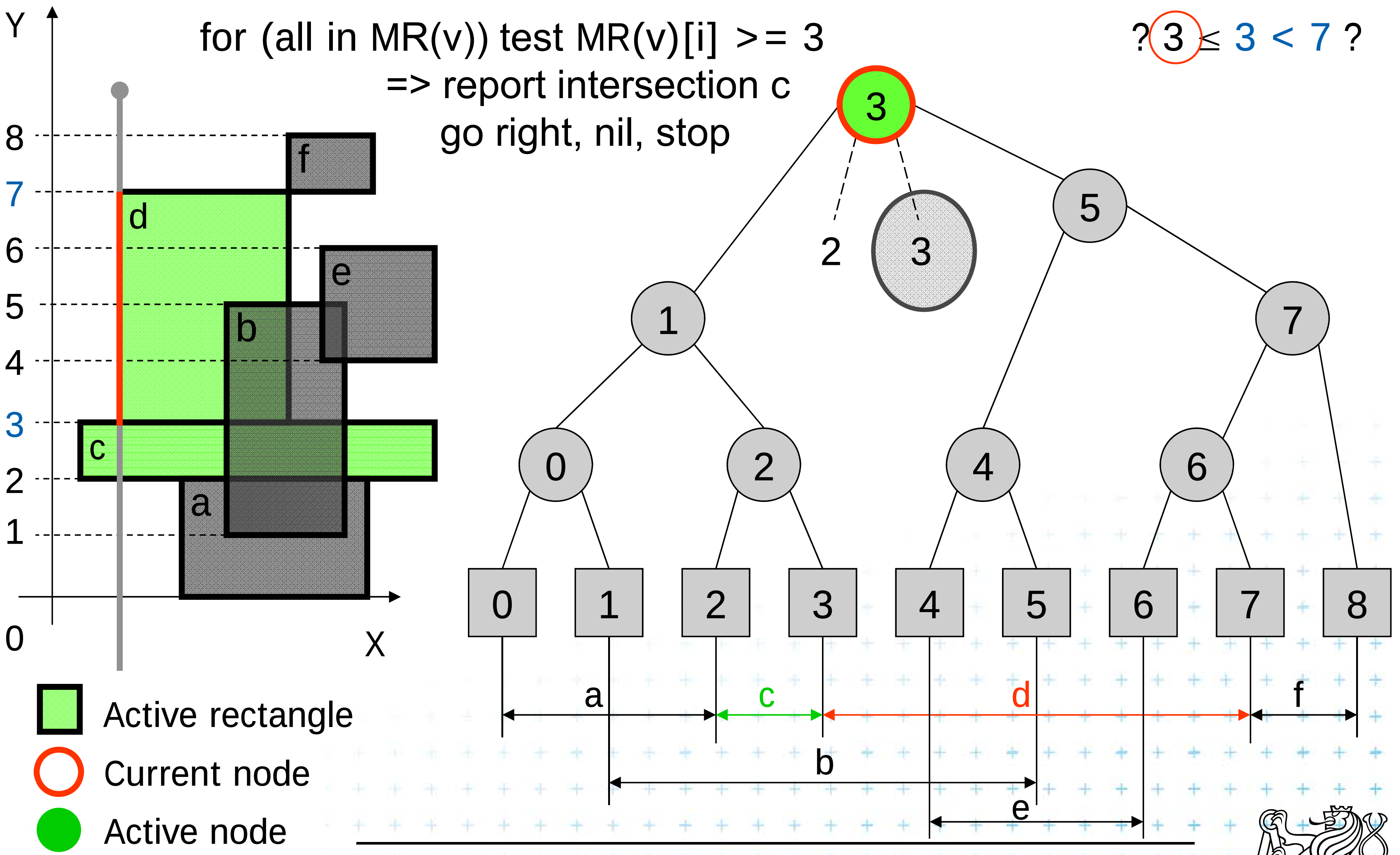
[Drtina]

Insert [3,7] a) Query Interval

$H(v) \le b < e$

for (all in MR(v)) test MR(v)[i] >= 3
=> report intersection c
go right, nil, stop

? 3 ≤ 3 < 7 ?



[Drtina]

Insert [3,7] b) Insert Interval

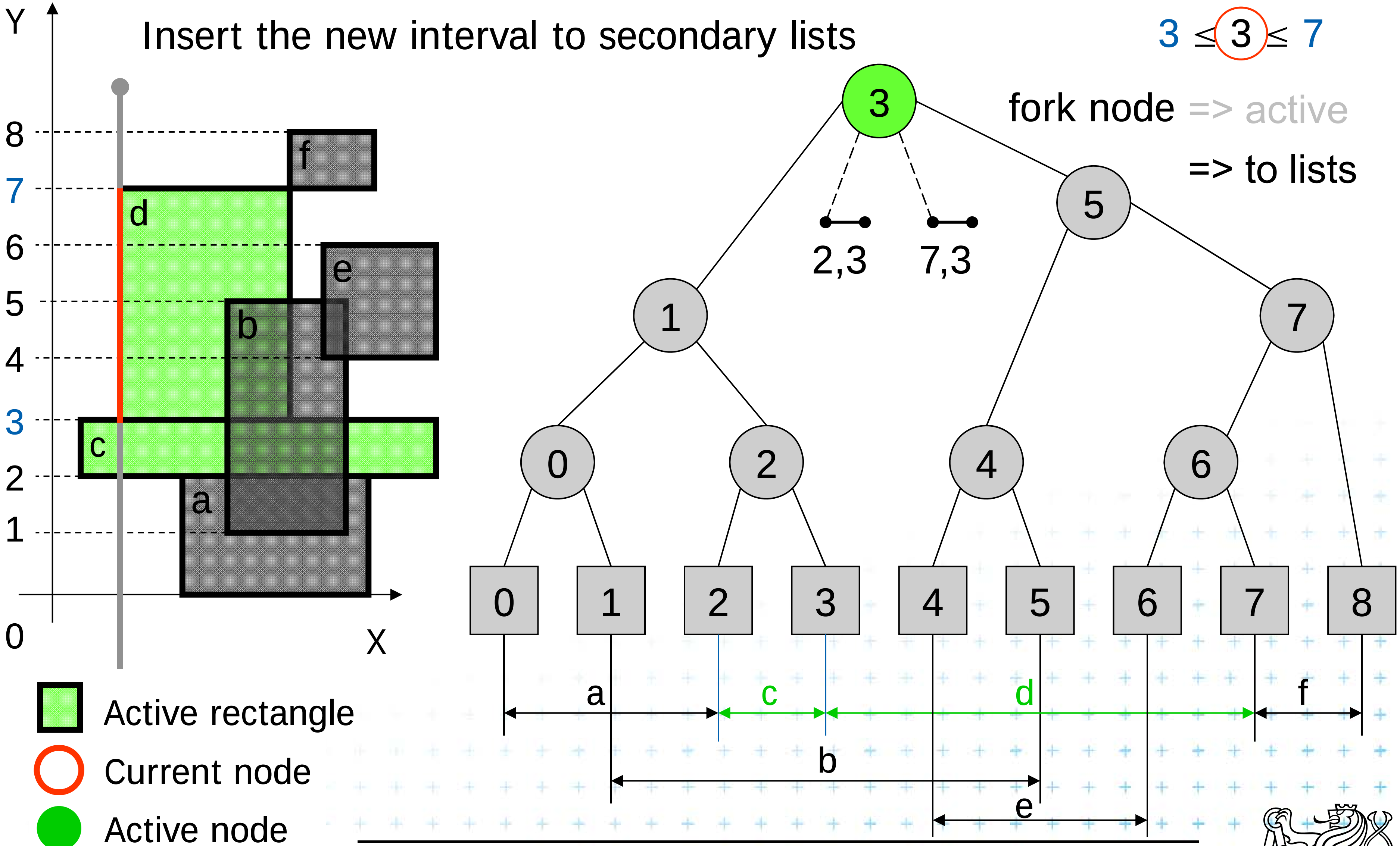
$b \leq H(v) \leq e$

Insert the new interval to secondary lists

$3 \leq 3 \leq 7$

fork node => active

=> to lists



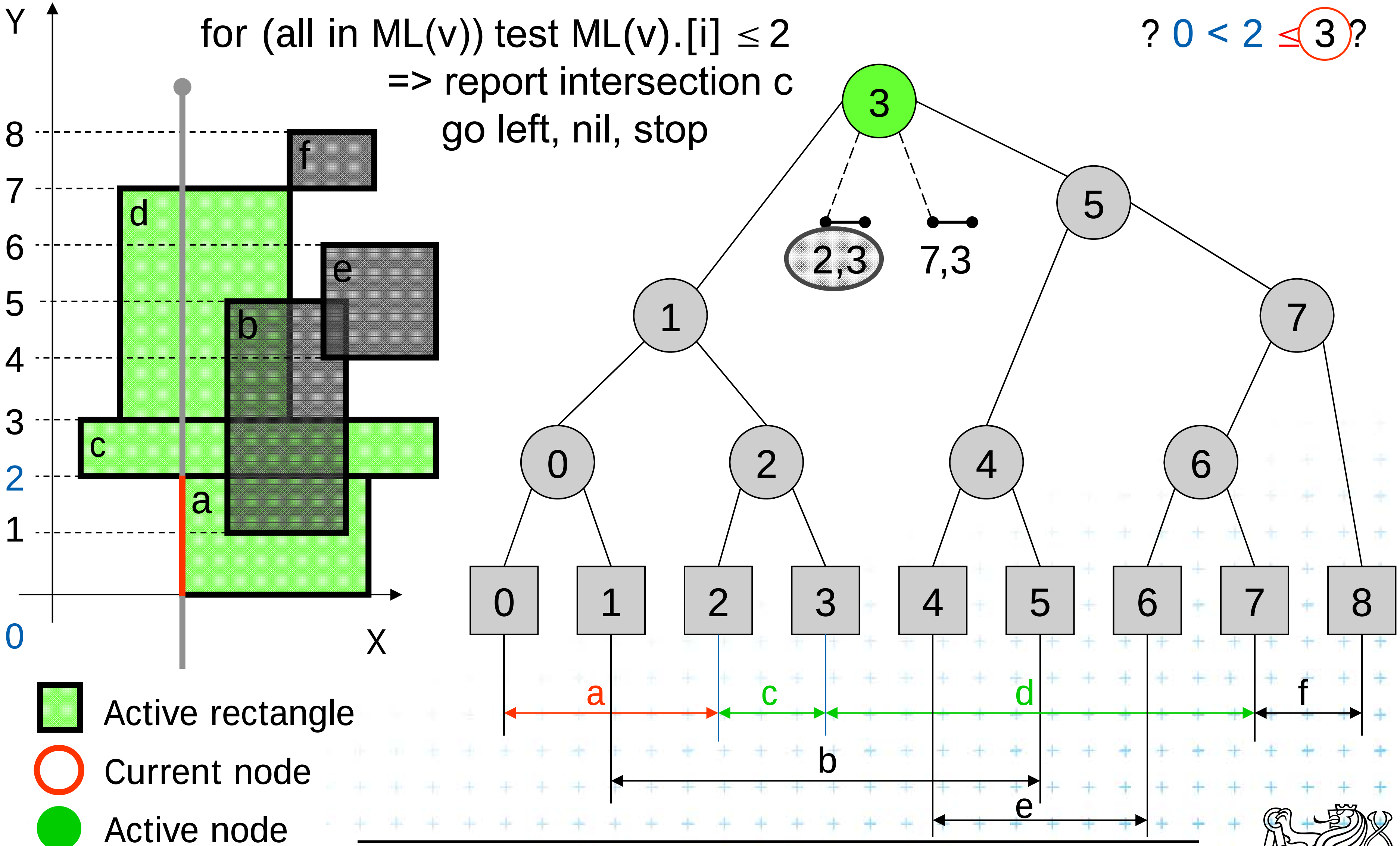
[Drtina]

