VORONOI DIAGRAM PART II

PETR FELKEL

FEL CTU PRAGUE

felkel@fel.cvut.cz

https://cw.felk.cvut.cz/doku.php/courses/a4m39vg/start

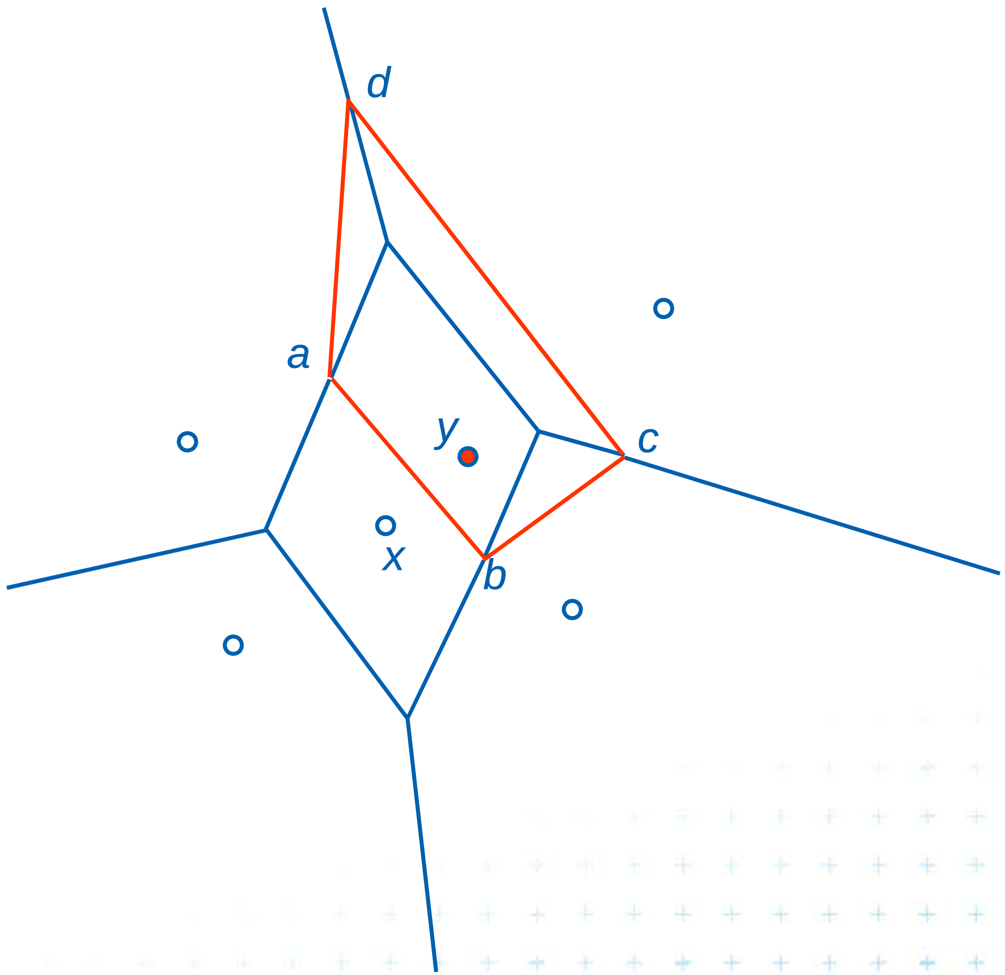
Based on [Berg], [Reiberg] and [Nandy]

Version from 8.11.2012

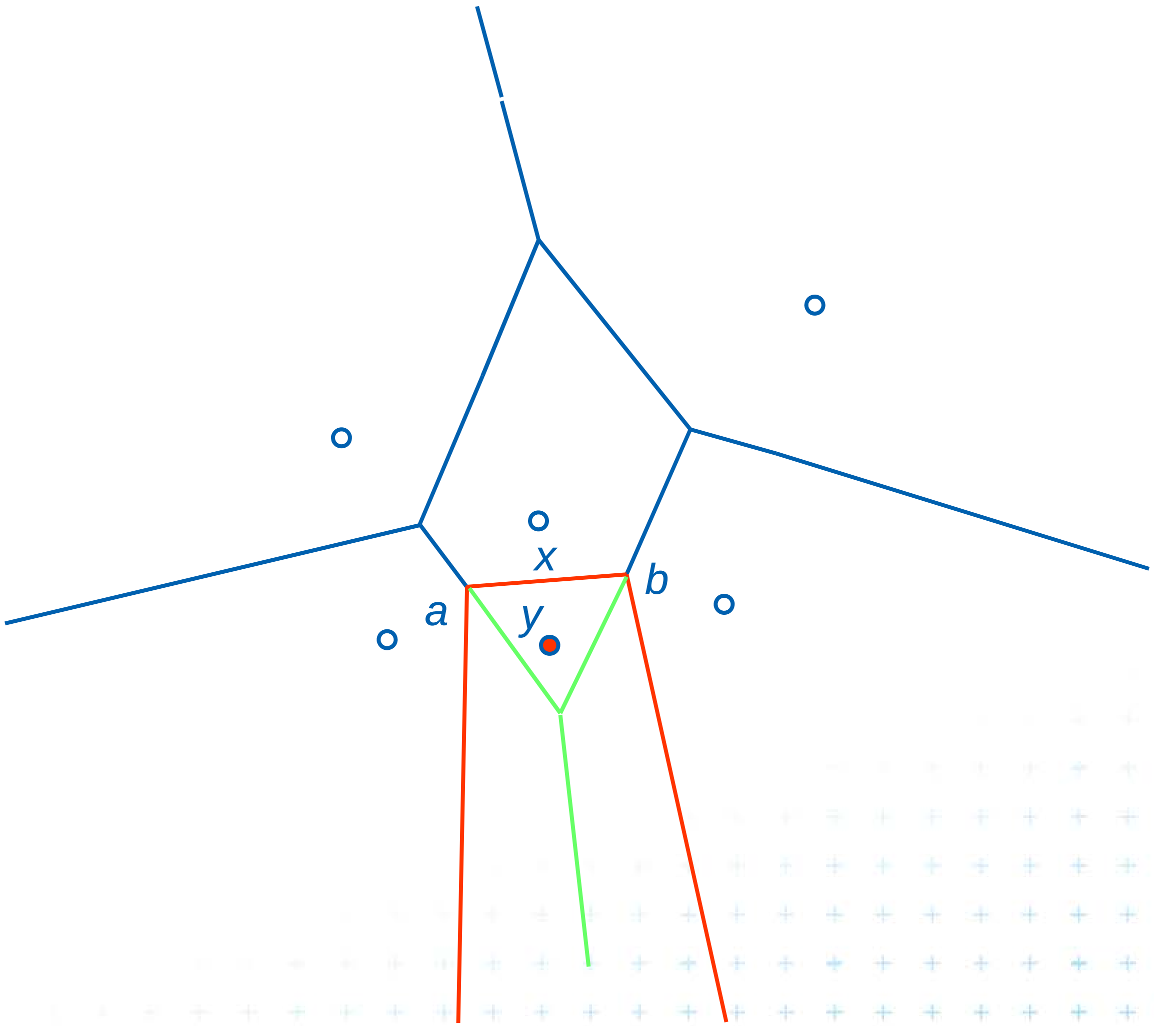
Talk overview

- Incremental construction
- Voronoi diagram of line segments
- VD of order k
- Farthest-point VD

Incremental construction – bounded cell



Incremental construction – unbounded cell



Incremental construction algorithm

InsertPoint(*S*, Vor(*S*), y) **… y = a new site**

Input: Point set *S*, its Voronoi diagram, and inserted point $y \notin S$

Output: VD after insertion of *y*

1. Find the cell V(*x*) in which *y* falls …O(log *n*)
2. Detect the intersections {*a*,*b*} of bisector *L*(*x*,*y*) with boundary of cell *V*(*x*)
   => * first edge *e* = *ab* on the border of cells of sites *x* and *y* …O(*n*)
3. *p* = *a*, site *z* = neighbor site across the border with point *a* …O(1)
4. **while**(exists*(p)* and z ≠ a) // trace the bisectors from *a* in one direction
   a. Detect the intersection *c* of bisector *L*(*z*,*y*) with *V*(*z*)
   b. Report Voronoi edge *pc*
   c. *p* = *c*

   …O(n^2)
5. **if**(c ≠ a) **then** p = b …O(1)
6. **while**(exists*(p)* and z ≠ a) // trace the bisectors from *b* in other direction
   1. Detect the intersection *c* of bisector *L*(*z*,*y*) with *V*(*z*)
   2. Report Voronoi edge *pc*
   3. *p* = *c*

   …O(n^2)

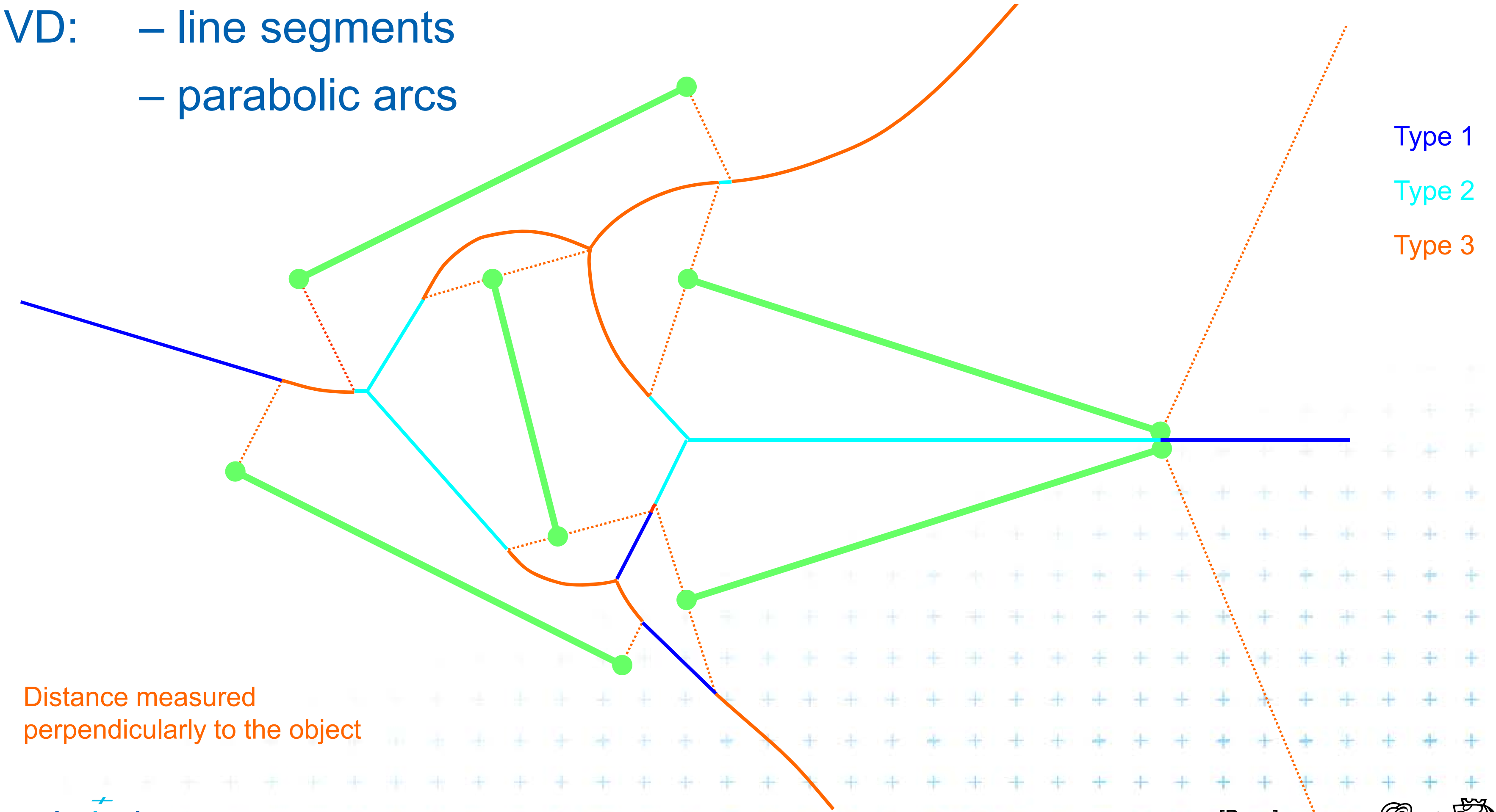
O(n^2) worst-case, O(*n*) expected time for some distributions

Voronoi diagram of line segments

Input: $S = \{s_1, \ldots, s_n\}$ = set of n disjoint line segments (sites)

