# VORONOI DIAGRAM PART II

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**https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start**

**Based on [Berg], [Reiberg] and [Nandy]**

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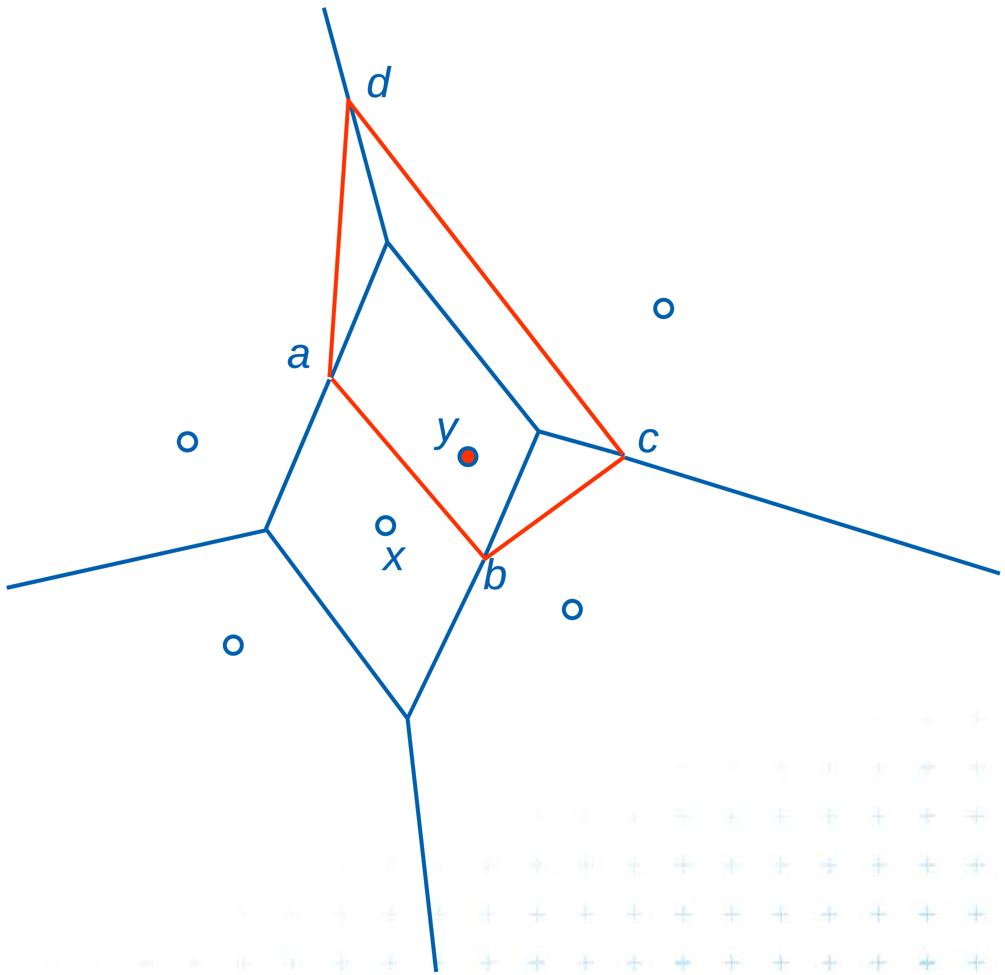
# Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

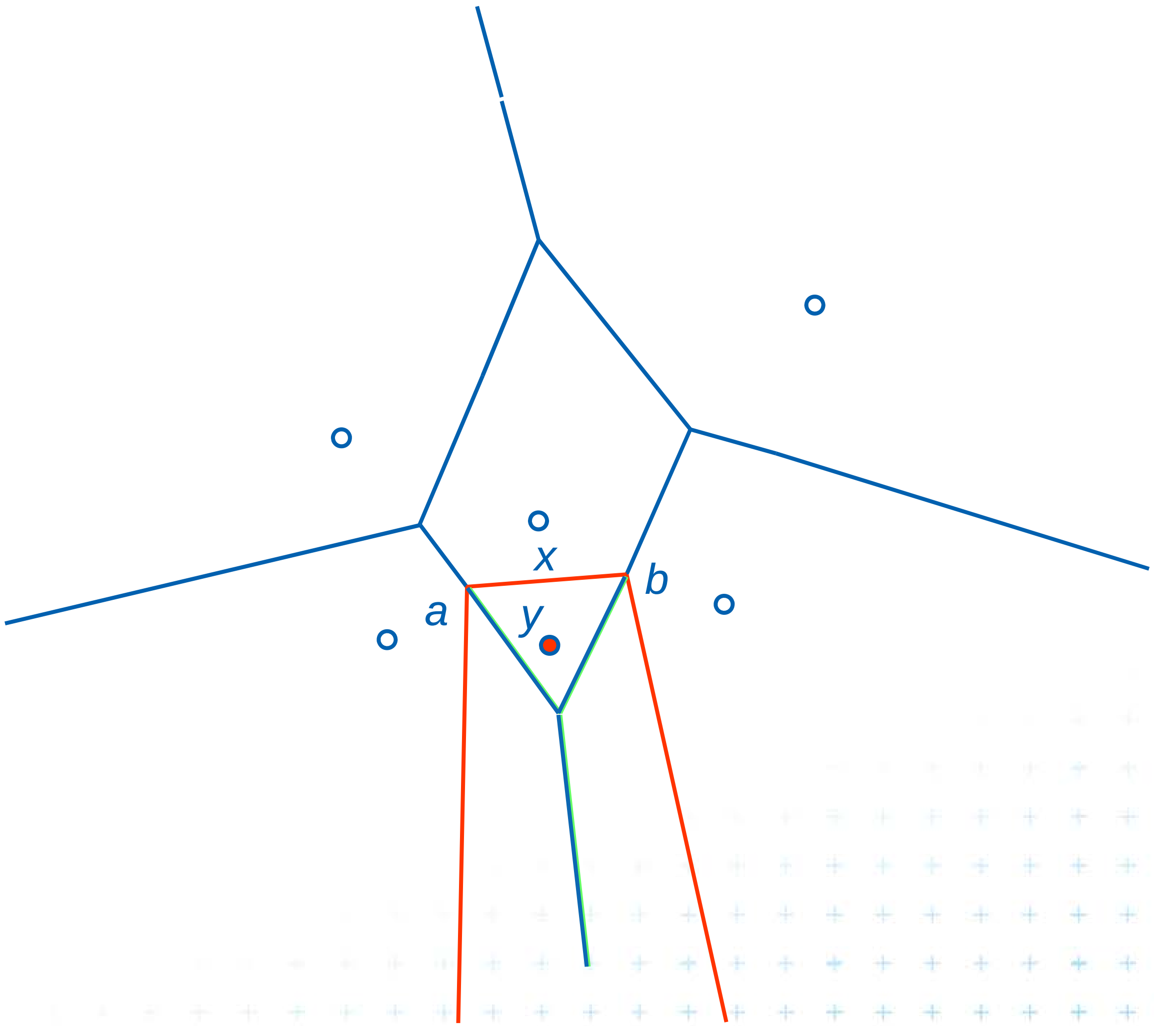
# Summary of the VD terms

- Site = input point, line segment, …
- Cell = area around the site, in $VD_1$ the nearest to site
- Edge, arc = part of Voronoi diagram (border between cells)
- Vertex = intersection of VD edges

# Incremental construction – bounded cell



# Incremental construction – unbounded cell



# Incremental construction algorithm

**InsertPoint($S$, Vor($S$), y )** … **y = a new site**

*Input:* Point set $S$, its Voronoi diagram, and inserted point $y \notin S$

*Output:* VD after insertion of $y$

1. Find the site $x$ in which cell point $y$ falls, …O(log $n$)
2. Detect the intersections $\{a,b\}$ of bisector $L(x,y)$ with cell $x$ boundary => create the first edge $e = ab$ on the border of site $x$ …O($n$)
3. Set start intersection point $p = b$, set new intersection $c$ = undef
4. site $z$ = neighbor site across the border with intersection $b$ …O(1)
5. **while**( exists($p$) *and* $c \neq a$ ) // trace the bisectors from $b$ in one direction
   a. Detect intersection $c$ of $L(y,z)$ with border of cell $z$
   b. Report Voronoi edge $pc$ …O($n^2$)
   c. $p = c$, $z$=neighbor site across border with intersec. $c$
5. **if**( $c \neq a$ ) **then** // trace the bisectors from $a$ in other direction
   a. $p = a$
   b. *Similarly as in steps 3,4,5 with* $a$

O($n^2$) worst-case, O($n$) expected time for some distributions

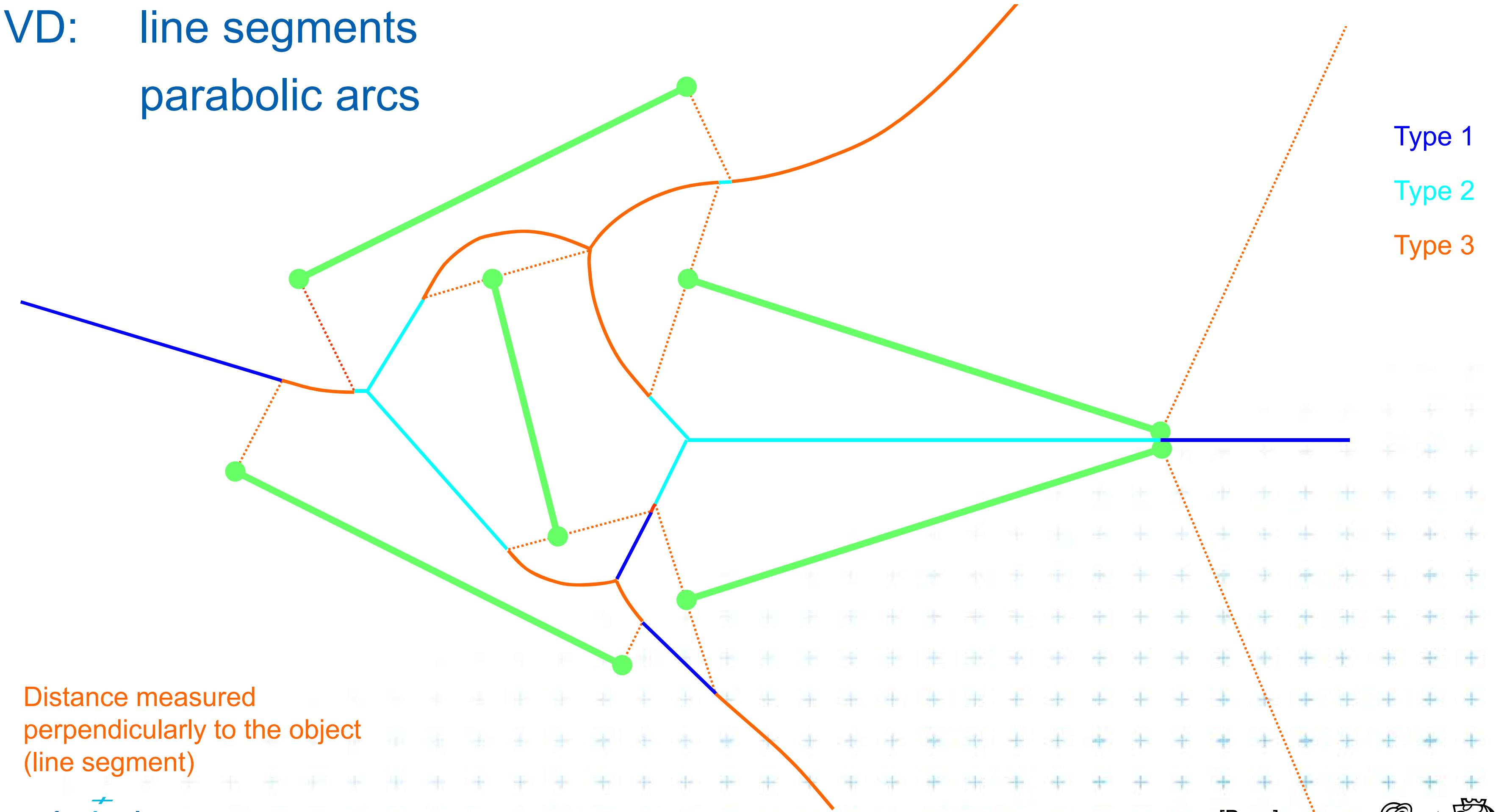
DCGI

# Voronoi diagram of line segments

Input: $S = \{s_1, \ldots, s_n\}$ = set of $n$ disjoint line segments (sites)