Insert [0,2] a) Query Interval

b < e ≤ H(v)

for (all in ML(v)) test ML(v).[i] ≤ 2
=> report intersection c
go left, nil, stop

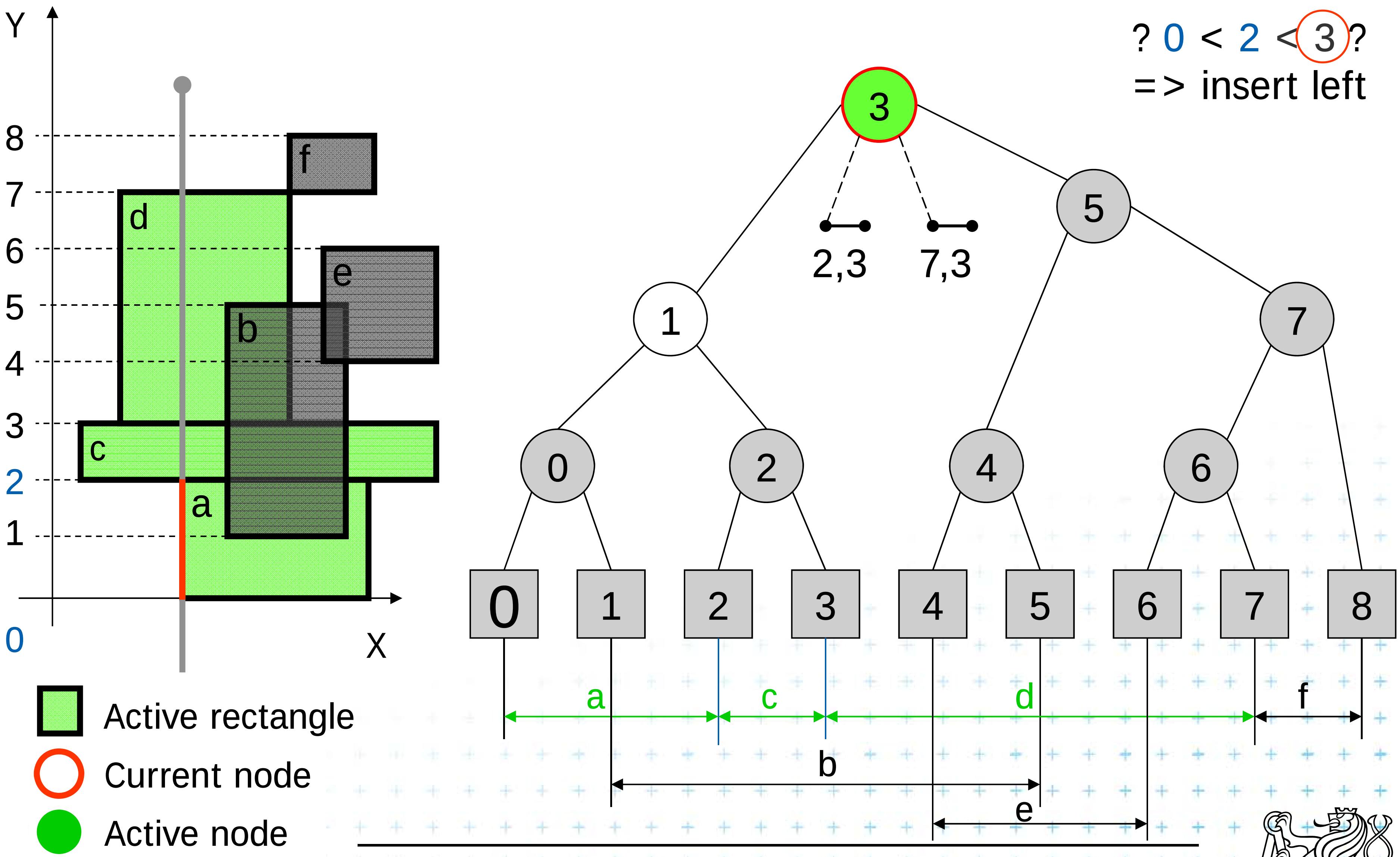
? 0 < 2 ≤ 3 ?



[Drtina]

Insert [0,2] b) Insert Interval 1/2

b < e <H(v)



[Drtina]

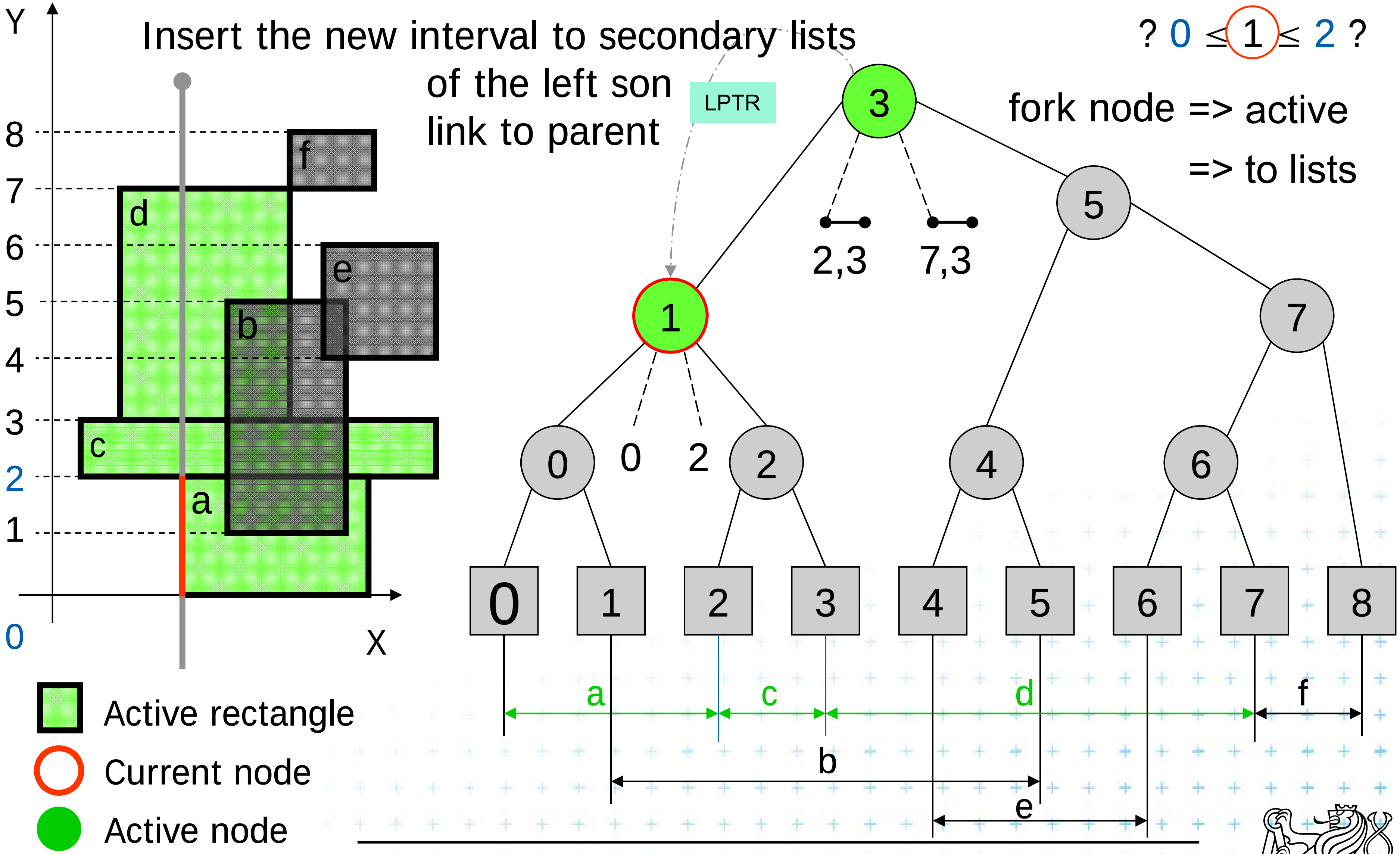
Insert [0,2] b) Insert Interval 2/2

b ≤ H(v) ≤ e

? 0 ≤ 1 ≤ 2 ?