VD: – line segments

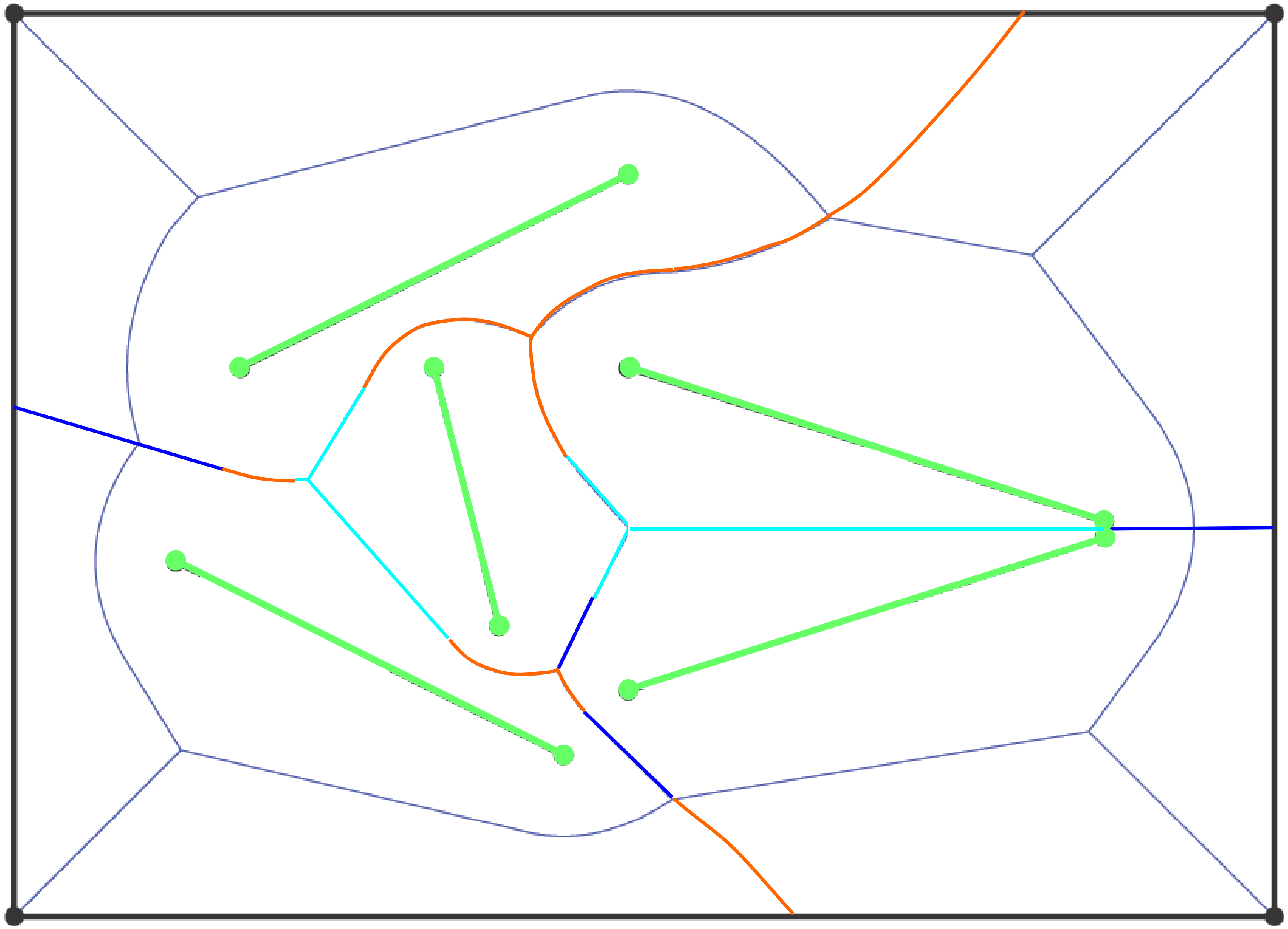
– parabolic arcs



Distance measured perpendicularly to the object

[Berg]

VD of line segments with bounding box



BBOX
=>
standard
DCEL

[Berg]

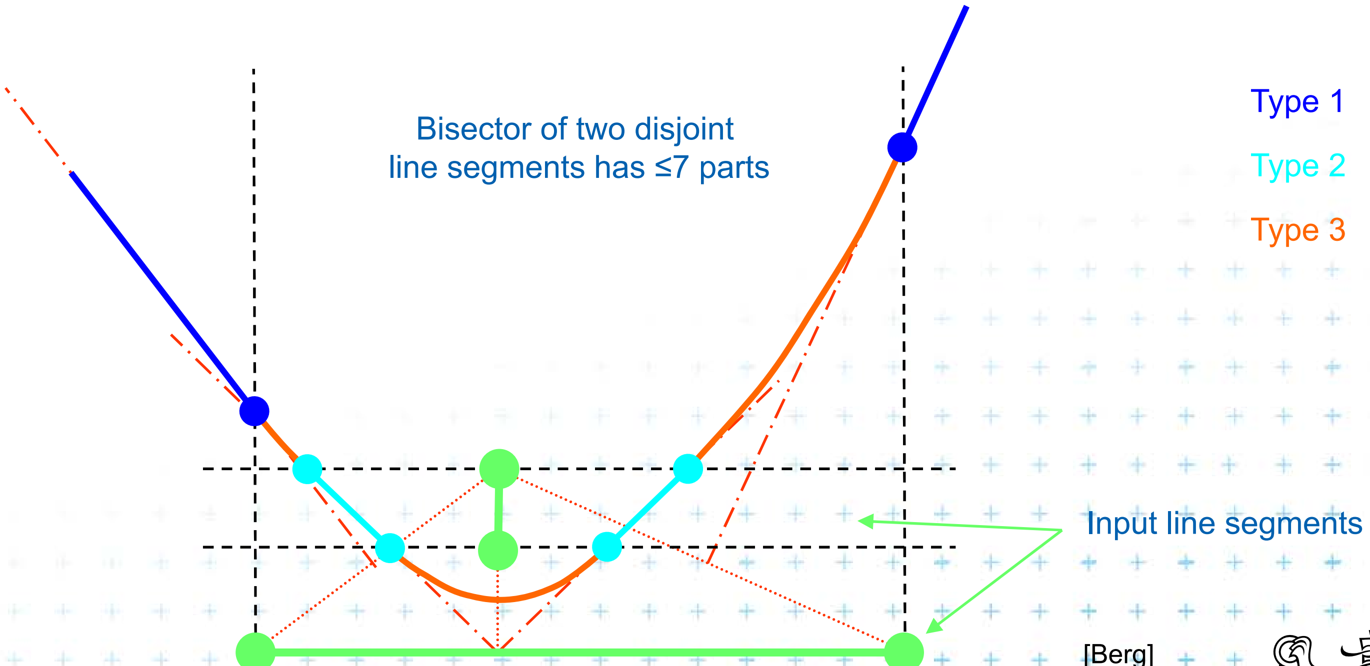
Bisector of 2 line-segments in detail

- Consists of line segments and parabolic arcs

  Distance from point-to-object is measured to the closest point on the object (perpendicularly to the object silhouette)

  – Line segment – bisector of end-points or of interiors

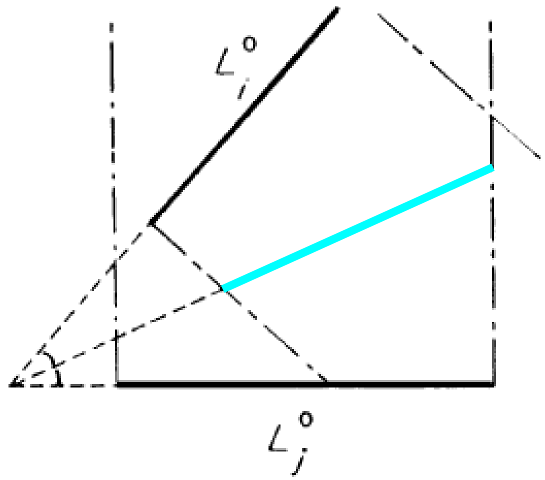
  – Parabolic arc – of point and interior of a line segment



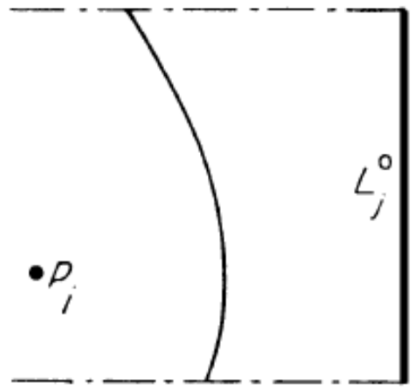
[Berg]

Bisector in greater details

Type 2



Type 3



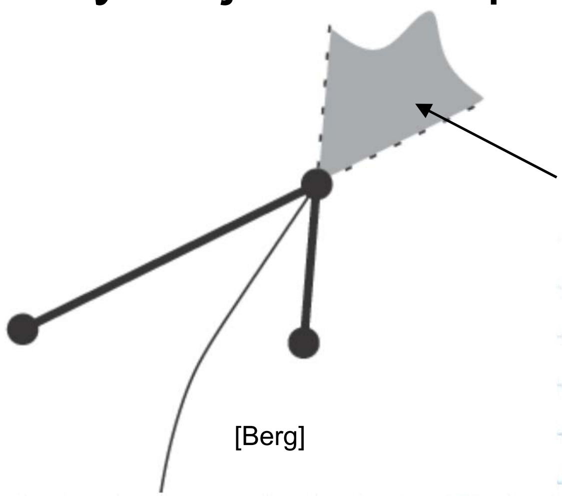
[Reiberg]

Bisector of two line segment interiors (in intersection of perpendicular slabs only)

Bisector of (end-)point and line segment interior

Voronoi diagram of line segments

- More complex bisectors of line segments
  - line segments and parabolic arcs
- Still combinatorial complexity of O(n)
- Assumptions on the input line segments:
  - non-crossing
  - strictly disjoint end-points (slightly shorten the segm.)



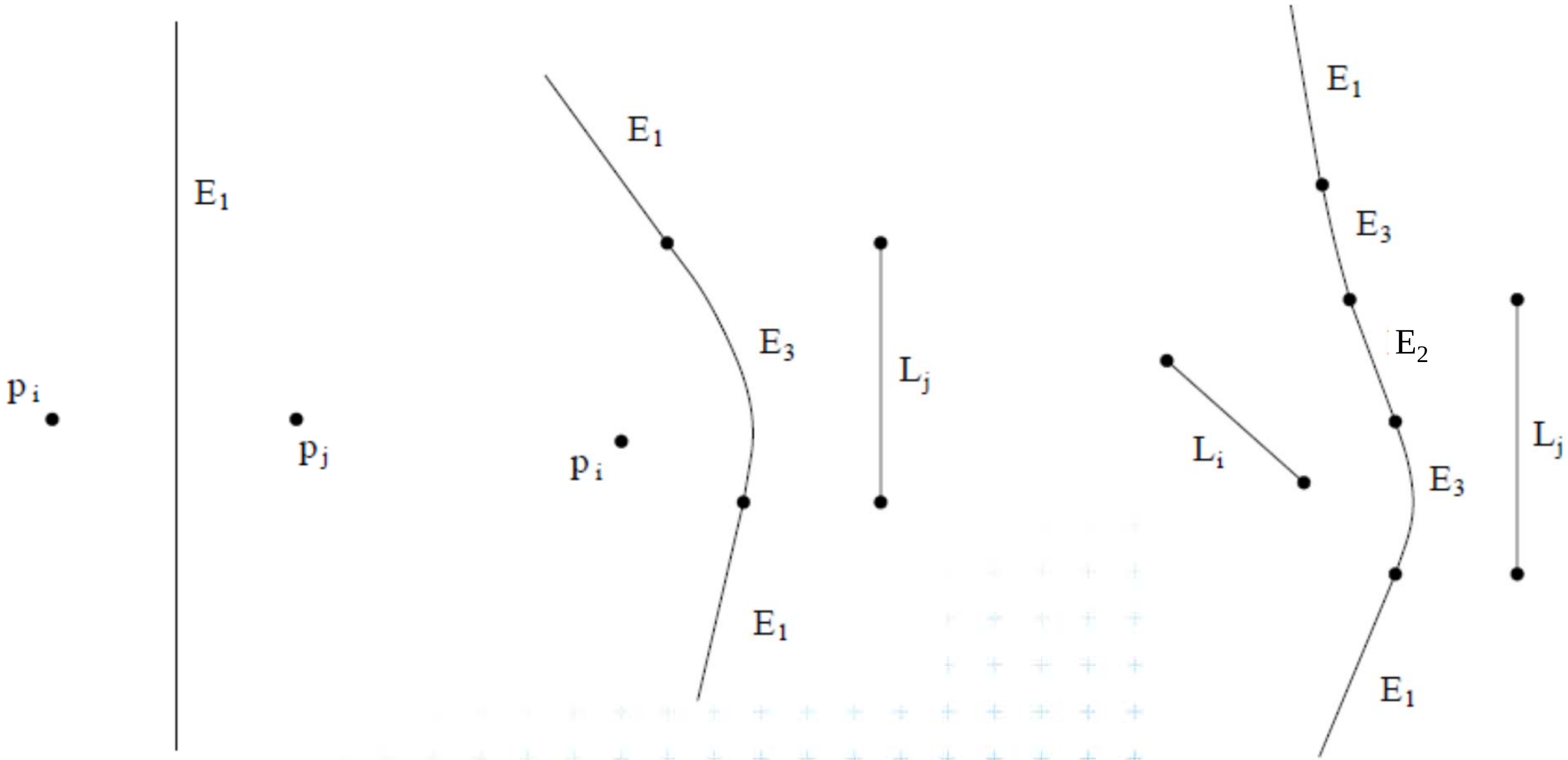
[Berg]

if(we allow touching segments)

Shared endpoints cause complication:
The whole region is equally close
to two line segments