VD: line segments

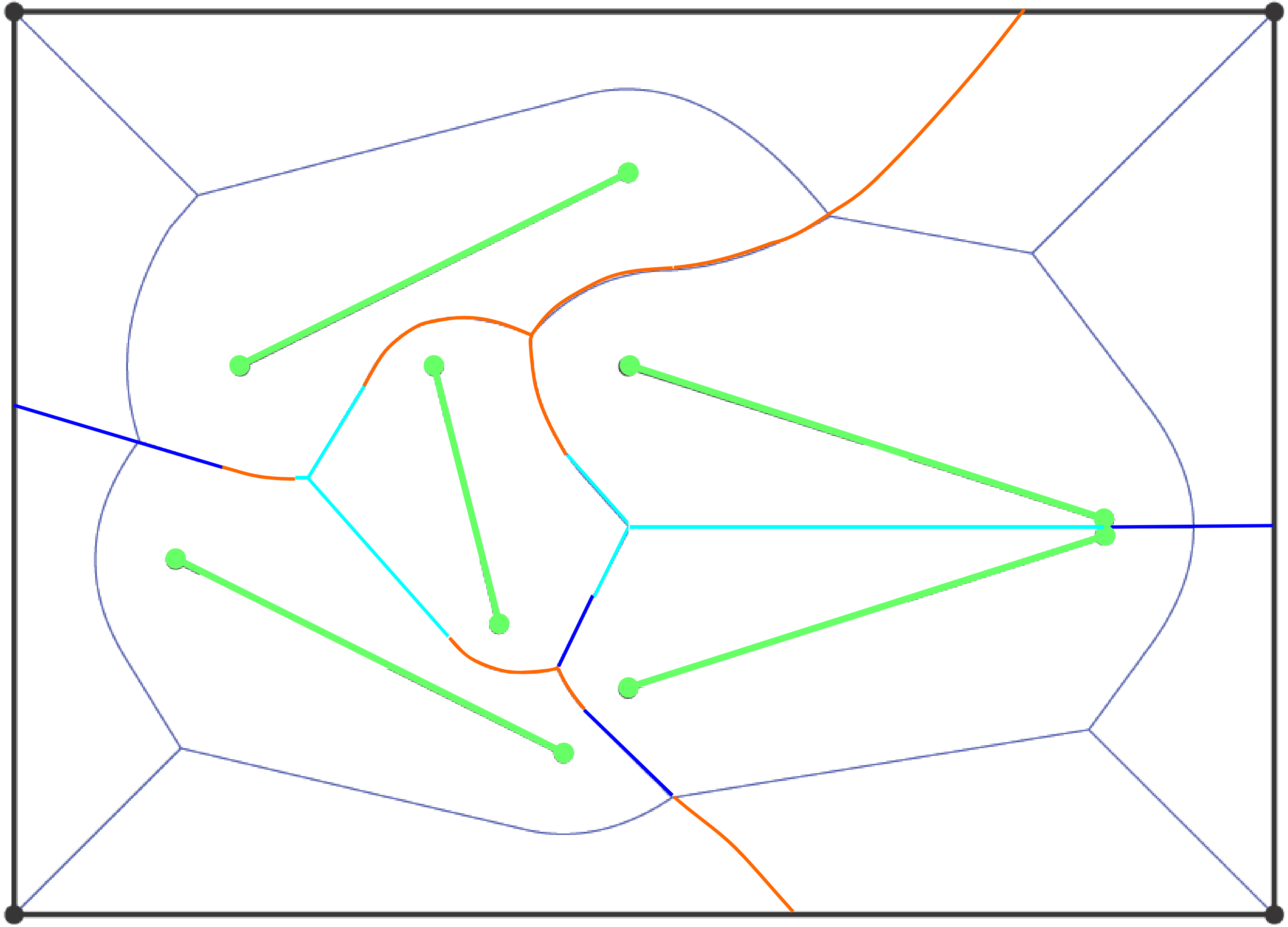
parabolic arcs




Distance measured perpendicularly to the object (line segment)

[Berg]

# VD of line segments with bounding box



BBOX
=>
standard
DCEL

[Berg]

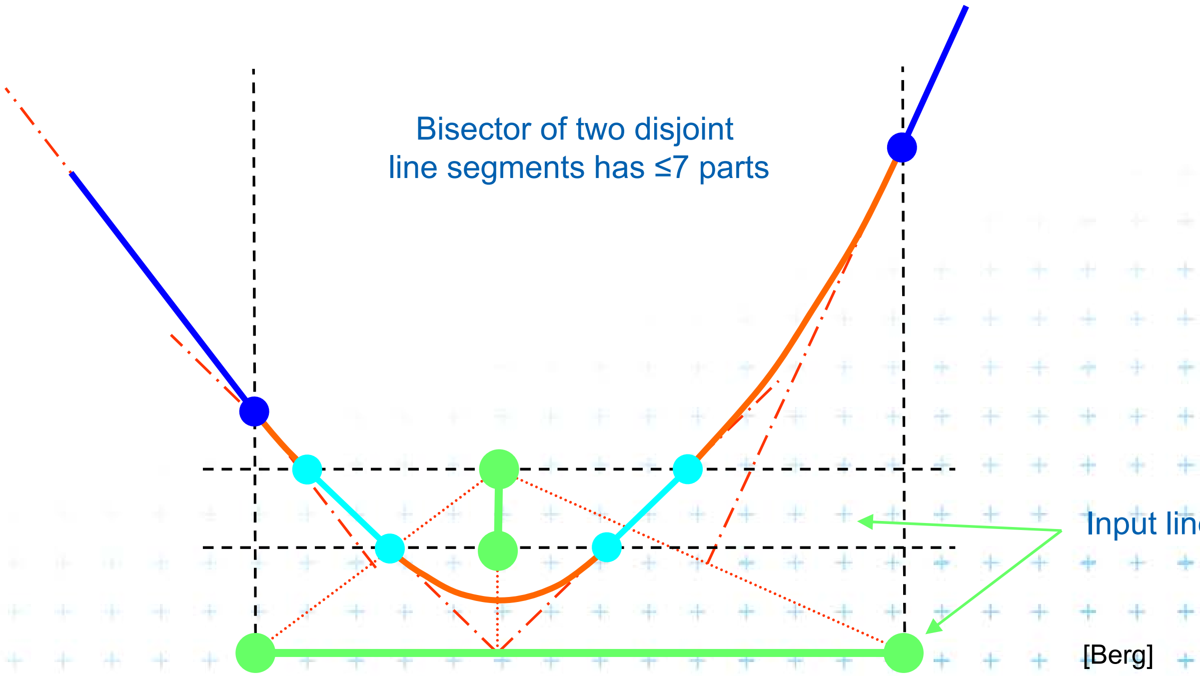
# Bisector of 2 line-segments in detail

- Consists of line segments and parabolic arcs

  Distance from point-to-object is measured to the closest point on the object (perpendicularly to the object silhouette)

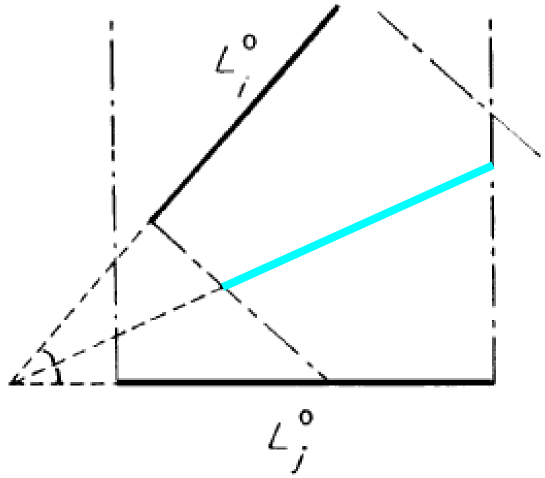
  – Line segment – bisector of end-points(1) or of interiors(2)

  – Parabolic arc – of point and interior(3) of a line segment

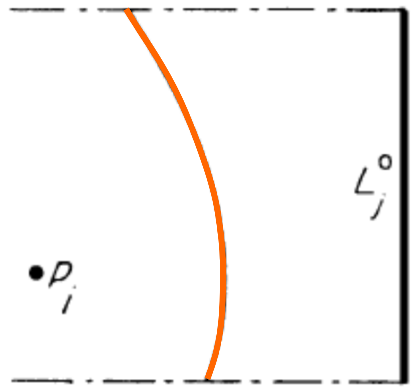


# Bisector in greater details

Type 2




Type 3



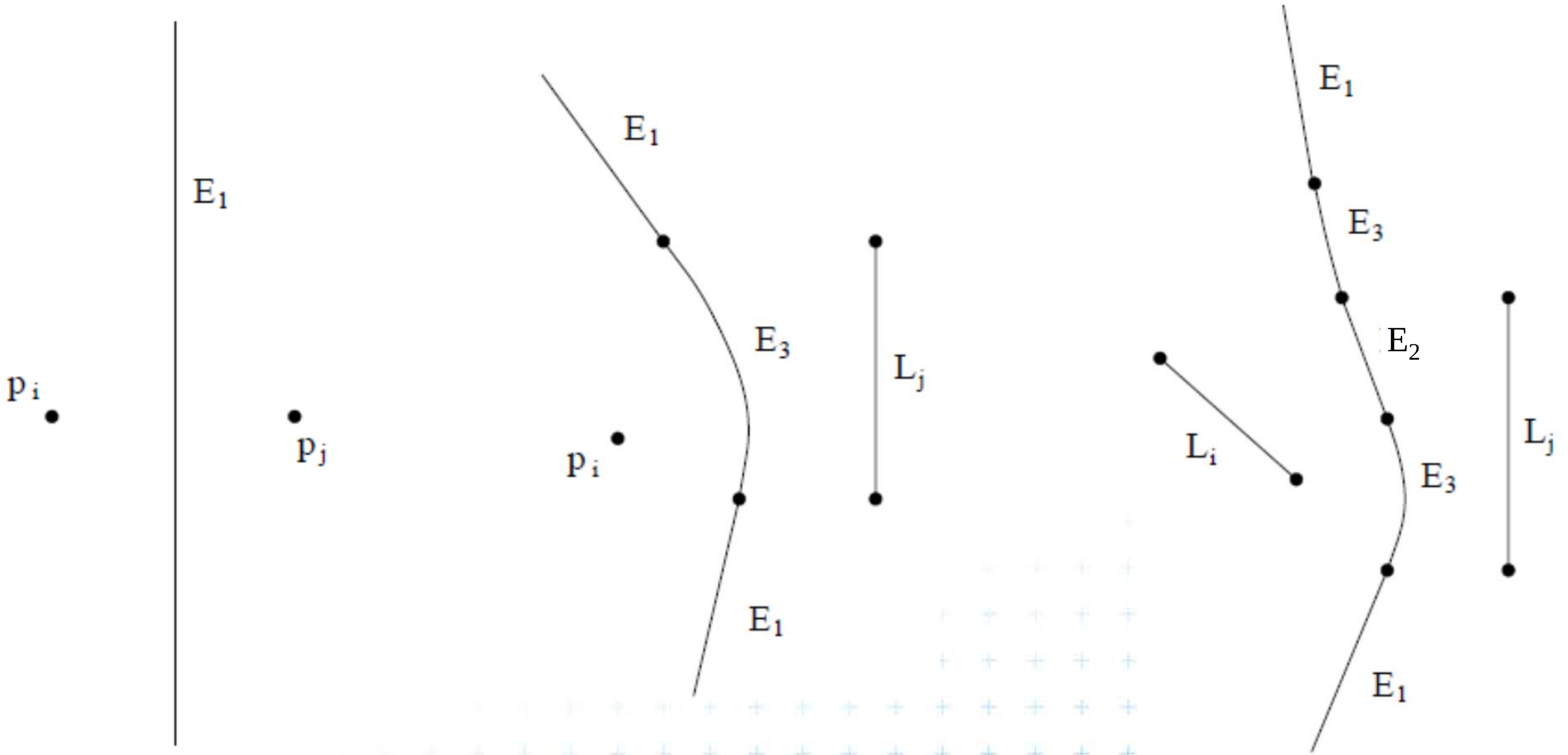

[Reiberg]

Bisector of two line segment interiors (in intersection of perpendicular slabs only)

Bisector of (end-)point and line segment interior

# VD of points and line segments examples

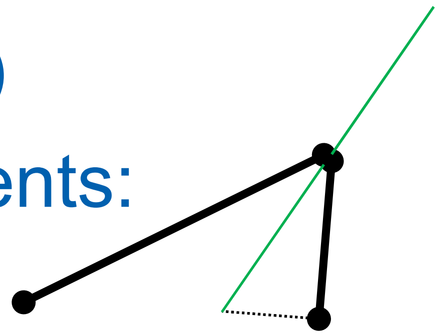
2 points    Point & segment    2 line segments

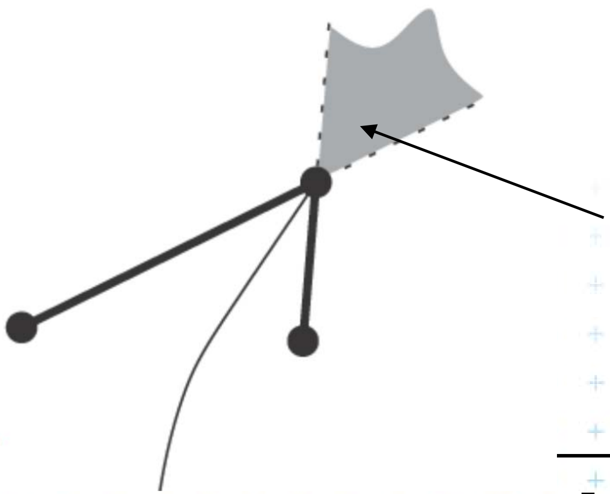



[Reiberg]

# Voronoi diagram of line segments

- More complex bisectors of line segments
  - VD contains line segments and parabolic arcs
- Still combinatorial complexity of O($n$)
- Assumptions on the input line segments:
  - non-crossing
  - strictly disjoint end-points (slightly shorten the segm.)