Insert the new interval to secondary lists of the left son link to parent

fork node => active => to lists

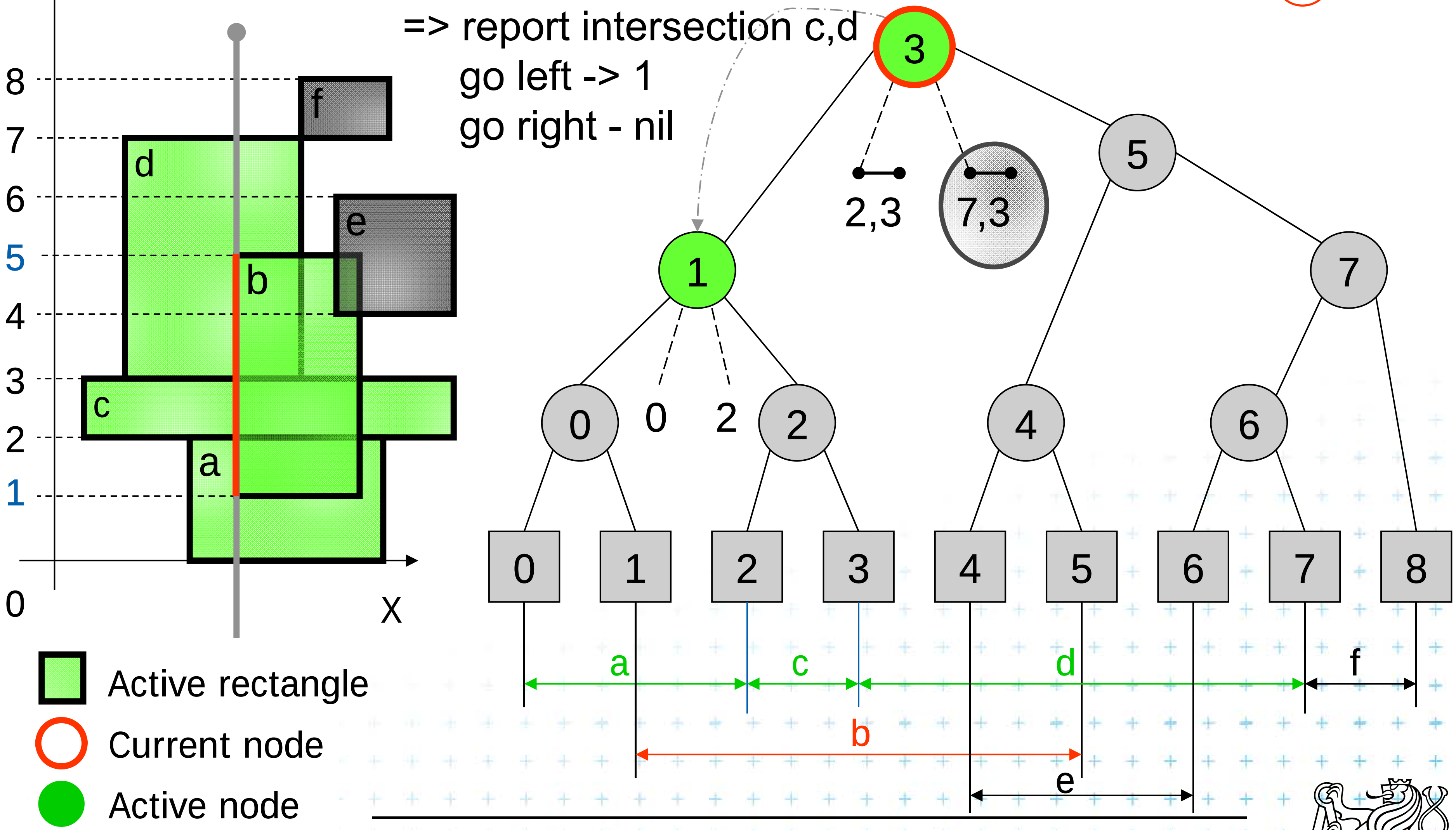


Insert [1,5] a) Query Interval 1/2

b < H(v) < e

for (all in MR(v))
=> report intersection c,d
go left -> 1
go right - nil

? 1 < 3 < 5 ?



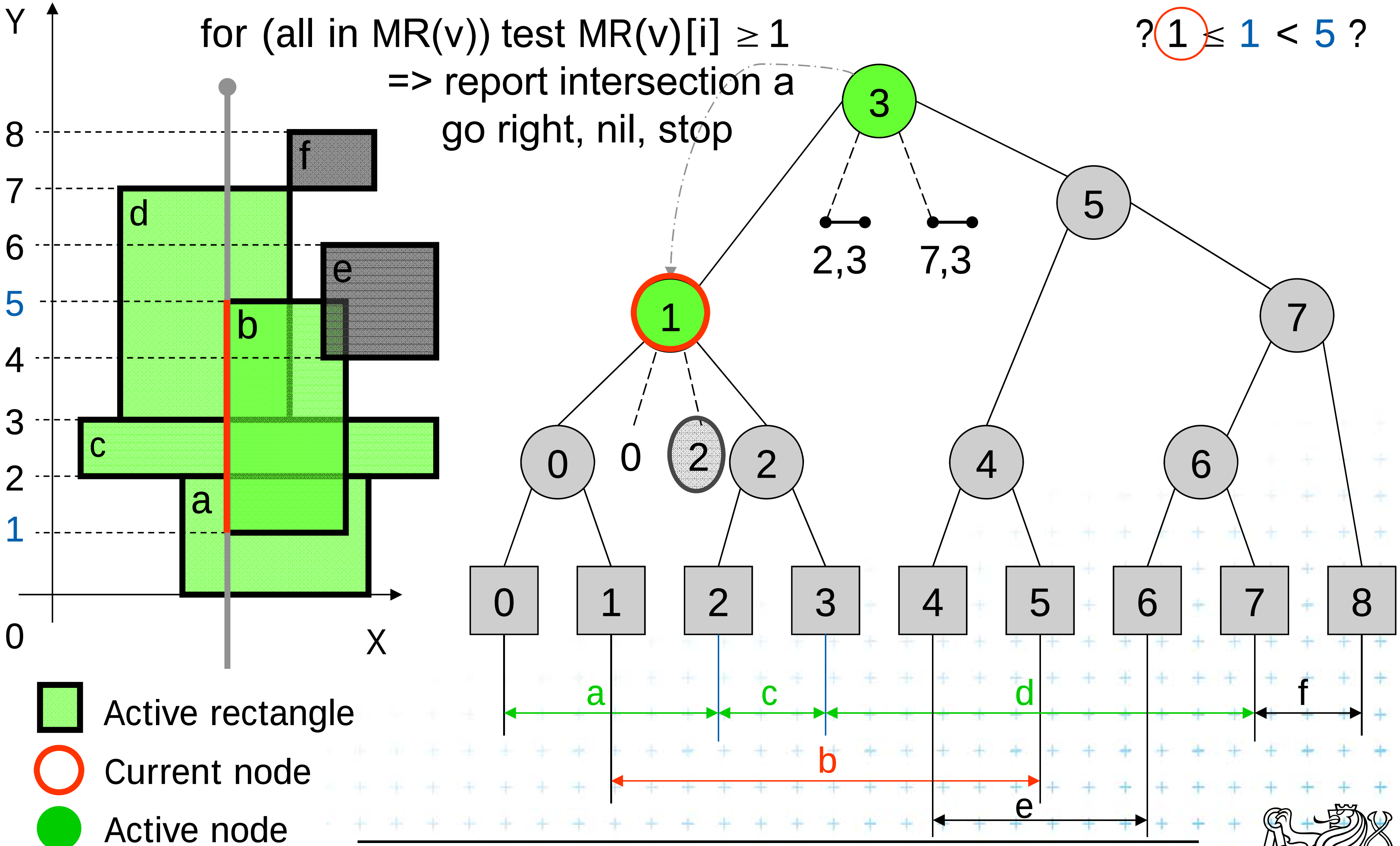
[Drtina]

Insert [1,5] a) Query Interval 2/2

H(v) ≤ b < e

for (all in MR(v)) test MR(v)[i] ≥ 1
=> report intersection a
go right, nil, stop

? 1 ≤ 1 < 5 ?



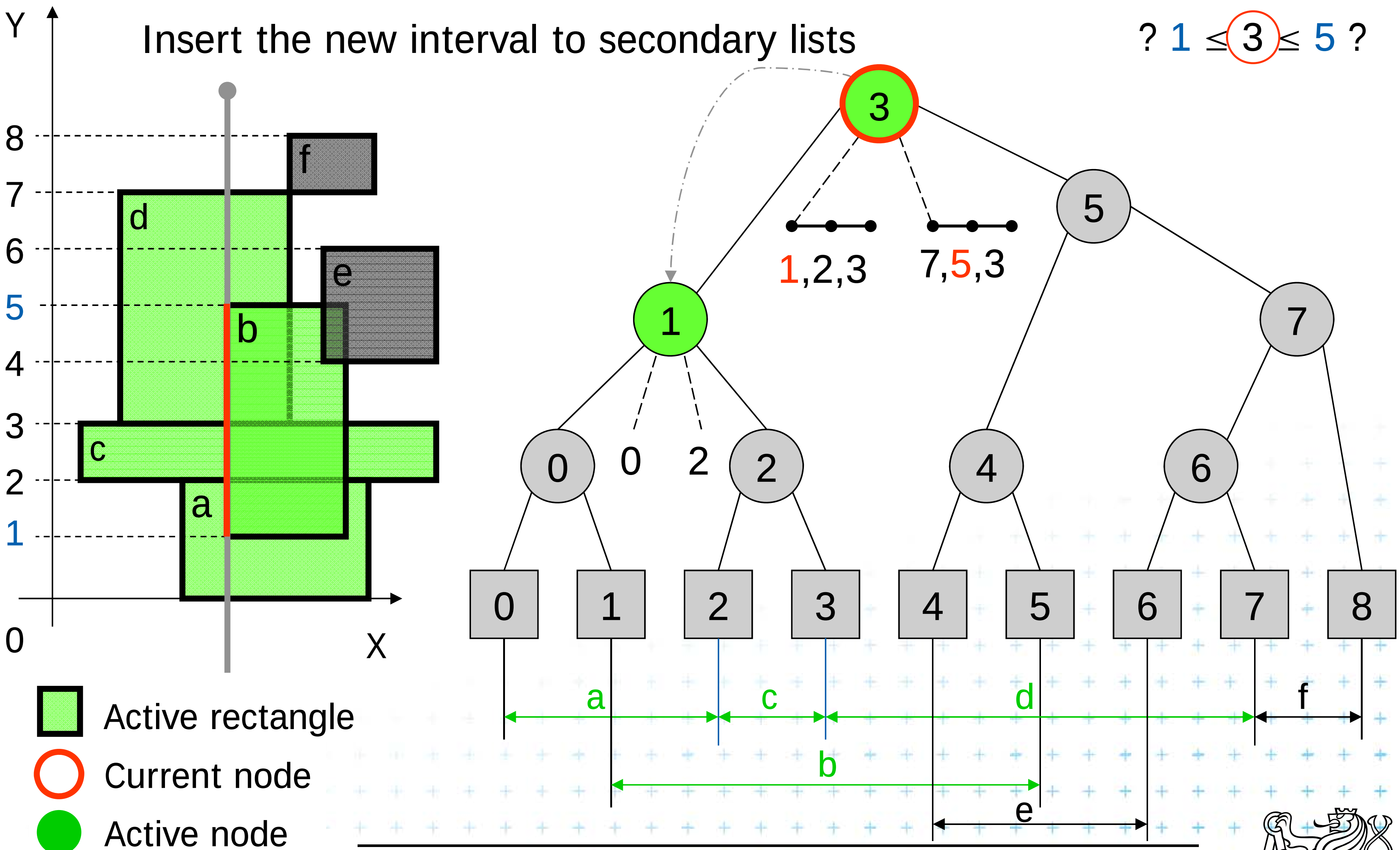
[Drtina]

Insert [1,5] b) Insert Interval

$b \leq H(v) \leq e$

Insert the new interval to secondary lists

? 1 ≤ 3 ≤ 5 ?