VD of points and line segments examples

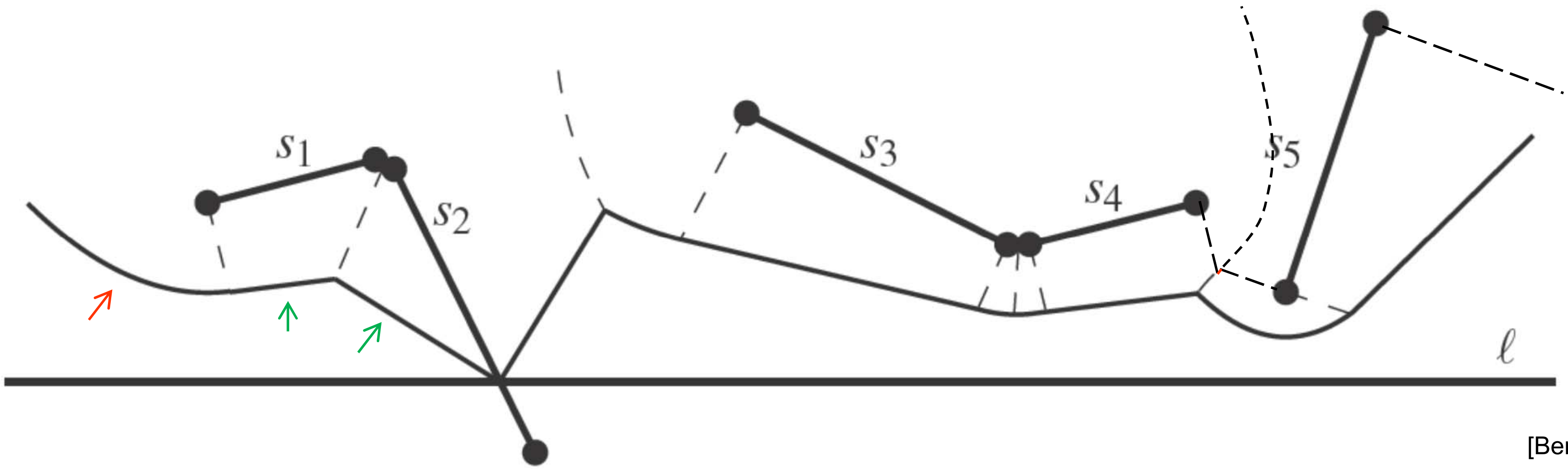
2 points Point & segment 2 line segments



[Reiberg]

Beach line

Note: site = line segment



[Berg]

= Points with distance to the closest site above sweep line l equal to the distance to l

- Beach line contains
  - *parabolic arcs* when closest to one site end-point
  - *straight line segments* when closest to a site interior (just the part of interior above l for intersection s with l)

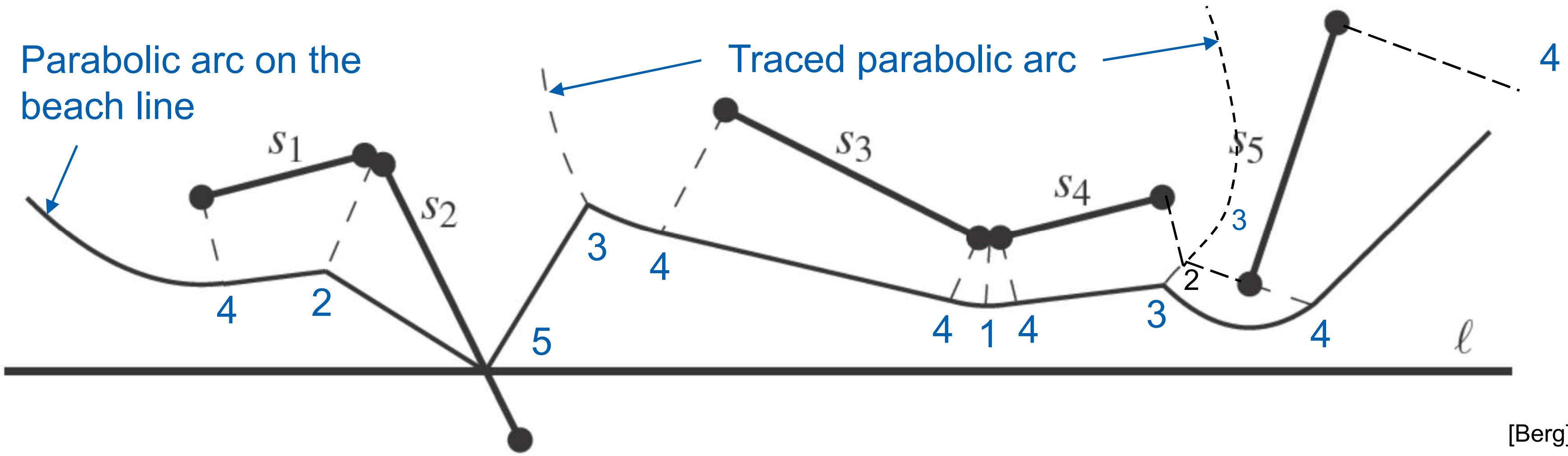
(This is the shape of the beach line)

Beach line breakpoints types

- Point p is equidistant from l and (equidistant and closest to)
  1. two site end-points => traces a VD line segment
  2. two site interiors => traces a VD line segment
  3. end-point and interior => traces a VD parabolic arc
  4. one site end-point => traces a line segment (border of the slab perpendicular to the site)
  5. site interior intersects the scan line l => intersection traces a line segment

Cases 4 and 5 involve only one site and therefore do not form a Voronoi diagram arc (used by alg. only)

Breakpoints types and what they trace



[Berg]

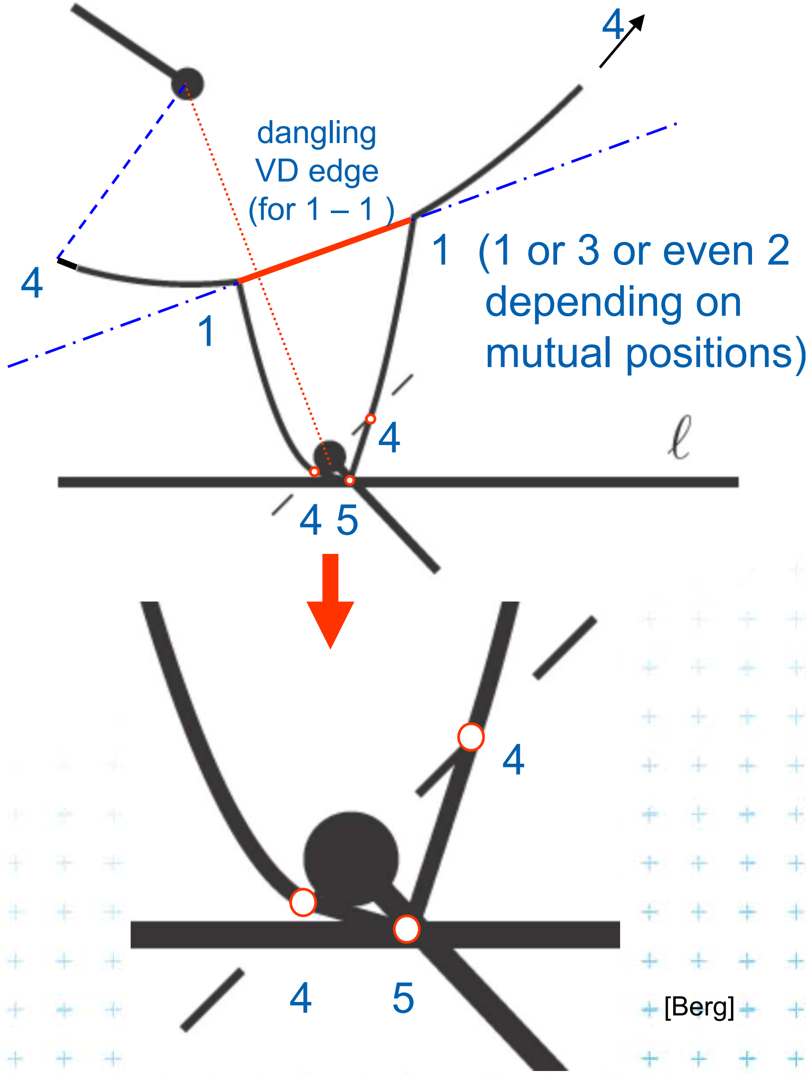
- 1,2 trace a line segment (part of VD edge) DRAW
- 3 traces a parabolic arc (part of VD edge) DRAW
- 4,5 trace a line segment (used only by the algorithm) MOVE
  - 4 limits the slab perpendicular to the line segment
  - 5 traces the intersection of input segment with a sweep line

(This is the shape of the traced VD arcs)

Site event – sweep line reaches an endpoint

I. At upper endpoint of ●—●

- Arc above is split into two
- 4 new arcs are created (2 segments + 2 parabolas)
- Breakpoints for 2 segments are of type 4-5-4
- Breakpoints for parabolas depend on the surrounding sites
  - Type 1 for two end-points
  - Type 3 for endpoint and interior
  - etc…

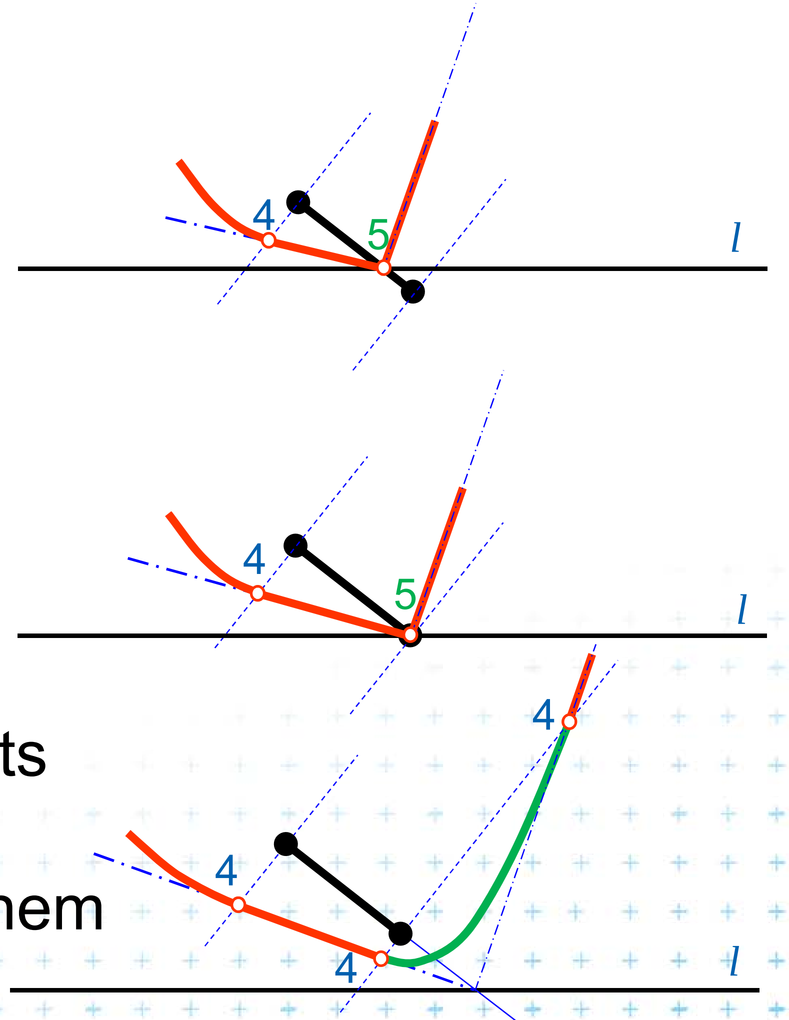


[Berg]

Site event – sweep line reaches an endpoint

II. At lower endpoint of

- Intersection with interior (breakpoint of type 5)
- is replaced by two breakpoints (of type 4) with parabolic arc between them

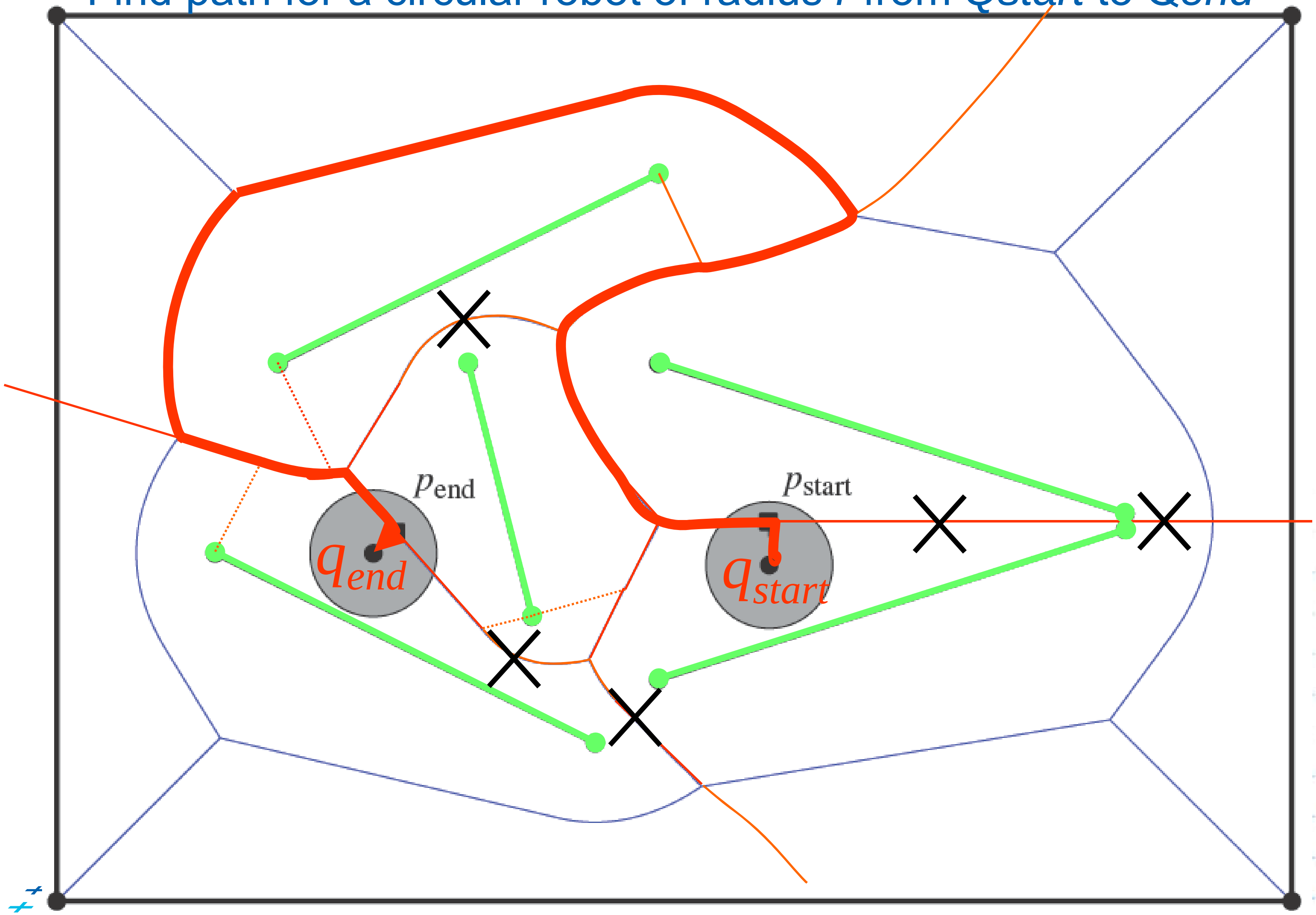


Circle event – lower point of circle of 3 sites

- Two breakpoints meet (on the beach-line)
- Solution depends on their type
  - Any of first three types meet
    - 3 sites involved – Voronoi vertex created
  - Type 4 with something else
    - two sites involved – breakpoint changes its type
    - Voronoi vertex not created
      (Voronoi edge may change its shape)
  - Type 5 with something else
    - never happens for disjoint segments
      (meet with type 4 happens before)