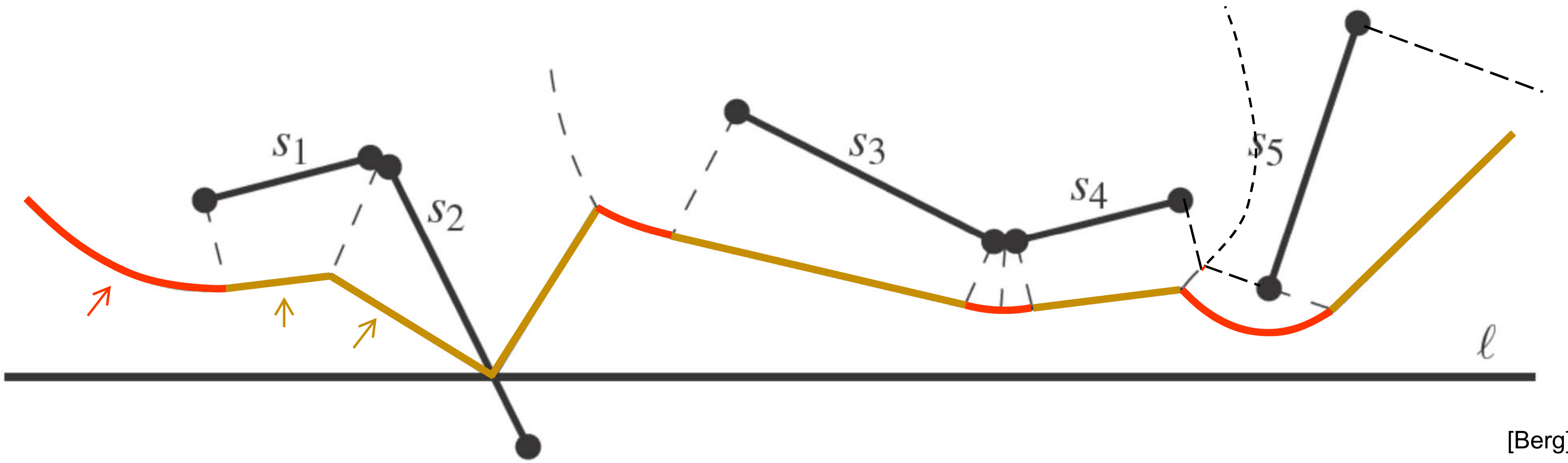




if(we allow touching segments)

Shared endpoints cause complication:
The whole region is equally close
to two line segments

# Shape of Beach line for line segments




[Berg]

= Points with distance to the closest site above sweep line *l* equal to the distance to *l*

- Beach line contains
  - *parabolic arcs* when closest to a site end-point
  - *straight line segments* when closest to a site interior (or just the part of the site interior above *l* if the site *s* intersects *l*)

(This is the shape of the beach line)

# Beach line breakpoints types

Breakpoint $p$ is equidistant from $l$ and equidistant and closest to:

points

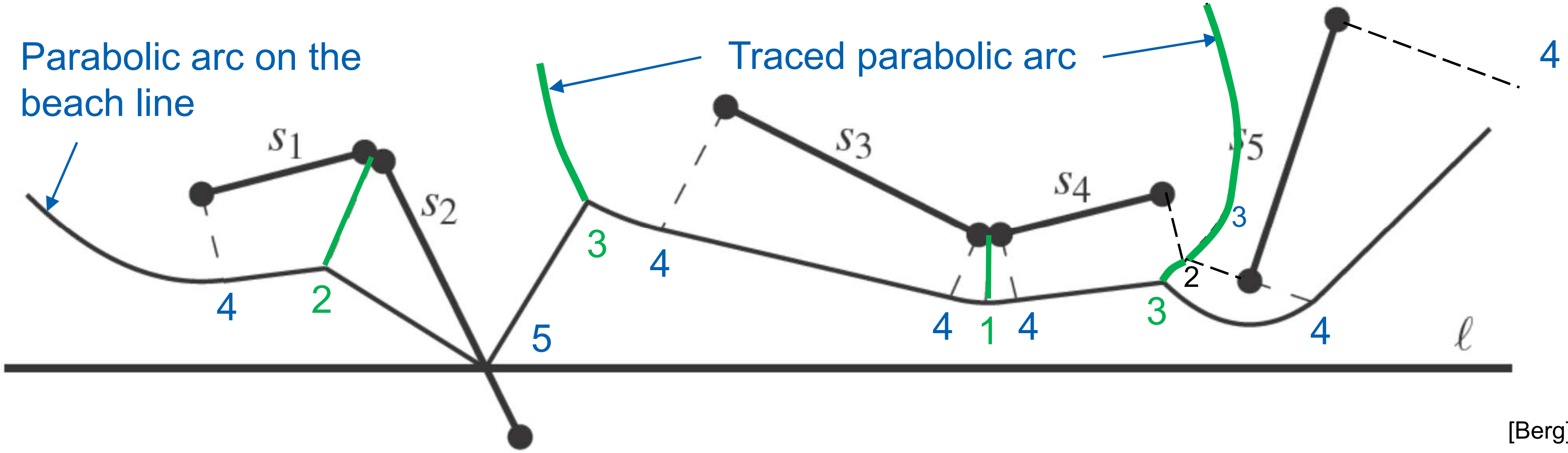
1. two site end-points => $p$ traces a VD line segment

segments

2. two site interiors => $p$ traces a VD line segment
3. end-point and interior => $p$ traces a VD parabolic arc
4. one site end-point => $p$ traces a line segment (border of the slab perpendicular to the site)
5. site interior intersects the scan line $l$ => $p$ = intersection, traces the input line segment

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram edge (are used by alg.only)

# Breakpoints types and what they trace




[Berg]

- 1,2 trace a Voronoi line segment (part of VD edge) DRAW
- 3 traces a Voronoi parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

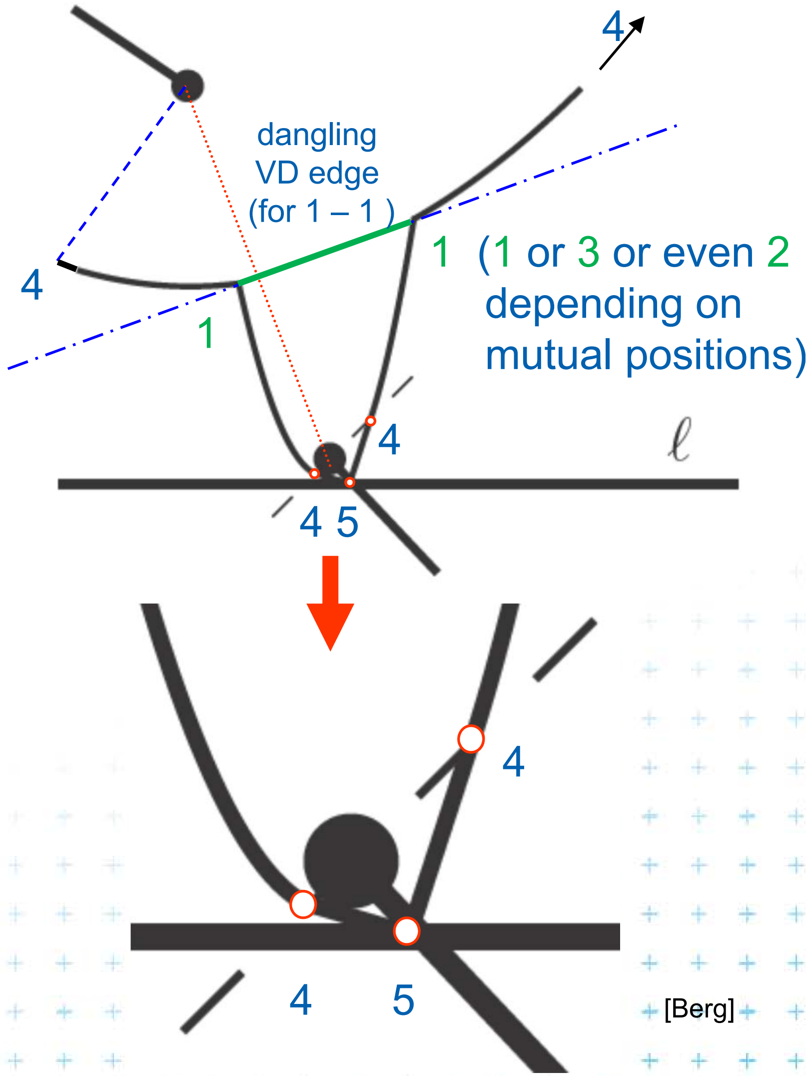
(This is the shape of the traced VD arcs)

DCGI

# Site event – sweep line reaches an endpoint

I. At upper endpoint of

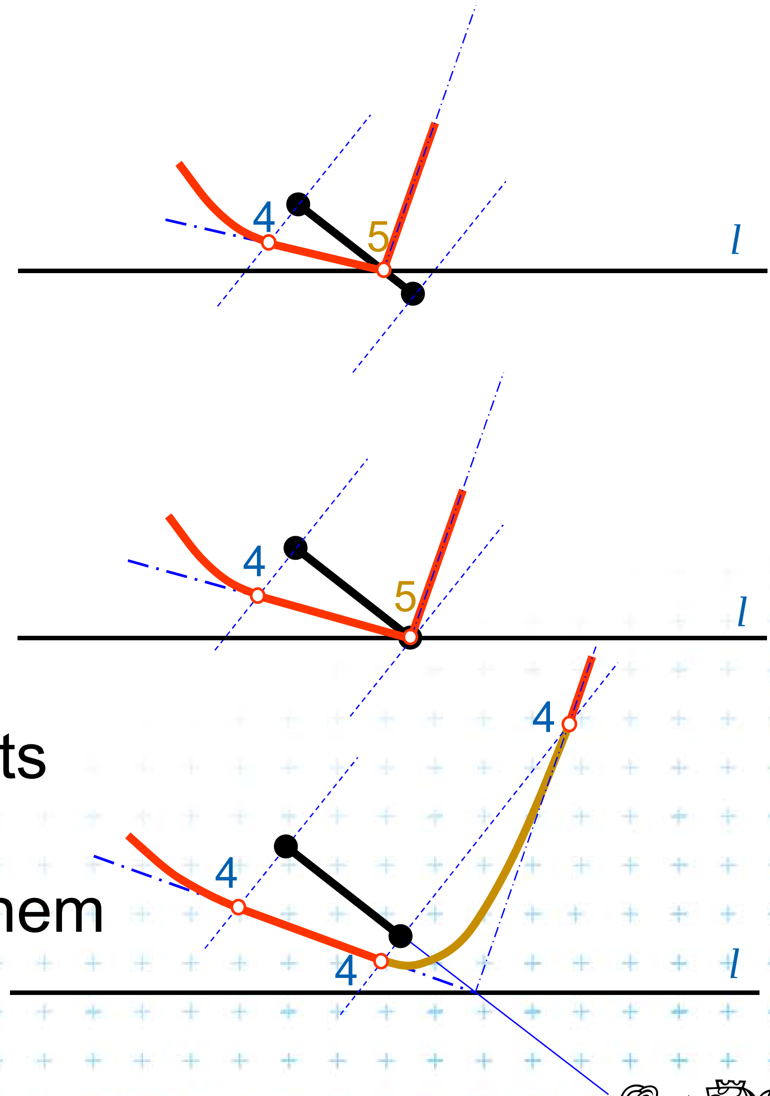
- Arc above is split into two
- four new arcs are created (2 segments + 2 parabolas)
- Breakpoints for two segments are of type 4-5-4
- Breakpoints for parabolas depend on the surrounding sites
  - Type 1 for two end-points
  - Type 3 for endpoint and interior
  - etc…



# Site event – sweep line reaches an endpoint

II. At lower endpoint of

- Intersection with interior (breakpoint of type 5)



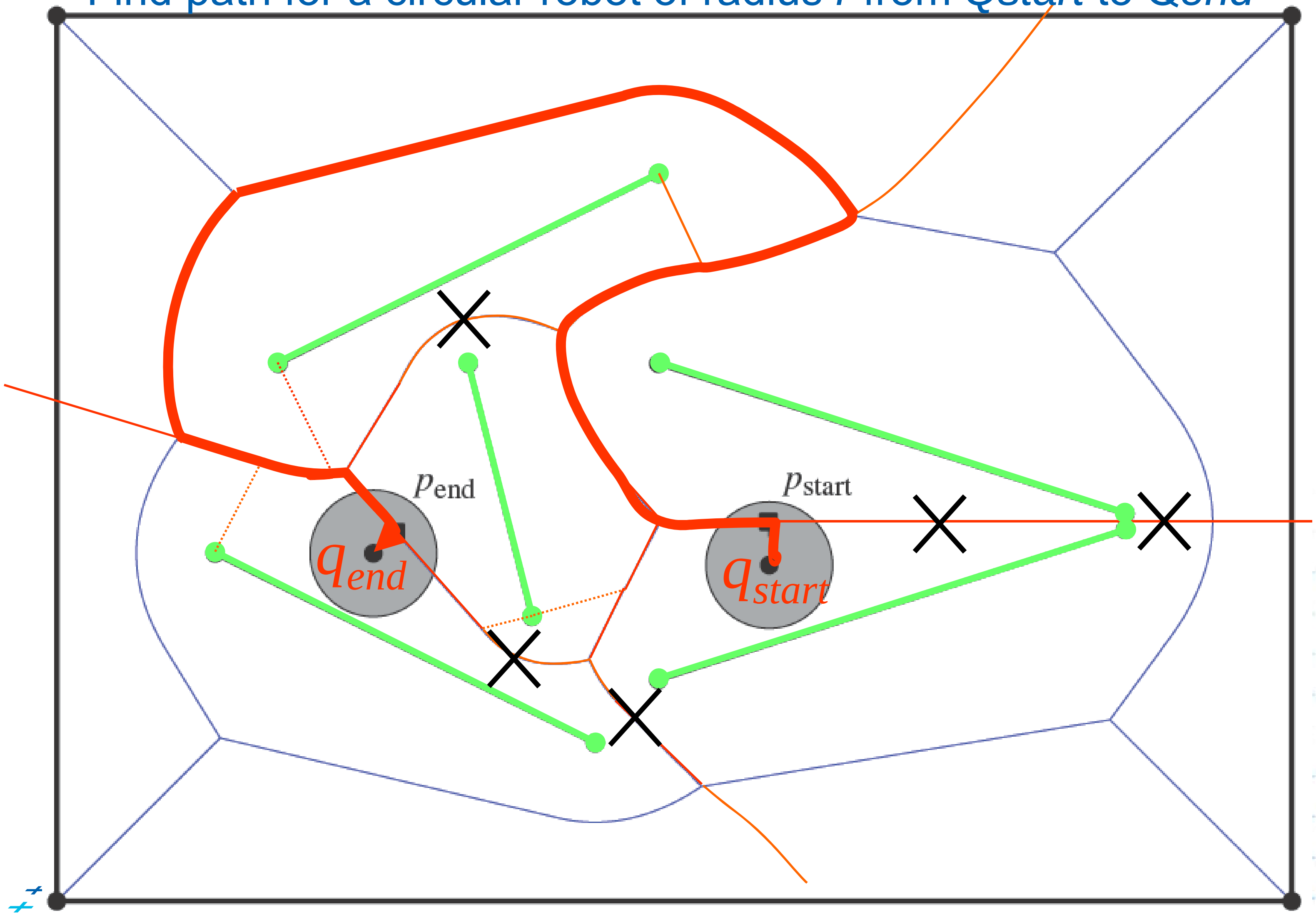

- is replaced by two breakpoints (of type 4) with parabolic arc between them

# Circle event – lower point of circle of 3 sites

- Two breakpoints meet (on the beach-line)
- Solution depends on their type
  - Any of first three types (1,2,or 3) meet
    - – 3 sites involved – Voronoi vertex created
  - Type 4 with something else
    - – two sites involved – breakpoint changes its type
    - – Voronoi vertex not created (Voronoi edge may change its shape)
  - Type 5 with something else
    - – never happens for disjoint segments (meet with type 4 happens before)

# Motion planning example - retraction Rušení hran

Find path for a circular robot of radius *r* from *Qstart* to *Qend*




[Berg]