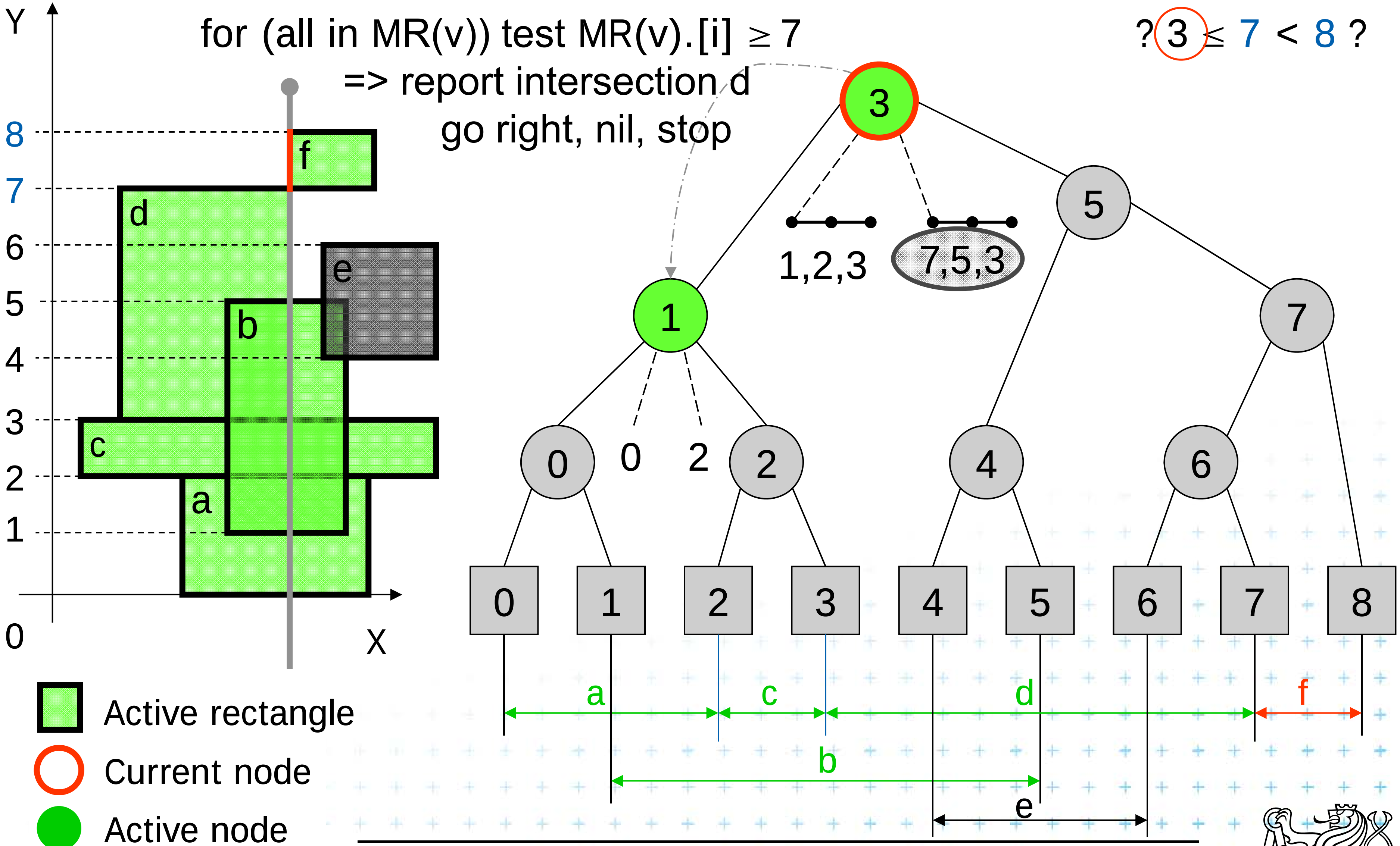


Insert [7,8] a) Query Interval

$H(v) \leq b < e$

for (all in MR(v)) test MR(v).[i] ≥ 7
=> report intersection d
go right, nil, stop

? 3 ≤ 7 < 8 ?



[Drtina]

Insert [7,8] b) Insert Interval

$b \le H(v) \le e$

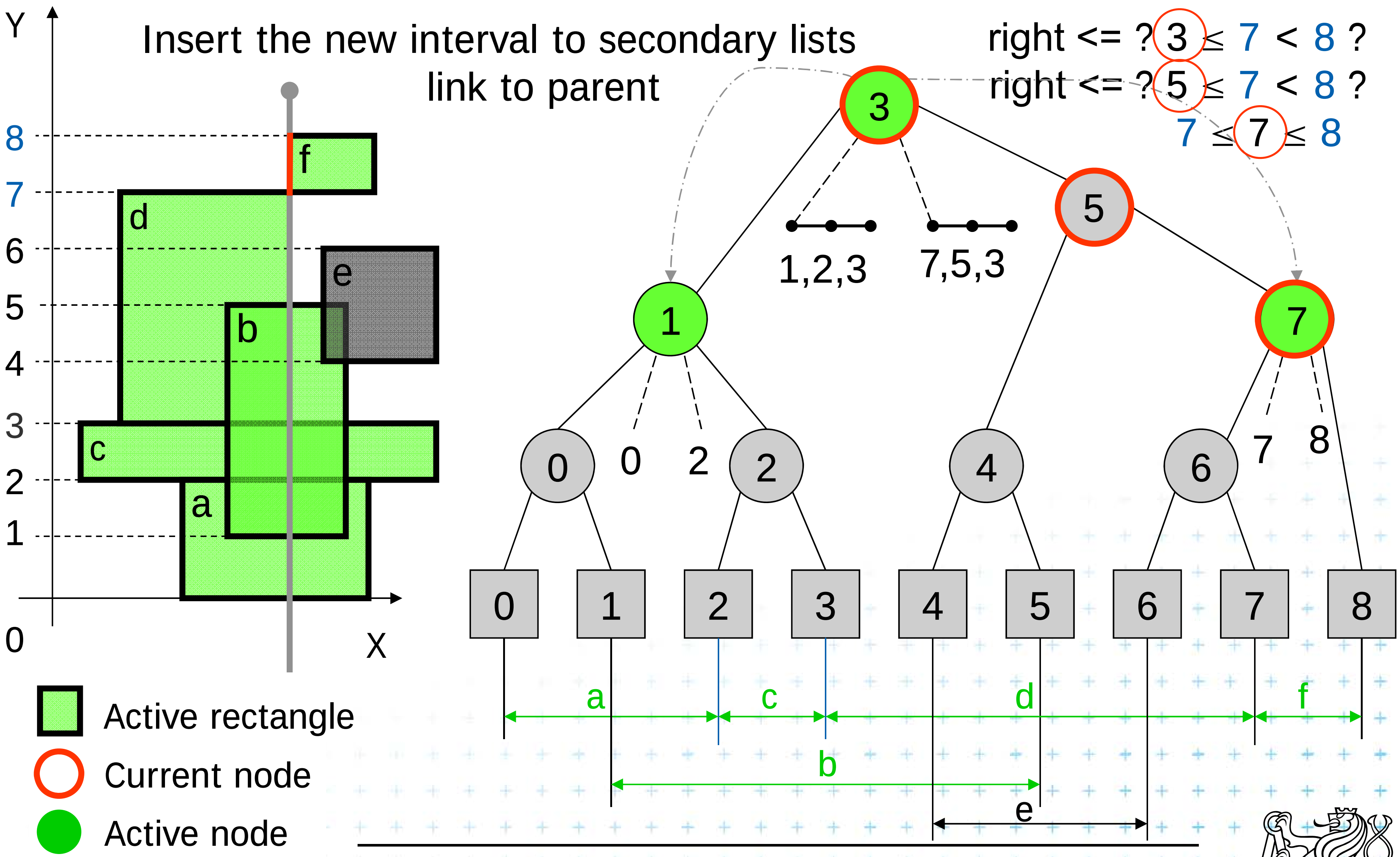
Insert the new interval to secondary lists

link to parent

right <= ? 3 ≤ 7 < 8 ?

right <= ? 5 ≤ 7 < 8 ?