Motion planning example - retraction

Find path for a circular robot of radius *r* from *Qstart* to *Qend*



[Berg]

Motion planning example - retraction **Rušení hran**

Find path for a circular robot of radius *r* from Q_{start} to Q_{end}

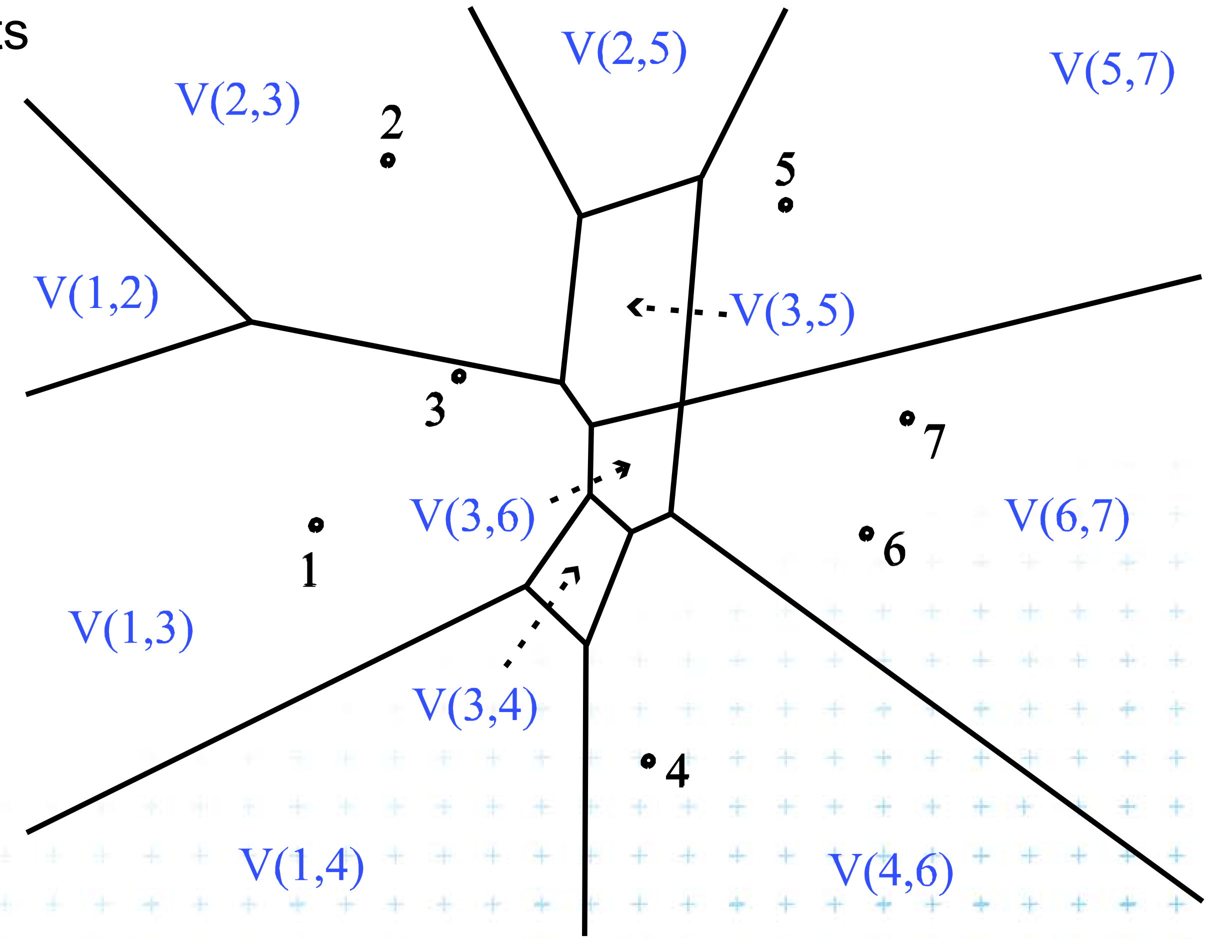
- Create Voronoi diagram of line segments,
  take it as a graph
- Project Q_{start} to P_{start} *on VD and* Q_{end} to P_{end}
- Remove segments with distance to sites smaller than radius *r* of a robot
- Depth first search if path from P_{start} to P_{end} *exists*
- Report path $Q_{start}\, P_{start}$…*path*… P_{end} to Q_{end}

- $O(n \log n)$ time using $O(n)$ storage

Order-2 Voronoi diagram

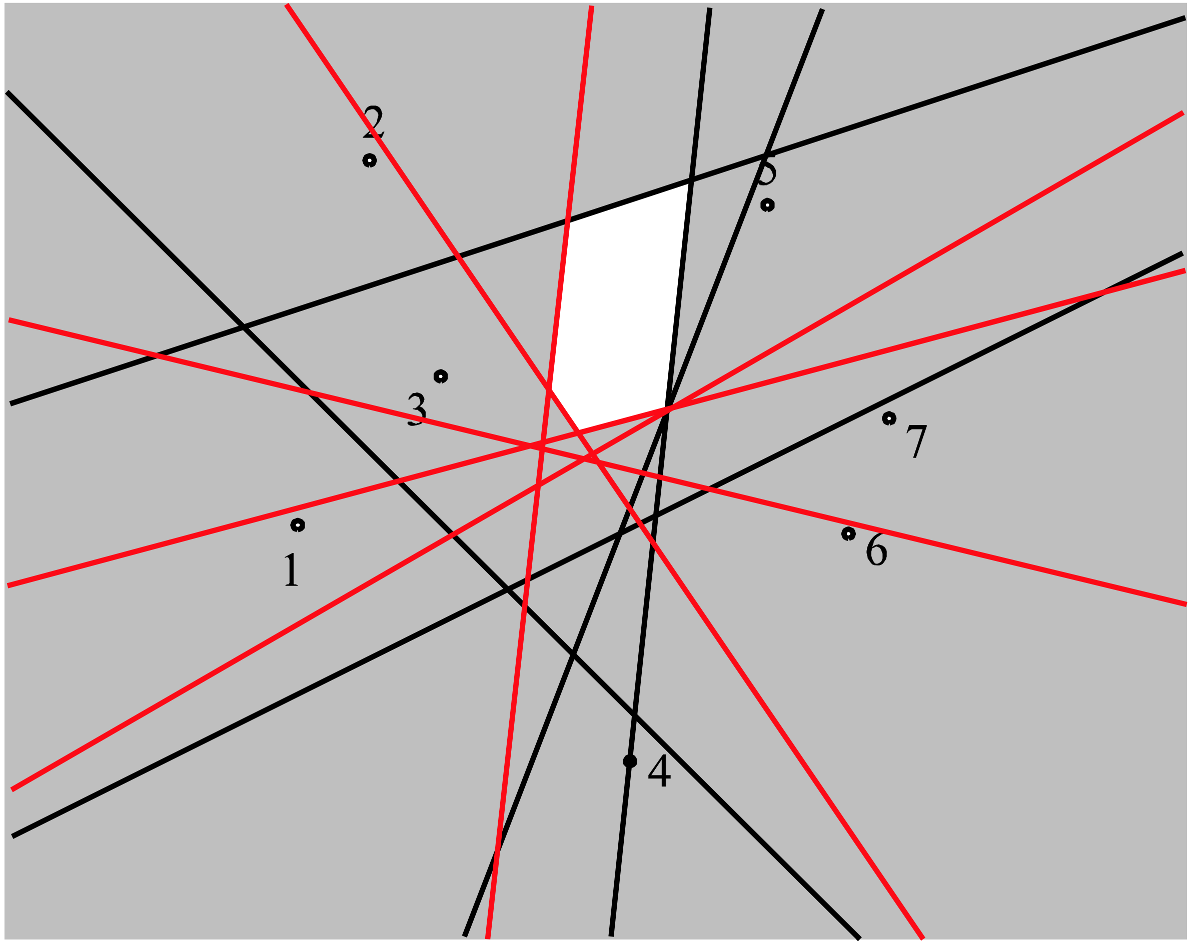
$V(p_i,p_j)$: the set of points of the plane closer to each of p_i and p_j than to any other site

Property
The order-2 Voronoi regions are convex



[Nandy]

Construction of V(3,5)



[Nandy]

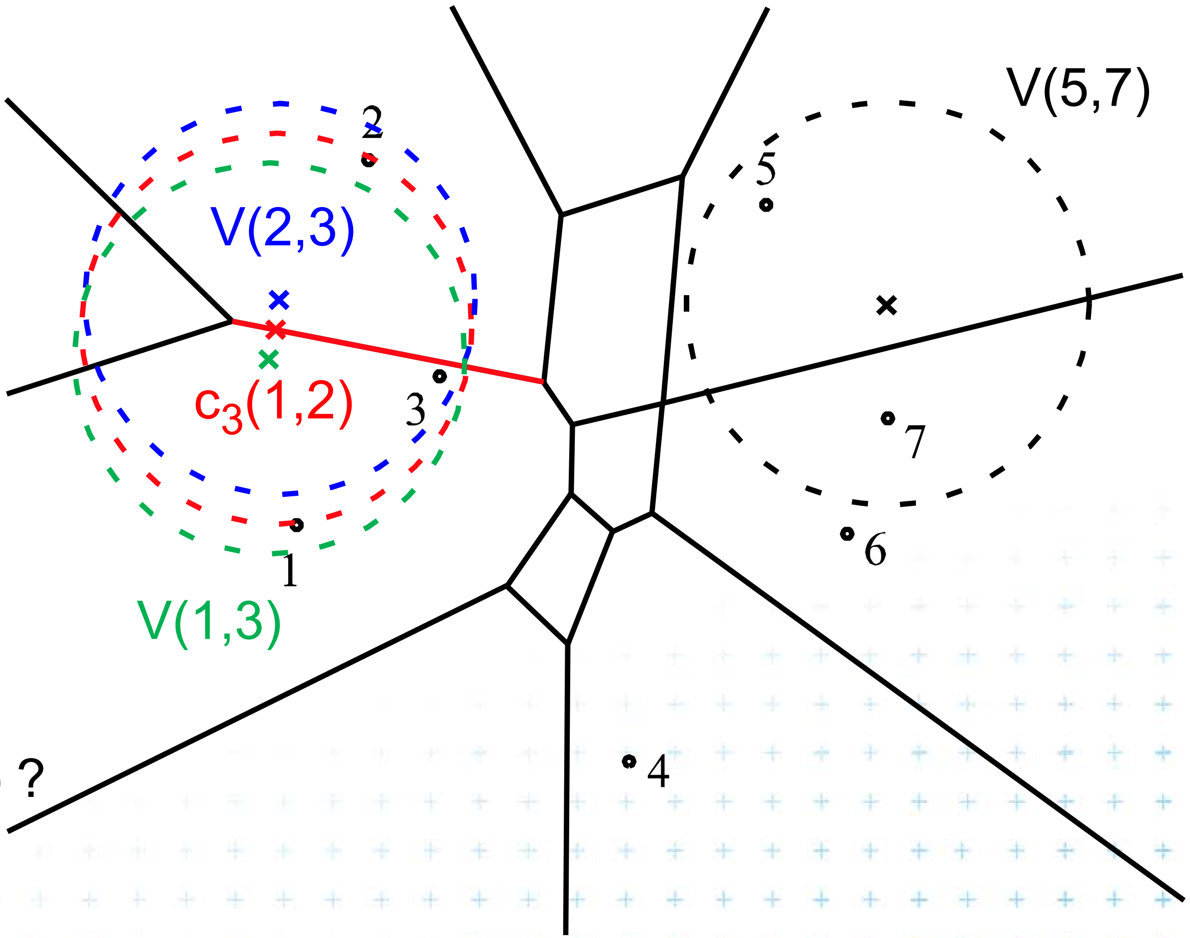
Intersection of all halfplanes except H(3,5) and H(5,3)

$$\underbrace{\bigcap_{x \neq 5} h(3,x)} \cap \underbrace{\bigcap_{x \neq 3} h(5,x)}$$

Order-2 Voronoi edges

edge : set of centers of circles passing through 2 sites s and t and containing 1 site p

=> $c_p(s,t)$



Question

Which are the regions on both sides of $c_p(s,t)$?