# Motion planning example - retraction Rušení hran

Find path for a circular robot of radius *r* from $Q_{start}$ to $Q_{end}$

- Create Voronoi diagram of line segments, take it as a graph
- Project $Q_{start}$ to $P_{start}$ *on VD and* $Q_{end}$ to $P_{end}$
- Remove segments with distance to sites smaller than radius *r* of a robot
- Depth first search if path from $P_{start}$ to $P_{end}$ *exists*
- Report path $Q_{start}\,P_{start}$*…path…* $P_{end}$ to $Q_{end}$

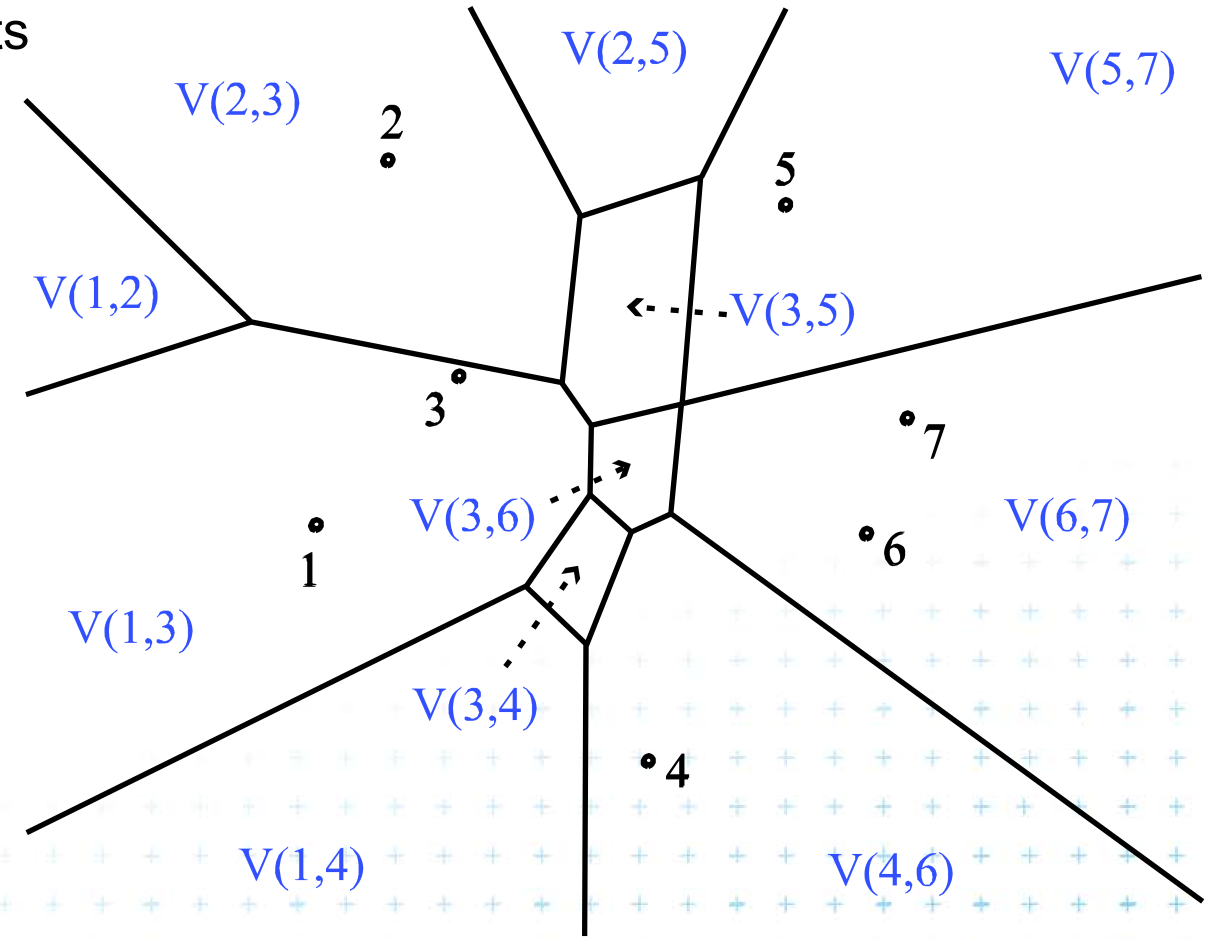
- $O(n \log n)$ time using $O(n)$ storage

# Order-2 Voronoi diagram

$V(p_i,p_j)$ : the set of points of the plane closer to each of $p_i$ and $p_j$ than to any other site

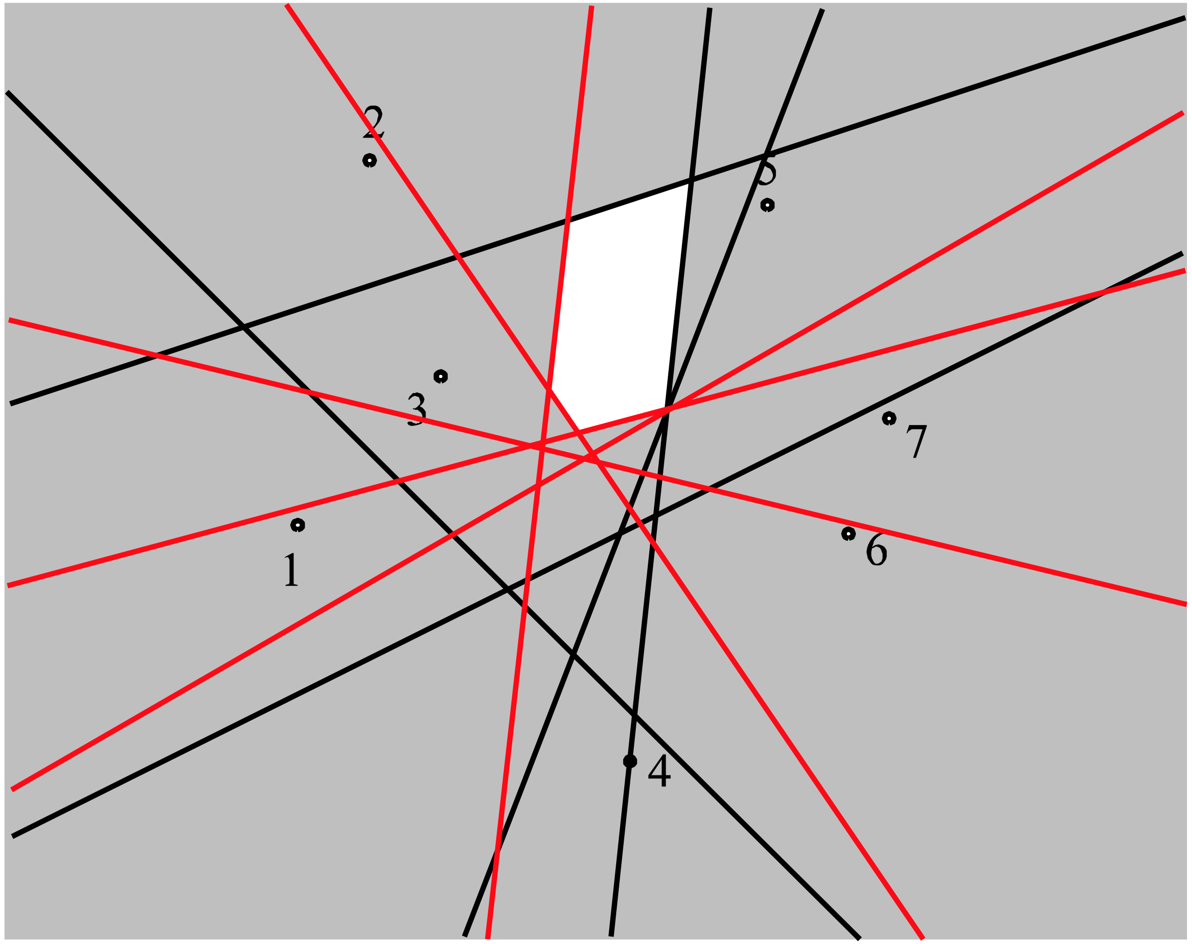
Property

The order-2 Voronoi regions are convex




[Nandy]

# Construction of V(3,5) = V(5,3)




[Nandy]

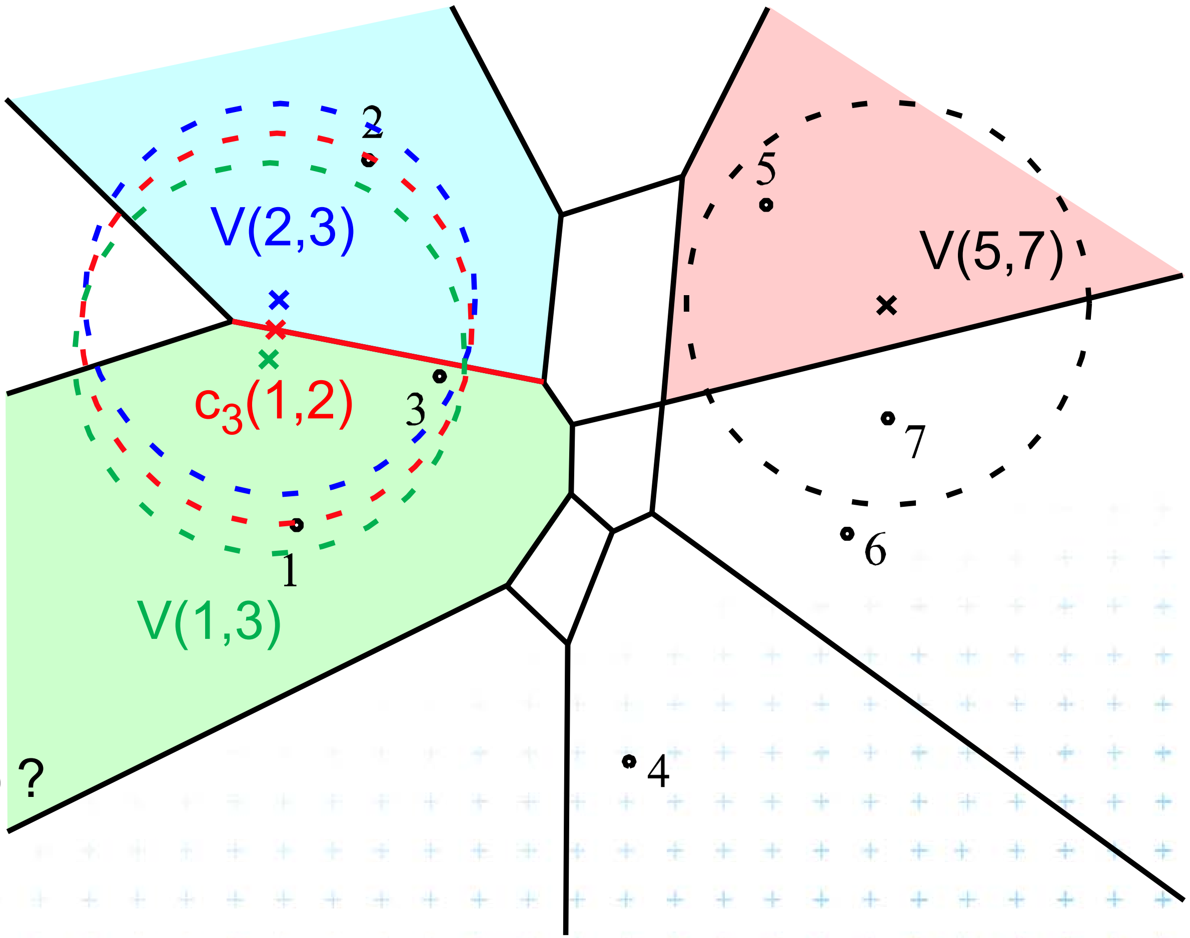
Intersection of all halfplanes except *h*(3,5) and *h*(5,3)

$$\underbrace{\bigcap_{x \neq 5} h(3,x)} \cap \underbrace{\bigcap_{x \neq 3} h(5,x)}$$

# Order-2 Voronoi edges

edge : set of centers of circles passing through 2 sites *s* and *t* and containing one site *p*

=> $c_p(s,t)$




Question

Which are the regions on both sides of $c_p(s,t)$ ?

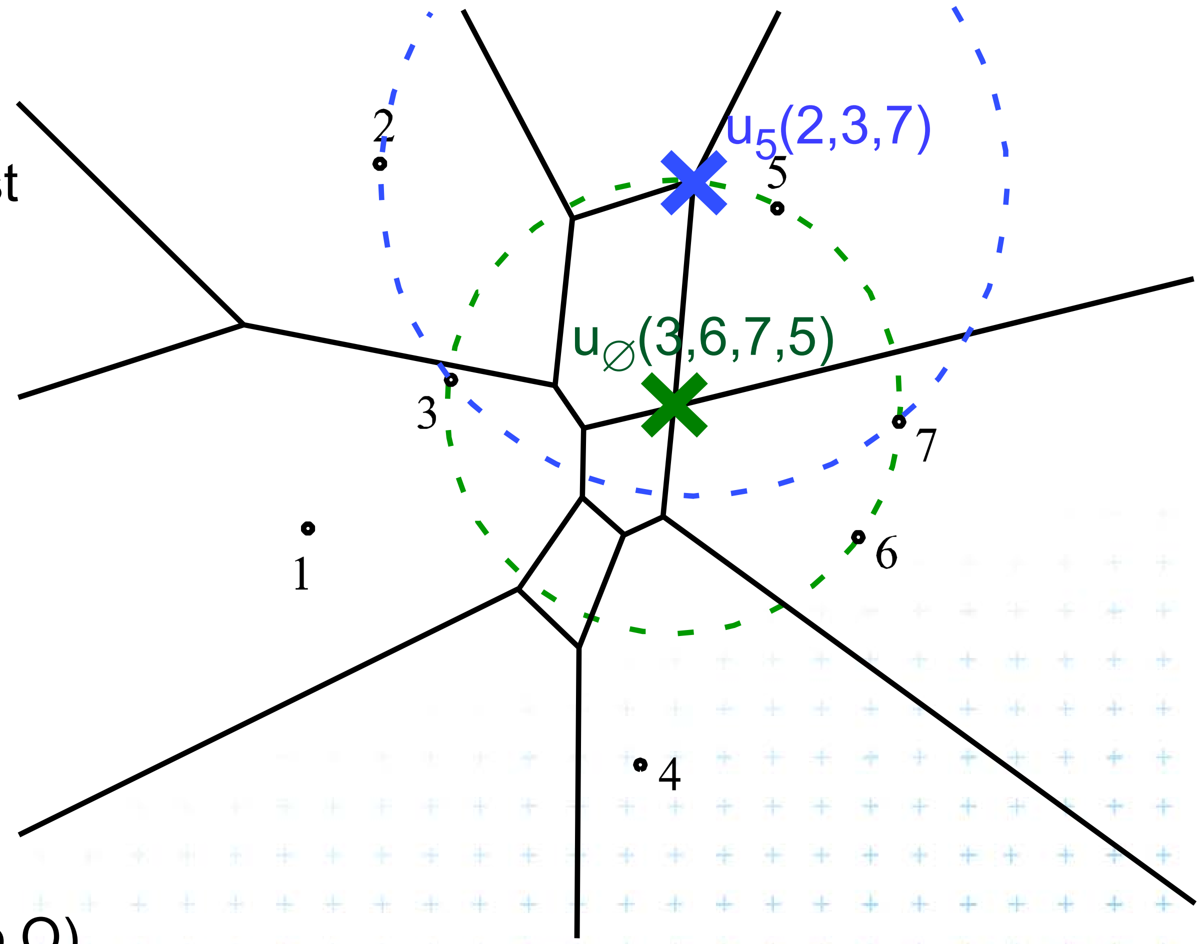
=> V(p,s) and V(p,t)