7 ≤ 7 ≤ 8



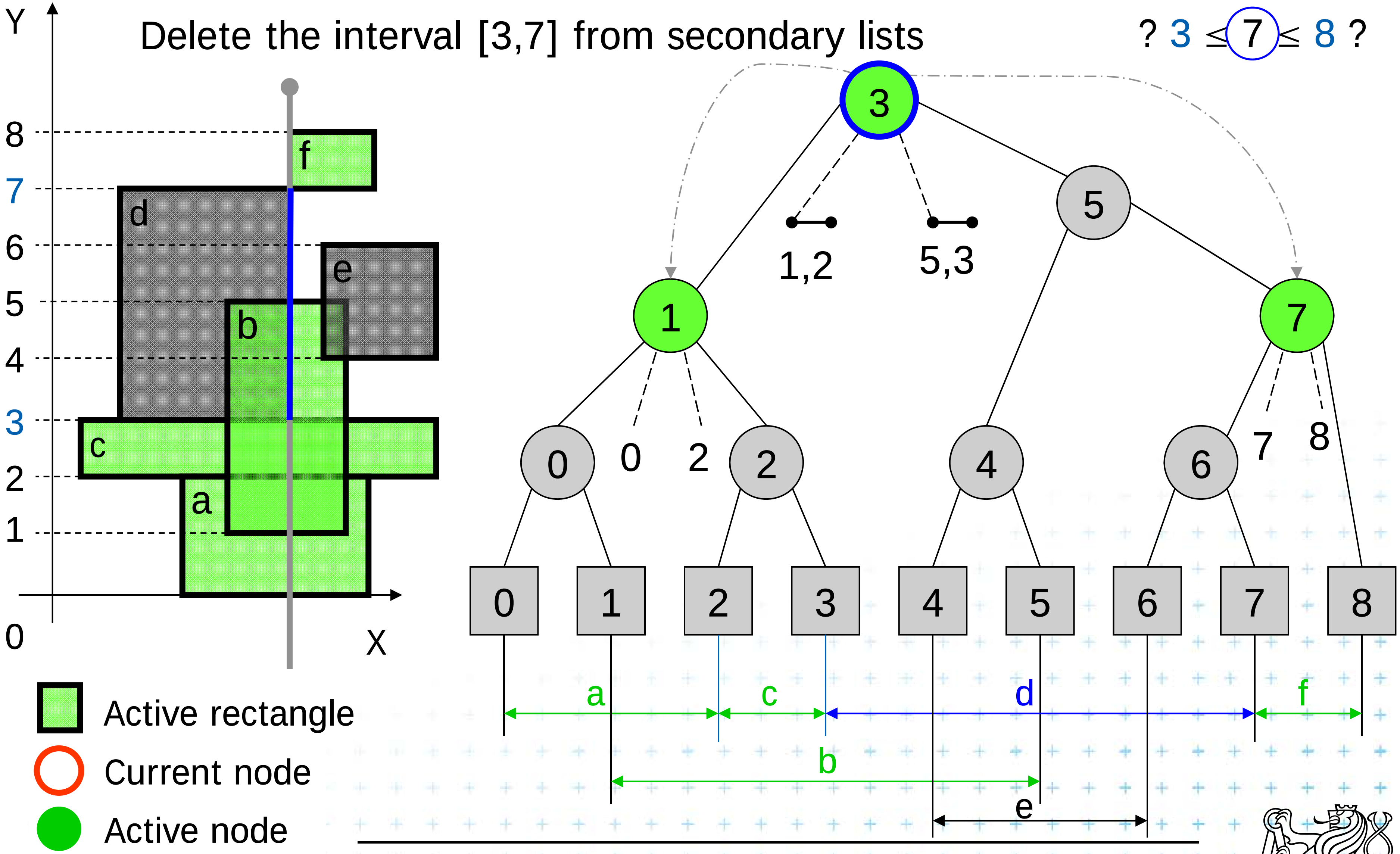
[Drtina]

Delete [3,7] Delete Interval

$b \le H(v) \le e$

Delete the interval [3,7] from secondary lists

? 3 ≤ 7 ≤ 8 ?



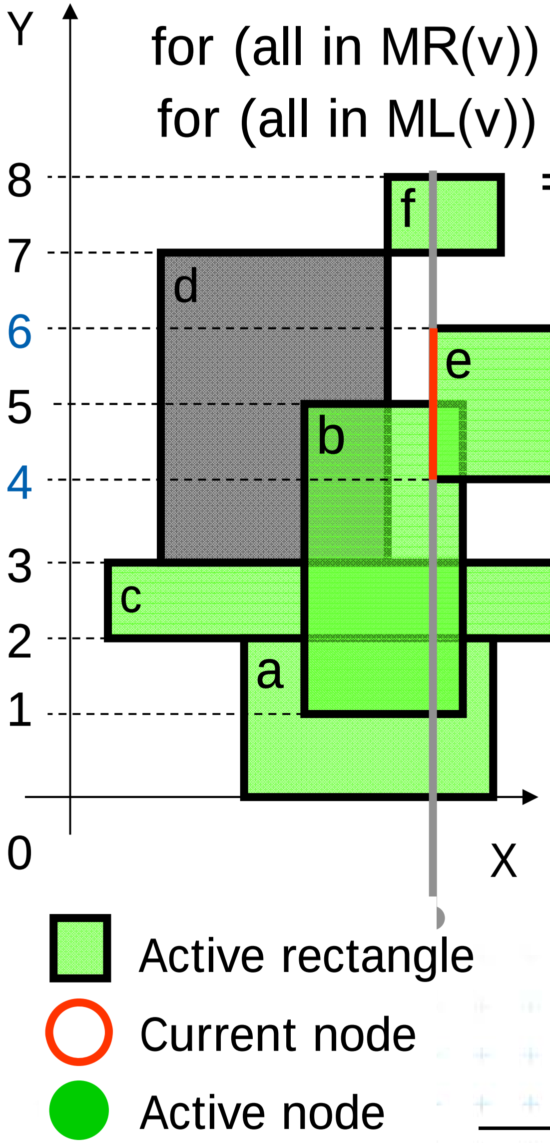
Insert [4,6] a) Query Interval

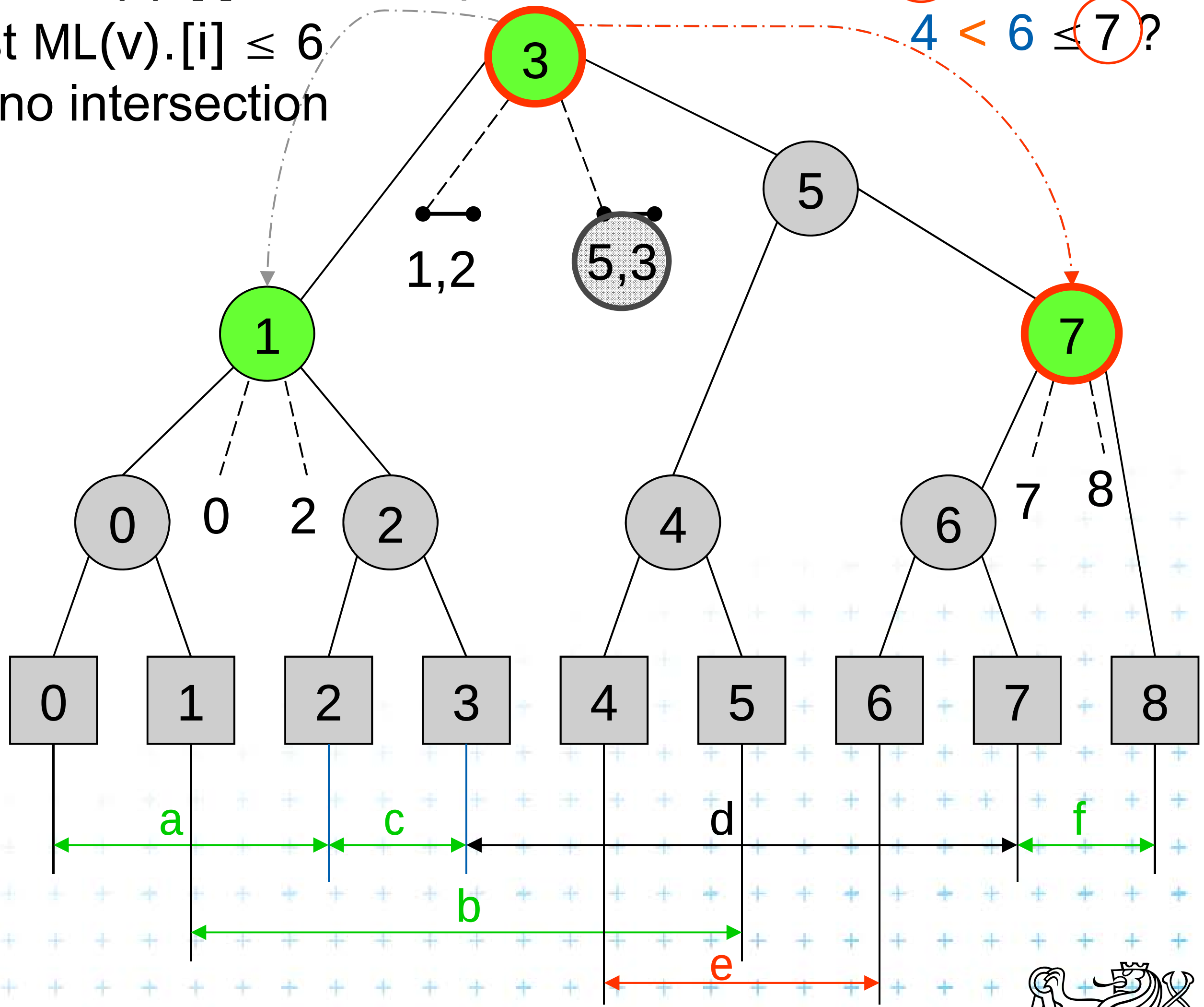
H(v) ≤ b < e

for (all in MR(v)) test MR(v).[i] ≥ 4 => report intersection b 3 ≤ 4 < 6 ?

for (all in ML(v)) test ML(v).[i] ≤ 6 4 < 6 ≤ 7 ?