=> V(p,s) and V(p,t)

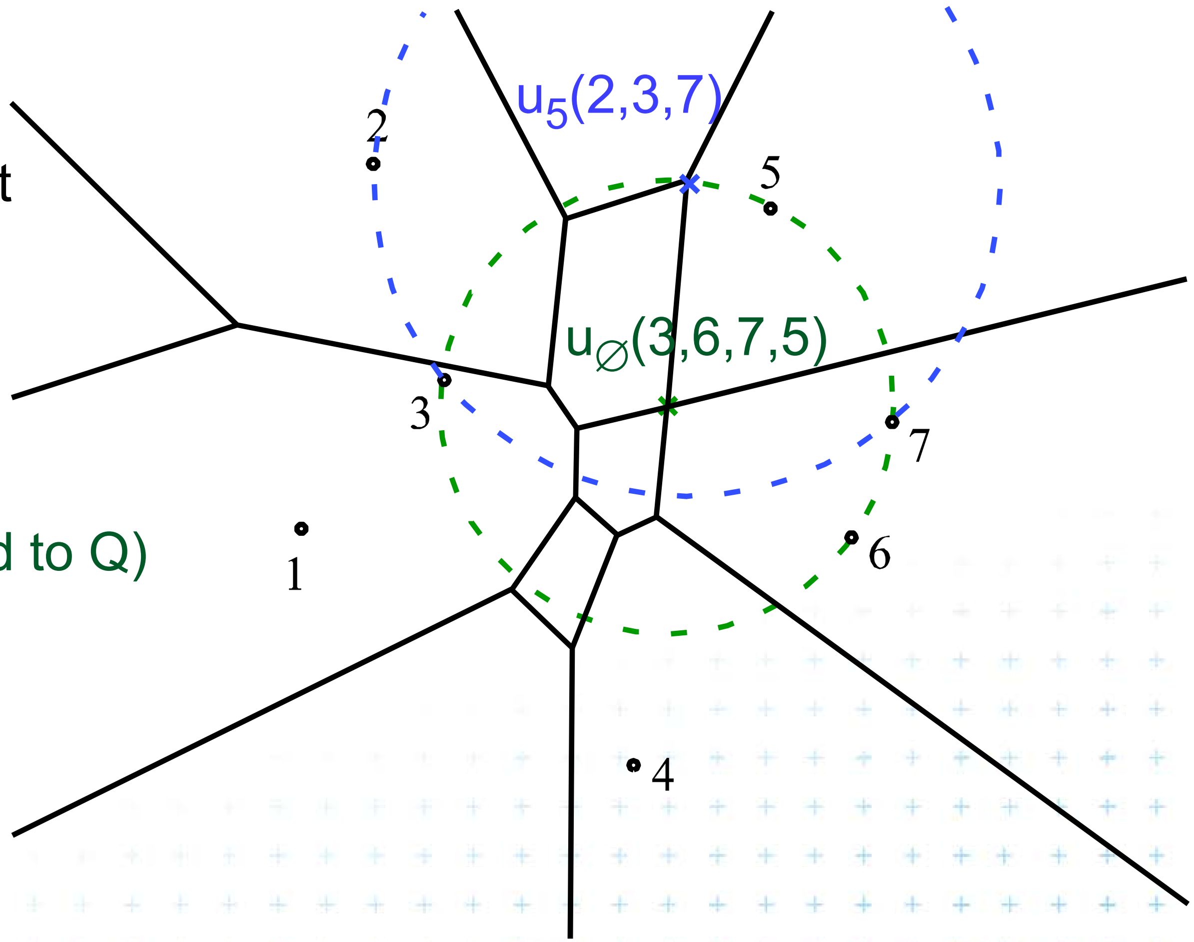
[Nandy]

Order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites and containing either 1 or 0 site

=> $u_p(Q)$ or $u_\varnothing(Q \cup p)$

(circle circumscribed to Q)



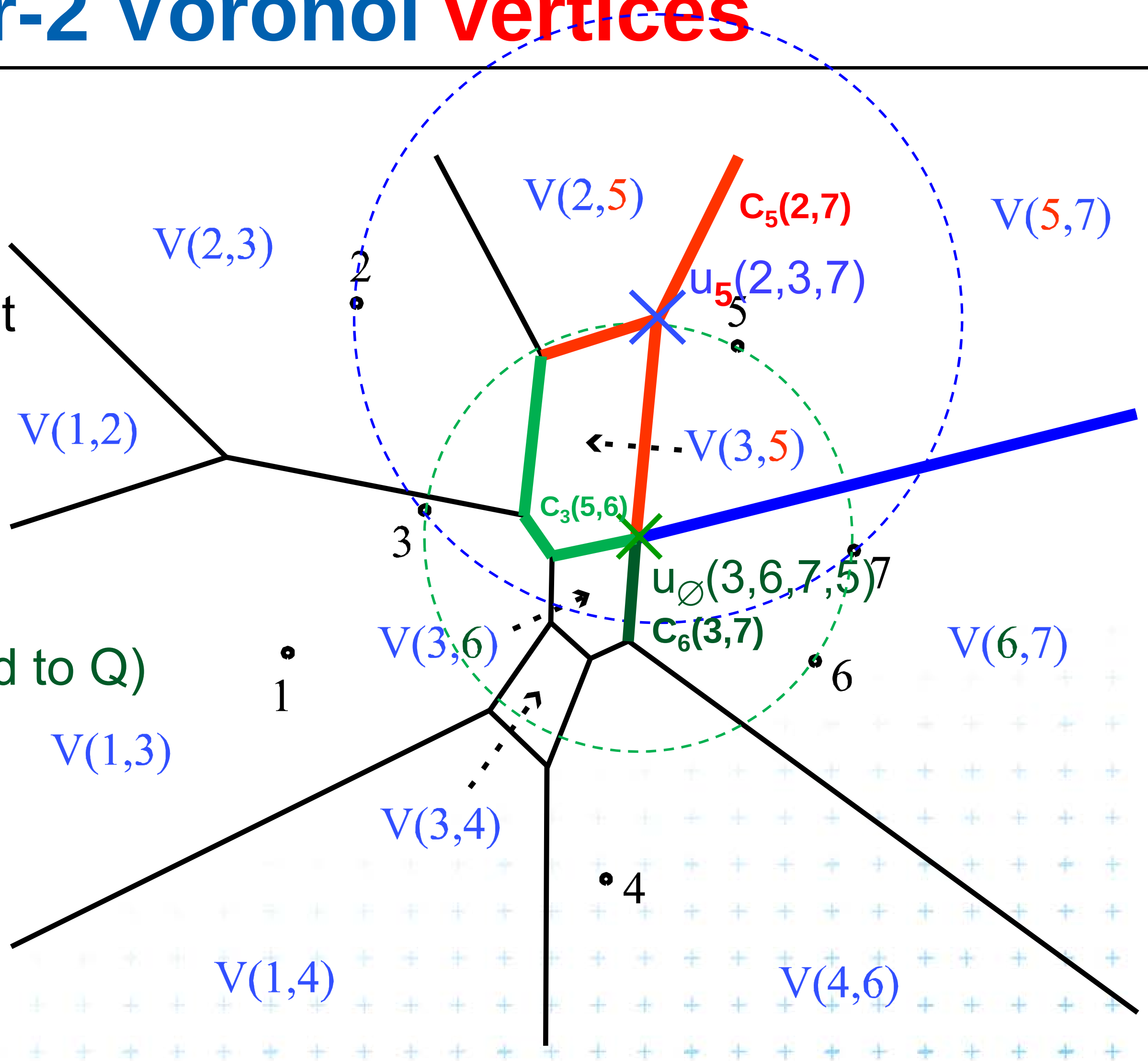
[Nandy]

Types of order-2 Voronoi vertices

vertex : center of a circle passing through at least 3 sites and containing either 1 or 0 site

=> $u_p(Q)$ or $u_\varnothing(Q \cup p)$

(circle circumscribed to Q)



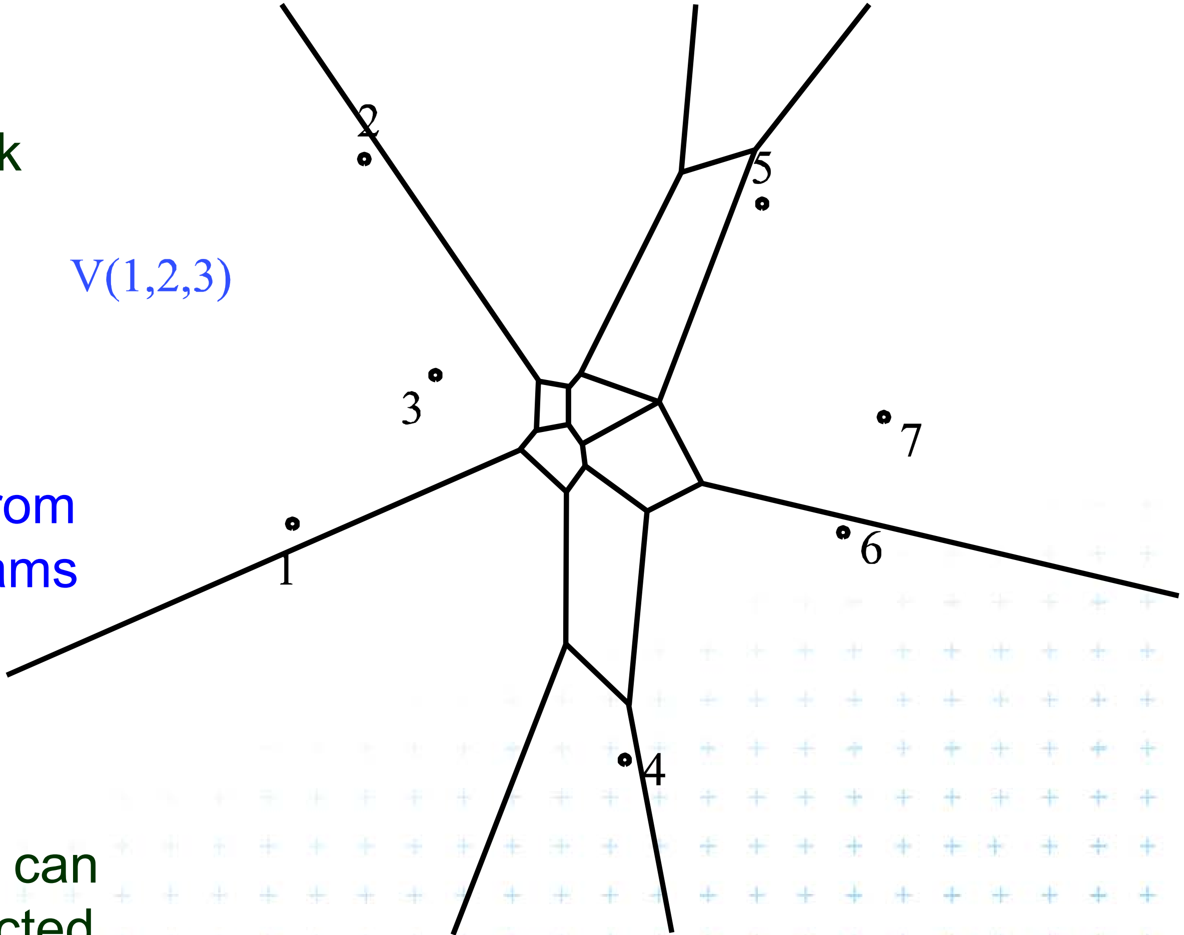
[Nandy]

Order-k Voronoi Diagram

Theorem
The size of the order-k diagrams is $O(k(n-k))$

Theorem
The order-k diagrams can be constructed from the order-(k-1) diagrams in $O(k(n-k))$ time

Corollary
The order-k diagrams can be iteratively constructed in $O(n \log n + k^2(n-k))$ time

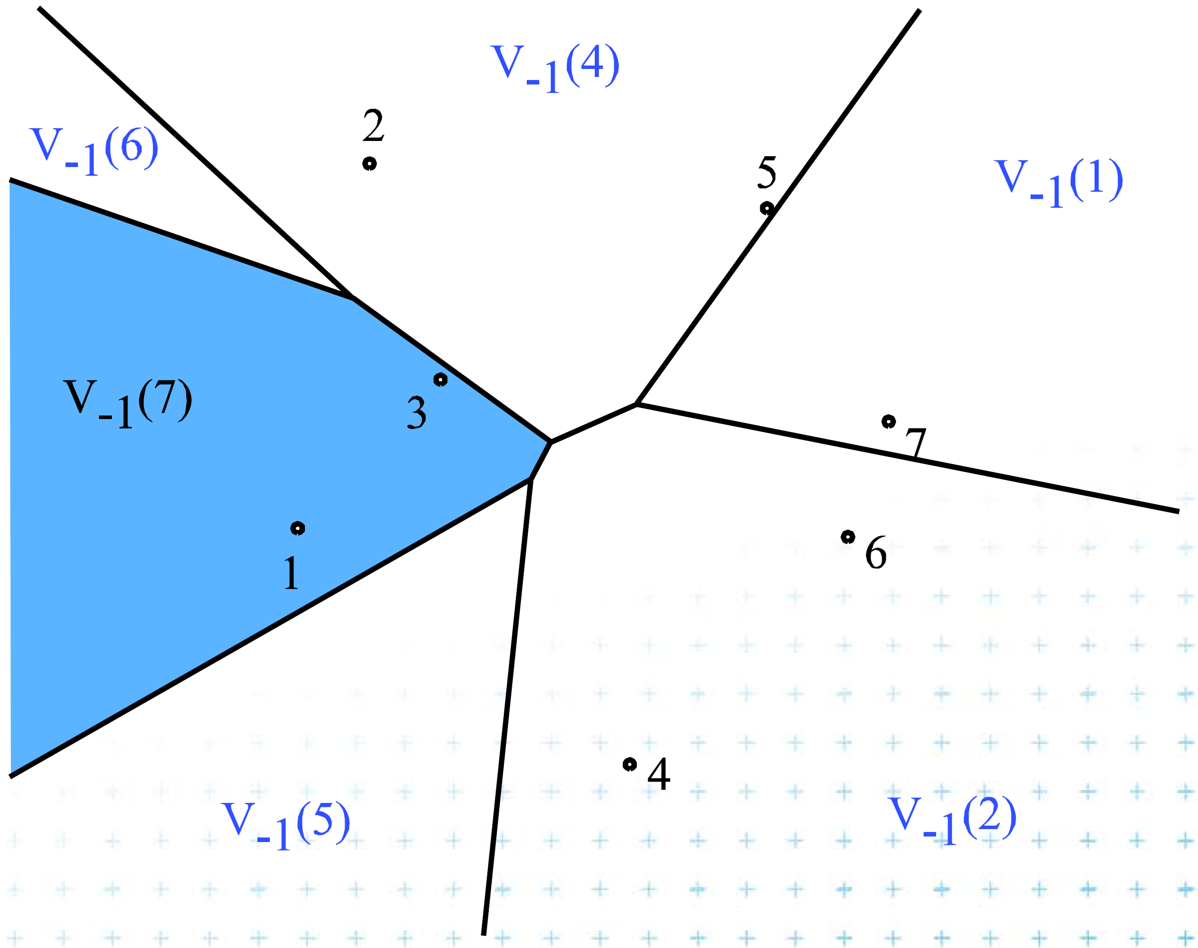


[Nandy]