[Nandy]

# Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

$\Rightarrow u_p(Q)$ or $u_\varnothing(Q \cup p)$
$u_5(2,3,7)$, $u_\varnothing(3,6,7)$




(circle circumscribed to Q)

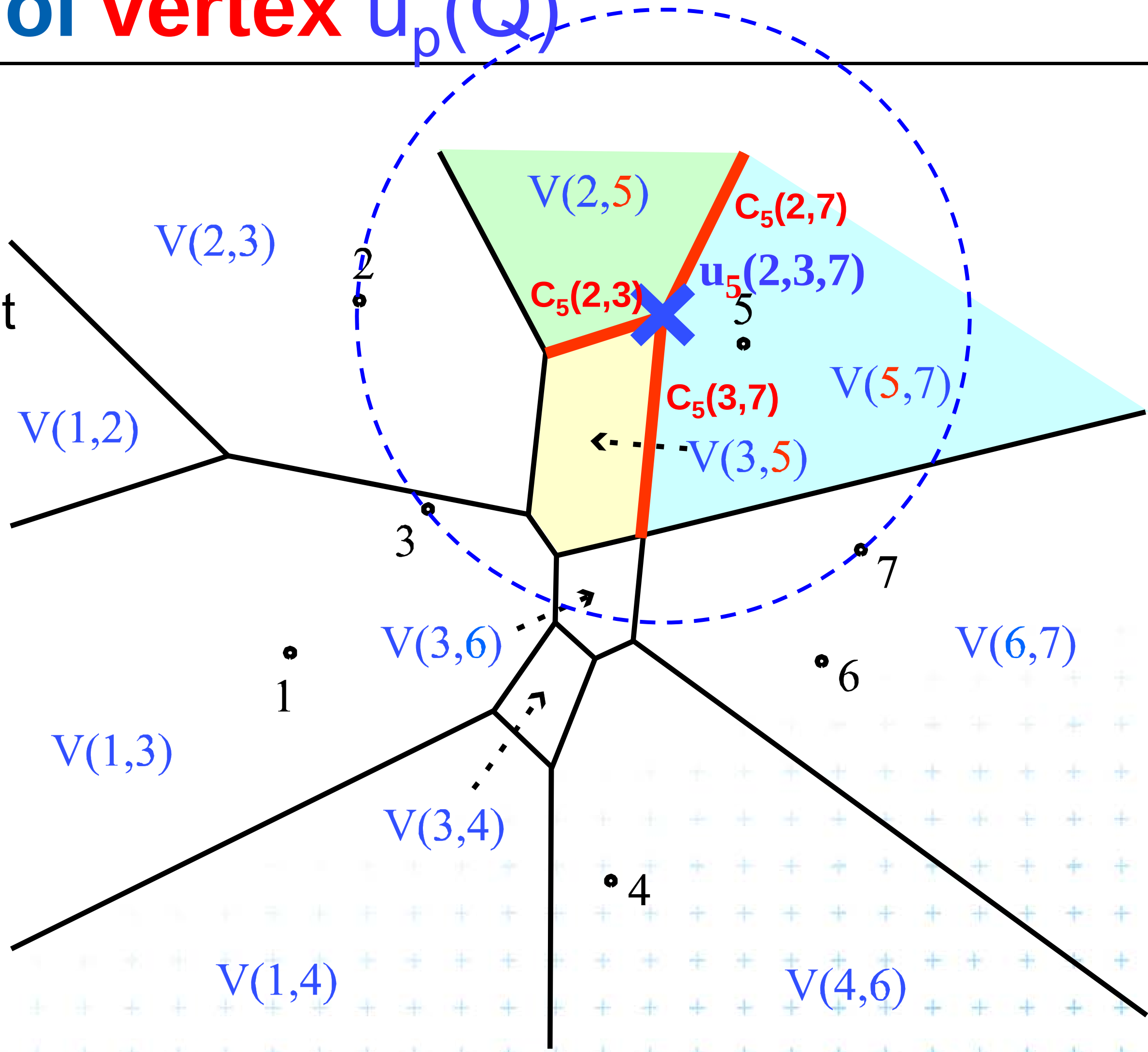
[Nandy]

# Order-2 Voronoi vertex $u_p(Q)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_p(Q)$
$u_5(2,3,7)$

5 is inside for all incident edges:
$C_5(2,3)$
$C_5(2,7)$
$C_5(3,7)$
=> is inside for circle with center in vertex

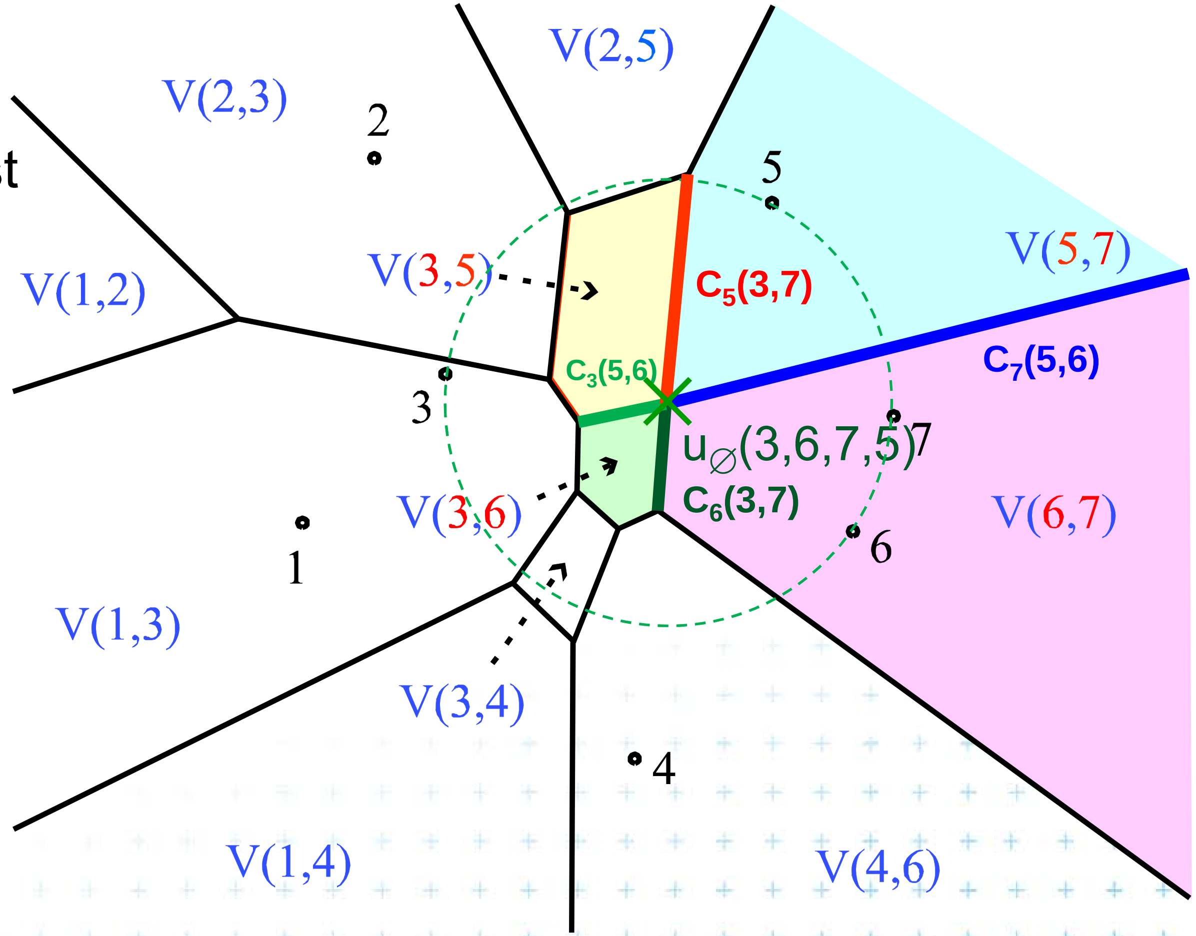



[Nandy]

# Order-2 Voronoi vertex $u_\varnothing(Q \cup p)$

vertex : center of a circle passing through at least 3 sites and containing either site p or nothing

Case $u_\varnothing(Q \cup p)$
$u_\varnothing(3,6,7,5)$




[Nandy]

# Order-k Voronoi Diagram

Theorem věta

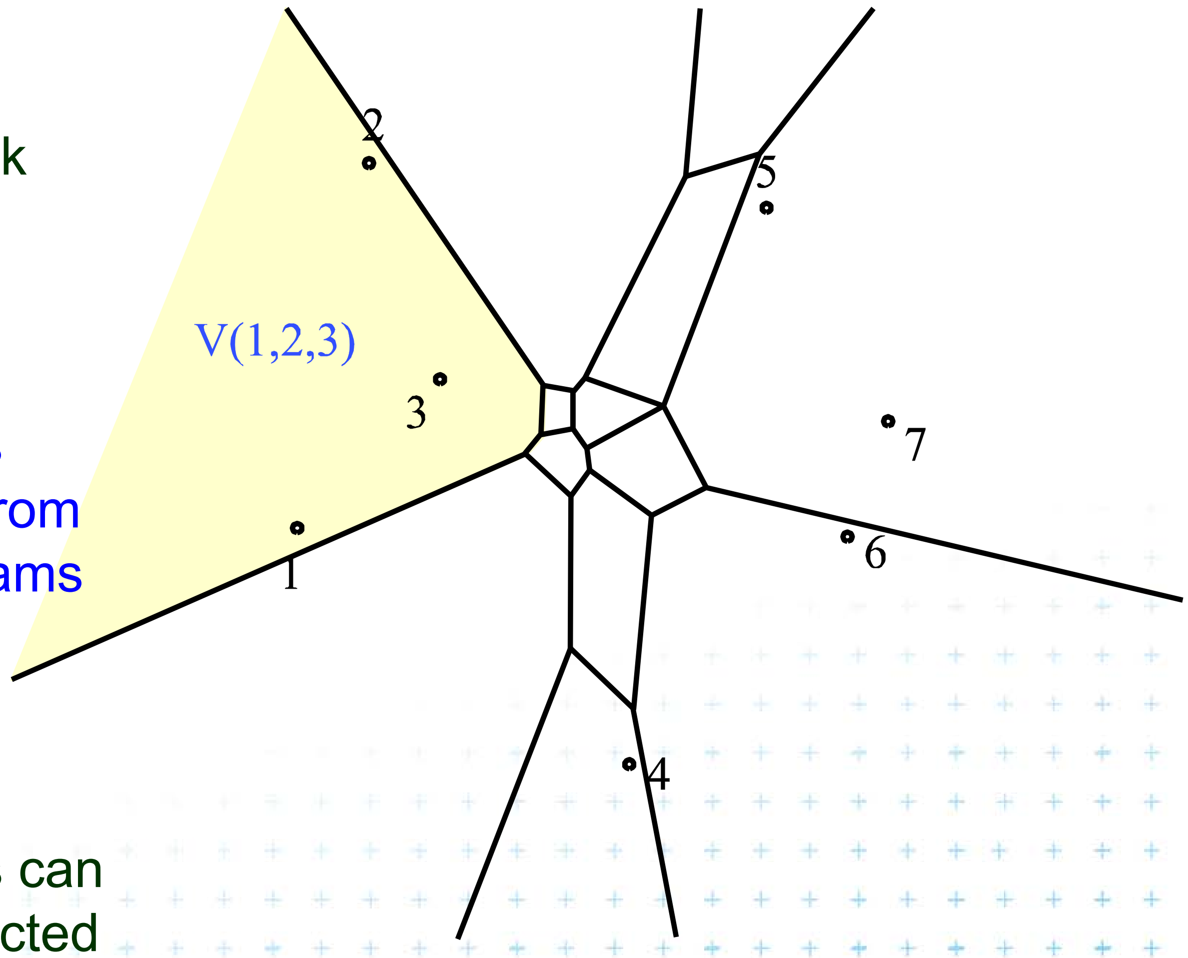
The size of the order-k diagrams is $O(k(n-k))$

Theorem věta

The order-k diagrams can be constructed from the order-(k-1) diagrams in $O(k(n-k))$ time