=> no intersection



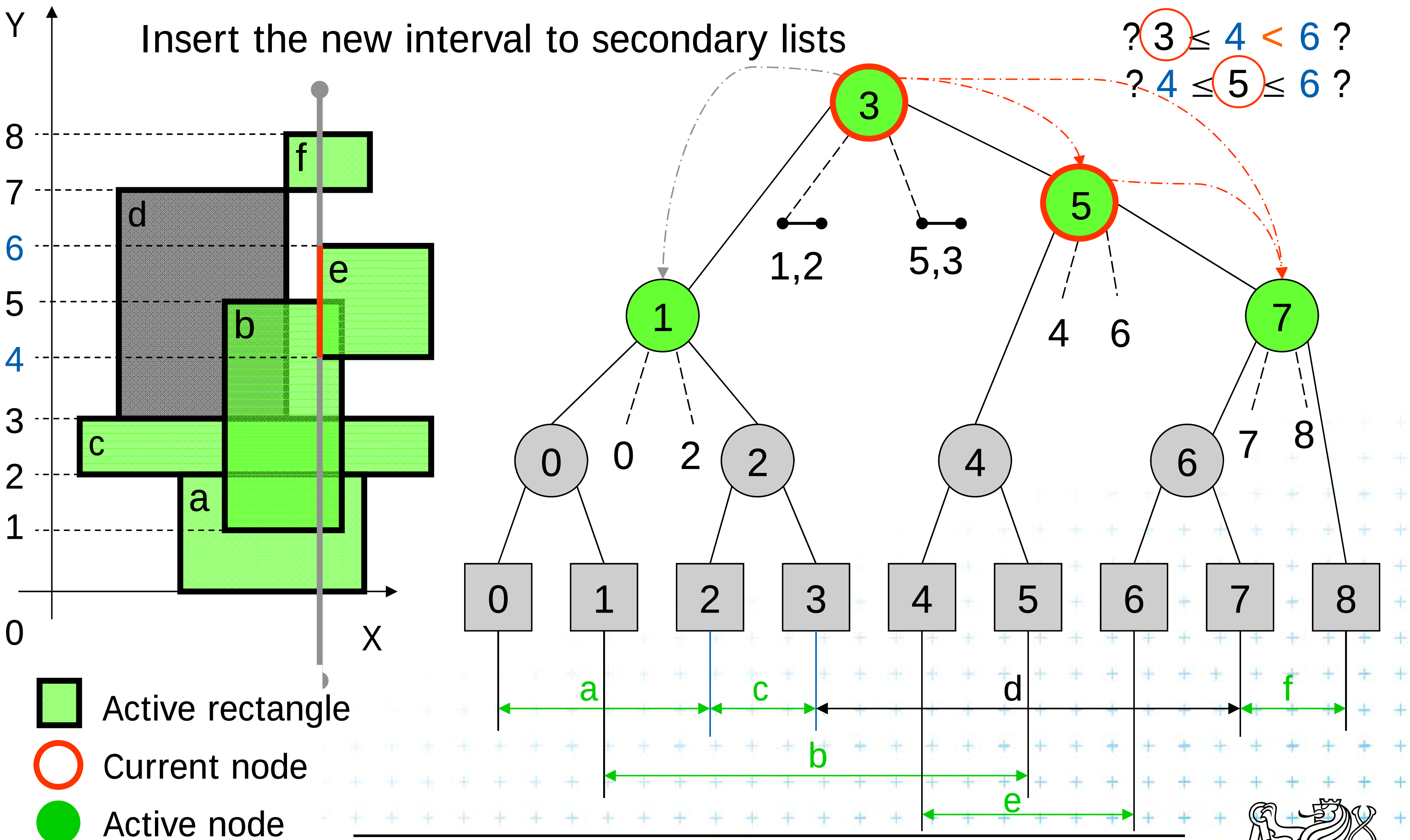


[Drtina]

Insert [4,6] b) Insert Interval

$H(v) \le b < e$

Insert the new interval to secondary lists

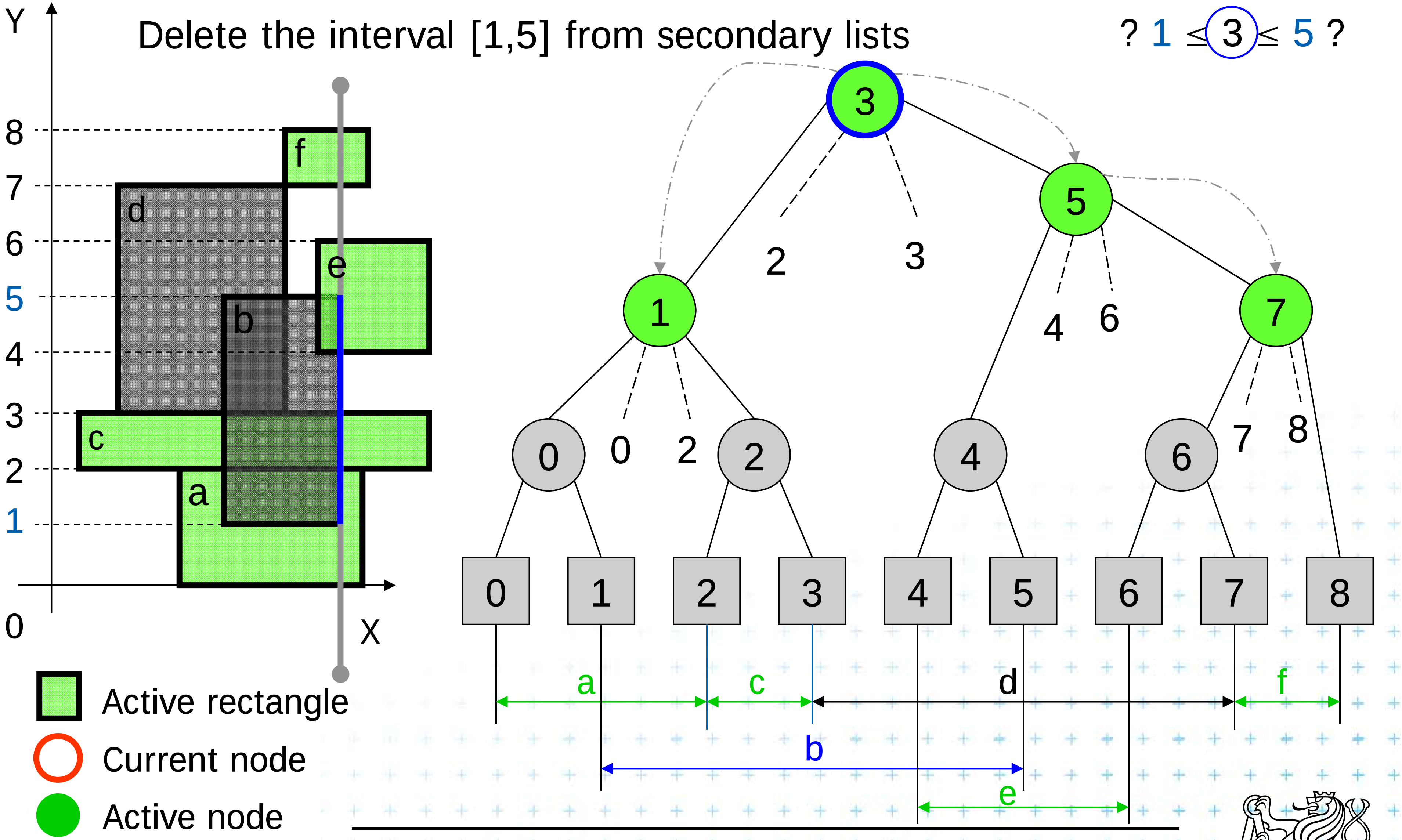


Delete [1,5] Delete Interval

$b \le H(v) \le e$

Delete the interval [1,5] from secondary lists

? 1 ≤ 3 ≤ 5 ?

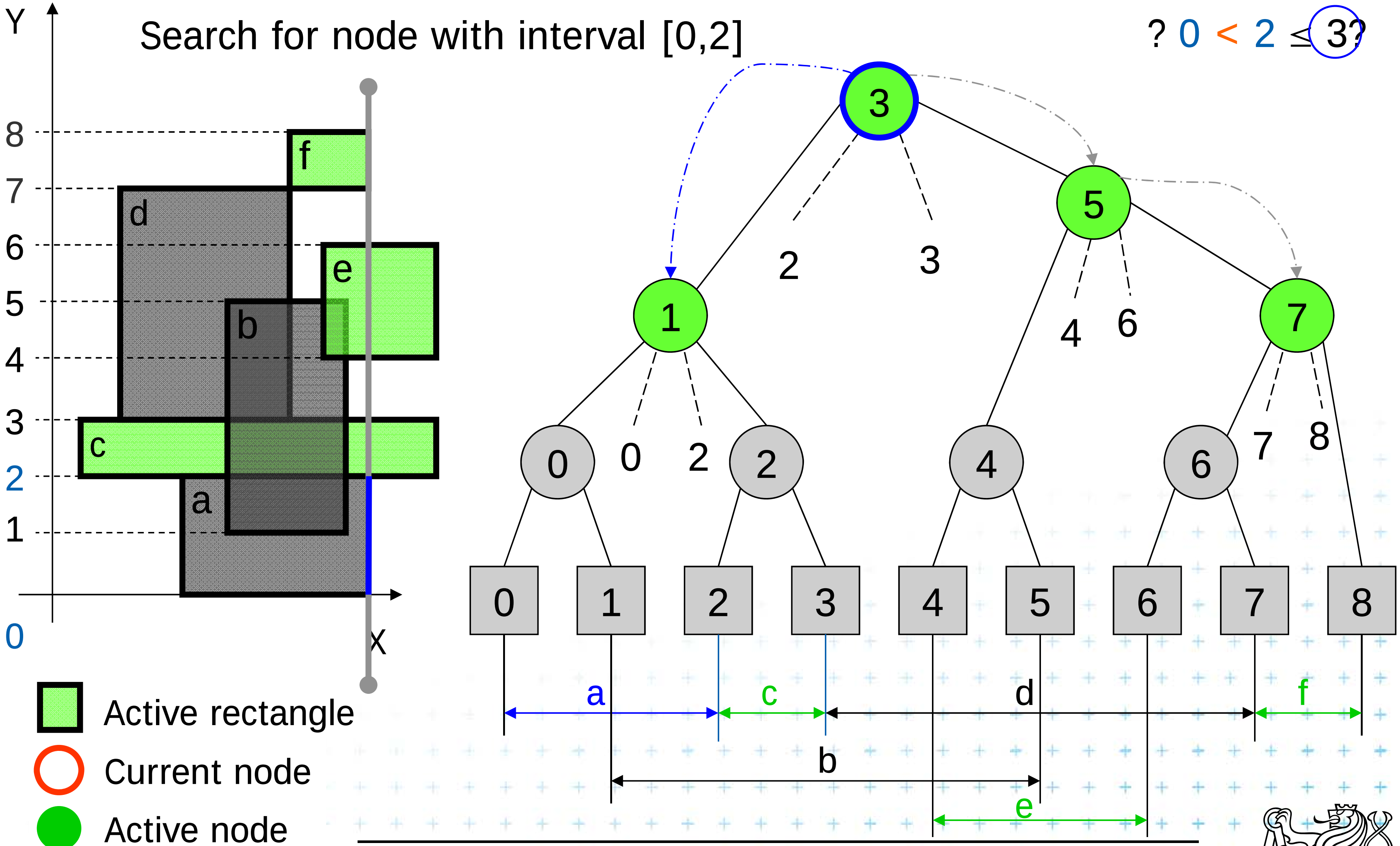


Delete [0,2] Delete Interval 1/2

$b < e \leq H(v)$

Search for node with interval [0,2]

? $0 < 2 \leq 3$?