Order n-1 = Farthest-point Voronoi diagram

cell $V_{-1}(7) = V_{n-1}(\{1,2,3,4,5,6\})$
= set of points in the plane farther from $p_i=7$ than from any other site

$Vor_{-1}(P) = Vor_{n-1}(P)$
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



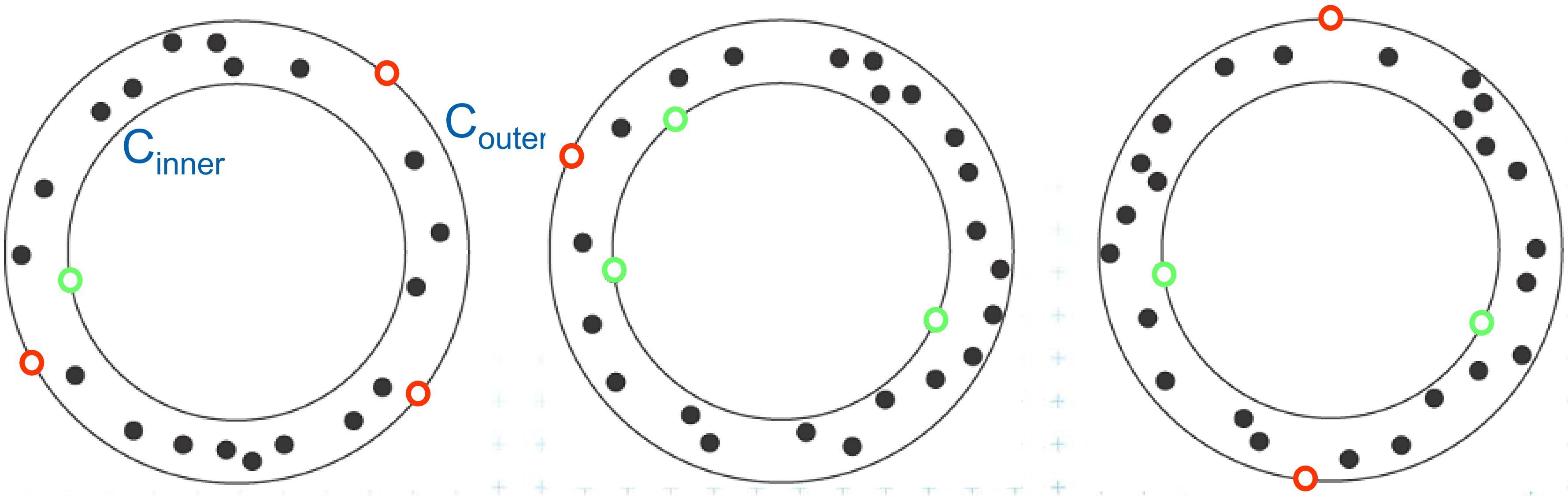
[Nandy]

Farthest-point Voronoi diagrams example

Roundness of manufactured objects

- Input: set of measured points in 2D
- Output: width of the smallest-width annulus
  (region between two concentric circles C_{inner} and C_{outer})

Three cases to test – one will win:



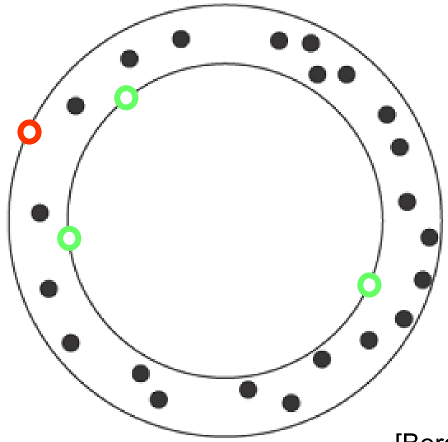
a) 1 point in – 3 out b) 3 in – 1 out c) 2 in – 2 out

[Berg]

Smallest width annulus – cases with 3 pts

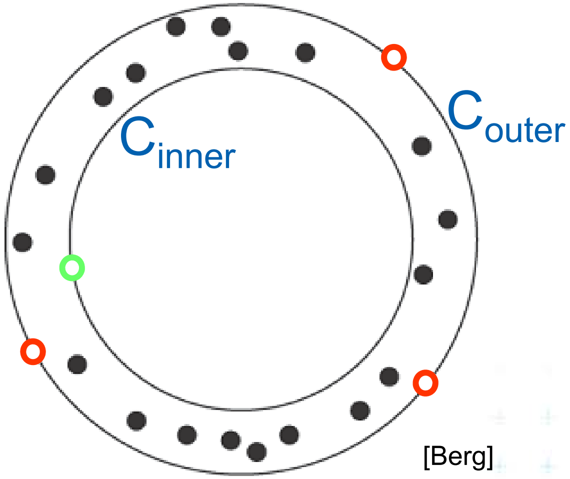
b) C_{inner} contains at least 3 points

- Center is the *vertex of normal Voronoi diagram* (1st order VD)
- The remaining point on C_{outer} in O(n) for each vertex => not the largest (inscribed) empty circle - Seminar [13] as we must test all vertices in combination with point on C outer



[Berg]

3 in – 1 out



[Berg]

1 point in – 3 out

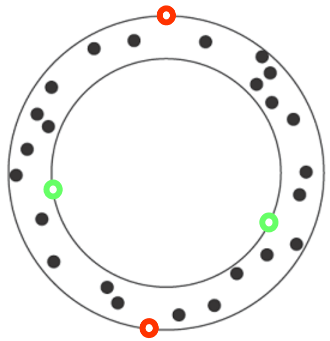
a) C_{outer} contains at least 3 points

- Center is the *vertex of the farthest Voronoi diagram*
- The remaining point on C_{inner} in O(n) => not the smallest enclosing circle - Seminar [12] as we must test all vertices **in combination** with point on C inner

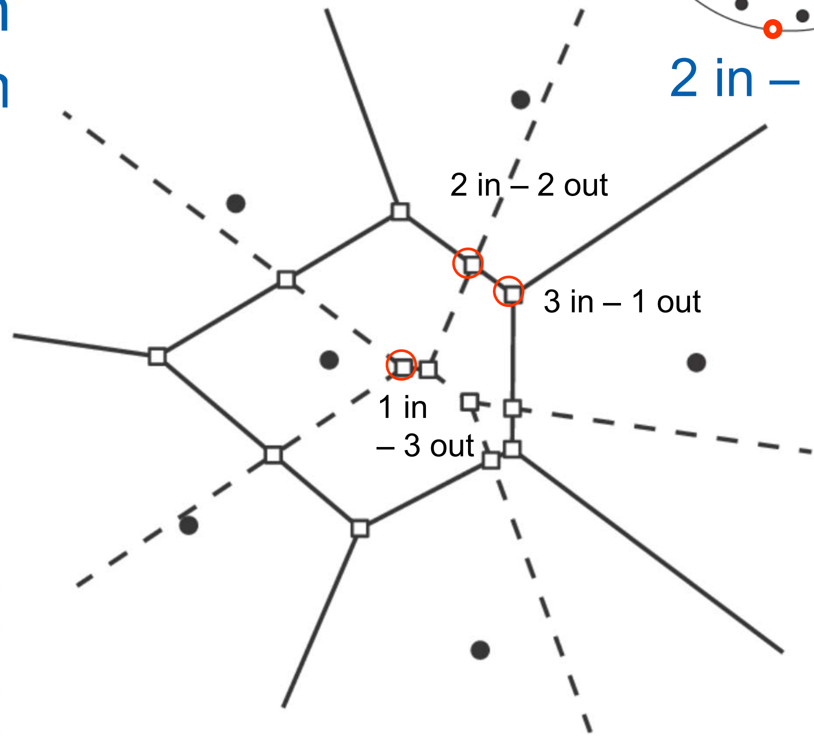
Smallest width annulus – case with 2+2 pts

c) C_{inner} and C_{outer} contain 2 points each

- Generate vertices of overlay of Voronoi and farthest-point Voronoi diagrams
  => $O(n^2)$ candidates for cen
  (we need vertices, not th
  whole overlay)
- annulus computed in O(1) from center and 4 points (same for all 3 cases)



2 in – 2 out



[Berg]

Smallest width annulus

Smallest-Width-Annulus
Input: Set P of n points in the plane
Output: Smallest width annulus center and radii r and R (roundness)

1. Compute Voronoi diagram Vor(P) and farthest-point Voronoi diagram $\mathrm{Vor}_{-1}(P)$ of P
2. For each vertex of $\mathrm{Vor}_{-1}(P)$ (R) determine the *closest point* (r) from P => $O(n)$ sets of four points defining candidate annuli
3. For each vertex of Vor(P) (r) determine the *farthest point* (R) from P => $O(n)$ sets of four points defining candidate annuli
4. For every pair of edges Vor(P) and $\mathrm{Vor}_{-1}(P)$ test if they intersect => another set of four points defining candidate annulus
5. For all candidates of all three types chose the smallest-width annulus

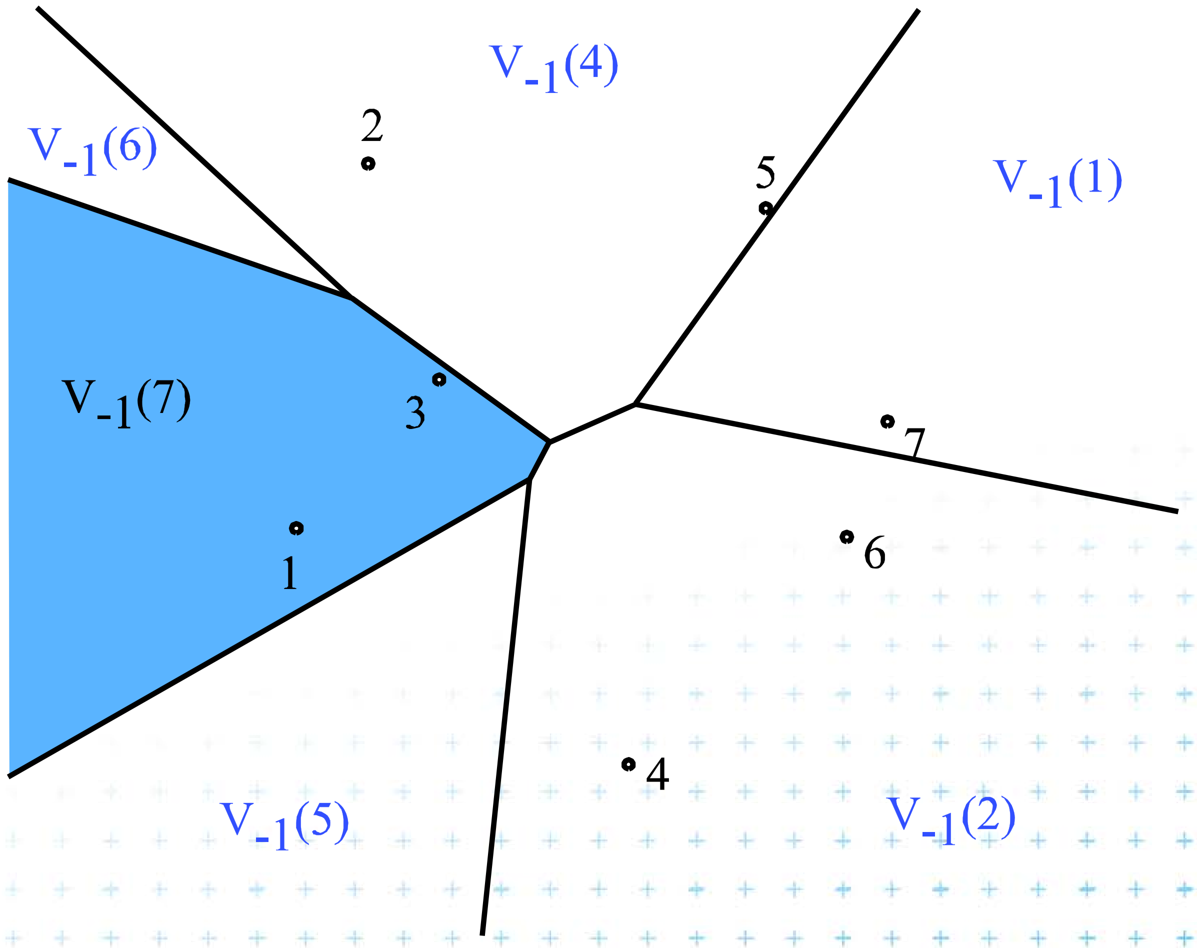
1. $O(n \log n)$
2. $O(n^2)$
3. $O(n^2)$
4. $O(n^2)$
5. $O(n^2)$