Corollary důsledek

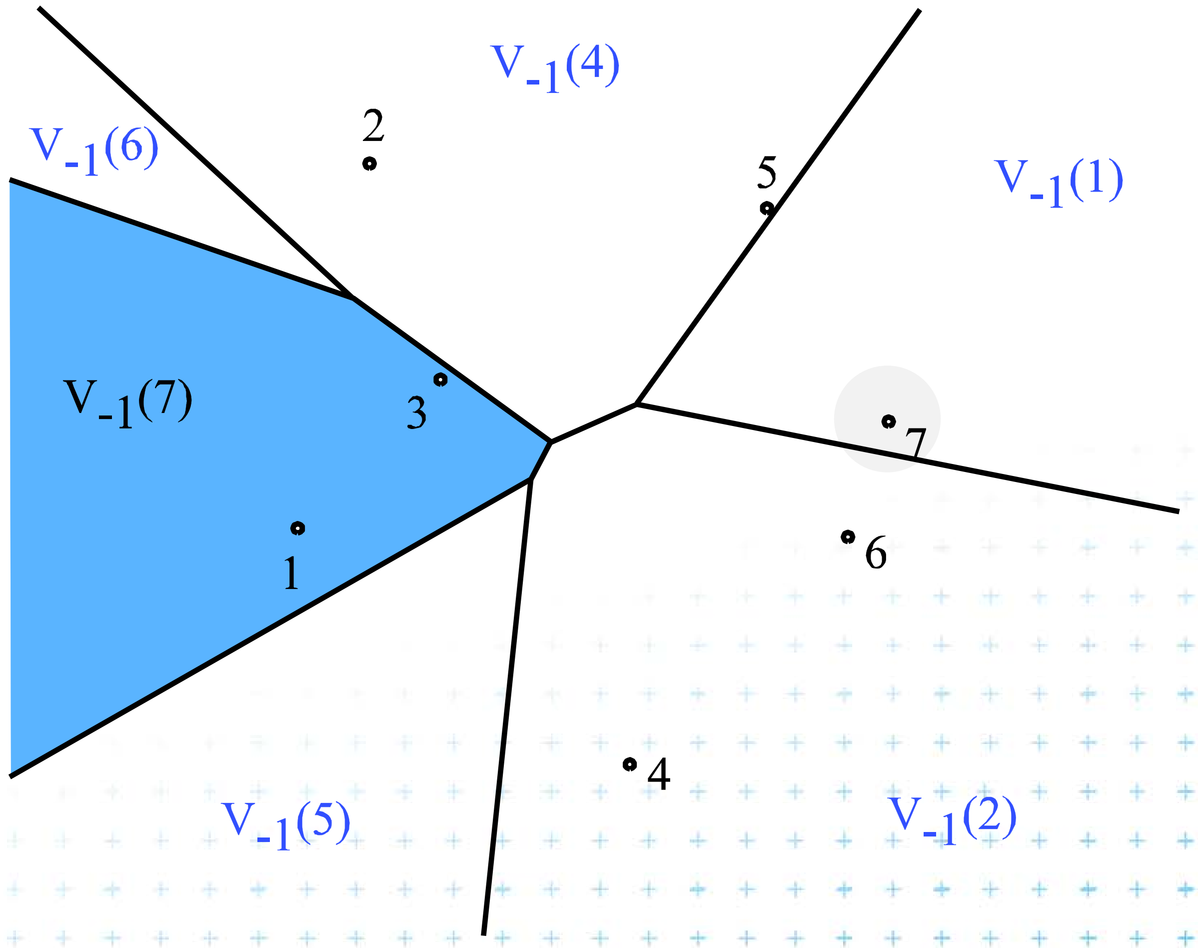
The order-k diagrams can be iteratively constructed in $O(n \log n + k^2(n-k))$ time




[Nandy]

# Order n-1 = Farthest-point Voronoi diagram

cell $V_{-1}(7) = V_{n-1}(\{1,2,3,4,5,6\})$
= set of points in the plane farther from $p_i=7$ than from any other site




$Vor_{-1}(P) = Vor_{n-1}(P)$
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices

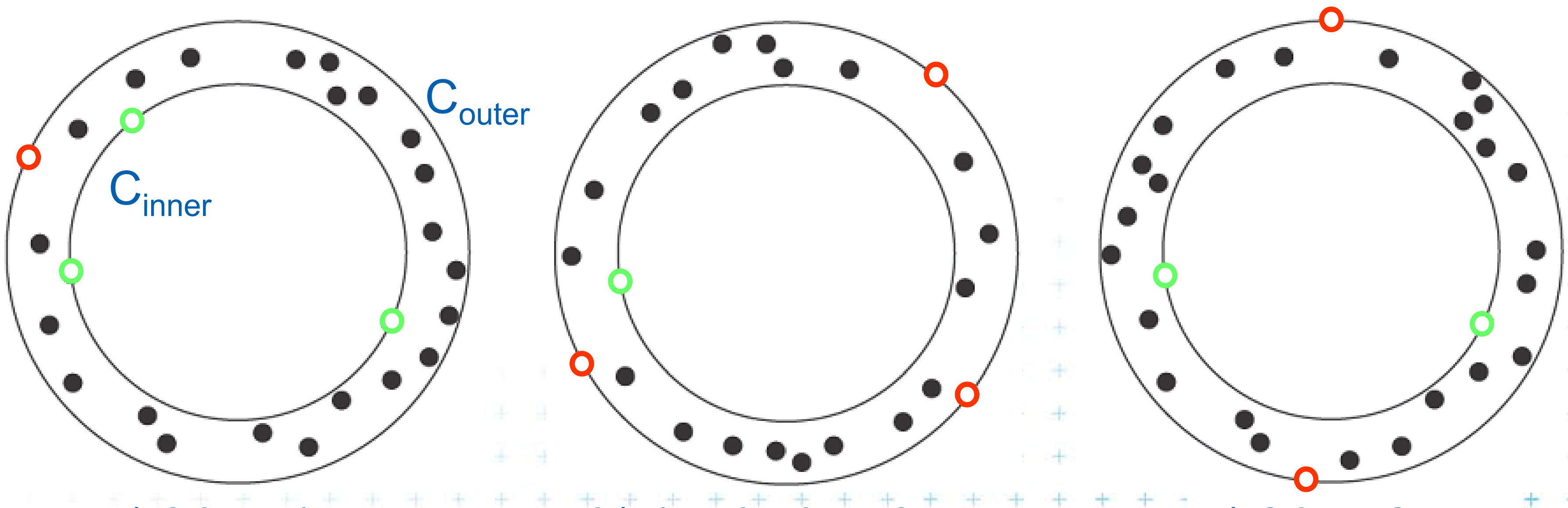
[Nandy]

# Farthest-point Voronoi diagrams example

## Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus mezikruží s nejmenší šířkou (region between two concentric circles $C_{inner}$ and $C_{outer}$)

Three cases to test – one will win:



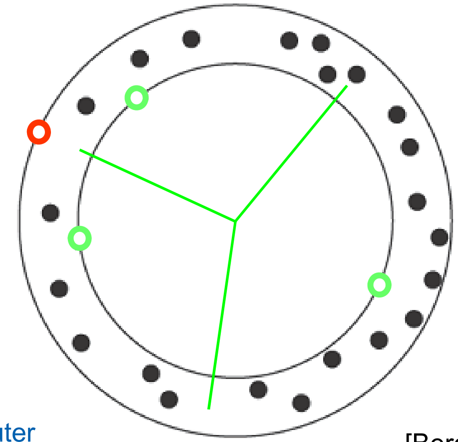

a) 3 in – 1 out    b) 1 point in – 3 out    c) 2 in – 2 out

[Berg]

# Smallest width annulus – cases with 3 pts

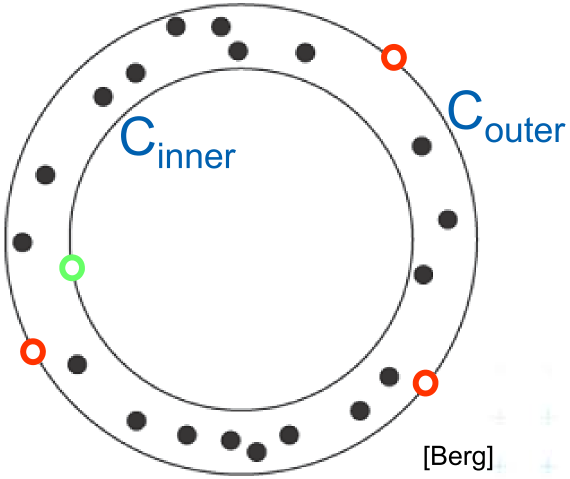
a) $C_{inner}$ contains at least 3 points

- Center is the *vertex of normal Voronoi diagram* (1st order VD)
- The remaining point on $C_{outer}$ in O(n) for each vertex
  - ⇒ not the largest (inscribed) empty circle - as discussed on seminar as we must test all VD vertices in combination with point on C outer
  - ⇒ $O(n^2)$



[Berg]

3 in – 1 out




[Berg]

1 point in – 3 out

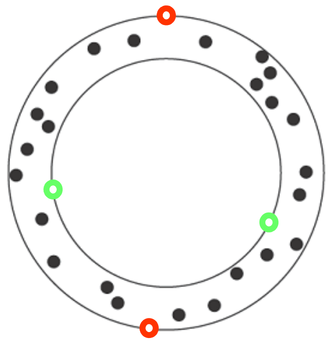
b) $C_{outer}$ contains at least 3 points

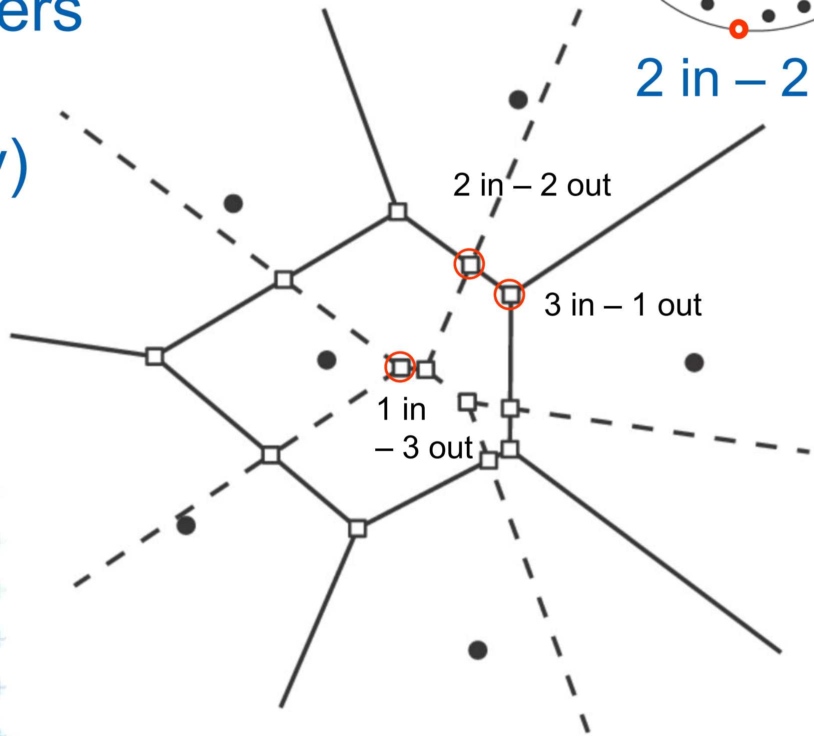
- Center is the *vertex of the farthest Voronoi diagram*
- The remaining point on $C_{inner}$ in O(n)
  - ⇒ not the smallest enclosing circle - as discussed on seminar as we must test all vertices **in combination** with point on C inner
  - ⇒ $O(n^2)$

# Smallest width annulus – case with 2+2 pts

c) $C_{inner}$ and $C_{outer}$ contain 2 points each

- Generate vertices of overlay of Voronoi (___) and farthest-point Voronoi (- - -) diagrams => $O(n^2)$ candidates for centers
  (we need only vertices,
  not the complete overlay)
- annulus computed in O(1) from center and 4 points (same for all 3 cases)
- $O(n^2)$




[Berg]

# Smallest width annulus

**Smallest-Width-Annulus**

*Input:* Set $P$ of $n$ points in the plane

*Output:* Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram Vor($P$) and farthest-point Voronoi diagram $\mathrm{Vor}_{-1}(P)$ of $P$
2. For each vertex of Vor($P$) ($r$) determine the *farthest point* ($R$) from $P$ => $O(n)$ sets of four points defining candidate annuli – case a)
3. For each vertex of $\mathrm{Vor}_{-1}(P)$ ($R$) determine the *closest point* ($r$) from $P$ => $O(n)$ sets of four points defining candidate annuli – case b)
4. For every pair of edges Vor($P$) and $\mathrm{Vor}_{-1}(P)$ test if they intersect => another set of four points defining candidate annulus – c)
5. For all candidates of all three types chose the smallest-width annulus

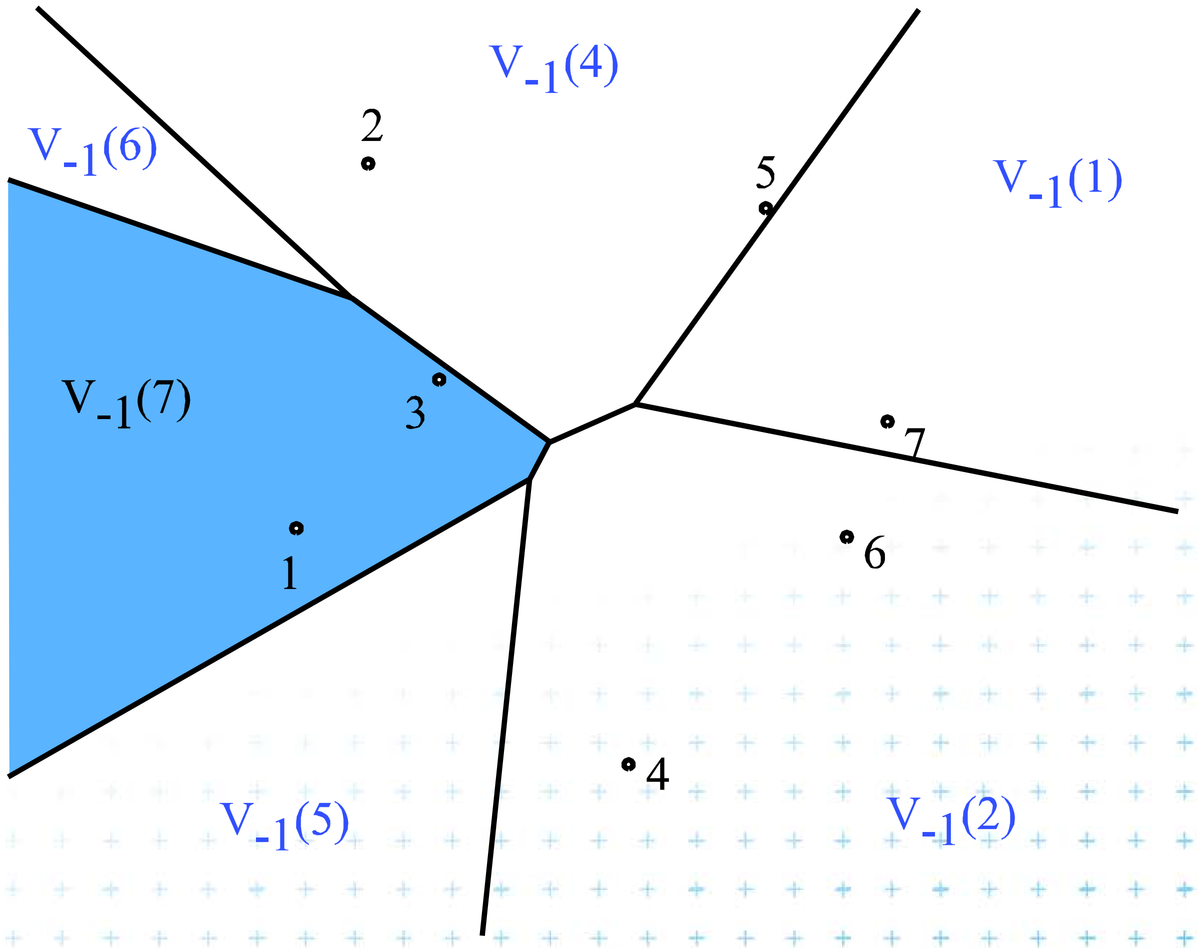
*1.* $O(n \log n)$
*2.* $O(n^2)$
*3.* $O(n^2)$
*4.* $O(n^2)$
*5.* $O(n^2)$