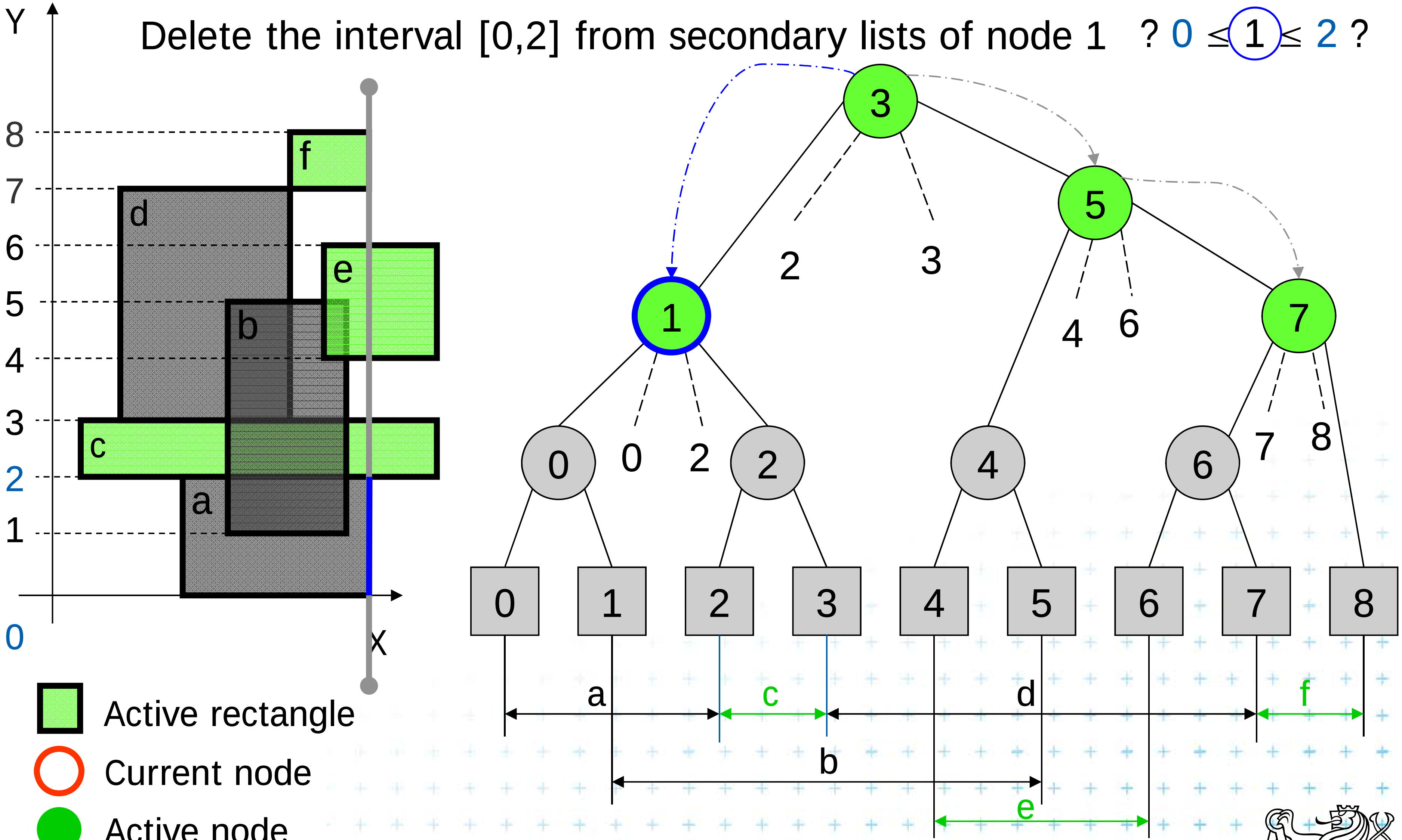


[Drtina]

Delete [0,2] Delete Interval 2/2

b ≤ H(v) ≤ e

Delete the interval [0,2] from secondary lists of node 1 ? 0 ≤ 1 ≤ 2 ?



Active rectangle

Current node

Active node

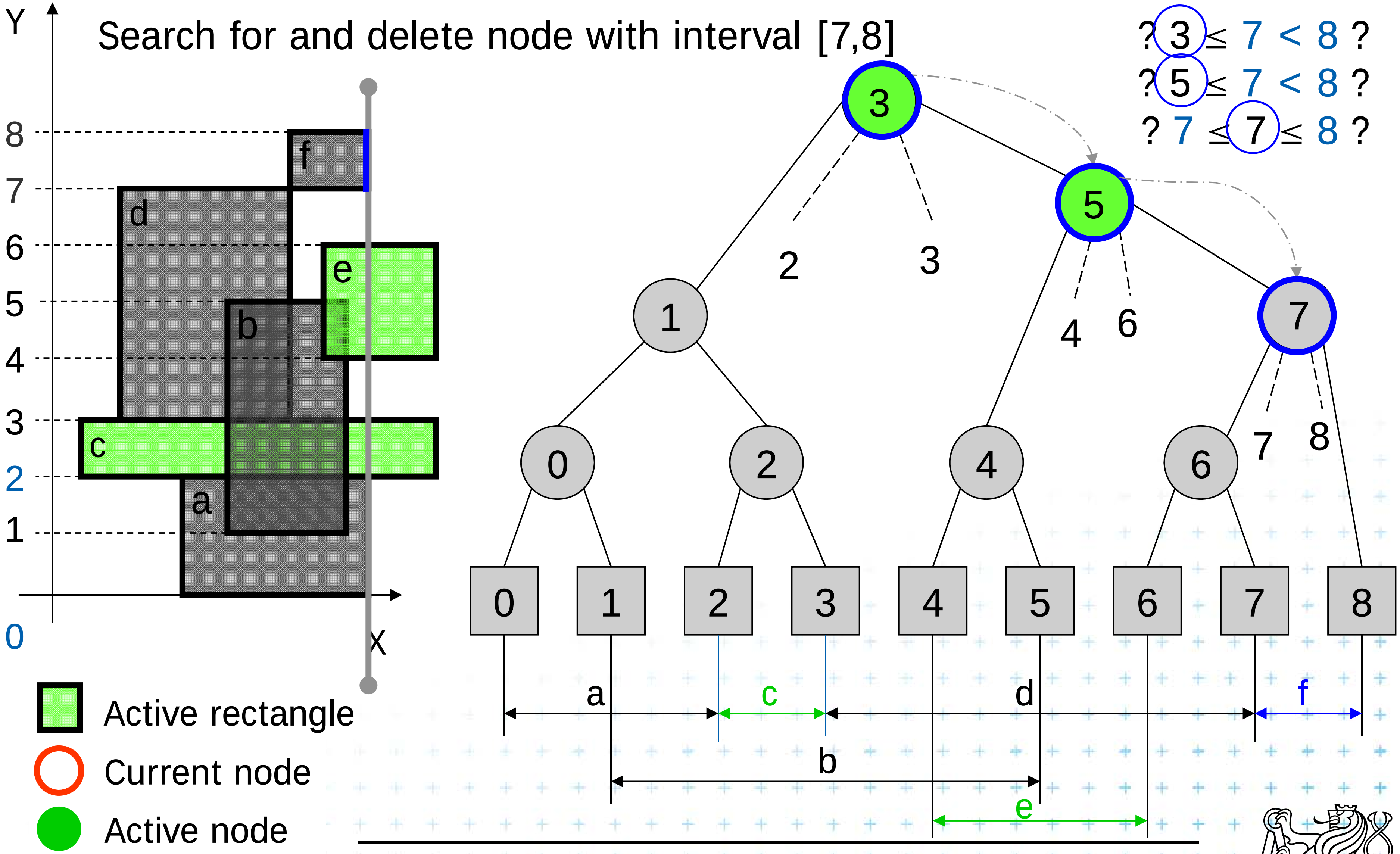
[Drtina]

Delete [7,8] Delete Interval

$b \le H(v) \le e$

Search for and delete node with interval [7,8]

? 3 ≤ 7 < 8 ?
? 5 ≤ 7 < 8 ?
? 7 ≤ 7 ≤ 8 ?



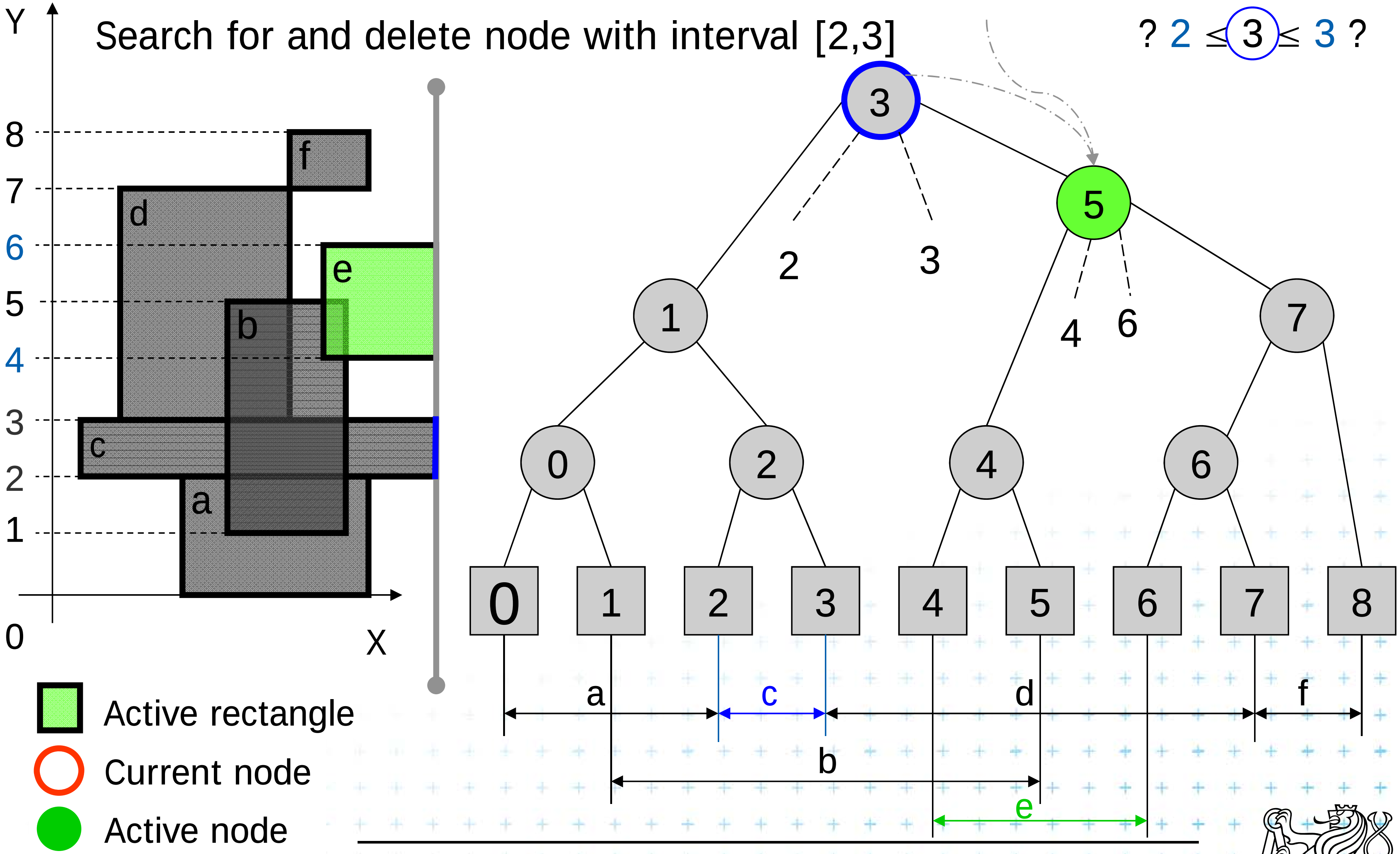
[Drtina]