$O(n^2)$ time using $O(n)$ storage

Farthest-point Voronoi diagram

$V_{-1}(p_i)$ cell
= set of points in the plane farther from p_i than from any other site

$Vor_{-1}(P)$ diagram
= partition of the plane formed by the farthest point Voronoi regions, their edges, and vertices



[Nandy]

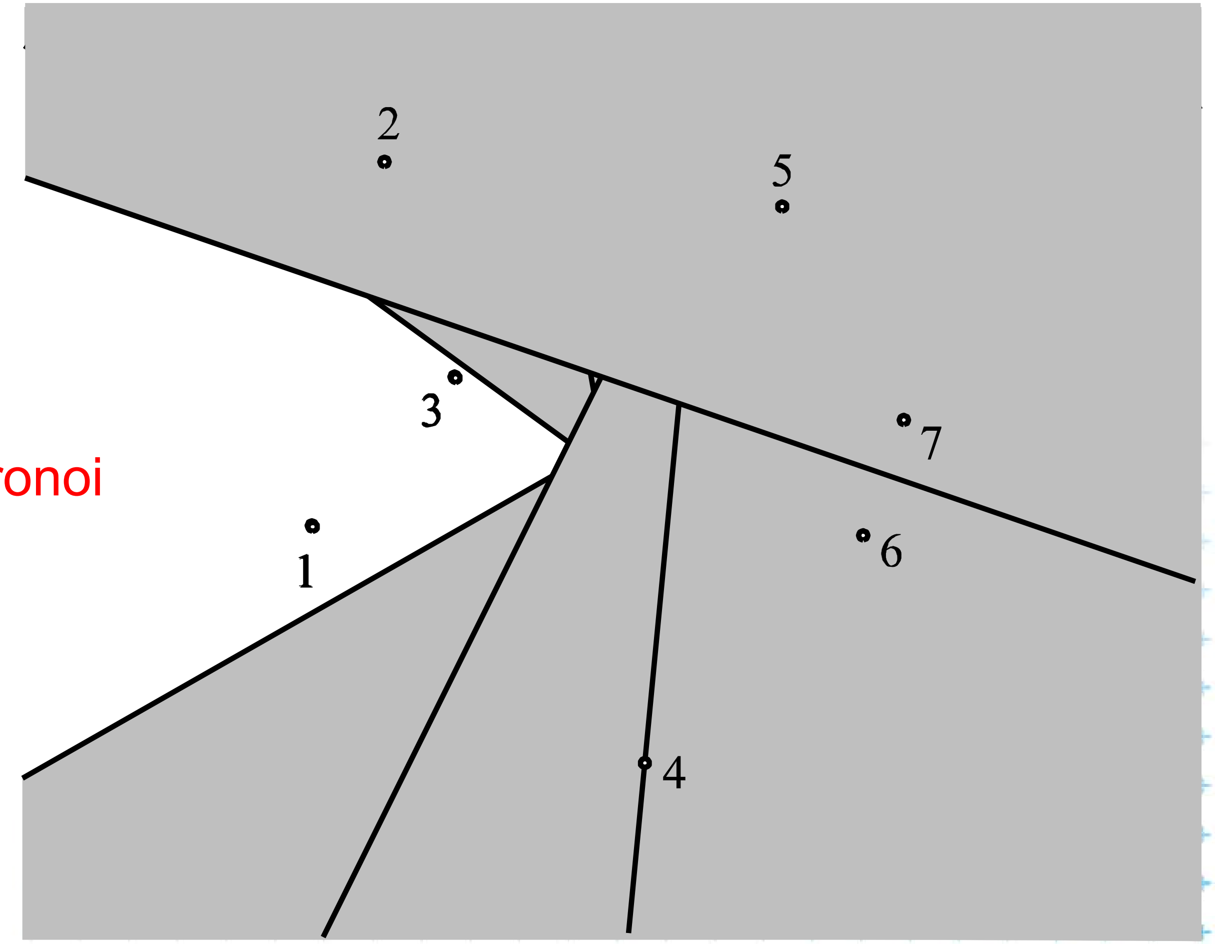
Farthest-point Voronoi region (cell)

Computed as intersection of halfplanes, but we take "other sides" of bisectors

Construction of $V_{-1}(7)$

Property

The farthest point Voronoi regions are convex and unbounded

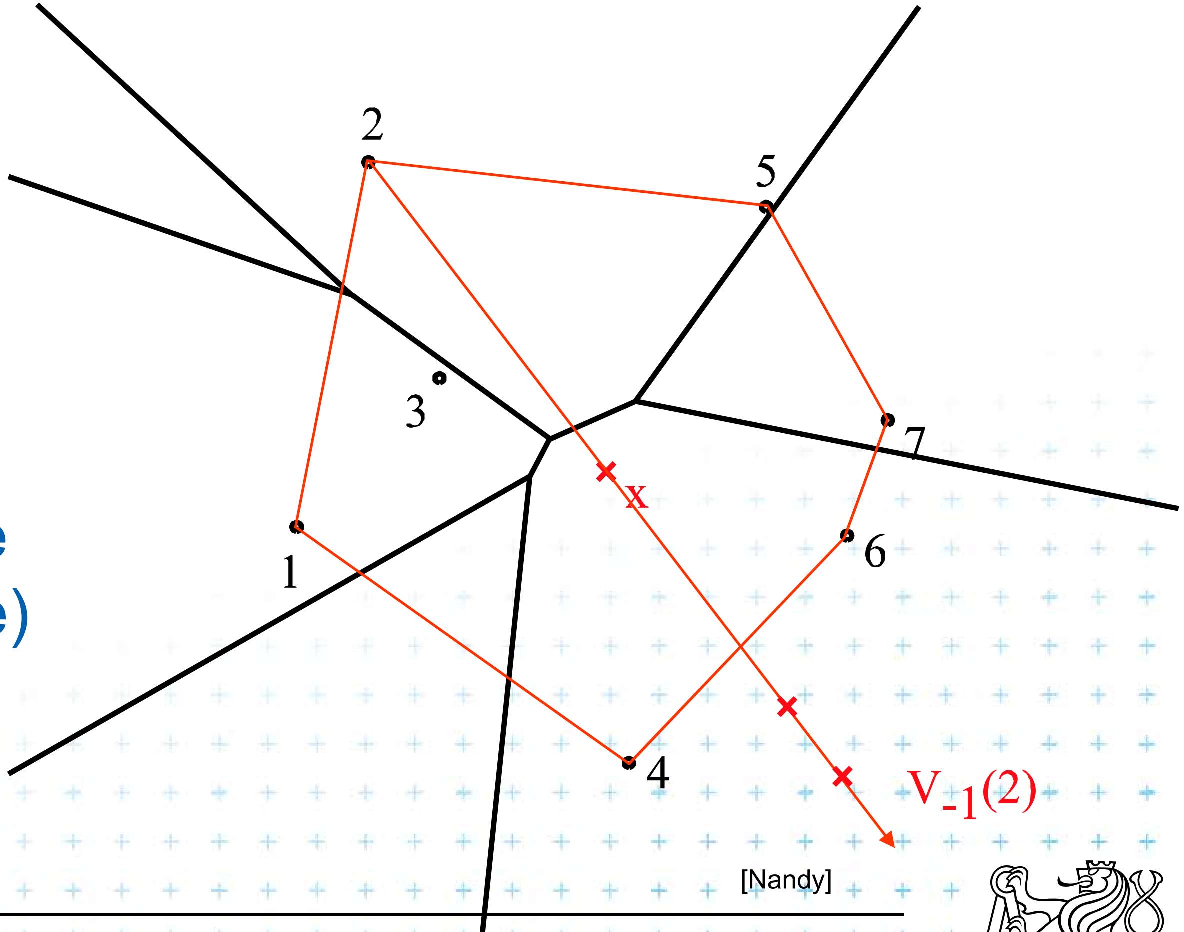


[Nandy]

Farthest-point Voronoi region

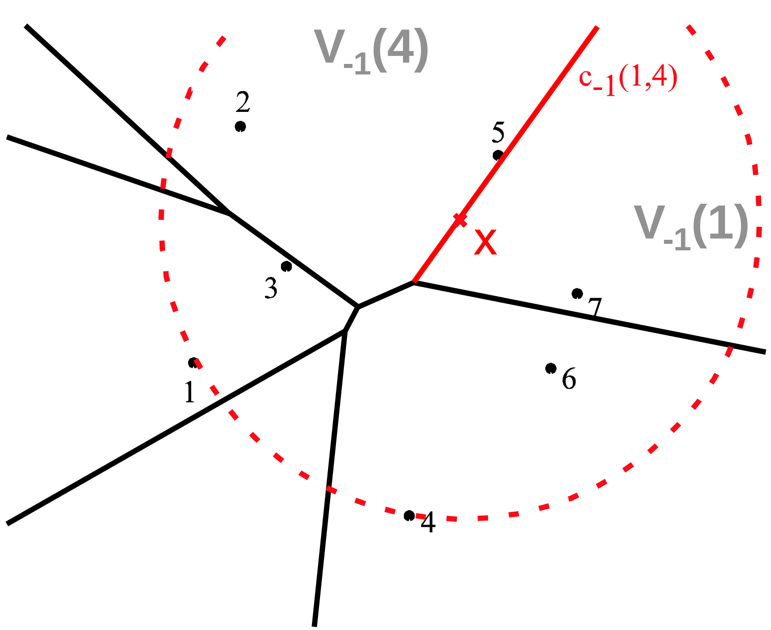
Properties:

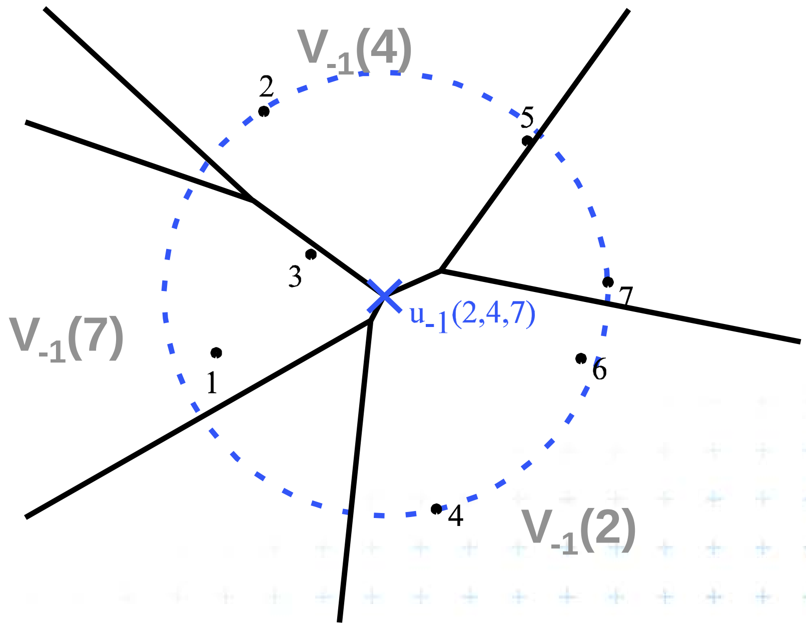
- Only vertices of the convex hull have their cells in farthest Voronoi diagram
- The farthest point Voronoi regions are unbounded
- The farthest point Voronoi edges and vertices form a tree (in the graph sense)



[Nandy]

Farthest point Voronoi edges and vertices





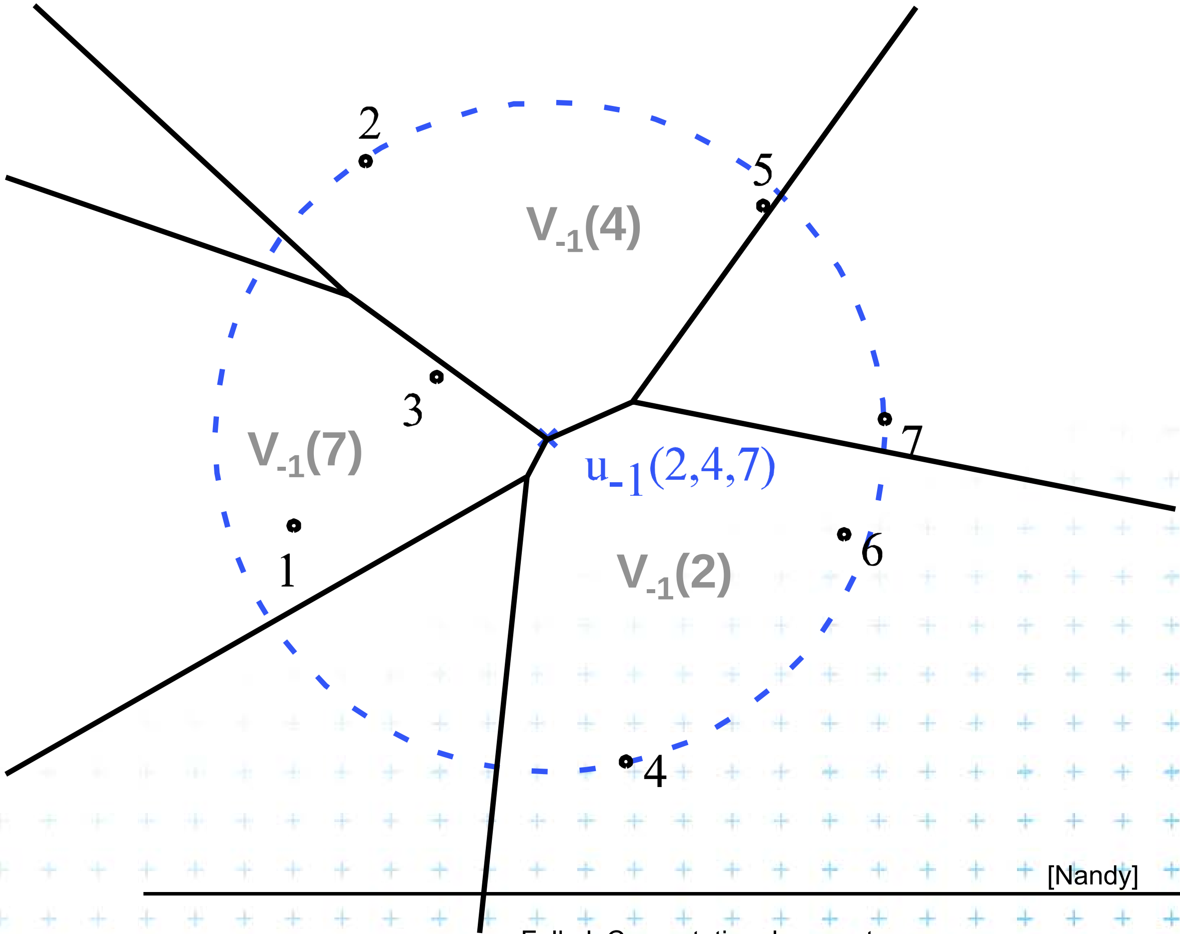
edge : set of points equidistant from 2 sites and closer to all the other sites

vertex : point equidistant from at least 3 sites and closer to all the other sites