$O(n^2)$ time using $O(n)$ storage

# Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell
= set of points in the plane farther from $p_i$ than from any other site

$Vor_{-1}(P)$ diagram
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices




[Nandy]

# Farthest-point Voronoi region (cell)

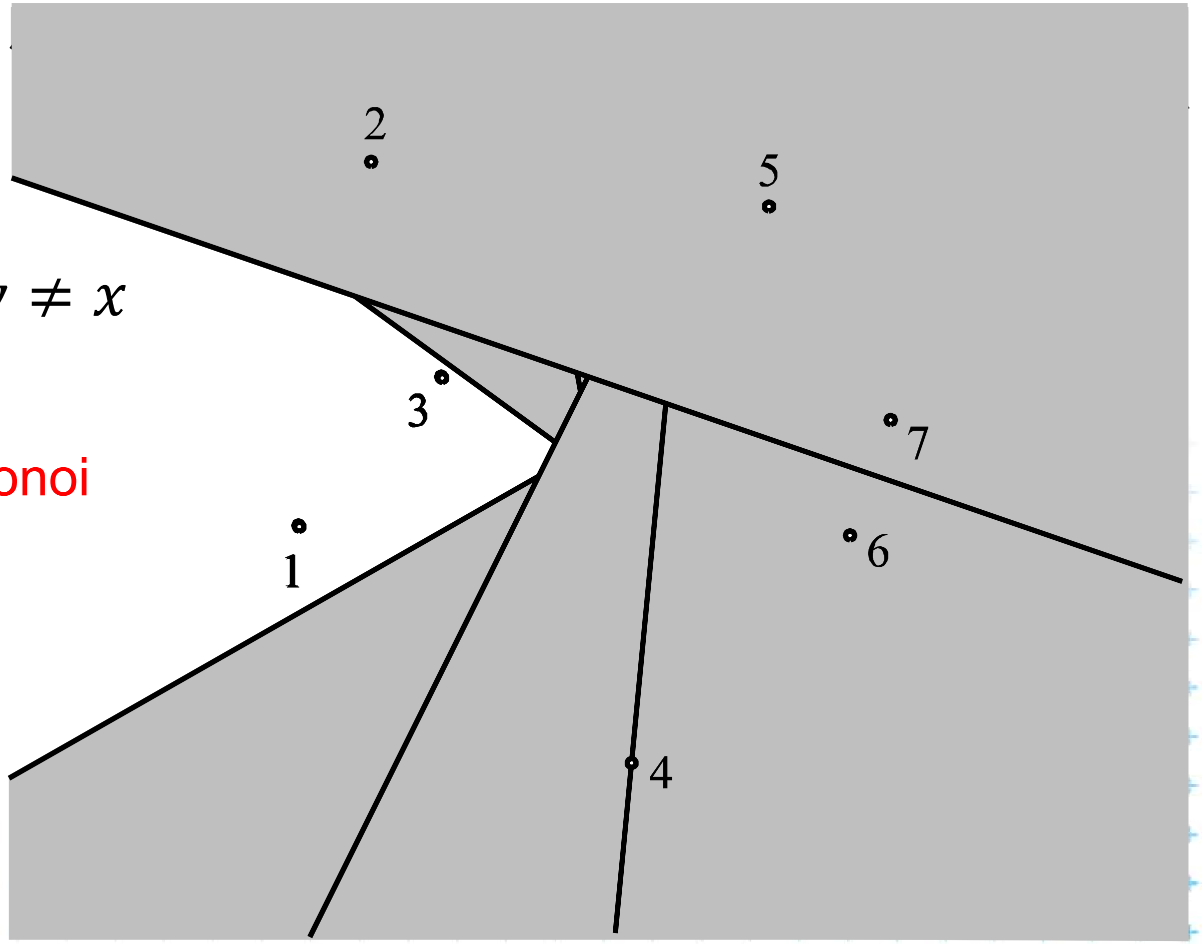
Computed as intersection of halfplanes, but we take "other sides" of bisectors

Construction of $V_{-1}(7)$

$$V_{-1} = \cap_{x=1}^{n} h(y,x),\ y \neq x$$

Property

The farthest point Voronoi regions are convex and unbounded

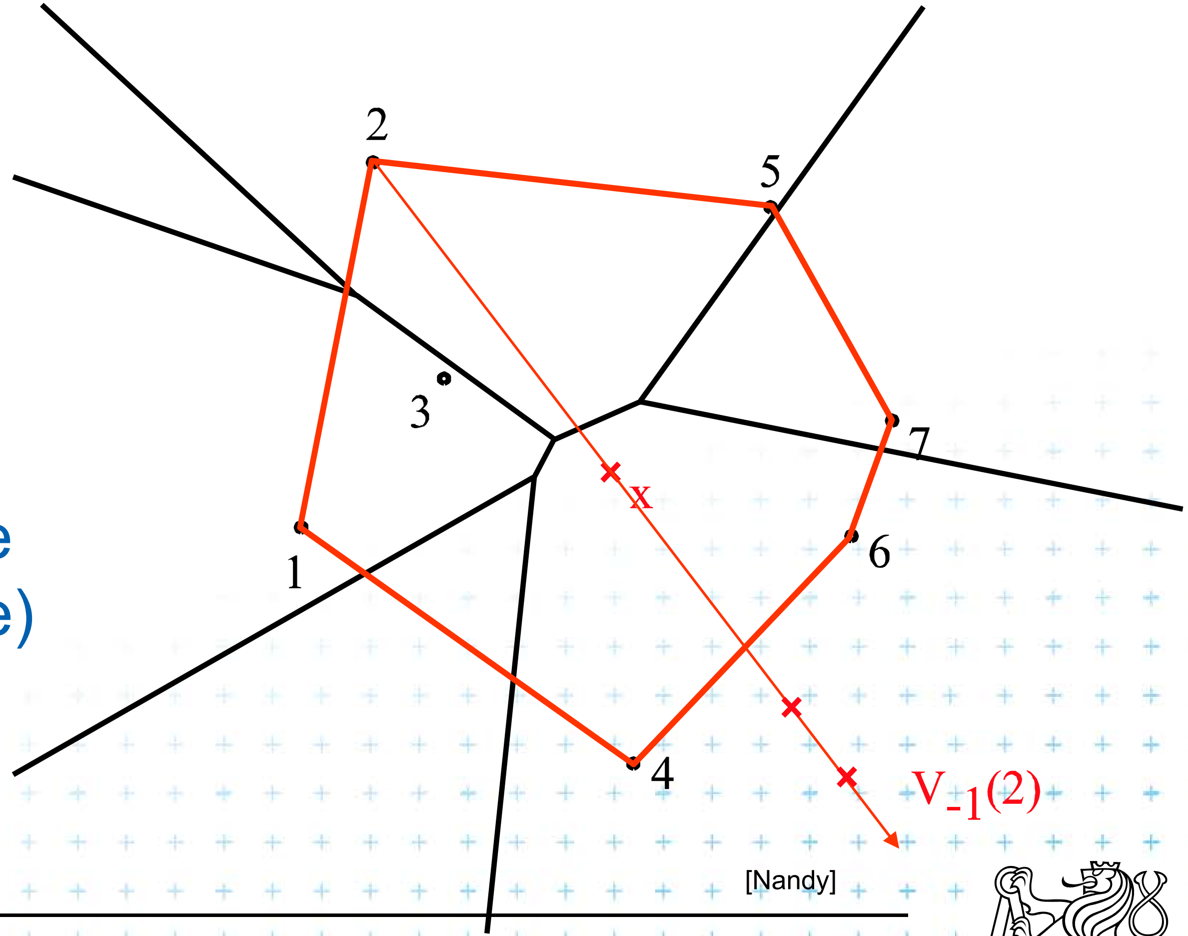



[Nandy]

# Farthest-point Voronoi region

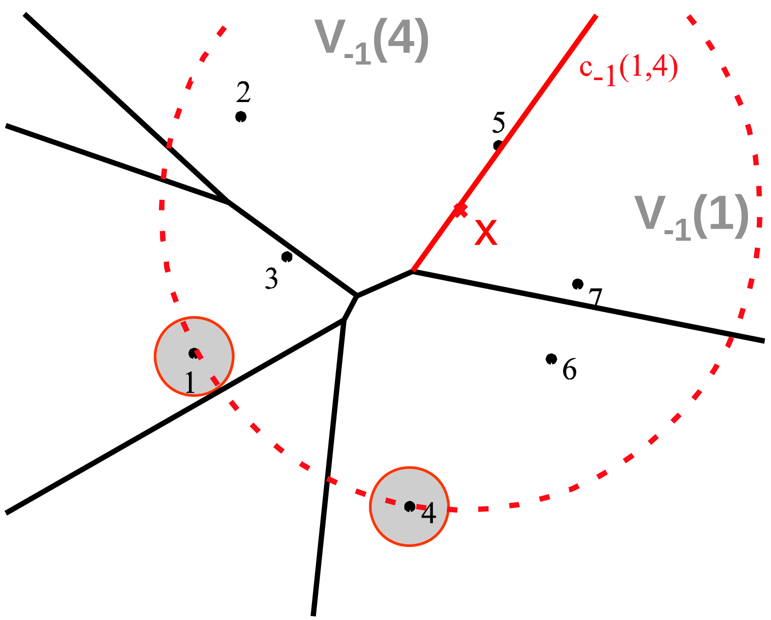
Properties:

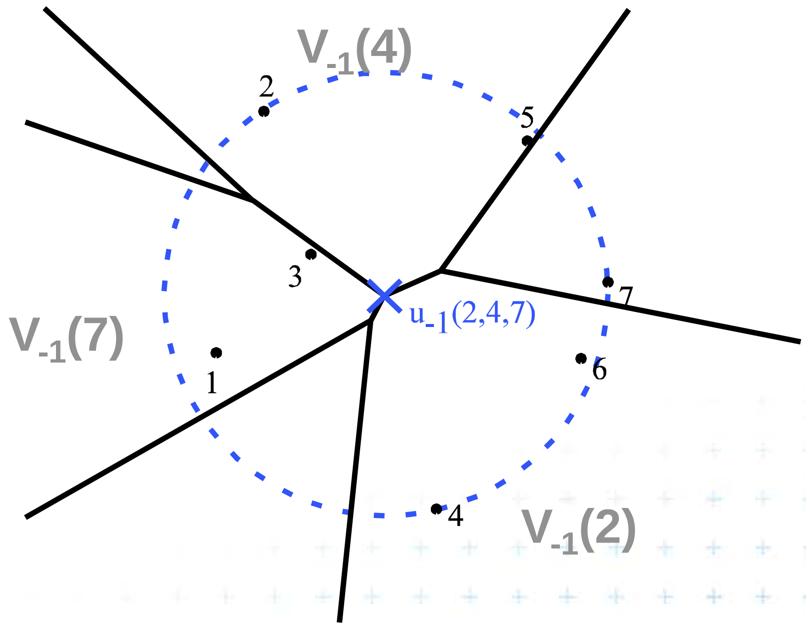
- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded
- The farthest point Voronoi edges and vertices form a tree (in the graph sense)




[Nandy]