Delete [2,3] Delete Interval

$b \le H(v) \le e$

Search for and delete node with interval [2,3]

? 2 ≤ 3 ≤ 3 ?

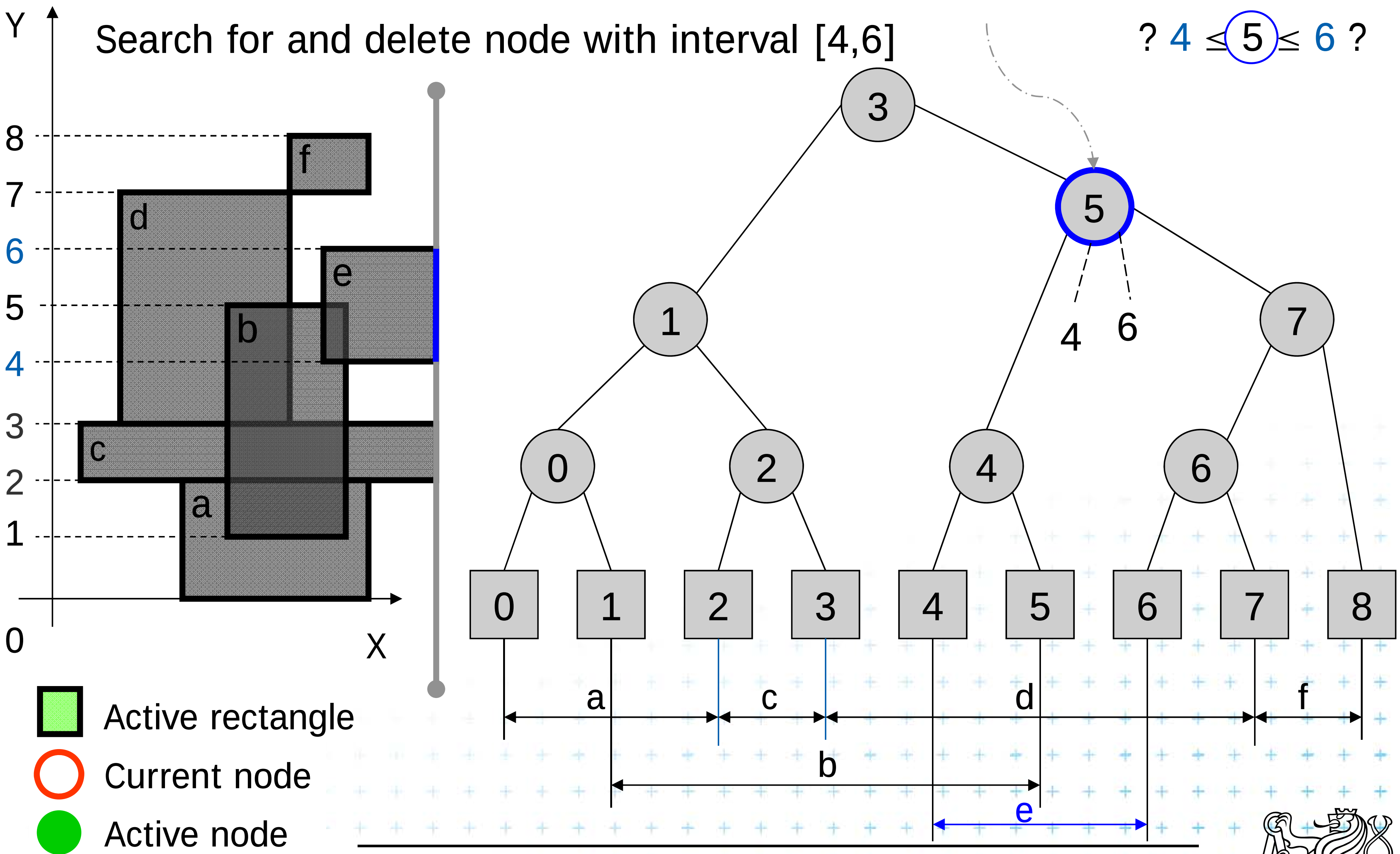


Delete [4,6] Delete Interval

$b \le H(v) \le e$

Search for and delete node with interval [4,6]

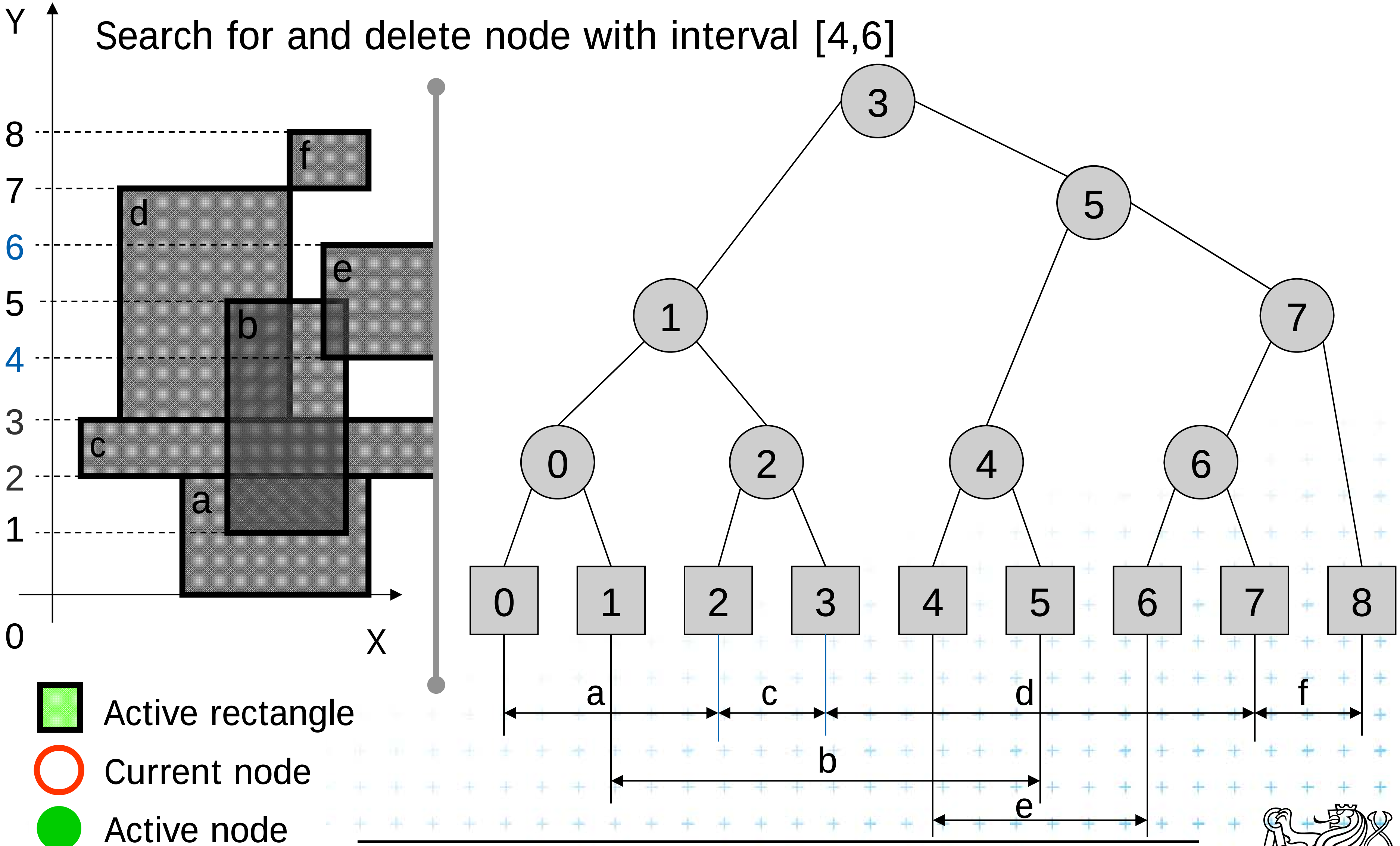
? 4 ≤ 5 ≤ 6 ?



[Drtina]

Empty tree

Search for and delete node with interval [4,6]



[Drtina]

Complexities of rectangle intersections

- *n* rectangles, *s* intersected pairs found
- O(*n* log *n*) preprocessing time to separately sort
  - x-coordinates of the rectangles for the plane sweep
  - the y-coordinates for initializing the interval tree.
- The plane sweep itself takes O(*n* log *n* + *s*) time, so the overall time is O(*n* log *n* + *s*)
- O(*n*) space
- This time is optimal for a decision-tree algorithm (i.e., one that only makes comparisons between rectangle coordinates).

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