[Nandy]

Application of $Vor_{-1}(P)$: Smallest enclosing circle

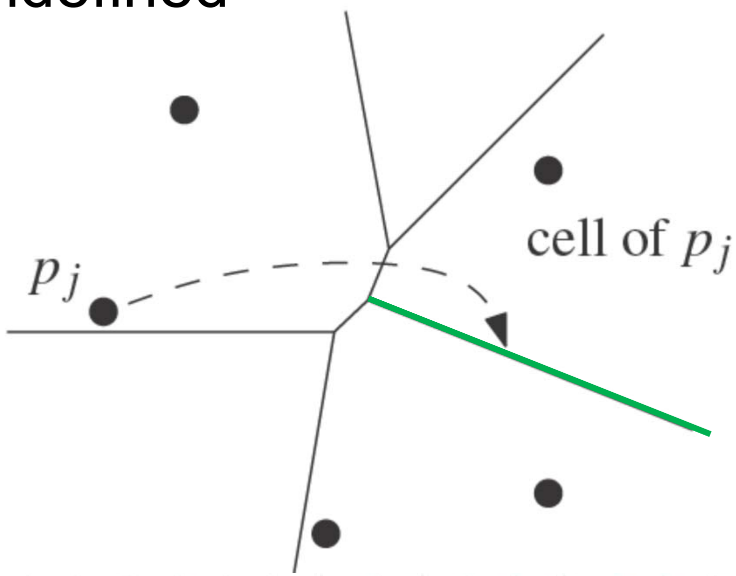
- Construct $Vor_{-1}(P)$ and find minimal circle with center in $Vor_{-1}(P)$ vertices or on edges



[Nandy]

Modified DCEL for farthest-point Voronoi d

- Half-infinite edges -> we adapt DCEL
- Half-edges with origin in infinity
  - Special vertex-like record for origin in infinity
  - Store direction instead of coordinates
  - Next(e) or Prev(e) pointers undefined
- For each inserted site p_j
  - store a pointer to the most CCW half-infinite half-edge of its cell in DCEL



Farthest-point Voronoi d. construction

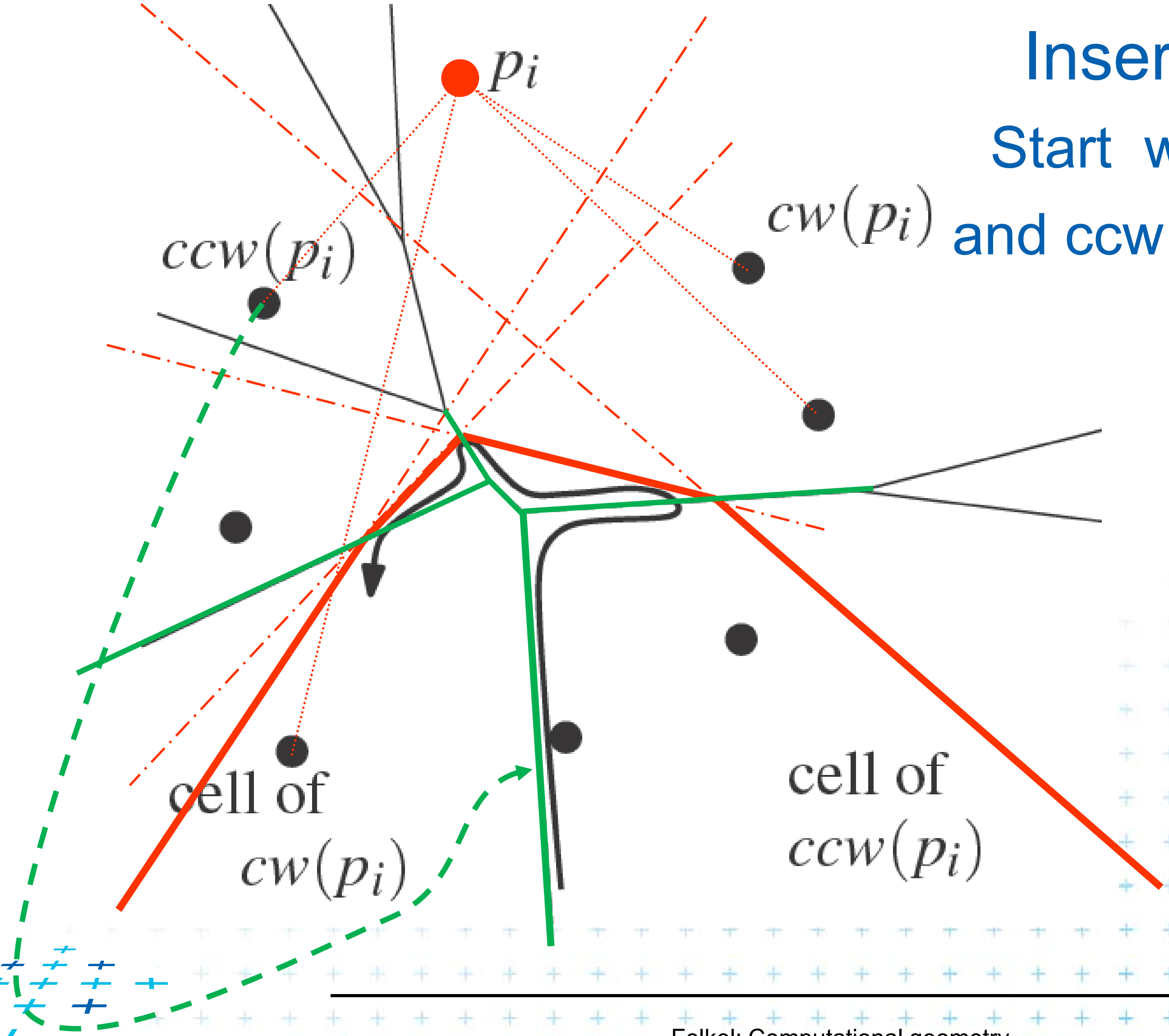
Farthest-pointVoronoi O(nlog n) time in O(n) storage

Input: Set of points P in plane

Output: Farthest-point VD $Vor_{-1}(P)$

1. Compute convex hull of P
2. Put points in CH(P) of P in random order $p_1,\ldots,p_h$
3. Remove $p_h, \ldots, p_4$ from the cyclic order (around the CH). When removing p_i, store the neighbors: $cw(p_i)$ and $ccw(p_i)$ at the time of removal. (This is done to know the neighbors needed in step 6.)
4. Compute $Vor_{-1}(\{ p_1, p_2, p_3 \})$ as init
5. **for** i = 4 **to** h **do**
6. Add site p_i to $Vor_{-1}(\{ p_1, p_2,\ldots, p_{i-1} \})$ between site $cw(p_i)$ and $ccw(p_i)$
7. - start at most CCW edge of the cell $ccw(p_i)$
8. - continue CW to find intersection with bisector($ccw(p_i)$, p_i)
9. - trace borders of Voronoi cell p_i in CCW order, add edges
10. - remove invalid edges inside of Voronoi cell p_i

Farthest-point Voronoi d. construction

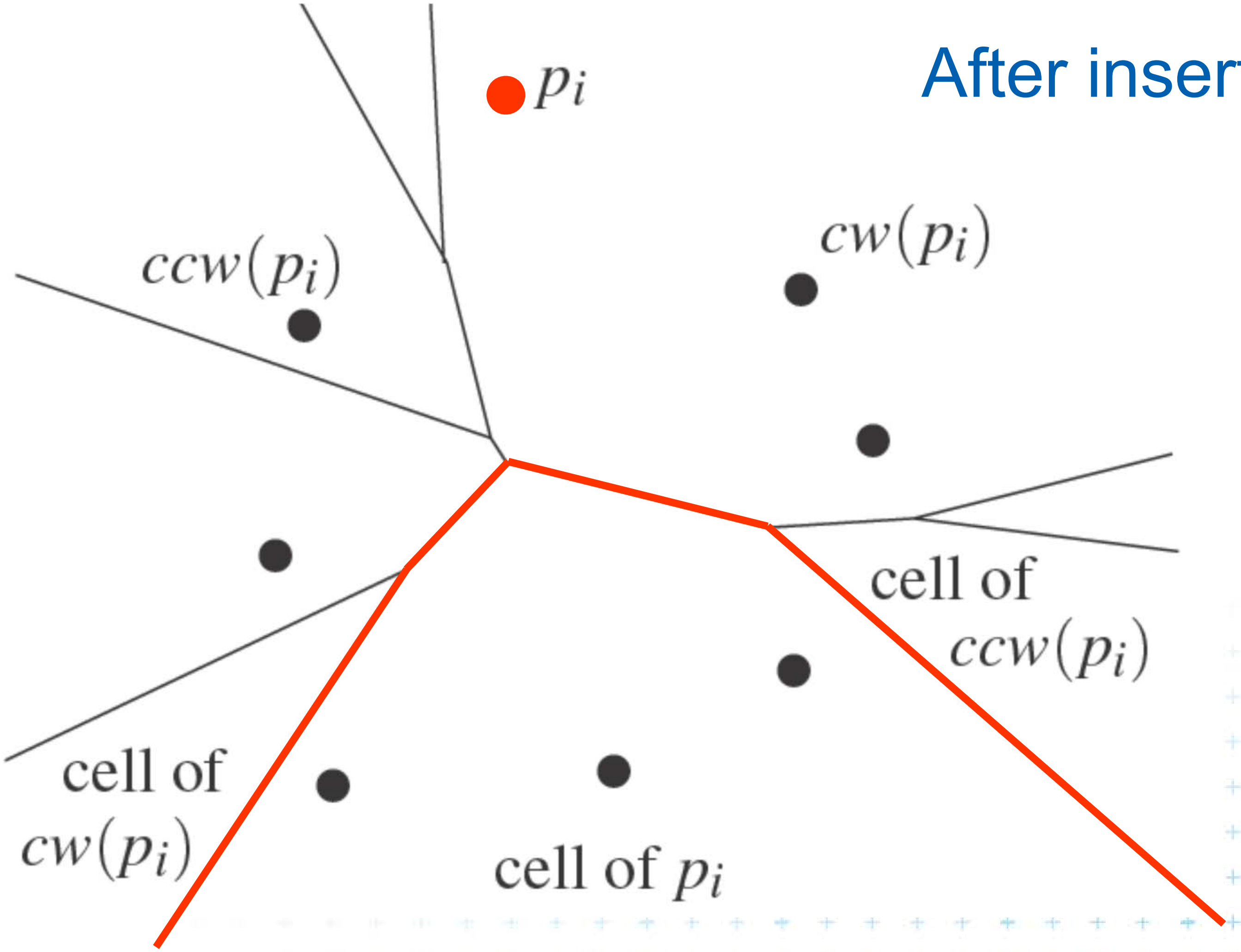


Insertion of site p_i

Start with site ccw(p_i) and ccw edge of its cell

Farthest-point Voronoi d. construction

After insertion of site p_i



References

[Berg] Mark de Berg, Otfried Cheong, Marc van Kreveld, Mark Overmars: Computational Geometry: *Algorithms and Applications*, Springer-Verlag, 3rd rev. ed. 2008. 386 pages, 370 fig. ISBN: 978-3-540-77973-5, Chapter 7, http://www.cs.uu.nl/geobook/

[Preparata] Preperata, F.P., Shamos, M.I.: *Computational Geometry. An Introduction.* Berlin, Springer-Verlag,1985. Chapters 5 and 6

[Reiberg] Reiberg, J: Implementierung Geometrischer Algoritmen. Berechnung von Voronoi Diagrammen fuer Liniensegmente. http://www.reiberg.net/project/voronoi/avortrag.ps.gz

[Nandy] Subhas C. Nandy: Voronoi Diagram – presentation. Advanced Computing and Microelectronics Unit. Indian Statistical Institute. Kolkata 700108 http://www.tcs.tifr.res.in/~igga/lectureslides/vor-July-08-2009.ppt

[CGAL] http://www.cgal.org/Manual/3.1/doc_html/cgal_manual/Segment_Voronoi_diagram_2/Chapter_main.html

[applets] http://www.personal.kent.edu/~rmuhamma/Compgeometry/MyCG/Voronoi/Fortune/fortune.htm a http://www.liefke.com/hartmut/cis677/