# Farthest point Voronoi edges and vertices



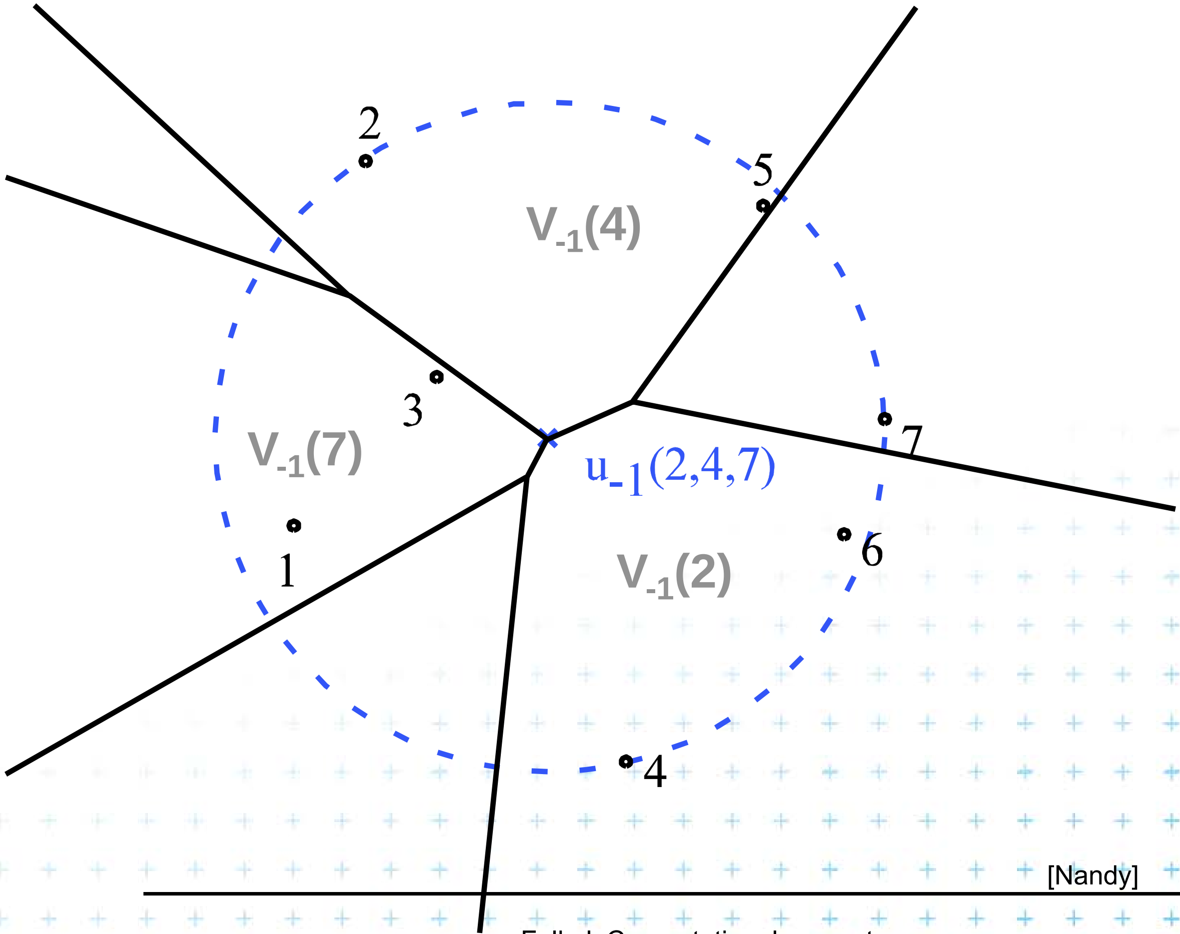

edge : set of points equidistant from 2 sites and closer to all the other sites

vertex : point equidistant from at least 3 sites and closer to all the other sites

[Nandy]

# Application of $Vor_{-1}(P)$ : Smallest enclosing circle

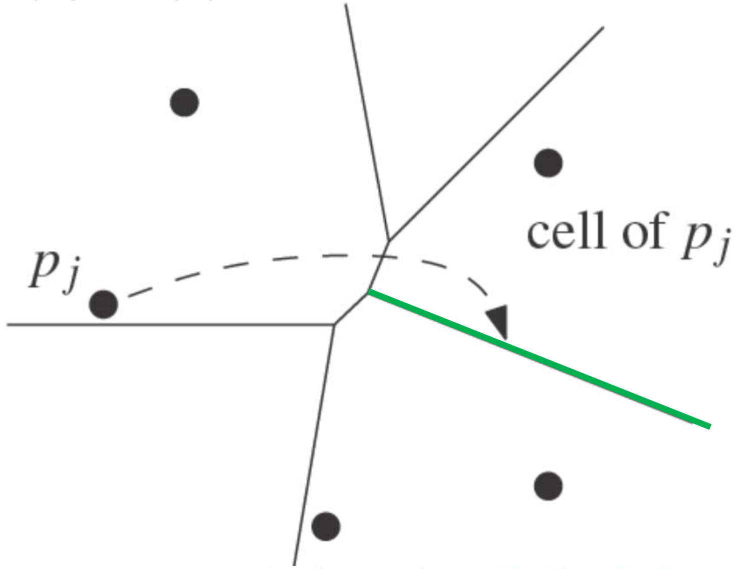
- Construct $Vor_{-1}(P)$ and find minimal circle with center in $Vor_{-1}(P)$ vertices or on edges




[Nandy]

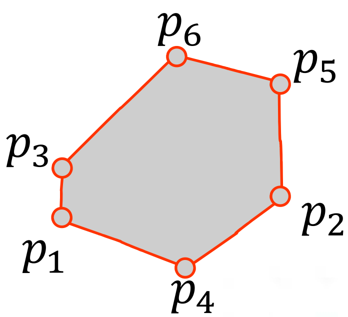
# Modified DCEL for farthest-point Voronoi d

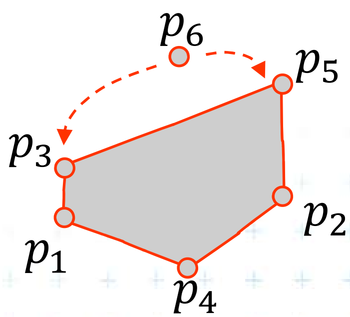
- Half-infinite edges -> we adapt DCEL
- Half-edges with origin in infinity
  - Special vertex-like record for origin in infinity
  - Store direction instead of coordinates
  - Next(e) or Prev(e) pointers undefined
- For each inserted site $p_j$
  - store a pointer to the most CCW half-infinite half-edge of its cell in DCEL



# Idea of the algorithm

1. Create the convex hull
   and number the CH points randomly
2. Remove the points starting in the last of this random order and store *cw*($p_i$) and *ccw*($p_i$) points at the time of removal.
3. Include the points back and compute $V_{-1}$




| $p_i$ | $ccw(p_i)$ | $cw(p_i)$ |
|---|---|---|
| $p_6$ | $p_3$ | $p_5$ |
| $p_5$ | $p_3$ | $p_2$ |
| … | | |

# Farthest-point Voronoi d. construction

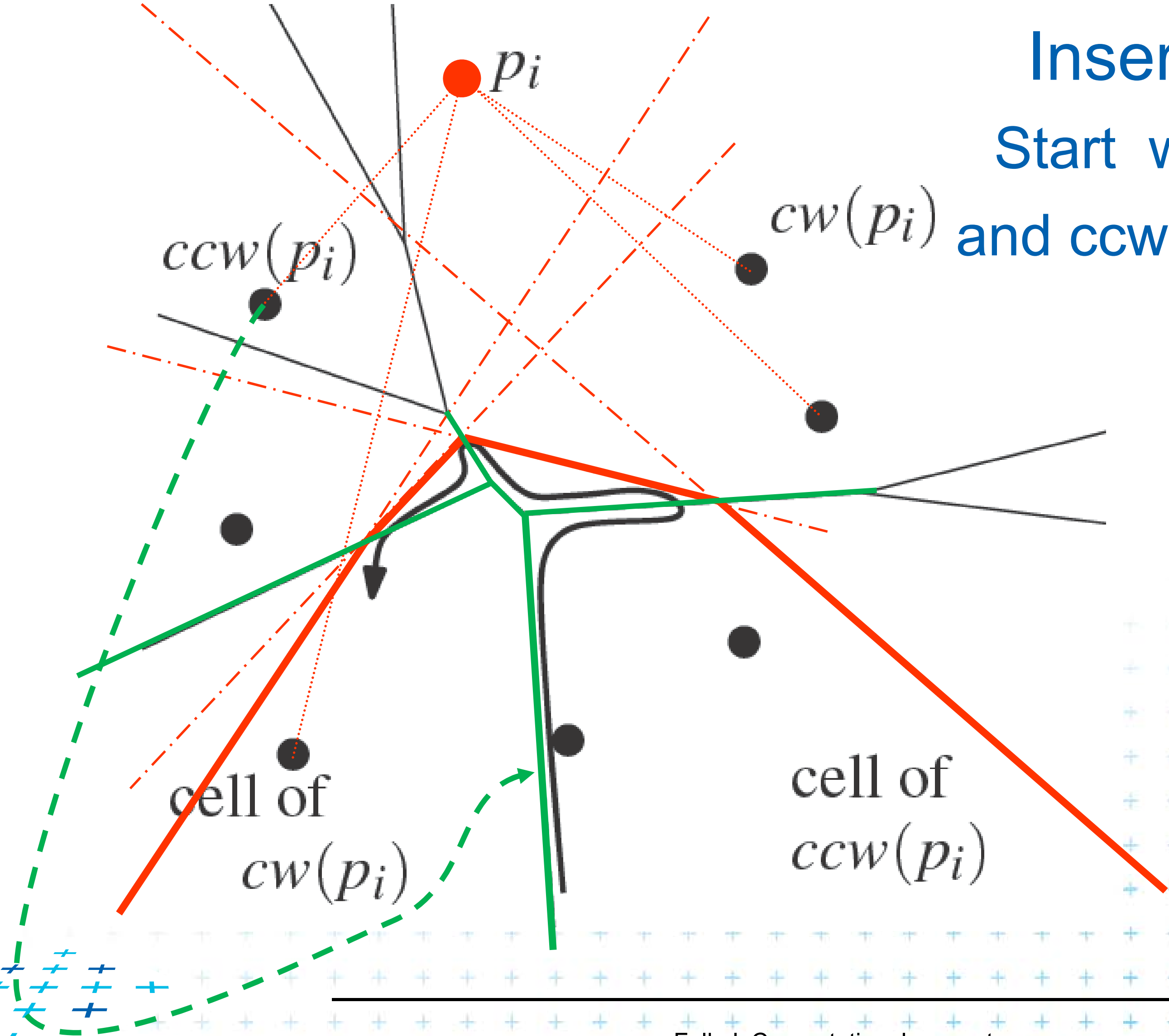
**Farthest-pointVoronoi** O(nlog n) time in O(n) storage

*Input:* Set of points *P* in plane

*Output:* Farthest-point VD $Vor_{-1}(P)$

1. Compute convex hull of *P*
2. Put points in CH(*P*) of *P* in random order $p_1,\ldots,p_h$
3. Remove $p_h, \ldots, p_4$ from the cyclic order (around the CH).
   When removing $p_i$, store the neighbors: $cw(p_i)$ and $ccw(p_i)$ at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $Vor_{-1}(\{ p_1, p_2, p_3 \})$ as init
5. **for** i = 4 **to** *h* **do**
6. Add site $p_i$ to $Vor_{-1}(\{ p_1, p_2,\ldots, p_{i-1} \})$ between site $cw(p_i)$ and $ccw(p_i)$
7. - start at most CCW edge of the cell $ccw(p_i)$
8. - continue CW to find intersection with bisector( $ccw(p_i), p_i$ )
9. - trace borders of Voronoi cell $p_i$ in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell $p_i$

# Farthest-point Voronoi d. construction

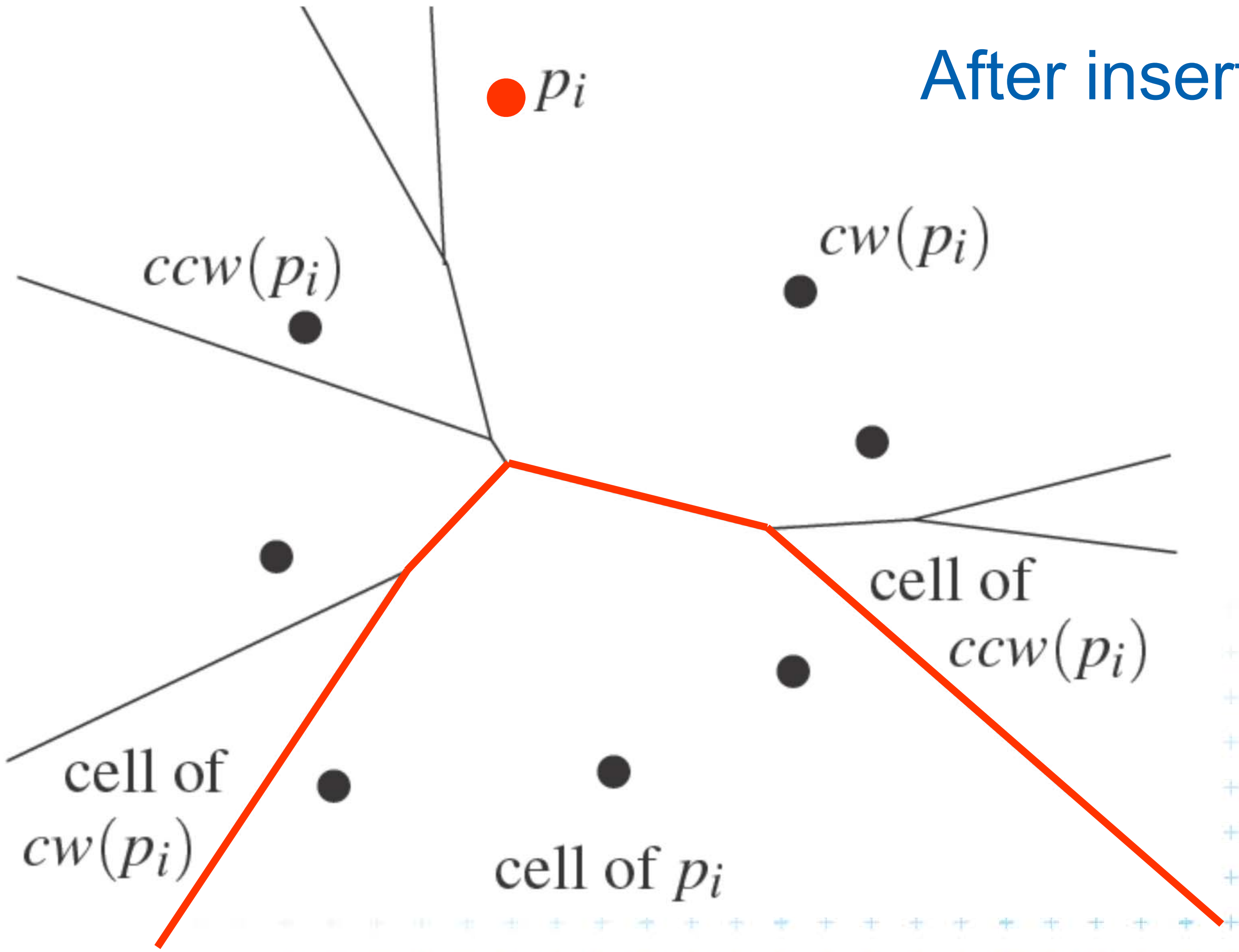



Insertion of site $p_i$

Start with site ccw($p_i$) and ccw edge of its cell

# Farthest-point Voronoi d. construction

After insertion of site $p_i$



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